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NUMERICAL SOLUTION OF P-SV WAVES WITH FREE SURFACE BOUNDARY CONDITIONS BY NETWORK METHOD

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Abstract. A numerical model based on network simulation method is presented for the numerical solution of P-SV waves in elastic media subjected to mixed boundary conditions: free surface and fixed displacements. As excitation, the standard form of Ricker pulse is used. After a detailed explanation of the model, it is run in suitable circuit simulation software. Boundary conditions are implemented by simple electrical components contained in the library of the software. An application is presented in order to show the efficiency of the proposed method. Tabulates result are processed by Matlab in order to present pictures of displacement for times along the transitory; displacements for sections of the medium, as a function of time, are also presented.

1 INTRODUCTION

Despite the electric analogy widely treated in many books for educational proposes (as an alternative representation of a problem in the context of a general analogy between equations), particularly in heat transfer problems [1-2], it goes far beyond the scope of this subject and can be used as a real numerical tool, making good use of the powerful mathematical algorithms implemented in the circuit simulation codes. Recently, it has been applied to the solution of well known problems in engineering: flow and transport problems [3], tribology [4], chemical corrosion [5], magnetohydrodinamic flows [1], inverse problems [6], heat transfer [6] and others; processes whose mathematical model is formed by a set of coupled, non-lineal partial differential equations. Despite the disadvantage of having to be familiar with the basic principles of electric circuit theory, in all these problems, the method has demonstrated itself to be an accurate and reliable tool for many researchers.

In this context we apply this method to obtain the solution of the P-SV wave equation: the emergence and propagation of elastic waves in a plane domain from the point source.

The design of the model starts from the finite-difference differential equation resulting from the spatial discretization of the mathematical model, the 2-D wave equation [7], retaining time as a continuous variable. The terms of this equation are implemented as electrical currents by using a special kind of device, named 'controlled source'. These currents, in turn, are balanced according to their signs at a common node, forcing to find the solution for the dependent variable. The controlled sources are capable of implementing any kind of nonlinearities or couplings between variables thanks to the possibility of specify their outputs by software, as a function of currents and voltages at other elements or nodes of the network, while lineal terms are directly implemented by simple devices (resistors, capacitors and coils). As regards derivative or higher order terms, auxiliary networks are required [8]. To complete the model, boundary conditions are also implemented be suitable components whose constitutive equation fits these requirements.

Once designed, the network is run in a standard code of circuit simulation such as PSpice [9]. The widespread use of this code and its new versions ORCAD attests to the applicability of the program to a large variety of circuit simulation problems and provides a valuable base of experience that demonstrates the advantages of the powerful, efficient and reliable numerical algorithms that are implemented therein. Instantaneous pictures of the perturbation along the domain are depicted by processing the tabulated output data with MATLAB.

To show the application of the proposed method we describe a scenario of 2-D, P-SV waves, in seismic problems subjected to a standard Ricker pulse [7], as excitation, applied to the center of the boundary side. Mixed boundary conditions are imposed: free surface at the upper side of the domain and fix displacement at the other sides.

2 MATHEMATICAL MODEL

The governing equations of P-SV waves, in absence of body forces, are [7]:

$$\rho\frac{\partial^2 u_x}{\partial t^2}=\left(\lambda+2\mu\right)\frac{\partial^2 u_x}{\partial x^2}+\mu\frac{\partial^2 u_x}{\partial z^2}+\left(\lambda+\mu\right)\frac{\partial^2 u_z}{\partial x\partial z} \tag{1a}$$

$$\rho\frac{\partial^2 u_z}{\partial t^2}=\mu\frac{\partial^2 u_z}{\partial x^2}+\left(\lambda+2\mu\right)\frac{\partial^2 u_z}{\partial z^2}+\left(\lambda+\mu\right)\frac{\partial^2 u_x}{\partial x\partial z} \tag{1b}$$

being u_x and u_z the displacement components, λ the Lamé's constant, μ the shear modulus and ρ the density. The relation between these parameters and elastic constants, E (Young's modulus) and v (Poisson's ratio), is:

$$\mu=\frac{E}{2\left(1+\nu\right)},\qquad \lambda=\frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)} \tag{2}$$

For the completed mathematic model, the mixed boundary conditions are:

$$u_i=u_i^b \qquad \text{on } S_u \tag{3a}$$

$$\sigma_{ij}n_j=t_i^b \qquad \text{on } S_t \tag{3b}$$

where S_u denotes boundary surface points where the displacement values are prescribed, while S_t refers to points where the traction values are also given. Note that $S = S_t + S_u$ represents the complete boundary surface, Figure 1.

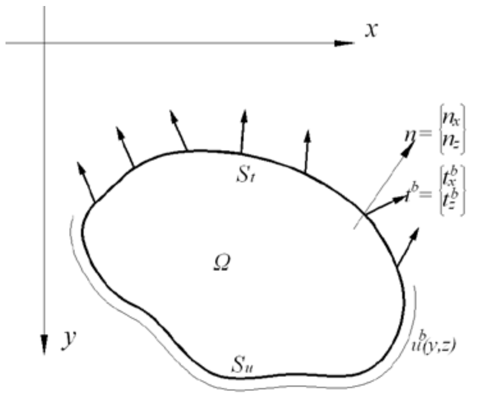


Figure 1: Mixed boundary conditions in a 2D domaine

For the P-SV waves described by Equation (1), the characteristic velocity are $\alpha = \left[(\lambda + 2\mu)/\rho\right]^{0.5}$ and $\beta = (\mu/\rho)^{0.5}$ for the P and S waves, respectively.

To apply the conditions (3b), in terms of the displacements, it is necessary to consider the Hooke´s law in term of displacements $\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$. For 2D problems, the relation reduces to:

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \tag{4a}$$

$$\sigma_{zz} = \lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} \tag{4b}$$

$$\sigma_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{4c}$$

Finally, tractions conditions are:

$$t_x^b = \left[(\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z}\right]n_x + \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)n_z \tag{5a}$$

$$t_z^b = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)n_x + \left[\lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z}\right]n_z \tag{5b}$$

As the excitation point source of the seismic waves, the classical Ricker pulse is considered, Figure 2. The amplitude of the Ricker's pulse is defined by the function:

$$A(t) = \left(a^2 - \frac{1}{2}\right)e^{-a^2} \tag{6}$$

with $a = \pi(t - t_s)/t_p$, being t_s y t_p the delay and width times, respectively.

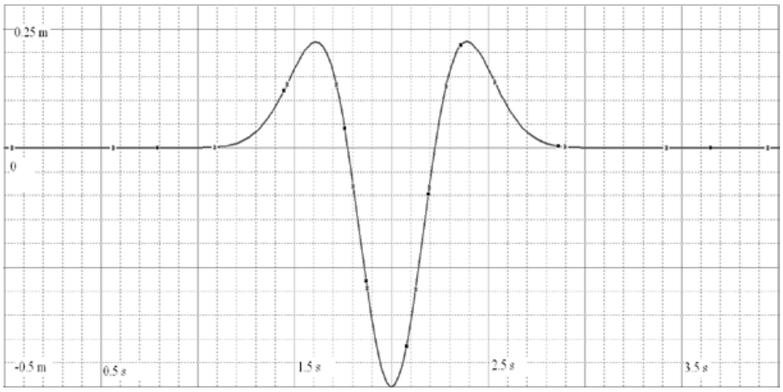


Figure 2: Ricker pulse for t_s = 2 s and t_p = 1 s

3 NETWORK MODEL

Following the rules of the Network Method, for a suitable electrical analogy, the first step is to establish the analogy between the displacements of the elastic problem and the electric voltages of the network model. After that, partial differential of governing equation (1), as well as boundary conditions (5), are written in finite-differences form discretizating the spatial variables. Times remains as continuous variable, as in the lines method. For example, using the nomenclature of Figure 3, the second partial derivative for one of the displacement components reduces to:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{k,0} \simeq \frac{\frac{u_{k,2}-u_{k,0}}{\Delta x/2}-\frac{u_{k,0}-u_{k,4}}{\Delta x/2}}{\Delta x} = \frac{u_{k,2}-2u_{k,0}+u_{k,4}}{\Delta x^2/2} = -\left(\frac{u_{k,0}-u_{k,2}}{\Delta x^2/2}+\frac{u_{k,0}-u_{k,4}}{\Delta x^2/2}\right) \qquad (7)$$

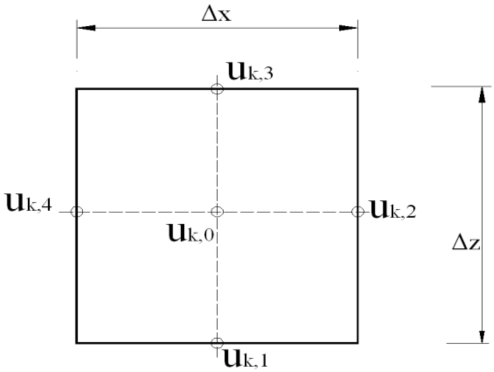


Figure 3: Nomenclature of a volume element

The resulting network model for each volume element that implements the governing equation is shown in Figure 4. Note that there are two similar circuits for each displacement components u_x and u_y, respectively.

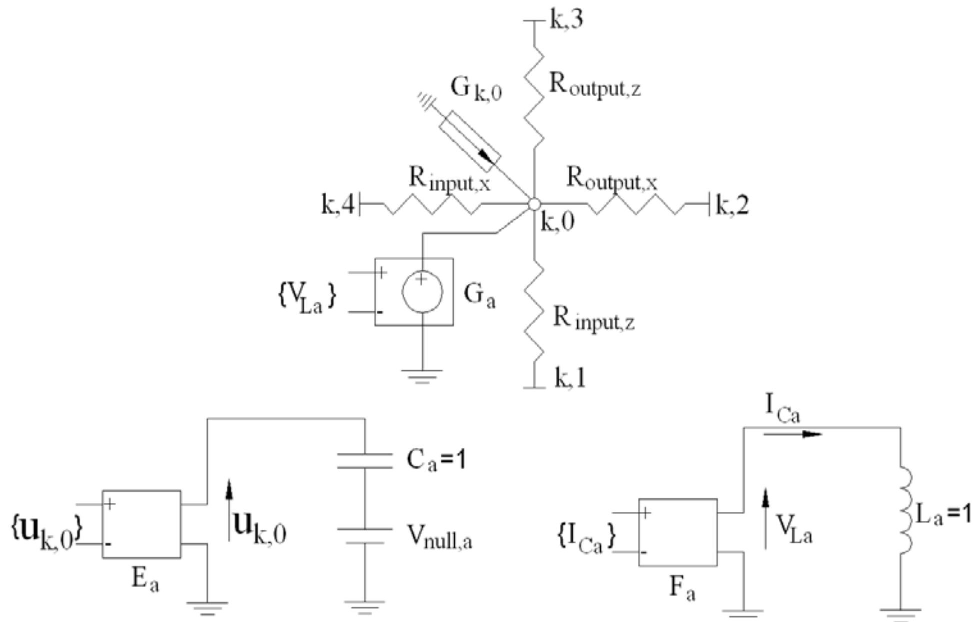


Figure 4: Network model for a volume element related with one of the displacement components of the P-SV wave equation

The network model for displacement boundary condition is a simple voltage source with the value of the imposed displacement, Figure 5(a).

For traction boundary conditions, controlled voltage source (defined by software) must be used. These devices are capable to implement the coupled relations between both displacement components. For example, Equation (5) applied to the free boundary condition, $t_x^b = t_z^b = 0$, at the upper side of a domain reduces to:

$$\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)=0 \qquad (8a)$$

$$\lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)+2\mu\frac{\partial u_z}{\partial z}=0 \qquad (8b)$$

Using the same nomenclature, the value of the controlled voltage source is:

$$\mathrm{ux}_{k,3}=\mathrm{ux}_{k,0}-\frac{\Delta z}{4\Delta x}\left(\mathrm{uz}_{kr,3}-\mathrm{uz}_{kl,3}\right) \qquad (9a)$$

$$\mathrm{uz}_{k,3}=\mathrm{uz}_{k,0}-\lambda\frac{\Delta z/2}{\lambda+2\mu}\cdot\frac{\mathrm{ux}_{kr,1}-\mathrm{ux}_{kl,1}}{2\Delta x} \qquad (9b)$$

For the circuit corresponding to the *x* displacement component, Equation (9a) defines the value of the controlled voltage that implements the horizontal component of the free boundary condition, Figure 5(b). The other part of the condition is defined by the Equation (9b) and implemented by a similar component in the circuit related with the *z* displacement component.

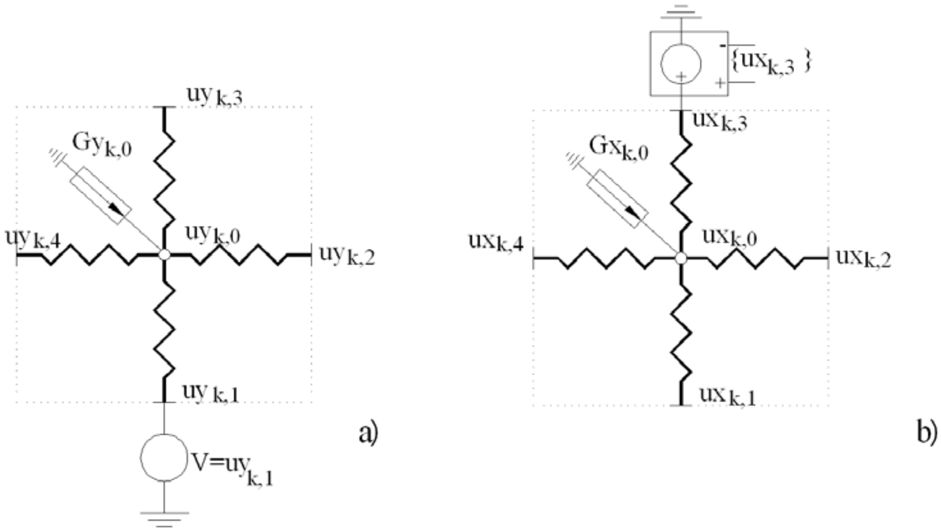


Figure 5: Network model of displacement (a) and traction (b) boundary conditions

The network model for the source of the seismic perturbation, that reproduces the standard Ricker pulse, is implemented by a generator with output signal obeying the Equation (6). In order to release the domain to excitation source, an electrical switch is imposing between the Ricker pulse generator and the node where it is applied. The network model for this condition is shown in Figure 6. The switch opens when pulse is finished, releasing the node allowing to move freely.

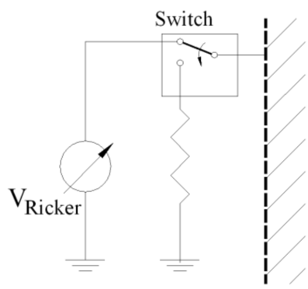


Figure 6: Network model for the implementation of the standard Ricker pulse

The connection between N_xxN_z cells, by ideal electrical contacts, for the *x* component of the displacement (Figure 4), and N_xxN_z cells for the other *z* component, defines the whole network of the physical domain to which mixed boundary conditions and Ricker pulse must be added, Figures 5 and 6.

Since very few components are required in most cases, including boundary conditions, very few rules are required for the design of the complete network model, which is run without other mathematical manipulations in a suitable code such as PSpice. The powerful computational algorithms implemented in such codes makes that the simulation reproduces the exact solution of the model so that the errors are only due to the chosen grid size.

4 THE CODE PSPICE

Once the complete network model is designed, its simulation is carried out using a standard circuit simulation analysis code such as PSpice [9] without any other mathematical requirements. This provides the exact solution of the model. PSpice was initially developed by the Integrated Circuit group of the Electronic Research Laboratory at the University of California, and the first version was finished in 1992. Since that time, thousands of copies of the new version of the program have been given to universities and companies in the electronic industry. The widespread use of PSpice and its new versions attests to the applicability of the program to a large variety of circuit simulation problems and provide a valuable base of experience that demonstrated the advantages and disadvantages of the powerful, efficient and reliable numerical algorithms that are employed in the program.

The system of equations that describes the complete circuit is determined by the model equations for each element and topological constraint, which are determined by interconnecting the elements, reflecting Kirchhoff's Current and Voltages Laws (Kirchhoff's Current law is a conservation principle while Voltages Kirchhoff's law ensures the uniqueness of the related variable).

For transient analysis, PSpice maintain an internal time-step, which is continuously adjusted to maintain accuracy, while not performing unnecessary steps. The time-step is reduced by the code during the simulation so that the integrated charges and currents are sufficiently accurate. During periods of inactivity, the internal time-step is increased, while during activity periods, it is decreased. The maximum internal step size can be controlled by using software to specify it. The minimum time-step is the overall run time divided by 1E12, while the initial time-step is fixed as a function of both the total time required for the simulation and the value of the relative accuracy, a data (parameter) also specified by the user. If convergence is not reached, the initial time-step is successively reduced in the simulation until the solution converges.

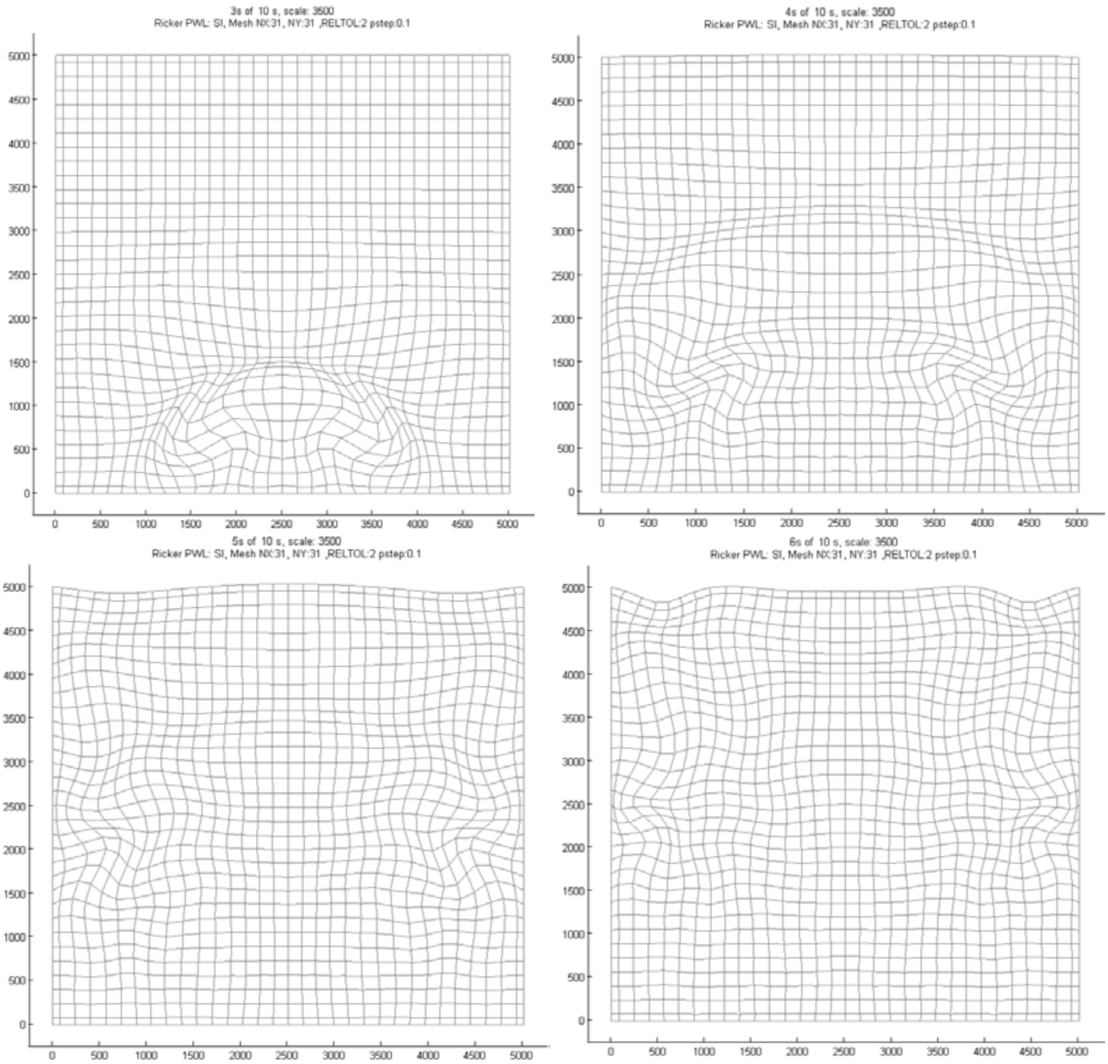
The circuit equations are a system of algebraic-differential equations of the form $F(\mathbf{x}, \mathbf{x}', t)=0$, where $\mathbf{x}$ is the vector of the unknown circuit variables, $\mathbf{x}'$ is the time derivative of x and F is, in general, a nonlinear operator. The equations formulation algorithm based on a combination of Cutset and Loop Analysis is a modified version of the classical Nodal Analysis. This method provides the same generality as other formulation methods, but produces a near-symmetric system of equations that are solved with an amount of computational effort that is comparable to that needed for the simplest Nodal Methods. The Markowitz algorithm is used for reordering the system of equations. Once the equations $F(\mathbf{x},\mathbf{x}',t)=0$ are reordered, direct eliminations methods, such as L>U factorization and sparse-matrix, are used to obtain the solution.

If the circuit contains elements that are modeled by nonlinear equations, then the solution is obtained by an iterative sequence of linearized solutions. The modified Newton-Raphson algorithm, which approximates each non-linearity by a Taylor series That is truncated after the first order term, is the most common method of linearization. The modified Newton-Raphson method contains the reliable and efficient simple-limiting of Colon (algorithm) that requires the lowest number of iterations for convergence.

For the implicit transient analysis method of numerical integration, stiffly-stable algorithms are used, such as Trapezoidal integration. The local truncation error of the integration is proportional to the time-step, which is successively reduced to make the error negligible.

5 APPLICATION

A square domain has been used to simulate P-SV seismic waves subjected to mixed boundary conditions. The length of the medium is $L_x = L_z = 5000$ m. Physical properties are: density $\rho = 1000$ kg/m^3, Poisson's ratio $\nu = 0.25$ and shear modulus $\mu = 1.33$ GPa. The corresponding propagations velocity of the P and S waves are $\alpha = 2000$ m/s and $\beta = 1155$ m/s, respectively. For the upper side, free boundary condition is prescribed. Zero displacements are imposed at the rest of the boundary. The perturbation Ricker pulse, Equation (6), is applied in the middle of de bottom side of the domain using $t_p = 1$ s and $t_s = 2$ s values. The chosen grid is 31x31 and the simulation time window 10 s. Figures 7 to 9 show the results using Matlab.



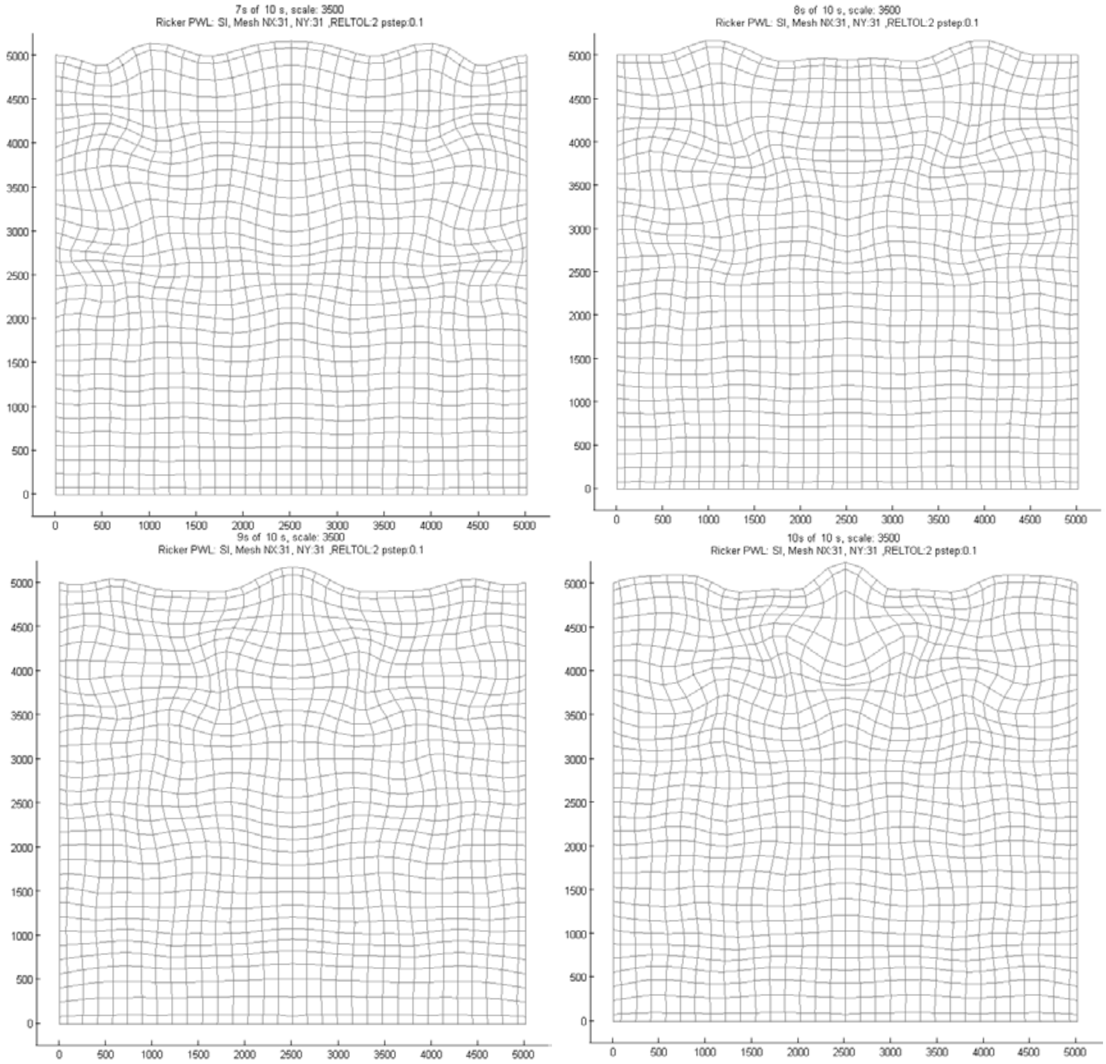


Figure 7: Deformed domain at t = 3, 4 , 5, 6, 7, 8, 9 and 10 s.

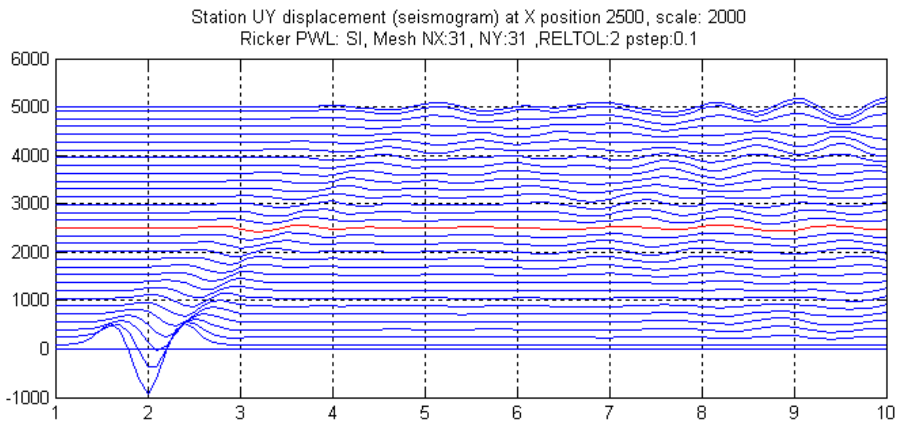


Figure 8: Vertical displacements at x=2500 m

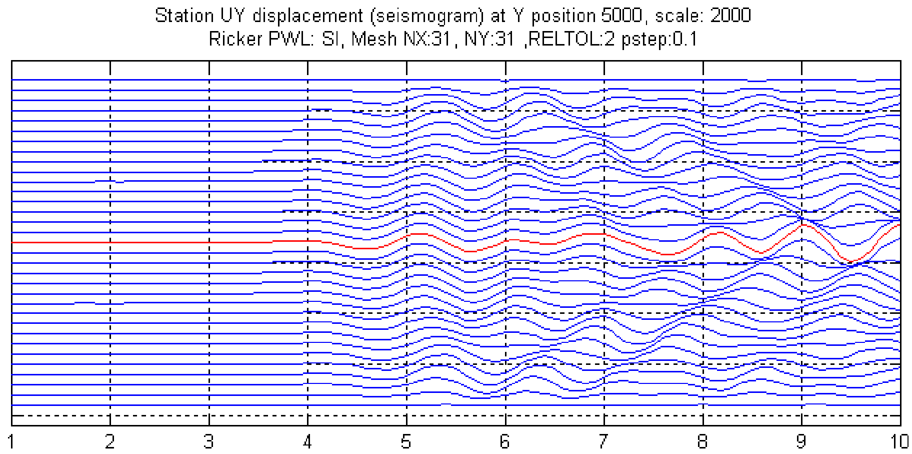


Figure 9: Vertical displacements at y=5000 m

6 CONCLUSIONS

- Application of the network method to solving the P-SV wave equation has been demonstrated to be an efficient tool for the numerical simulation of this problem. The design of the network as well the implementation of the mixed boundary conditions, whatever they are, has been made in an easy way thanks to: i) the rules needed for the design are very few and ii) the existence of the especial controlled sources defined in the libraries of the circuit simulation codes; these sources are capable of implementing by software any kind of non-linearity and coupled condition involved in the problem.
- No mathematical manipulation other than the spatial discretization of the partial differential equation is required since this work is made by the software PSpice and the data treatment routines.
- The application of the free boundary condition is also direct by using the above mentioned controlled source that is suitable for solving the coupling between the two equations.

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