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SPATIAL AND TEMPORAL INSTABILITY OF SLIGHTLY CURVED PARTICLE-LADEN SHALLOW MIXING LAYERS

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Abstract. In the present paper we present linear and weakly nonlinear models for the analysis of stability of particle-laden slightly curved shallow mixing layers. The corresponding linear stability problem is solved using spatial stability analysis. Growth rates of the most unstable mode are calculated for different values of the parameters of the problem. The accuracy of Gaster's transformation away from the marginal stability curve is analyzed. Two weakly nonlinear methods are suggested in order to analyze the development of instability analytically above the threshold. One method uses parallel flow assumption. If a bed-friction number is slightly smaller than the critical value then it is shown that the evolution of the most unstable mode is governed by the complex Ginzburg-Landau equation. The second method assumes that the base flow is slightly changing downstream. Applying the WKB method we derive the first-order amplitude evolution equation for the amplitude.

1 INTRODUCTION

Shallow mixing layers are widespread in nature. Typical examples include flows in compound and composite channels and flows at river junctions. Three widely used methods of analysis of shallow flows include experimental investigation, numerical modelling and stability analysis [1]. Experimental analyses in [2]-[5] showed that (a) bottom friction suppresses the growth of perturbations and (b) shallow mixing layer grows at a smaller rate than a free shear layer. Several papers are devoted to linear stability analysis of shallow mixing layers [6]-[9]. It is shown in [8] that rigid-lid assumption can be used for stability analysis of shallow flows for small Froude numbers. The effect of Froude number of stability characteristics of shallow mixing layers is analysed in [9]. Theoretical calculations in [6]-[9] support experimental observations: bed friction stabilizes the flow and reduces the growth of a mixing layer width.

The effect of small curvature on the stability characteristics of a free mixing layer is analysed in [10] where it is shown that curvature has a destabilizing effect on an unstably curved mixing layer and stabilizing effect on a stably curved mixing layer. Linear stability of two-phase flows where a fluid contains solid particles is investigated in [11] under some simplifying assumptions. It is shown in [11] that a particle loading parameter has a stabilizing influence on the flow

In the present paper linear stability analysis of slightly curved shallow mixing layers for the case where a fluid contains solid particles is performed. Two basic methods are usually used in practice for linear stability analysis: (a) spatial stability analysis and (b) temporal stability analysis. The second approach is more convenient from a computational point of view since the corresponding generalized eigenvalue problem is linear and can easily be solved by standard software packages. The first approach is more convenient for the purpose of comparison with experiments but requires more computational efforts since the problem is nonlinear with respect to unknown eigenvalues. Well-known Gaster's transformation [12] is often used to simplify stability calculations. However, Gaster's transformation is valid only in the vicinity of a marginal stability curve where the growth rates are small. In other regions of the parameter space the difference between spatial and temporal growth rates can be quite large as is illustrated in the paper.

Linear stability gives only conditions of instability but it cannot describe the evolution of the unstable mode above the threshold. Weakly nonlinear theories are used in such cases in order to analyse the development of instability analytically. Two such methods are briefly described in the paper. The first approach is based on a parallel flow assumption and can be applied for the case where the bed-friction number is slightly smaller than the critical value (see, for example, [13]). Using the method of multiple scales an amplitude evolution equation for the most unstable mode is derived. It is shown that for particle-laden slightly curved shallow mixing layers the amplitude equation is the complex Ginzburg-Landau equation. The coefficients of the equation are calculated explicitly in terms of integrals containing linear stability characteristics of the flow.

The second approach takes into account slow longitudinal variation of the base flow. The analysis is based on weakly non parallel WKBJ approximation [14]. A first-order amplitude evolution equation is derived. The solution of the amplitude equation is then used to obtain the first-order approximation to the perturbation field.

2 LINEAR STABILITY ANALYSIS

The two-dimensional shallow water equations under the rigid-lid assumption have the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=B(u^p-u), \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}+\frac{1}{R}u^2=B(v^p-v), \tag{3}$$

where u and v are the depth-averaged velocity components in the x and y directions, respectively, u^p and v^p are the components of the particle velocities, B is the particle loading parameter [11], R is the radius of curvature, p is the pressure, h is water depth and c_f is the friction coefficient. The following simplifying assumptions are used in order to derive (1)-(3): (a) rigid-lid assumption is used (in other words, free surface acts as a rigid lid so that water depth is assumed to be constant); (b) Chezy formula [13] is used to model bottom friction; (c) curvature is assumed to be small $(1/R \ll 1)$; (d) the distribution of particles in a carrier fluid is assumed to be uniform; (e) no dynamic interaction between carrier fluid and particles is assumed. Assumption (a) is verified in [8] where it is shown that from a linear stability point of view rigid-lid assumption works well for small Froude numbers. Assumptions (d) and (e) are discussed in [11] where it is shown that these assumptions are reasonable for the case of large Stokes number of the flow.

Introducing the stream function by the relations

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x} \tag{4}$$

and eliminating the pressure the following equation is obtained from (1)-(3):

$$\begin{aligned}&(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy}\\&+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} + \frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) + B\Delta\psi = 0\end{aligned} \tag{5}$$

Consider a perturbed solution to (5) of the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + \ldots \tag{6}$$

Here $\psi_0(y)$ is the base flow solution and $\psi_1(x,y,t)$ is a small unsteady perturbation. Using the method of normal modes we represent $\psi_1(x,y,t)$ in the form

$$\psi_1(x,y,t) = \varphi(y)\exp[i(\alpha x - \beta t)], \tag{7}$$

where both α and β can be complex. However, two widely used approaches in the theory of linear stability are (a) temporal stability analysis and (b) spatial stability analysis. In case (a) the wave number α is assumed to be real while β, in general, is complex: $\beta = \beta_r + i\beta_i$. For spatial stability analysis the usual assumptions are as follows: $\beta = \beta_r$ is real while the parameter α is complex: $\alpha = \alpha_r + i\alpha_i$.

Substituting (6) and (7) into (5) and linearizing the resulting equation in the neighborhood of the base flow we obtain the following ordinary differential equation for the amplitude $\varphi(y)$ of the normal perturbation:

$$\begin{aligned}&\varphi_{yy}(\alpha u_0 - \beta - iSu_0 - iB) - iSu_{0y}\varphi_y + 2u_0\alpha / R\varphi_y\\&+\varphi\left(\alpha^2\beta - \alpha^3u_0 - \alpha u_{oyy} + i\alpha^2u_0S/2 + i\alpha^2B\right) = 0\end{aligned} \tag{9}$$

with the boundary conditions

$$\varphi(\pm\infty) = 0 \tag{10}$$

where $S = c_f b/h$ is the bed-friction number [6] and b is a characteristic length scale (for mixing layers b is usually the width of a mixing layer). Note that (9), (10) is an eigenvalue problem where for temporal stability problem β is an eigenvalue (it is seen from (9), (10) that the problem is linear in β) and for spatial stability problem α is an eigenvalue (problem (9), (10) is nonlinear in α).

3 NUMERICAL RESULTS FOR SPATIAL AND TEMPORAL INSTABILITY

Problem (9), (10) is solved numerically by means of a pseudospectral collocation method based on Chebyshev polynomials (the details of the method are given, for example, in [13]). The corresponding discretized linear generalized eigenvalue problem in β (temporal stability analysis) can be solved using one of subroutines in IMSL package. The base flow is said to be linearly stable if all eigenvalues β have negative imaginary parts and linearly unstable if at least of β_i is positive.

The following procedure is suggested to solve spatial stability problem. Assuming that both parameters α and β are complex of the form $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$, for each set of the parameters R, S, B, α_r and β_r we find α_i such that $\beta_i = 0$ (using a bisection method). The condition of instability is $\alpha_i < 0$.

Calculated spatial growth rates for the case *1/R* = 0.025 are shown in Fig. 1 for the case of a base velocity profile of the form

$$u_0(y) = \frac{1}{2}(1 + \tanh y). \tag{11}$$

The bed-friction number is S = 0.1. The three curves in Fig. 1 corrspond to the following values of the particle loading parameter B (from top to bottom) : 0,0.01 and 0.02. As can be seen from the graph, the increase in B leads to a more stable flow (the growth rates are getting smaller as the parameter B increases).

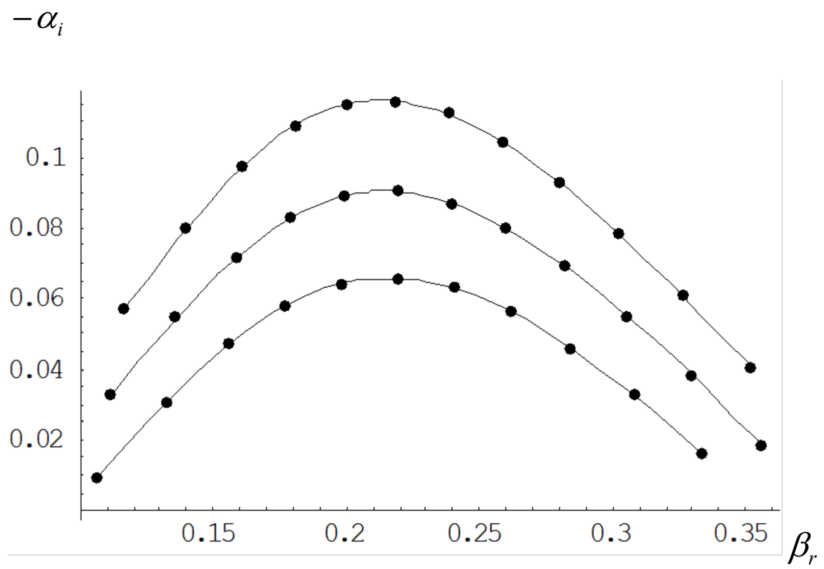

Figure 1: Spatial growth rates for the case *1/R* = 0.025.

Spatial growth rates for the case *1/R* = 0.05 are shown in Fig. 2. The other values of the parameters are the same as in Fig. 1. It is seen from the comparison of Figs. 1 and 2 that both parameters (*B* and *1/R*) have a stabilizing influence on the base flow. The growth rates for the case *1/R* = 0.05 are smaller than for the case *1/R* = 0.025.

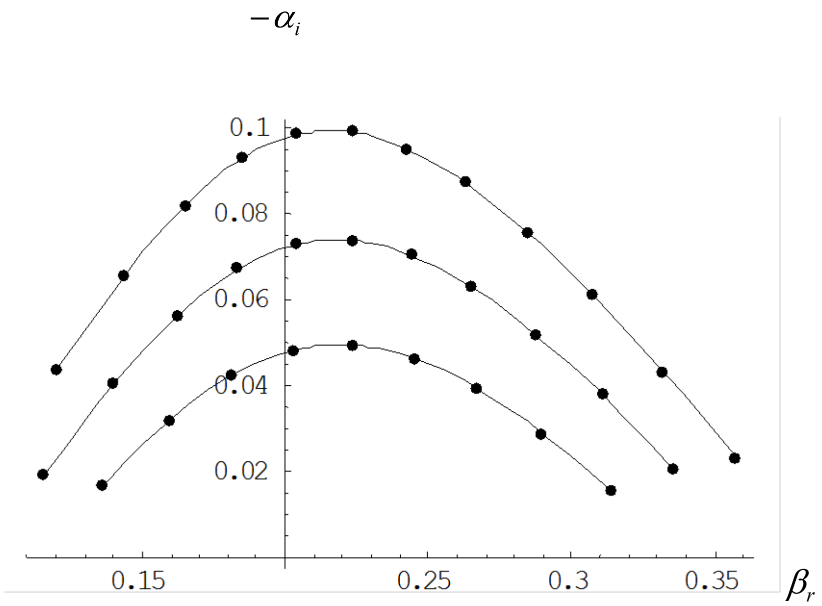

Figure 2: Spatial growth rates for the case *1/R* = 0.05.

Following Gaster [12] we denote by (*T*) and (*Sp*) the solutions to (9), (10) correspondign to temporal and spatial problems, respectively. It is shown in [12] that near the marginal stability curve

$$\alpha_r(T) = \alpha_r(Sp), \ \beta_r(T) = \beta_r(Sp), \ \alpha_i(Sp) = -\beta_i(T)/c(T), \tag{12}$$

where $c(T) = \beta_r(T)/\alpha_r(T)$. It follows from the Gaster's transformation that on the stability boundary either spatial or temporal stability analyses can be used since in this case $\alpha_i(Sp) = \beta_i(T) = 0.$ If the objective of the analysis is to construct a marginal stability curve then it is recommended to use temporal stability analysis (which is a simpler method from a computational point of view than spatial stability analysis). However, the use of the Gaster's transformation away from the marginal stability curve can result in relatively large errors. We have computed temporal and spatial growth rates for the case S = 0.05, B = 0 and *1/R* = 0. The relative percentage errors δ in using Gaster's transformation are shown in the Table 1.

Table 1: Relative errors in using Gaster's transformation

α_r	$\delta(\%)$
0.1	11.6
0.2	15.4
0.3	16.3
0.4	15.0
0.5	12.7

It is seen from Table 1 that errors in using Gaster's transformation for the calculation of growth rates away from the marginal stability curve can be quite large.

4 WEAKLY NONLINEAR ANALYSIS

In this section we briefly describe applications of weakly nonlinear theory to the analysis of development of instability above the threshold when the base flow loses stability.

The first approach is based on a parallel flow assumption. Using the method of multiple scales with the "slow" variables $\xi = \varepsilon(x - c_g t)$ and $\tau = \varepsilon^2 t$, where c_g is the group velocity, we assume that the evolution of the most unstable mode (in a small neighborhood of the critical value of the parameter *S*) can be described by the formula

$$\psi_1(x, y, t) = A(\xi, \tau)\varphi(y)\exp[i(\alpha x - \beta t)], \qquad (13)$$

where $A(\xi, \tau)$ is a slowly varying amplitude. Applying the method of multiple scales to (5), (6) and (13) and using solvability conditions at order two we obtain the group velocity c_g. Using the solvability condition at order three we obtain an amplitude evolution equation of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \kappa \frac{\partial^2 A}{\partial \xi^2} - \mu A \,|\, A \,|, \qquad (14)$$

where σ, κ and μ are complex coefficients which are calculated in terms on integrals containing the characteristics of linear stability problems. Equation (14) is known as the Ginzburg-Landau equation in the hydrodynamic stability literature. It is shown (see, for example, [15]) that it has a rich variety of solutions from deterministic to chaotic depending on the values of the coefficients. In fact, (14) is used in two ways in the literature: first, as a phenomenological equation (that is, it is assumed that a certain phenomenon can be modeled by (14) where the coefficients are usually determined from experimental data), and second, it can be derived (in some cases) from the equations of motion. We have shown that (14) is derived from (5).

The second approach uses a slow longitudinal variation of the base flow under the assumption that the wave length of the most unstable mode λ is much smaller than the typical length scale L of the longitudinal variation of the base flow. In this case a small parameter $\varepsilon = \lambda / L$ is used to measure non-parallelism of the base flow. Using the WKB method we assume that a perturbation stream function can be represented in the form

$$\psi(x,y,t)=\varphi(y,X)e^{i(\theta(X)/\varepsilon-\omega t)} \quad (15)$$

where $\varphi(y,X)$ is the amplitude which is represented in the form

$$\varphi(y,X)=\varphi_1(y,X)+\varepsilon\varphi_2(y,X)+\ldots \quad (16)$$

and $\varphi_1(y,X)=A(X)\Phi(y,X,\omega)$, where $A(X)$ is an amplitude function and $\Phi(y,X,\omega)$ is the eigenfunction of the linear stability problem. It is shown that the amplitude evolution equation for this case has the form

$$M(X)\frac{dA}{dX}+N(X)A=0, \quad (17)$$

where the functions $M(X)$ and $N(X)$ are calculated in terms of integrals containing the characteristics of linear stability problem.

5 CONCLUSIONS

Linear and weakly nonlinear analyses of particle-laden slightly curved shallow mixing layers are presented in the paper. Spatial stability problem under the rigid-lid assumption is solved numerically by pseudospectral collocation method based on Chebyshev polynomials. Spatial growth rates are calculated for different values of the parameters of the problem. Two weakly nonlinear approaches leading to amplitude evolution equations for the most unstable mode are described. Experimental and/or numerical data are needed in order to assess the applicability of weakly nonlinear models to the analysis of instability of particle-laden shallow mixing layers.

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