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FUNDAMENTAL SOLUTIONS FOR A COUPLED FORMULATION OF POROUS BIPHASIC MEDIA WITH COMPRESSIBLE SOLID AND FLUID PHASES

R. SERPIERI*, F. TRAVASCIO† AND S. ASFOUR†

*Dipartimento di Ingegneria (DING)
Università degli Studi del Sannio
Piazza Roma, 21 - I, 82100, Benevento, Italy
e-mail: roberto.serpieri@unisannio.it, web page: http://www.ding.unisannio.it/

†Biomechanics Research Laboratory, College of Engineering
University of Miami
1251 Memorial Drive, MCA268, Coral Gables, FL, 33130, USA
e-mail: f.travascio@miami.edu, sasfour@miami.edu - web page: http://www.ie.miami.edu/brl/

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Summary. A general biphasic poroelastic formulation at finite strains with intrinsic compressibility of phases, whose governing equations are inferred on account of a least-action variational principle, has been recently proposed (TMCPM). Hereby, a theoretical, analytical, and numerical assessment is presented on the capability of linearized TMCPM to recover, in the limit of vanishing porosity, a traditional single phase continuum model.

1 INTRODUCTION

During the past sixty years, significant effort has been made for describing the mechanical behavior of fluid saturated poroelastic media [5]. Interest in this research field is motivated by the capability of this theory to describe the mechanics of several physical systems (e.g., consolidation of soils, biological tissues, polymeric foams, etc.) [9].

In applications where phases can be reasonably modeled as intrinsically incompressible continua, Incompressible Theory of Mixtures (ITM) [4, 2, 12] have found widespread deployment. In contrast, the achievement of a general consistent theory of poroelasticity, capable of addressing media with any degree and range of compressibility of the constituent phases, turns out to be a still intensively debated issue [6, 14, 11].

In biphasic theories, the two phases are most frequently treated as superposed continua each endowed with a separate energy potential. As shown by several authors (e.g. [17]), this general approach requires, alongside of traditional linear momentum and mass balances, an additional governing equation, simply referred to henceforth as *closure equation*, for which very differentiated candidates have been proposed.

In a study specifically addressing intrinsically compressible phases, tracing back to Cosserat's theory [10], Bowen [3] identified this equation in momentum of momentum balance. Porosity was added as an additional independent state field by Wilmansky: in [19, 1] several additional balance equations have been investigated in the form of a porosity balance, or as an integrability condition for the deformation of the solid skeleton. In [8] and [6], a geometric saturation constraint is employed as a closure equation, and is combined with a multiplicative decomposition of the deformation gradient.

Alternatively, the lacking closure equations has been also inferred tracing back to the fundamental least-action principle of Hamiltonian mechanics. This approach was exploited in [13] to derive a porosity-associated additional governing equation by pairing an additional porosity field introduced as an independent state field.

A least-action principle was also exploited recently in [15, 16] to derive a biphasic formulation at finite strains that fully accounts for the compressibility of both phases. This framework, termed Theory of Microscopically Compressible Porous Media (TMCPM), presents the follwing distinctive features: 1) the employment of an additional macroscopic state field of effective volumetric strain; 2) the identification of the lacking additional equation as the stationarity condition that is Action-conjugated with such effective volumetric strain; 3) the explicit presence of the fluid mass balance among the governing PDEs, as an equation completely independent from the remaining ones.

The present study addresses a systematization of the linearized form of TMCPM within the range of infinitesimal perturbations. The linearized model is specifically investigated within quasi stationary conditions that address biphasic behavior under slow consolidation phenomena in soil mechanics or fast deformations in soft hydrated tissues. A reduced $\boldsymbol{u}-p$ form of governing equations is thus obtained, which differs from its ITM counterpart by the presence of a dimensionless microstructural parameter, termed $\bar{k}_r$.

A simple homogenization procedure is proposed for retrieving estimates and bounds for this parameter. The estimates obtained are exploited to investigate the general behavior of the quasi stationary-system in the limit of vanishing porosity, to assess the consistent recovery of single-phase behavior of a solid continuum filling the entire space. This study is supported by analytical and numerical solutions of a set of stress relaxation test problems where the full range of porosity is spanned.

2 THEORETICAL BACKGROUND

2.1 Kinematics

Linearized TMCPM is briefly recalled. Volumetric fractions are customarily defined as $\phi^\alpha = \dfrac{\hat{V}^{(\alpha)}}{\bar{V}}$ $(\alpha = s,\ f)$, where $\bar{V}$, $\hat{V}^{(s)}$, and $\hat{V}^{(f)}$ represent, respectively, the volumes of the mixture, of the solid phase, and of the fluid phase contained in a Representative Volume

Element (RVE). At reference configuration, the mass distribution is defined by the fields $\bar{\rho}_o^\alpha$ and $\hat{\rho}_o^\alpha$, which represent apparent and true mass densities of the components of the mixture. The complete saturation hypothesis implies the two following relations:

$$\phi_o^f + \phi_o^s = 1, \qquad \bar{\rho}_o^\alpha = \phi_o^\alpha \hat{\rho}_o^\alpha, \qquad \alpha = s,\ f \tag{1}$$

A second configuration of the solid phase, infinitesimally displaced from the reference one, is macroscopically described by the displacement field $\bar{\mathbf{u}}^{(s)}$. For this configuration, linearization of equations (1) immediately yields:

$$d\phi^f = -d\phi^s, \qquad d\bar{\rho}^\alpha = d\phi^\alpha \hat{\rho}_o^\alpha + \phi_o^\alpha d\hat{\rho}^\alpha, \qquad \alpha = s,\ f \tag{2}$$

Characteristic feature of TMCPM is the presence of two volumetric strain measures associated with each phase composing the mixture: the apparent volumetric dilatation $\bar{e}^\alpha$, and the effective volumetric dilatation $\hat{e}^\alpha$. The latter is introduced as a primary state field completely independent from the former. In linearized TMCPM, as a result of linearization of finite volumetric strains, $\bar{e}^s$ and $\hat{e}^s$ are defined as follows:

$$\bar{e}^s = \frac{d\bar{V}}{\bar{V}}, \qquad \hat{e}^s = \frac{d\hat{V}^{(s)}}{\hat{V}^{(s)}} \tag{3}$$

In particular, the apparent solid volumetric dilatation is the trace of the solid strain tensor: $\bar{e}^s = \mathrm{tr}\bar{\boldsymbol{\varepsilon}}^{(s)} = \nabla \cdot \bar{\mathbf{u}}^{(s)}$. Volumetric strains are related to density differentials by:

$$\bar{e}^\alpha = -\frac{d\bar{\rho}^\alpha}{\bar{\rho}_o^\alpha}, \qquad \hat{e}^\alpha = -\frac{d\hat{\rho}^\alpha}{\hat{\rho}_o^\alpha}, \qquad \alpha = s,\ f \tag{4}$$

In addition, a relationship among the variation of the fluid volumetric fraction, and the apparent and intrinsic solid deformation has been reported [16]:

$$d\phi^f = (\bar{e}^s - \hat{e}^s)\phi_o^s \tag{5}$$

Combination of equations (2), (4) and (5) allows deriving a fundamental kinematic relation in linearized TMCPM relating apparent and intrinsic deformations of the mixture:

$$\phi_o^f \bar{e}^f + \phi_o^s \bar{e}^s = \phi_o^f \hat{e}^f + \phi_o^s \hat{e}^s \tag{6}$$

This equation, derived in [16] for the special case $\phi_o^f = \phi_o^s = 0.5$, represents a dimensionless saturation constraint for a biphasic compressible medium.

2.2 Mechanics

A hyperelastic isotropic formulation is recalled. A macroscopic solid strain energy density is introduced as a function of the infinitesimal strain measures of solid $\bar{\psi}^{(s)} =$

$\bar{\psi}^{(s)}\left(\bar{\boldsymbol{\varepsilon}}^{(s)}, \hat{e}^s\right)$. The stress measures are the *extrinsic* stress tensor, $\breve{\boldsymbol{\sigma}}^{(s)}$, conjugated to $\bar{\boldsymbol{\varepsilon}}^{(s)}$, and a single scalar stress conjugated to $\hat{e}^s$, termed *intrinsic solid stress*, $\hat{\sigma}^s$:

$$\breve{\boldsymbol{\sigma}}^{(s)} = \frac{\partial \bar{\psi}^{(s)}}{\partial \bar{\boldsymbol{\varepsilon}}^{(s)}}, \qquad \hat{\sigma}^s = \frac{\partial \bar{\psi}^{(s)}}{\partial \hat{e}^s} \tag{7}$$

As previously reported [16], a volumetric-deviatoric stress-strain uncoupling yields the following representation for solid stress in the isotropic case

$$\breve{\boldsymbol{\sigma}}^{(s)} = 2\bar{\mu}\bar{\boldsymbol{\varepsilon}}^{(s)} + \bar{Z}\bar{e}^s\boldsymbol{I} + \bar{k}_{sM}\hat{e}^s\boldsymbol{I}, \qquad \hat{\sigma}^s = \bar{k}_{sM}\bar{e}^s + \bar{k}_s\hat{e}^s \tag{8}$$

where $\bar{\mu}$ is a shear modulus while $\bar{Z}$, $\bar{k}_{sM}$ and $\bar{k}_s$ are all volumetric moduli.

Introducing intrinsic solid pressure as $\hat{p}^{(s)} = -\frac{\partial \bar{\psi}^{(s)}}{\partial \hat{e}^s} = -\hat{\sigma}^s$, with the aid of a customary volumetric-deviatoric split and of relationship $\bar{Z} = \bar{\lambda} + \frac{\bar{k}_{sM}^2}{\bar{k}_s}$ [16], the elastic constitutive law for the solid phase can thus be conveniently represented in the alternate uncoupled form in terms of Lamé coefficients $\bar{\mu}$ and $\bar{\lambda}$:

$$\breve{\boldsymbol{\sigma}}^{(s)}_{dev} = 2\bar{\mu}\bar{\boldsymbol{\varepsilon}}^{(s)}_{dev} \tag{9}$$

$$\begin{bmatrix} \breve{p}^{(s)} \\ \hat{p}^{(s)} \end{bmatrix} = -\left[\bar{\boldsymbol{K}}^{(s)}_{iso}\right]\begin{bmatrix} \bar{e}^s \\ \hat{e}^s \end{bmatrix}, \qquad \text{with} \left[\bar{\boldsymbol{K}}^{(s)}_{iso}\right] = \begin{bmatrix} \left(\frac{2}{3}\bar{\mu} + \bar{\lambda} + \frac{\bar{k}_{sM}^2}{\bar{k}_s}\right) & \bar{k}_{sM} \\ \bar{k}_{sM} & \bar{k}_s \end{bmatrix} \tag{10}$$

The macroscopic strain energy density of the fluid is:

$$\bar{\psi}^{(f)} = \phi^f_o\hat{\psi}^{(f)} = \phi^f_o\hat{k}_f\left(\hat{e}^f\right)^2 \tag{11}$$

where $\hat{\psi}^{(f)}$ is the *interstitial* strain energy density and $\hat{k}_f$ is the fluid bulk modulus.

Momentum equations are hereby recalled by directly neglecting inertia terms under the assumption of quasi-stationarity. The linear momentum balance of solid is reported assuming two simplificative hypotheses: the medium is homogeneous in space, so that the porosity gradient vanishes and, as a consequence, also the volume force term responsible for buoyancy forces as shown in [16]. Moreover, all external body forces, such as gravitational, are neglected. Consequently, the drag $\boldsymbol{\pi}^{fs}$ exerted by the fluid phase over the solid is the only nonvanishing volume force applied on the solid phase:

$$\nabla \cdot \breve{\boldsymbol{\sigma}}^{(s)} + \boldsymbol{\pi}^{fs} = 0 \tag{12}$$

In particular, a simple linear Darcy law $\boldsymbol{\pi}^{fs} = K(\boldsymbol{v}^f - \frac{\partial \bar{\mathbf{u}}^{(s)}}{\partial t})$, with K being the friction coefficient, is considered.

Characteristic feature of TMCPM is the presence of an additional momentum balance equation which, on account of a least-action principle, stems from imposing stationarity of the Action functional with respect to the effective solid volumetric strain $\hat{e}^s$:

$$\frac{\partial \bar{\psi}^{(s)}}{\partial \hat{e}^s} + \frac{\partial \bar{\psi}^{(f)}}{\partial \hat{e}^s} + \rho^*_0\frac{\partial^2 \hat{e}^s}{\partial t^2} = 0 \tag{13}$$

In (13), ρ_0^* is an additional inertia term associated with the acceleration of the parameter $\hat{e}^s$. The previous equation comes alongside of the classical linear solid momentum balance (see equation (12)) which, instead, expresses stationarity with respect to infinitesimal variations of the displacement field of solid. The interested reader is referred to [15] for a complete account of these theoretical issues. Balance (13) accounts for volumetric coupling between phases. Rewritten in terms of pressure-like quantities with neglected inertia forces it reads:

$$-\hat{p}^{(s)} - \hat{p}^{(fs)} = 0 \tag{14}$$

where $\hat{p}^{(s)}$ has been previously defined, and $\hat{p}^{(fs)} = -\frac{\partial \bar{\psi}^{(f)}}{\partial \hat{e}^s}$ is an interaction term between the two phases. On account of (6), (11), and of the chain rule, this last term is:

$$\hat{p}^{(fs)} = -\phi_o^f \frac{\partial \hat{\psi}^{(f)}}{\partial \hat{e}^f} \frac{\partial \hat{e}^f}{\partial \hat{e}^s} = -\phi_o^s p \tag{15}$$

Thus, due to the second of (10), (14) achieves the following form:

$$\bar{k}_{sM} \bar{e}^s + \bar{k}_s \hat{e}^s - \phi_o^s \hat{k}_f \hat{e}^f = 0 \tag{16}$$

Concerning the momentum balance of the fluid phase, this is expressed with respect to the reference frame of the solid phase. Consistent with the assumption of neglecting body forces other than drag (for the fluid $\boldsymbol{\pi}^{sf} = -\boldsymbol{\pi}^{fs}$), as well as inertia terms, this reads:

$$\phi_o^f \nabla p + \boldsymbol{\pi}^{fs} = 0 \tag{17}$$

Concerning mass balances, mass generation/depletion is neglected for simplicity. Notice that mass balance of solid phase is not included in the main set of governing equations since solid density variation $d\bar{\rho}^s$ is a secondary variable, whose value is related to $\bar{e}^s$ by $d\bar{\rho}^s = -\bar{\rho}_o^s \bar{e}^s$. Conversely, fluid mass balance is included among the governing equations and, as a consequence of linearization, admits the simple dimensionless form [16]:

$$\frac{\partial \bar{e}^f}{\partial t} = \nabla \cdot \boldsymbol{v}^f \tag{18}$$

The set of governing equations so far derived for the quasi-stationary system amounts to 8 independent scalar PDEs in the primary fields $\bar{\mathbf{u}}^{(s)}$, $\hat{e}^s$, $\boldsymbol{v}^f$, p (8 scalar fields overall).

However, a simpler form of the governing set, where all quantities are expressed in terms of solid displacement $\bar{\mathbf{u}}^{(s)}$ and fluid pressure p fields, can be conveniently achieved by means of few manipulations hereby synthesized. This equivalent formulation will be referred to as reduced quasi-stationary $\boldsymbol{u} - p$ form. To this end, a dimensionless ratio of the solid moduli $\bar{k}_r$ and a further modulus $\hat{k}_s$ are introduced as $\bar{k}_r = \frac{\phi_o^s \bar{k}_{sM}}{\bar{k}_s}$ and $\hat{k}_s = \frac{\bar{k}_s}{\phi_o^s}$.

A first equation, that may be viewed as a combined linear momentum balance of the mixture, is obtained adding (12) and (17), whereby drag forces cancel each other:

$$\nabla \cdot \breve{\boldsymbol{\sigma}}^{(s)} - \phi_o^f \nabla p = 0 \tag{19}$$

Moreover, upon performing a partial Legendre transform of the constitutive equation (10) with respect to $\hat{e}^s$, and by also recalling relation $\bar{Z} = \bar{\lambda} + \frac{\bar{k}_{sM}^2}{\bar{k}_s}$ together with equations (8), a constitutive law is obtained, in which $\hat{e}^s$ is replaced by p:

$$\check{\boldsymbol{\sigma}}^{(s)}\left(\bar{\boldsymbol{\varepsilon}}^{(s)},\, \hat{e}^s\right) \longrightarrow \check{\boldsymbol{\sigma}}^{(s)}\left(\bar{\boldsymbol{\varepsilon}}^{(s)},\, p\right) = 2\bar{\mu}\bar{\boldsymbol{\varepsilon}}^{(s)} + \bar{\lambda}\bar{e}^s\boldsymbol{I} - \bar{k}_r p\boldsymbol{I} \tag{20}$$

Using (20), the following restated form of the linear momentum balance of the whole mixture is obtained, which is the first equation of the quasi-stationary reduced $\boldsymbol{u} - p$ set:

$$\nabla \cdot \boldsymbol{\sigma}_D^{(s)} - \left(\phi_o^f + \bar{k}_r\right)\nabla p = 0, \qquad \text{where } \boldsymbol{\sigma}_D^{(s)} = \bar{\lambda}\bar{e}^s\boldsymbol{I} + 2\bar{\mu}\bar{\boldsymbol{\varepsilon}}^{(s)} \tag{21}$$

The second equation of the reduced $\boldsymbol{u} - p$ set is obtained by substituting (18) and (16) into (6). The resulting equation is finally combined with the divergence of the fluid momentum balance (17) to yield an equation expressing, altoghether, *saturation constraint, fluid mass balance, $\hat{e}^s$-associated momentum balance and fluid linear momentum balance*:

$$\left(1 + \bar{k}_r\right)\frac{\partial}{\partial t}\nabla \cdot \bar{\mathbf{u}}^{(s)} + \left(\frac{\phi_o^s}{\hat{k}_s} + \frac{\phi_o^f}{\hat{k}_f}\right)\frac{\partial p}{\partial t} - \frac{(\phi_o^f)^2}{K}\nabla^2 p = 0 \tag{22}$$

The sought reduced $\mathbf{u} - p$ form of equations governing the quasi-stationary behavior in linearized TMCPM is composed of (21) and (22).

3 ESTIMATES OF THE MODULI

For a meaningful assessment of the compressible biphasic model at hand, it is fundamental to dispose of estimates of the moduli $\bar{k}_s$, $\bar{k}_{sM}$ in (10) and of $\bar{k}_r$ in (21)-(22). Basic estimates are hereby obtained by introducing simple geometrical/topological assumptions on the microstructural geometry of the RVE. With this perspective, hypotheses of spherical symmetry of the RVE, and isotropy of the constituent material are now introduced.

A simple hollow spherical geometry, with exterior radius R_e and interior radius R_i, is considered. Figure 1a shows the diametral section of the hollow sphere. The solid domain is subjected to an internal, p_i, and an external pressure p_e. The solution for this elastostatic problem is available in spherical coordinates [18]. Denoting by r the radial coordinate, the outward radial displacement u is:

$$u = -\frac{(1-2\nu)}{2(1+\nu)\mu}\frac{p_e R_e^3 - p_i R_i^3}{R_e^3 - R_i^3}r - \frac{1}{4\mu}\frac{(p_e - p_i)R_e^3 R_i^3}{R_e^3 - R_i^3}\frac{1}{r^2} \tag{23}$$

where μ and ν are the shear modulus and Poisson ration of the material that constitutes the solid phase. Setting $r = R_e$ and $r = R_i$ in (23), one respectively obtains the displacements on the exterior, u_e, and interior, u_i, boundaries. Since the relation between the pressure vector $\mathbf{p} = [p_e,\, p_i]^t$ and the displacements on the external and internal surfaces of the sphere $\mathbf{u} = [u_e,\, u_i]^t$ is linear, this can be written in matrix form as:

$$\mathbf{u} = \mathbf{D_{up}}\mathbf{p} \tag{24}$$

where, upon setting $\alpha = 4\mu\,(1+\nu)\,(R_e^3 - R_i^3)$, $\mathbf{D_{up}}$ turns out to be the following matrix:

$$\mathbf{D_{up}} = -\frac{1}{\alpha}\begin{bmatrix} R_e\left[R_i^3(1+\nu)+2R_e^3(1-2\nu)\right] & 3(\nu-1)R_eR_i^3 \\ -3(\nu-1)R_iR_e^3 & R_i\left[R_e^3(1+\nu)+2R_i^3(1-2\nu)\right] \end{bmatrix} \tag{25}$$

Volumetric strains are collected in the vector $\mathbf{e} = [\bar{e}^s,\, \hat{e}^s]$. A relation between $\mathbf{e}$ and $\mathbf{u}$ is now sought. Suitable expressions for $\bar{e}^s$ and $\hat{e}^s$ are inferred from definition (3), by considering that $\bar{V}$ is the volume of the external sphere while $\hat{V}^{(s)}$ is the volume of the hollow sphere (i.e. of the subdomain containing only the solid phase), so that $\bar{V} = \frac{4}{3}\pi R_e^3$ and $\hat{V}^{(s)} = \frac{4}{3}\pi\,(R_e^3 - R_i^3)$. The infinitesimal variations $d\bar{V}$ and $d\hat{V}^{(s)}$ can be computed as function of u_e and u_i by retaining, as usual in infinitesimal displacements, only first-order terms. Accordingly (3) provide:

$$\bar{e}^s = \frac{(R_e+u_e)^3}{R_e^3} \simeq 3\frac{u_e}{R_e}, \qquad \hat{e}^s = \frac{(R_e+u_e)^3-(R_i+u_i)^3}{(R_e^3-R_i^3)} \simeq 3\frac{R_e^2u_e - R_i^2u_i}{(R_e^3-R_i^3)} \tag{26}$$

As a check of the formulas so far introduced, observe incidentally that when equal external and internal pressures are applied (i.e. setting $\mathbf{p} = [p,\, p]^t$ in (24)), the cell experiments a homothety. In this case, the external and internal displacements are:

$$u_e = -R_e\frac{1}{2\mu}\frac{1-2\nu}{1+\nu}p, \qquad u_i = -R_i\frac{1}{2\mu}\frac{1-2\nu}{1+\nu}p \tag{27}$$

and, according to (26), the strains are $\bar{e}^s = \hat{e}^s = \frac{p}{k_s}$, where $k_s = \frac{2}{3}\frac{1+\nu}{1-2\nu}\mu$ is the bulk modulus of the solid constituent material. It is convenient to express (26) in the matrix form $\mathbf{e} = \mathbf{A_{eu}}\mathbf{u}$, where:

$$\mathbf{A_{eu}} = \begin{bmatrix} 3\frac{1}{R_e} & 0 \\ 3\frac{R_e^2}{\left(R_e^3-R_i^3\right)} & -3\frac{R_i^2}{\left(R_e^3-R_i^3\right)} \end{bmatrix} \tag{28}$$

The vector of stress measures that is energy-conjugated with $\mathbf{e}$ is $\mathbf{p}^{(s\mathbf{e})} = -\left[\check{p}^{(s)},\, \hat{p}^{(s)}\right]^t$ (recall (7) and (10)). Energy conjugation implies that strain energy of the finite spherical cell $\bar{\psi}^{(s)}_{finite}$ can be expressed in the form:

$$\bar{\psi}^{(s)}_{finite} = \frac{1}{2}\mathbf{p}^{(s\mathbf{e})}\cdot\mathbf{e} \tag{29}$$

It is also convenient to use, in place of the displacements $\mathbf{u}$, a couple of displacement parameters $\tilde{\mathbf{u}} = [4R_e^2\pi u_e,\, 4R_i^2\pi u_i]^t$ that includes the relevant surface area. In this case, energy conjugation with the pressures $\mathbf{p}$ can be written in the form:

$$\bar{\psi}^{(s)}_{finite} = \frac{1}{2}\tilde{\mathbf{I}}\mathbf{p}\cdot\tilde{\mathbf{u}} \tag{30}$$

where $\tilde{\mathbf{I}}$ is a matrix that accounts for the opposite orientation of tractions associated with positive pressures on the external and internal surfaces $\tilde{\mathbf{I}} = [-1 \quad 0; \quad 0 \quad 1]$. As one may check, the relationship between $\mathbf{e}$ and $\tilde{\mathbf{u}}$ is

$$\mathbf{e} = \tilde{\mathbf{A}}_{\mathbf{eu}}\tilde{\mathbf{u}} \qquad \text{with } \tilde{\mathbf{A}}_{\mathbf{eu}} = \frac{3}{4\pi}\begin{bmatrix} \frac{1}{R_e^3} & 0 \\ \frac{1}{\left(R_e^3 - R_i^3\right)} & -\frac{1}{\left(R_e^3 - R_i^3\right)} \end{bmatrix} \tag{31}$$

Accordingly, the stress strain relation in terms of $\tilde{\mathbf{u}}$ and its inverse are:

$$\tilde{\mathbf{u}} = \tilde{\mathbf{D}}_{\mathbf{up}}\mathbf{p}, \qquad \mathbf{p} = \tilde{\mathbf{D}}_{\mathbf{up}}^{-1}\tilde{\mathbf{u}} \tag{32}$$

where $\tilde{\mathbf{D}}_{\mathbf{up}}$ is the following matrix:

$$\tilde{\mathbf{D}}_{\mathbf{up}} = \frac{4\pi}{\alpha}\begin{bmatrix} R_e^3\left[R_i^3(1+\nu) + 2R_e^3(1-2\nu)\right] & 3(\nu-1)R_e^3R_i^3 \\ -3(\nu-1)R_i^3R_e^3 & R_i^3\left[R_e^3(1+\nu) + 2R_i^3(1-2\nu)\right] \end{bmatrix} \tag{33}$$

Thus, on account of (31), the pressure vector turns out to be:

$$\mathbf{p} = \tilde{\mathbf{D}}_{\mathbf{up}}^{-1}\tilde{\mathbf{A}}_{\mathbf{eu}}^{-1}\mathbf{e} \tag{34}$$

Equating (29) to (30) provides $\frac{1}{2}\mathbf{p}^{(se)} \cdot \mathbf{e} = \frac{1}{2}\tilde{\mathbf{I}}\mathbf{p} \cdot \tilde{\mathbf{u}}$. A substitution of (34) in this last identity provides $\mathbf{p}^{(se)} \cdot \mathbf{e} = \tilde{\mathbf{I}}\tilde{\mathbf{D}}_{\mathbf{up}}^{-1}\tilde{\mathbf{A}}_{\mathbf{eu}}^{-1}\mathbf{e} \cdot \tilde{\mathbf{u}}$. Upon transposing $\tilde{\mathbf{I}}\tilde{\mathbf{D}}_{\mathbf{up}}^{-1}\tilde{\mathbf{A}}_{\mathbf{eu}}^{-1}$, one has:

$$\mathbf{p}^{(se)} \cdot \mathbf{e} = \tilde{\mathbf{A}}_{\mathbf{eu}}^{-t}\tilde{\mathbf{D}}_{\mathbf{up}}^{-t}\tilde{\mathbf{I}}^t\tilde{\mathbf{u}} \cdot \mathbf{e} \tag{35}$$

and since (35) must hold for any $\mathbf{e}$, one infers $\mathbf{p}^{(se)} = \tilde{\mathbf{A}}_{\mathbf{eu}}^{-t}\tilde{\mathbf{D}}_{\mathbf{up}}^{-t}\tilde{\mathbf{I}}^t\tilde{\mathbf{u}}$. Finally, on account of (31), one has:

$$\mathbf{p}^{(se)} = \bar{\mathbf{K}}_{iso/finite}^{(s)}\mathbf{e}, \qquad \text{with } \bar{\mathbf{K}}_{iso/finite}^{(s)} = \tilde{\mathbf{A}}_{\mathbf{eu}}^{-t}\tilde{\mathbf{D}}_{\mathbf{up}}^{-t}\tilde{\mathbf{I}}^t\tilde{\mathbf{A}}_{\mathbf{eu}}^{-1} \tag{36}$$

where $\bar{\mathbf{K}}_{iso/finite}^{(s)}$ is the estimate of $\bar{\mathbf{K}}_{iso}^{(s)}$ for a finite sphere, defined as:

$$\left[\bar{\mathbf{K}}_{iso/finite}^{(s)}\right] = \frac{16\pi\mu}{9}\begin{bmatrix} \frac{R_e^3\left(R_e^3 - R_i^3\right)}{R_i^3} & -\frac{R_e^3\left(R_e^3 - R_i^3\right)}{R_i^3} \\ -\frac{R_e^3\left(R_e^3 - R_i^3\right)}{R_i^3} & \frac{\left(R_e^3 - R_i^3\right)\left[R_i^3(1+\nu) + 2R_e^3(1-2\nu)\right]}{2R_e^3R_i^3(1-2\nu)} \end{bmatrix} \tag{37}$$

This stiffness matrix has to be homogenized in space. Accordingly, it is divided by the volume of the external sphere $\frac{4}{3}\pi R_e^3$. Expressing the resulting matrix in terms of volumetric fractions, which for the spherical cell the volumetric fractions are trivially related to the radii by $\frac{R_e^3 - R_i^3}{R_e^3} = \phi_o^s$ and $\frac{R_i^3}{R_e^3} = 1 - \phi_o^s$, one finally obtains:

$$\left[\bar{\boldsymbol{K}}_{iso}^{(s)}\right] = \frac{4}{3}\mu\frac{\phi_o^s}{1-\phi_o^s}\begin{bmatrix} 1 & -1 \\ -1 & \frac{(3-3\nu-\phi_o^s(1+\nu))}{2(1-2\nu)} \end{bmatrix} = \frac{\phi_o^s}{1-\phi_o^s}\begin{bmatrix} \frac{4}{3}\mu & -\frac{4}{3}\mu \\ -\frac{4}{3}\mu & \frac{4}{3}\mu + (1-\phi_o^s)k_s \end{bmatrix} \tag{38}$$

where the last equality holds since, from the theory of elasticity, one has $\frac{1+\nu}{2(1-2\nu)} = \frac{3}{4}\frac{k_s}{\mu}$. The ratio $\bar{k}_r$ and modulus $\hat{k}_s$ turn out to be:

$$\bar{k}_r = -\frac{\phi_o^s \frac{4}{3}\mu}{\frac{4}{3}\mu + k_s(1-\phi_o^s)}, \qquad \hat{k}_s = \frac{1}{1-\phi_o^s}\left[\frac{4}{3}\mu + k_s(1-\phi_o^s)\right] \tag{39}$$

The contour plot of $\bar{k}_r$ as function of ϕ_o^s and ν is shown in Figure 1 whereby the bounds $-1 \leq \bar{k}_r \leq 0$ can be also visualized. In particular, the upper bound 0 is attained in the limit of vanishing solidity $\phi_o^s = 0$, and when the solid constituent material has $\nu = 0.5$ (i.e. is volumetrically incompressible). In contrast, the lower bound is attained at $\phi_o^s = 1$.

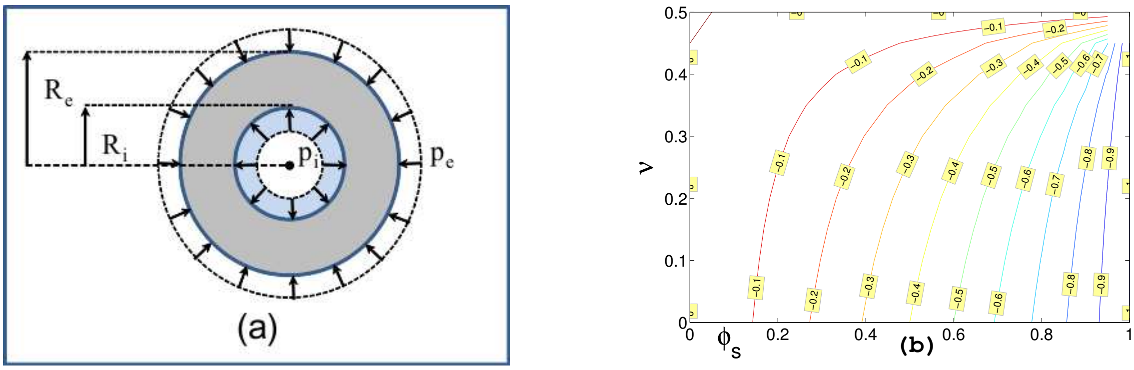

Figure 1: (a) diametral section of the solid spherical cell; (b) contour plot of $\bar{k}_r$ as function of ν and ϕ_o^s.

4 LIMIT BEHAVIOR AT VANISHING POROSITY

Estimates (39) for $\bar{k}_r$ and $\hat{k}_s$ are now deployed to analyze the behavior of the $\boldsymbol{u} - p$ set in the limit of vanishing porosity (LVP) (i.e. as $\phi_o^s \to 1$). This situation amounts to achievement of the limit of dilute suspension of the fluid phase in the solid matrix, and needs to be handled with special attention since all elastic coefficients in matrix (38) tend to infinity and, furthermore, equation (22) becomes clearly an identity $0 = 0$. However, as shown below, it can be deduced with few logical passages that the behavior of a single continuum made by the sole solid phase is duly recovered.

Actually, upon introducing the compliance coefficient $\frac{1}{\hat{k}_{sf}} = \left(\frac{\phi_o^s}{\hat{k}_s} + \frac{\phi_o^f}{\hat{k}_f}\right)$, the order of the infinitesimal factors in (22) can be compared by taking the ratio of (22) over $\frac{1}{\hat{k}_{sf}}$:

$$\hat{k}_{sf}\left(1 + \bar{k}_r\right)\frac{\partial}{\partial t}\nabla \cdot \bar{\mathbf{u}}^{(s)} + \frac{\partial p}{\partial t} - \hat{k}_{sf}\frac{(\phi_o^f)^2}{K}\nabla^2 p = 0 \tag{40}$$

Since, as $\phi_o^f \to 0$, one has $\bar{k}_r \to -1$ and $\hat{k}_{sf} \to \infty$, the first and third coefficients in (40) have both undetermined limit of the form $[\infty \cdot 0]$. However, with (39) one computes:

$$1 + \bar{k}_r = (1-\phi_o^s)\frac{\frac{4}{3}\mu + k_s}{\frac{4}{3}\mu + (1-\phi_o^s)\,k_s}, \qquad \hat{k}_{sf} = \frac{1}{(1-\phi_o^s)}\frac{\left[\frac{4}{3}\mu + k_s\right]\hat{k}_f}{\frac{4}{3}\mu + (1-\phi_o^s)\,k_s + \phi_o^s\hat{k}_f} \tag{41}$$

so that the undetermined forms can be evaluated, and the following limits are computed:

$$\hat{k}_{sf}\left(1+\bar{k}_r\right)=\frac{\left(\frac{4}{3}\mu+k_s\right)^2\hat{k}_f}{\left(\frac{4}{3}\mu+\hat{k}_f\right)\frac{4}{3}\mu},\qquad \hat{k}_{sf}\frac{(\phi_o^f)^2}{K}=0 \tag{42}$$

Hence $\hat{k}_{sf}\left(1+\bar{k}_r\right)$ holds a finite value, and (40) becomes $\frac{\partial}{\partial t}\left[\hat{k}_{sf}\left(1+\bar{k}_r\right)\nabla\cdot\bar{\mathbf{u}}^{(s)}+p\right]=0$. Assuming that, at initial time t_0, the body is in an undeformed state with zero pressure, the previous equation can be integrated and solved, providing:

$$p=-\hat{k}_{sf}\left(1+\bar{k}_r\right)\bar{e}^s \tag{43}$$

Substitution of (43) into (19) yields the classical equilibrium equation for Cauchy continuum $\nabla\cdot\check{\boldsymbol{\sigma}}^{(s)}=0$. Moreover, taking the divergence of (17), upon setting $\phi_o^f=0$ in (17), one infers $\bar{e}^s=\bar{e}^f$. Substitution of this last relation and of (43) into the dimensionless saturation constraint (6) yields:

$$\hat{e}^s=\frac{1}{\phi_o^s}\left[1-\frac{\phi_o^f}{\hat{k}_f}\hat{k}_{sf}\left(1+\bar{k}_r\right)\right]\bar{e}^s \tag{44}$$

Setting $\phi_o^s=1$ in (44), it is thus recognized that in the limit of vanishing porosity the only admissible path for volumetric strains of the solid phase is $\hat{e}^s=\bar{e}^s$.

On the other hand, from (38) one infers that at limit vanishing porosity $\bar{k}_s=-\bar{k}_{sM}$ and, consequently, in such a limit the elastic law (8) specializes to

$$\check{\boldsymbol{\sigma}}^{(s)}=2\bar{\mu}\bar{\boldsymbol{\varepsilon}}^{(s)}+\left[\left(\bar{\lambda}-\bar{k}_{sM}\right)\bar{e}^s+\bar{k}_{sM}\hat{e}^s\right]\boldsymbol{I} \tag{45}$$

On account of the property at LVP just found ($\bar{e}^s=\hat{e}^s$), the limit of (45) can be computed. It is then recognized that a traditional isotropic law is also recovered.

5 NUMERICAL AND ANALYTICAL EXAMPLES

The quasi-stationary behavior of the biphasic compressible medium was investigated as function of porosity also numerically. Solutions for uni-axial compressive stress relaxation tests problems were computed to obtain a family of solid stress curves parametric with ϕ_o^s. Figs. 2a and 2b show the experimental setup and the relevant boundary conditions.

Note that, for the specific simulated testing conditions, the reduced $\boldsymbol{u}-p$ form simplifies to a system of two scalar PDEs in the unknown functions $\bar{u}_x^{(s)}(x,t)$, $p_x(x,t)$ with $x\in[0,\,L]$:

$$\frac{\partial\check{\sigma}_x^{(s)}}{\partial x}-\left(\phi_o^f+\bar{k}_r\right)\frac{\partial p}{\partial x}=0,\qquad \left(1+\bar{k}_r\right)\frac{\partial^2\bar{u}_x^{(s)}}{\partial t\partial x}+\left(\frac{\phi_o^s}{\hat{k}_s}+\frac{\phi_o^f}{\hat{k}_f}\right)\frac{\partial p}{\partial t}-\frac{(\phi_o^f)^2}{K}\frac{\partial^2 p}{\partial x^2}=0 \tag{46}$$

where $\check{\sigma}_x^{(s)}=(\lambda+2\mu)\frac{\partial\bar{u}_x^{(s)}}{\partial x}$ whereas $\check{\sigma}_x^{(s)}$ and $\bar{u}_x^{(s)}$ are the components of the drained solid stress and the solid displacement in the x-direction, respectively. Analytical solutions for the system of equations (46) were obtained in the complex Laplace space. Subsequently, solutions in the real space were achieved by numerical computation of Laplace anti-transforms via de Hoog et al's algorithm [7]. The time dependent behavior of the solid stress is reported in Fig. 2, parametric with the solid volumetric fraction ϕ_o^s. Note that, for the specific problem at hand, the term $\breve{\sigma}_x^{(s)}(L,t)$ corresponds to the stress at the plug. Numerical results indicate that, in the LVP ($\phi_o^f = 0$), the biphasic system recovers the ramp-and-hold behavior of a Cauchy solid. This is in agreement with the theoretical predictions of Section 4.

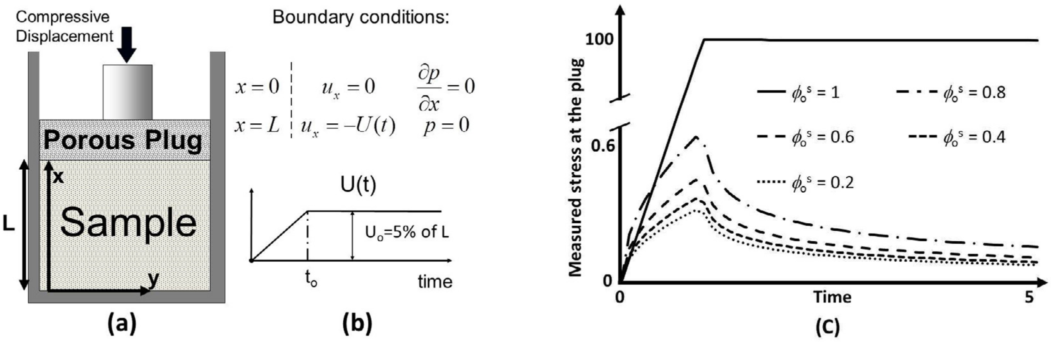

Figure 2: (a) Schematic of the simulated experimental setup for the uni-axial compressive stress-relaxation test: a biphasic sample is confined in an impermeable chamber and compressed axially by a porous plug allowing for fluid exudation. (b) Boundary conditions for the simulated experiment: compression of the biphasic sample is performed by applying a compression of 5 % of the sample via a ramp-and-hold displacement of the plug. (c) Time dependent response of solid stress at the plug parametric with ϕ_o^f.

6 CONCLUSIONS

In this contribution, an assessment of the consistency of TMCPM was provided showing that linearized TMCPM successfully recovers, in the limit of vanishing porosity, a traditional single phase continuum Cauchy model. This result was obtained with the aid of a homogenization technique that, although simple, provides insights on the coefficients $\bar{k}_r$ and $\hat{k}_s$ appearing in the $\boldsymbol{u} - p$ form of the quasi-stationary linearized theory. Overall, this contribution shows that TMCPM is a viable theoretical framework for modeling the mechanical behavior of intrinsically compressible biphasic systems.

REFERENCES

[1] Albers, B., Wilmanski, K. Influence of coupling through porosity changes on the propagation of acoustic waves in lin. poroelastic materials. *Arch. Mech.* (2006) **58**:313-325.

[2] Ateshian, G. A., Warden, W. H., Kim, J. J., Grelsamer, R. P., Mow, V.C. Finite deformation biphasic material properties of bovine articular cartilage from confined compression experiments. *J. Biomech.* (1997) **30**:1157–1164.

[3] Bowen, R. M. Compressible porous media models by use of the theory of mixtures. *Int. Jour. Eng. Scien.* (1982) **20**(6):697–735.

[4] de Boer, R., Ehlers, W. The development of the concept of effective stresses. *Acta Mech.* (1990) **83**:77–92.

[5] de Boer, R. Theory of Porous Media-Past and Present. *ZAMM* (1998) **78**(7):441–466.

[6] de Boer, R. Theoretical poroelasticity-a new approach. *Chaos. Solit. Fract.* (2005) **25**:861-878.

[7] De Hoog, F.R., Knight, J.H., Stokes, A.N. An improved method for numerical inversion of Laplace transforms. *SIAM J. Sci. Stat. Comput.* (1982) **3**:357–366.

[8] Diebels, S. A Micropolar Theory of Porous Media: Constitutive Modelling. *Transp. Por. Media* (1999) **34**, 193-208.

[9] Ehlers, W., Bluhm, J. (Eds.), *Porous Media.* Springer-Verlag, (2002).

[10] Eringen, A.C., Kafadar, C.B. Polar field theories, in *Continuum Physics IV*, A.C.Eringen (ed.), Academic Press, (1976).

[11] Gajo, A. A general approach to isothermal hyperelastic modeling of saturated porous media at finite strains with compressible solid. *Pr.Roy. Soc. A* (2010) **466**:3061–87.

[12] Markert, B., Ehlers, W. Dynamic wave propagation in infinite saturated porous media half spaces, *Comput. Mech.* (2012) **49**:319–336.

[13] Lopatnikov, S.L., Cheng, A.H.D. Macroscopic Lagrangian formulation of poroelasticity with porosity dynamics. *Jour. Mech. Phis. Sol.* (2004) **52**:2801–2839.

[14] Lopatnikov, S.L., Gillespie, J.W. Poroelasticity-I: Governing Equations of the Mechanics of Fluid-Saturated Porous Materials. *Trans. Por. Med.* (2010) **84**(2):471–492.

[15] Serpieri, R., Rosati, L. Formulation of a finite deformation model for the dynamic response of open cell biphasic media. *Jour. Mech. Phis. Sol.* (2011) **59**:841–862.

[16] Serpieri, R. A Rational Procedure for the Experimental Evaluation of the Elastic Coefficients in a Linearized Formulation of Biphasic Media with Compressible Constituents, *Transp. Por. Media* (2011) **90**:479-508.

[17] Svendsen, B., Hutter, K. On the thermodynamics of a mixture of isotropic materials with constraints. *Int. Jour. Eng. Scien.* (1995) **33**(14):2021–2054.

[18] Timoshenko, S.P., Goodier, J.N. *Theory of elasticity.* McGraw-Hill, (1987).

[19] Wilmanski, K. A Thermodynamic Model of Compressible Porous Materials with the Balance Equation of Porosity. *Transp. Por. Media* (1998) **32**:21–47.