The Generalized Hierarchical Product of Graphs

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Abstract

A generalization of both the hierarchical product and the Cartesian product of graphs is introduced and some of its properties are studied. We call it the generalized hierarchical product. In fact, the obtained graphs turn out to be subgraphs of the Cartesian product of the corresponding factors. Thus, some well-known properties of this product, such as a good connectivity, reduced mean distance, radius and diameter, simple routing algorithms and some optimal communication protocols, are inherited by the generalized hierarchical product. Besides some of these properties, in this paper we study the spectrum, the existence of Hamiltonian cycles, the chromatic number and index, and the connectivity of the generalized hierarchical product.

Keywords: Graph, Cartesian product, Hierarchical product, Diameter, Spectrum, Hamiltonian cycle, Coloring, Connectivity.

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1 Introduction

Some classical graphs, modeling real-life complex networks [14], present a modular or hierarchical structure [15]. This is the case, for instance, of networks with nodes having high degree, which are known as *hubs* [1]. These nodes usually play a critical role in the information flow of the system because many of the other nodes send and receive information through them. In [2] the authors introduced the hierarchical product of graphs which produces graphs with a strong (connectedness) hierarchy in their vertices. In fact, the obtained graphs turn out to be subgraphs of the Cartesian product of the corresponding factors. In particular, when each factor is the complete graph on two vertices, the resulting graph is a spanning tree of the hypercube, the so-called *binomial tree*, which is a data structure very useful in the context of algorithm analysis and design [7]. As it was shown in [3], an appealing property of this structure is that all its eigenvalues are distinct, a fact that has some structural consequences, such as the Abelianity of its automorphism group [13].

In this work we propose a new product of graphs, which in the extreme cases gives the hierarchical product and the Cartesian product. We call it the generalized hierarchical product. As before, the obtained graphs are again subgraphs of the Cartesian product. Hence, some well-known properties of the Cartesian product, such as a high connectivity, reduced mean distance and diameter, simple routing algorithms and some optimal communication protocols [10] are shared by the generalized hierarchical product.

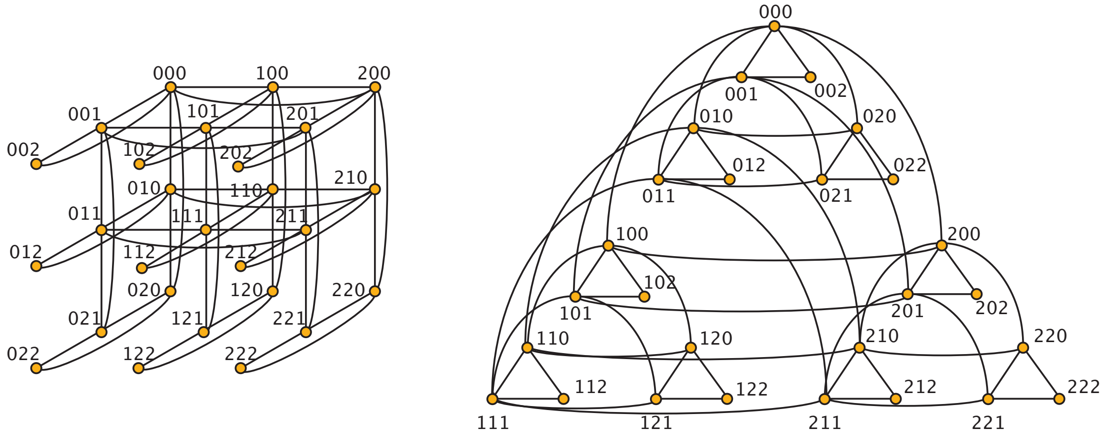

Figure 1: Two views of a generalized hierarchical product K_3^3 with $U_1 = U_2 = \{0, 1\}$.

Here we study some of this properties and also the following: the spectrum (through the characteristic polynomial), sufficient conditions for the existence of Hamiltonian cycles, the chromatic number and index, and, finally, the connectivity of the generalized hierarchical product.

In our study we use techniques from graph theory. For the basic concepts, notation and results about graphs, see for instance [5, 6].

2 The generalized hierarchical product

A natural generalization of the hierarchical product, proposed in [2], is as follows: Given N graphs $G_i = (V_i, E_i)$ and (non-empty) vertex subsets $U_i \subseteq V_i$, $i = 1, 2, \ldots, N-1$, the *generalized hierarchical product* $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$ is the graph with vertex set $V_N \times \cdots \times V_2 \times V_1$ and adjacencies:

$$x_N \ldots x_3 x_2 x_1 \sim \begin{cases} x_N \ldots x_3 x_2 y_1 & \text{if} \quad y_1 \sim x_1 \text{ in } G_1, \\ x_N \ldots x_3 y_2 x_1 & \text{if} \quad y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 \in U_1, \\ x_N \ldots y_3 x_2 x_1 & \text{if} \quad y_3 \sim x_3 \text{ in } G_3 \text{ and } x_i \in U_i,\ i = 1, 2, \\ \quad \vdots & \quad \vdots \\ y_N \ldots x_3 x_2 x_1 & \text{if} \quad y_N \sim x_N \text{ in } G_N \text{ and } x_i \in U_i,\ i = 1, 2, \ldots, N-1. \end{cases}$$

As an example, Fig. 1 shows two drawings of the generalized hierarchical product $K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, where $V(K_3) = \{0, 1, 2\}$ and $U_1 = U_2 = \{0, 1\}$.

In particular, the two "extreme" cases are the following:

- If all the subsets U_i are *singletons* (that is, the trivial graph with only one vertex), then the resulting graph is the (standard) hierarchical product [2].
- If $U_i = V_i$ for all $1 \leq i \leq N-1$, then the graph obtained is the Cartesian product of the graphs G_i.

2.1 Basic properties

Let us first list some basic properties on the degrees of the vertices in the generalized hierarchical product. The proofs are direct consequences of the definition.

- The degree of a vertex $v = x_N x_{N-1} \dots x_2 x_1$ in the generalized hierarchical product $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$ is

  $$\partial_H(v) = \partial_{G_1}(x_1) + \chi_{U_1}(x_1)\partial_{G_2}(x_2) + \cdots + [\chi_{U_1}(x_1) \cdots \chi_{U_{N-1}}(x_{N-1})]\partial_{G_N}(x_N),$$

  where ∂ and χ_{U_i} denotes, respectively, the degree and the characteristic function of the set U_i.

- The minimum and maximum degree of H are

  $$\delta_H = \min\{\delta_{G_1(\overline{U}_1)}, \delta_{G_1(U_1)} + \delta_{G_2(\overline{U}_2)}, \dots, \delta_{G_1(U_1)} + \cdots + \delta_{G_{N-1}(U_{N-1})} + \delta_{G_N}\},$$
  $$\Delta_H = \max\{\Delta_{G_1(\overline{U}_1)}, \Delta_{G_1(U_1)} + \Delta_{G_2(\overline{U}_2)}, \dots, \Delta_{G_1(U_1)} + \cdots + \Delta_{G_{N-1}(U_{N-1})} + \Delta_{G_N}\},$$

  where, for $i = 1, 2, \dots, N-1$, $\delta_{G_i(\overline{U}_i)} = \min_{x_i \notin U_i} \partial_{G_i}(x_i)$, $\delta_{G_i(U_i)} = \min_{x_i \in U_i} \partial_{G_i}(x_i)$, and, similarly, $\Delta_{G_i(\overline{U}_i)} = \max_{x_i \notin U_i} \partial_{G_i}(x_i)$, $\Delta_{G_i(U_i)} = \max_{x_i \in U_i} \partial_{G_i}(x_i)$, while δ_{G_N} and Δ_{G_N} are, respectively, the minimum and the maximum degrees of G_N.

- If, for every $i = 1, 2, \dots, N$, the graph G_i is ∂_i-regular, then the product graph $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$ contains exactly

  - $n_N(n_{N-1} - |U_{N-1}|)$ vertices of degree ∂_N;
  - $n_N|U_{N-1}|(n_{N-2} - |U_{N-2}|)$ vertices of degree $\partial_N + \partial_{N-1}$;

    $\vdots$

  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_2|(n_1 - |U_1|)$ vertices of degree $\partial_N + \partial_{N-1} + \cdots + \partial_2$;
  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_1|$ vertices of degree $\partial_N + \partial_{N-1} + \cdots + \partial_1$.

In the following proposition we show that, as in the case of the hierarchical product [2], the generalized hierarchical product is associative provided that the subsets U_i are appropriately chosen.

Proposition 2.1 *For $i = 1, 2, 3$, let G_i be a graph and, for $i = 1, 2$, $U_i \subseteq V_i$. The generalized hierarchical product satisfies*

$$G_3 \sqcap G_2(U_2) \sqcap G_1(U_1) = G_3 \sqcap \big(G_2 \sqcap G_1(U_1)\big)(U_2 \times U_1) = \big(G_3 \sqcap G_2(U_2)\big) \sqcap G_1(U_1).$$

Proof. To prove the first equality, we only need to show that in the generalized hierarchical product $G_3 \sqcap \big(G_2 \sqcap G_1(U_1)\big)(U_2 \times U_1)$ vertex $x_3(x_2x_1)$ has the same adjacencies as vertex $x_3x_2x_1$ in $G_3 \sqcap G_2(U_2) \sqcap G_1(U_1)$. Indeed,

$$x_3(x_2x_1) \sim \begin{cases} x_3(y_2y_1) & \text{if} \quad (y_2y_1) \sim (x_2x_1) \text{ in } G_2 \sqcap G_1(U_1); \text{ that is,} \\ & \text{if} \left\{ \begin{array}{l} y_1 \sim x_1 \text{ in } G_1 \text{ and } y_2 = x_2, \text{ or} \\ y_2 \sim x_2 \text{ in } G_2 \text{ and } y_1 = x_1 \in U_1, \end{array} \right. \\ y_3(x_2x_1) & \text{if} \quad y_3 \sim x_3 \text{ in } G_3 \text{ and } (x_2, x_1) \in U_2 \times U_1. \end{cases}$$

This is equivalent to

$$x_3(x_2x_1) \sim \begin{cases} x_3(x_2y_1) & \text{if} \quad y_1 \sim x_1 \text{ in } G_1; \\ x_3(y_2x_1) & \text{if} \quad y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 \in U_1; \\ y_3(x_2x_1) & \text{if} \quad y_3 \sim x_3 \text{ in } G_3,\ x_2 \in U_2 \text{ and } x_1 \in U_1. \end{cases}$$

Thus, the required isomorphism is simply $x_3(x_2x_1) \mapsto x_3x_2x_1$.

Analogously, we can prove the second equality by showing that in the generalized hierarchical product $(G_3 \sqcap G_2(U_2)) \sqcap G_1(U_1)$ vertex $(x_3x_2)x_1$ has the same adjacencies as vertex $x_3x_2x_1$ in $G_3 \sqcap G_2(U_2) \sqcap G_1(U_1)$. This completes the proof. $\square$

Corollary 2.2 *For $i = 1, 2, \ldots, N$, let G_i be a graph and, for $i = 1, 2, \ldots, N-1$, $U_i \subseteq V_i$. The generalized hierarchical product satisfies*

$$\begin{aligned} G_N \sqcap \cdots \sqcap G_1(U_1) &= \big(G_N \sqcap \cdots \sqcap G_2(U_2)\big) \sqcap G_1(U_1) \\ &= G_N \sqcap \big(G_{N-1} \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)\big)(U_{N-1} \times \cdots \times U_1). \end{aligned}$$

We have seen that the generalized hierarchical product is associative. Thus, for some of its properties, it suffices to study the case of two factors. With this aim, let $G_i = (V_i, E_i)$ be two graphs with vertex sets V_i, $i = 1, 2$, and consider a fixed (or *root*) subset $U_1 \subset V_1$. Then, the *generalized hierarchical product* $G_2 \sqcap G_1(U_1)$ is the graph with vertices x_2x_1, $x_i \in V_i$, and edges $\{x_2x_1, y_2y_1\}$ where either $y_2 = x_2$ and $y_1 \sim x_1$ in G_1, or $y_1 = x_1 \in U_1$ and $y_2 \sim x_2$ in G_2.

Thus, $G_2 \sqcap G_1(U_1)$ has $|V_2||V_1|$ vertices and $|U_1||E_2| + |V_2||E_1|$ edges. Also, notice that $G_2 \sqcap G_1(U_1)$ is a (spanning) subgraph of the Cartesian (or direct) product $G_2 \square G_1$. As a consequence, since clearly $K_1 \sqcap G(U) = G \sqcap K_1(u) = G$, the set of graphs with the binary operation $\sqcap$ is a *semigroup* with identity element K_1 (that is, a *monoid*). A simple consequence of the above is the following result, which generalizes a result given in [2].

Lemma 2.3 *Let $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$. For a fixed string $\boldsymbol{z}$ of appropriate length (for instance $\boldsymbol{z} = \boldsymbol{0} = 00\ldots0$), let $H\langle \boldsymbol{z}x_k \ldots x_1\rangle$ denote the subgraph of H induced by the vertex set $\{\boldsymbol{z}x_k \ldots x_1 \,|\, x_i \in V_i, 1 \le i \le k\}$. Let $H\langle x_N \ldots x_k\boldsymbol{z}\rangle$ be defined analogously. Then,*

(*a*) *$H\langle \boldsymbol{z}x_k \ldots x_1\rangle = G_k \sqcap G_{k-1}(U_{k-1}) \sqcap \cdots \sqcap G_1(U_1)$ for any fixed $\boldsymbol{z}$;*

(*b*) *$H\langle x_N \ldots x_k\boldsymbol{z}\rangle = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_k(U_k)$, for $\boldsymbol{z} \in U_{k-1} \times \cdots \times U_1$;*

(*c*) *$H\langle x_N \ldots x_k\boldsymbol{z}\rangle = m\,K_1$ (that is, a set of $m = n_N \cdots n_k$ singletons) where $n_i = |V_i|$, $k \le i \le N$, for $\boldsymbol{z} \notin U_{k-1} \times \cdots \times U_1$.*

Proof. We only need to notice that, for a fixed $\boldsymbol{z}$ of appropriate length,

- $\boldsymbol{z}x_k \ldots x_1 \sim \boldsymbol{z}y_k \ldots y_1$ in $H\langle x_N \ldots x_k\boldsymbol{z}\rangle$ if and only if $x_k \ldots x_1 \sim y_k \ldots y_1$ in $G_k \sqcap G_{k-1}(U_{k-1}) \sqcap \cdots \sqcap G_1(U_1)$; and
- $x_N \ldots x_k\boldsymbol{z} \sim y_N \ldots y_k\boldsymbol{z}$ in $H\langle x_N \ldots x_k\boldsymbol{z}\rangle$ if and only if $x_N \ldots x_k \sim y_N \ldots y_k$ in $G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_k(U_k)$ and $\boldsymbol{z} \in U_{k-1} \times \cdots \times U_1$.

This implies that the mapping $\boldsymbol{z}x_k \ldots x_1 \mapsto x_k \ldots x_1$ is an isomorphism between $H\langle \boldsymbol{z}x_k \ldots x_1\rangle$ and $G_k \sqcap G_{k-1}(U_{k-1}) \sqcap \cdots \sqcap G_1(U_1)$, and the mapping $x_N \ldots x_k\boldsymbol{z} \mapsto x_N \ldots x_k$ is an isomorphism between $H\langle x_N \ldots x_k\boldsymbol{z}\rangle$ and $G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_k(U_k)$ if $\boldsymbol{z} \in U_{k-1} \times \cdots \times U_1$. Moreover, if $\boldsymbol{z} \notin U_{k-1} \times \cdots \times U_1$, $H\langle x_N \ldots x_k\boldsymbol{z}\rangle$ consists of m independent vertices. □

3 Metric parameters

In this section we study some of the most relevant metric parameters of the generalized hierarchical product. Because of the associative property (Prop. 2.1), it is enough to study the product of two factors $H = G_2 \sqcap G_1(U_1)$.

We begin defining the distance through a vertex subset and some related concepts. Given a graph $G = (V, E)$ and a (non-empty) vertex subset $U \subset V$, a *path between vertices x and y through U*, denoted by $p_{G(U)}(x, y)$, is simply a x-y path of G containing some vertex $z \in U$ (vertex z could be the vertex x or y). Then, the distance *through*

U $\mathrm{dist}_{G(U)}(x,y)$ between x and y is the length of the shortest path $p_{G(U)}(x,y)$. Observe that, in general, this distance is not a metric in the usual sense because, for instance, $\mathrm{dist}_{G(U)}(x,x)$ is not necessarily 0. From this concept, we can define the metric parameters mean distance $d_{G(U)}$, eccentricity $\mathrm{ecc}_{G(U)}(x)$ of vertex x, radius $r_{G(U)}$ and diameter $D_{G(U)}$ all of them through U in the following way:

$$\begin{aligned} d_{G(U)} &= \frac{1}{n^2}\sum_{x,y\in V} d_{G(U)}(x,y), \\ \mathrm{ecc}_{G(U)}(x) &= \max_{y\in V} \mathrm{dist}_{G(U)}(x,y), \\ r_{G(U)} &= \min_{x\in V} \mathrm{ecc}_{G(U)}(x), \\ D_{G(U)} &= \max_{x\in V} \mathrm{ecc}_{G(U)}(x). \end{aligned}$$

Observe that the metric parameters through U coincide with the standard metric parameters if $U=V$: $d_{G(U)} \equiv d_G, \mathrm{ecc}_{G(U)}(x) \equiv \mathrm{ecc}_G(x)$, etc.

Let us consider two generic vertices $\boldsymbol{x}=(x_2,x_1)$ and $\boldsymbol{y}=(y_2,y_1)$ in the generalized hierarchical product $H=G_2 \sqcap G_1(U_1)$. Then,

$$\mathrm{dist}_H(\boldsymbol{x},\boldsymbol{y}) = \mathrm{dist}_{G_2}(x_2,y_2) + \mathrm{dist}_{G_1(U_1)}(x_1,y_1).$$

Indeed, if a shortest x_1-y_1 path through U_1 in G_1 is

$$x_1, v_1, \ldots, v_i, \ldots, v_{r-1}, y_1, \qquad (1)$$

where, say, $v_i \in U_1$ and a shortest x_2-y_2 path in G_2 is

$$x_2, w_1, \ldots, w_{s-1}, y_2, \qquad (2)$$

then a shortest $\boldsymbol{x}$-$\boldsymbol{y}$ path in H is

$$(x_2,x_1),(x_2,v_1),\ldots,(x_2,v_i),(w_1,v_i),\ldots,(y_2,v_i),(y_2,v_{i+1}),\ldots,(y_2,y_1). \qquad (3)$$

Theorem 3.1 *Let $H=(V,E)=G_2 \sqcap G_1(U_1)$ be the generalized hierarchical product of the graphs $G_1=(V_1,E_1)$, with vertex subset $U_1 \subset V_1$, and $G_2=(V_2,E_2)$, $n_2=|V_2|$, and metric parameters denoted as above. Then, the mean distance, eccentricity of a vertex $\boldsymbol{x}=(x_2,x_1)\in V$, radius and diameter of H are the following:*

(*a*) *Mean distance:*

$$d_H = d_{G_2} + \frac{1}{n_2}\left(d_{G_1} + (n_2-1)d_{G_1(U_1)}\right).$$

(*b*) *Eccentricity:*

$$\mathrm{ecc}_H(\boldsymbol{x}) = \mathrm{ecc}_{G_2}(x_2) + \mathrm{ecc}_{G_1(U_1)}(x_1).$$

(*c*) *Radius:*

$$r_H = r_{G_2} + r_{G_1(U_1)}.$$

(*d*) *Diameter:*

$$D_H = D_{G_2} + D_{G_1(U_1)}.$$

Proof. To prove (a) it is useful to consider the random variable X corresponding to the distance in H between the ordered pair of (not necessarily different) vertices $(\boldsymbol{x},\boldsymbol{y})$ chosen with uniform distribution. Let A be the event "the vertices $(\boldsymbol{x},\boldsymbol{y})$ belong to the same copy of G_1", with probability $\mathsf{P}(A)=\frac{1}{n_2}$. Now, d_H is simply the expected value of X, $E(X)$, which can be computed using the law of total expectation:

$$
\begin{aligned}
d_H &= E(X)=E(X|A)\,\mathsf{P}(A)+E(X|\overline{A})\,\mathsf{P}(\overline{A})\\
&= d_{G_1}\frac{1}{n_2}+\left(d_{G_1(U_1)}+d_{G_2}\frac{n_2^2}{n_2(n_2-1)}\right)\left(1-\frac{1}{n_2}\right)\\
&= \frac{1}{n_2}\left(d_{G_1}+(n_2-1)d_{G_1(U_1)}\right)+d_{G_2},
\end{aligned}
$$

where $E(X|\overline{A})$ has been computed by considering that the generic shortest path (3) is constructed from the shortest paths (1) in G_1 and (2) in G_2, with average values $d_{G_1(U_1)}$ and $d'_{G_2}=d_{G_2}\frac{n_2^2}{n_2(n_2-1)}$, respectively. Note that d'_{G_2} corresponds to the average distance between two different vertices x_2,y_2 in G_2 (since vertices $\boldsymbol{x},\boldsymbol{y}$ are in different copies of $G_1\cong H\langle \boldsymbol{z}x_1\rangle$, see Lemma 2.3).

Regarding the eccentricity, we have

$$
\mathrm{ecc}_H=\max_{y\in V} dist_H(x,y)=\max_{y_2\in V_2} dist_{G_2}(x_2,y_2)+\max_{y_1\in V_1} dist_{G_1}(x_1,y_1).
$$

Finally, the formulas (c) and (d) for the radius and the diameter are obtained from (b). $\square$

With respect to the mean distance, notice that when $U_1=V_1$, we have the Cartesian product $H=G_2\square G_1$, then $d_{G_1(U_1)}=d_{G_1}$ and (a) becomes $d_H=d_{G_2}+d_{G_1}$, as expected. Similar results hold for the eccentricity, radius and diameter.

4 Algebraic properties

The adjacency matrix of the generalized hierarchical product $H=G_2\sqcap G_1(U_1)$ can be written in terms of the adjacency matrices $\boldsymbol{A}_i$ of the factors G_i, $i=1,2$. To this end, first recall that the *Kronecker product* of two matrices $\boldsymbol{A}=(a_{ij})$ and $\boldsymbol{B}$, usually denoted by $\boldsymbol{A}\otimes\boldsymbol{B}$, is the matrix obtained by replacing each entry a_{ij} by the matrix $a_{ij}\boldsymbol{B}$ for every i and j. Then, if $V(G_1)=\{0,1,\dots,n_1-1\}$ and assuming that $U_1=\{0,1,\dots,r-1\}$, $1\le r\le n_1$, the adjacency matrix of the generalized hierarchical product $H=G_2\sqcap G_1(U_1)$ is (under the natural indexing of the rows and columns of the adjacency matrices):

$$
\boldsymbol{A}_H=\boldsymbol{A}_2\otimes\boldsymbol{D}_1+\boldsymbol{I}_2\otimes\boldsymbol{A}_1\cong\boldsymbol{D}_1\otimes\boldsymbol{A}_2+\boldsymbol{A}_1\otimes\boldsymbol{I}_2, \tag{4}
$$

where $\boldsymbol{D}_1=\mathrm{diag}(1,\overset{r}{\dots},1,0,\dots,0)$ and $\boldsymbol{I}_2$ (the identity matrix) have size $n_1\times n_1$ and $n_2\times n_2$, respectively. See [2] for the case $r=1$, corresponding to the hierarchical product. In the other extreme case, when $r=n_1$, then $\boldsymbol{D}_1=\boldsymbol{I}_1$ and $\boldsymbol{A}_H$ is the adjacency matrix of the Cartesian product $H=G_2\square G_1$.

For instance, when $G_1=G_2=K_3$ and $U_1=\{0,1\}$, as in the construction of Fig. 1, the adjacency matrix $\boldsymbol{A}_H$ of the generalized hierarchical product $H=K_3\sqcap K_3(U_1)$ turns out to be

$$
\boldsymbol{A}_H=\boldsymbol{D}_1\otimes\boldsymbol{A}_2+\boldsymbol{A}_1\otimes\boldsymbol{I}_2=\begin{pmatrix}\boldsymbol{A}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2\\ \boldsymbol{I}_2 & \boldsymbol{A}_2 & \boldsymbol{I}_2\\ \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{O}\end{pmatrix},
$$

where

$$\boldsymbol{D}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_1 = \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $\boldsymbol{A}_H$ is a 3×3 matrix of 3×3 blocks.

The next results provide a way to compute the spectrum of $H = G_2 \sqcap G_1(U_1)$. With this aim and for every eigenvalue λ of $\boldsymbol{A}_2$, we consider the $n_1 \times n_1$ matrix $\boldsymbol{A}(\lambda) = \lambda \boldsymbol{D}_1 + \boldsymbol{A}_1$. Note that this 'condensed' matrix is obtained from $\boldsymbol{A}_H$ by replacing every block $\boldsymbol{O}$ by 0, every block $\boldsymbol{I}_2$ by 1 and every block $\boldsymbol{A}_2$ by λ. Namely, every block is replaced for one of its eigenvalues.

Theorem 4.1 *Let λ be an eigenvalue of $\boldsymbol{A}_2$ with eigenvector $\boldsymbol{u}$, and let $\lambda_0, \lambda_1, \ldots, \lambda_{n_1-1}$ be the eigenvalues of $\boldsymbol{A}(\lambda)=\lambda\boldsymbol{D}_1+\boldsymbol{A}_1$, with corresponding eigenvectors $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n_1-1}$. Then, the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$ has the same eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n_1-1}$, with corresponding eigenvectors $\boldsymbol{w}_0 \otimes \boldsymbol{u}, \boldsymbol{w}_1 \otimes \boldsymbol{u}, \ldots, \boldsymbol{w}_{n_1-1} \otimes \boldsymbol{u}$.*

Proof. Using (4) giving $\boldsymbol{A}_H$, and with the fact that the Kronecker product satisfies $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u} \otimes \boldsymbol{B}\boldsymbol{v}$ (see, for instance, [11]), we get

$$\begin{aligned} \boldsymbol{A}_H(\boldsymbol{w}_i \otimes \boldsymbol{u}) &= (\boldsymbol{D}_1 \otimes \boldsymbol{A}_2 + \boldsymbol{A}_1 \otimes \boldsymbol{I}_2)(\boldsymbol{w}_i \otimes \boldsymbol{u}) \\ &= (\boldsymbol{D}_1 \otimes \boldsymbol{A}_2)(\boldsymbol{w}_i \otimes \boldsymbol{u}) + (\boldsymbol{A}_1 \otimes \boldsymbol{I}_2)(\boldsymbol{w}_i \otimes \boldsymbol{u}) \\ &= \boldsymbol{D}_1\boldsymbol{w}_i \otimes \boldsymbol{A}_2\boldsymbol{u} + \boldsymbol{A}_1\boldsymbol{w}_i \otimes \boldsymbol{u} \\ &= (\lambda\boldsymbol{D}_1 + \boldsymbol{A}_1)\,\boldsymbol{w}_i \otimes \boldsymbol{u} \\ &= \lambda_i(\boldsymbol{w}_i \otimes \boldsymbol{u}), \end{aligned}$$

so that λ_i is an eigenvalue of $\boldsymbol{A}_H$ with eigenvector $\boldsymbol{w}_i \otimes \boldsymbol{u}$ for every $0 \le i \le n_1 - 1$. Note that the eigenvectors $\boldsymbol{w}_0 \otimes \boldsymbol{u}, \boldsymbol{w}_1 \otimes \boldsymbol{u}, \ldots, \boldsymbol{w}_{n_1-1} \otimes \boldsymbol{u}$ are linearly independent because so are the eigenvectors $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n_1-1}$. □

Moreover, from the above result, we can give a formula for the characteristic polynomial of $H = G_2 \sqcap G_1(U_1)$ in terms of the eigenvalues of G_2 and the characteristic polynomials of some of the induced subgraphs of G_1. First, we introduce the following notation: Given a vertex subset $I \subset U_1 = \{0, 1, \ldots, r-1\}$, let $G_1^I = G_1 - I$ be the graph obtained from G_1 by removing the vertices in I, and let $\phi_1^I(x)$ be its characteristic polynomial. By convention, if $I = \emptyset$ we take $\phi_1^I(x) = \phi_1(x)$, and if $I = U_1 = V_1$ then $\phi_1^I(x) = 1$.

Theorem 4.2 *Given the graph G_1 with vertex subset $U_1 \subset V_1$, and the graph G_2 with eigenvalues $\operatorname{ev} G_2$, the characteristic polynomial of their generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$ is*

$$\phi_H(x) = \prod_{\lambda \in \operatorname{ev} G_2} \phi_\lambda(x), \tag{5}$$

where $\phi_\lambda(x)$ is the characteristic polynomial of $\boldsymbol{A}(\lambda)$ given by

$$\phi_\lambda(x) = \sum_{I \subset U_1} (-\lambda)^{|I|} \phi_1^I(x). \tag{6}$$

Proof. For every eigenvalue λ of G_2, the eigenvalues of H given by Theorem 4.1 are the roots of the characteristic polynomials $\phi_\lambda(x)$. Therefore, (5) holds since all its corresponding eigenvectors $\boldsymbol{w}_i \otimes \boldsymbol{u}$ of H, when varying the pair $(\lambda, \boldsymbol{u})$, are linearly independent.

The proof of Equation (6) is by induction on r. Let us consider the following matrix with rows and columns indexed by the elements of $V_1 = \{0, 1, \dots, n_1 - 1\}$:

$$\boldsymbol{M} = x\boldsymbol{I}_1 - \boldsymbol{A}(\lambda) = x\boldsymbol{I}_1 - \lambda\boldsymbol{D}_1 - \boldsymbol{A}_1 = \begin{pmatrix} x-\lambda & & & & & \\ & \ddots & & & & \\ & & x-\lambda & & & \\ & & & x & & \\ & & & & \ddots & \\ & & & & & x \end{pmatrix},$$

where, for simplicity, we have only written the diagonal entries omitting the elements of $-\boldsymbol{A}_1$. Given $i \in U_1$, let $\boldsymbol{M}^{\{i\}}$ be the matrix obtained from $\boldsymbol{M}$ by removing the row and column i and let $\boldsymbol{M}_{[i]}$ be the matrix obtained from $\boldsymbol{M}$ by changing the diagonal element with index i from $x - \lambda$ to x.

For $r = 1$, and expanding by the first row, we get

$$\begin{aligned} \phi_\lambda(x) &= \det \boldsymbol{M} = \det \begin{pmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{pmatrix} - \lambda \det \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \\ &= \det \boldsymbol{M}_{[0]} - \lambda \det \boldsymbol{M}^{\{0\}} = \phi_1(x) - \lambda\, \phi_1^{\{0\}}(x), \end{aligned}$$

and (6) holds.

Now, by the induction hypothesis, assume that the result holds for some $r > 1$. Then, if $|U_1| = r + 1$, we expand by the row r and we get

$$\begin{aligned} \phi_\lambda(x) &= \det \boldsymbol{M} = \det \boldsymbol{M}_{[r]} - \lambda \det \boldsymbol{M}^{\{r\}} \\ &= \sum_{I \subset U_1 \setminus \{r\}} (-\lambda)^{|I|} \phi^I(x) - \lambda \sum_{I \subset U_1 \setminus \{r\}} (-\lambda)^{|I|} \phi^{I \cup \{r\}}(x) \\ &= \sum_{I \subset U_1;\, r \notin I} (-\lambda)^{|I|} \phi^I(x) + \sum_{I \subset U_1;\, r \in I} (-\lambda)^{|I|} \phi^I(x) = \sum_{I \subset U_1} (-\lambda)^{|I|} \phi^I(x). \end{aligned}$$

This completes the proof. □

In particular, let us notice that, when the generalized hierarchical product coincides with the Cartesian product, namely when $U_1 = V_1$, the characteristic polynomial of $\boldsymbol{A}(\lambda) = \lambda\boldsymbol{I}_1 + \boldsymbol{A}_1$ is

$$\phi_\lambda(x) = \det\big((x - \lambda)\boldsymbol{I}_1 - \boldsymbol{A}_1\big) = \phi_1(x - \lambda), \tag{7}$$

for every eigenvalue λ of G_2. Thus, as it is well known (see, for instance, [8]), the eigenvalues of $H = G_2 \square G_1$ are $\lambda + \mu$, for each $\lambda \in \operatorname{ev} G_2$, $\mu \in \operatorname{ev} G_1$.

Moreover, as a by-product, for a generic graph $G_1 = G$ with vertex set V, $|V| = n$, and characteristic polynomial $\phi(x)$, we obtain

$$\begin{aligned} \phi(x - \lambda) &= \sum_{|I| \le n} (-\lambda)^{|I|} \phi^I(x) \\ &= \phi(x) - \Big(\phi^{\{0\}}(x) + \cdots + \phi^{\{n-1\}}(x)\Big)\lambda \\ &+ \Big(\phi^{\{0,1\}}(x) + \cdots + \phi^{\{n-2,n-1\}}(x)\Big)\lambda^2 + \cdots + (-1)^n \lambda^n, \end{aligned}$$

which, actually, is the Mac-Laurin decomposition of the polynomial $\psi(\lambda) \equiv \phi(x-\lambda)$. Therefore, the coefficient of λ is $\psi'(0) = -\phi'(x)$ giving the known formula $\phi'(x) = \sum_{u\in V} \phi^{\{u\}}(x)$ (see, for instance, [9]).

Going back to our study, the above reasonings can be used to derive an alternative expression for the characteristic polynomial of the generalized hierarchical product.

Theorem 4.3 *The characteristic polynomial of the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$ is:*

$$\phi_H(x) = \det\left(\sum_{I\subset U_1} (-\boldsymbol{A}_2)^{|I|} {\phi_1}^I(x)\right). \tag{8}$$

Proof. Working with the adjacency matrix of H, we have

$$\phi_H(x) = \det(x\boldsymbol{I} - \boldsymbol{A}_H) = \det \begin{pmatrix} x\boldsymbol{I}_2 - \boldsymbol{A}_2 & & & & & \\ & \ddots & & & & \\ & & x\boldsymbol{I}_2 - \boldsymbol{A}_2 & & & \\ & & & x\boldsymbol{I}_2 & & \\ & & & & \ddots & \\ & & & & & x\boldsymbol{I}_2 \end{pmatrix}.$$

Again, for simplicity, we have only written the diagonal entries. Thus, the n_1^2 blocks are of the types: $x\boldsymbol{I}_2 - \boldsymbol{A}_2$, $x\boldsymbol{I}_2$, $-\boldsymbol{I}_2$ or $\boldsymbol{O}$. Since every block commutates with each other, the result of Silvester [17] holds, and we can obtain $\phi_H(x)$ by computing the determinant in $\mathbb{R}^{n_2\times n_2}$, as in the previous theorem $\big($compare Eqs. (8) and (6)$\big)$. □

According to the cardinality r of the subset U_1, we next discuss some cases of the above result:

- $r = 1$: This corresponds to the hierarchical product $H = G_2 \sqcap G_1$. Thus, $\phi_1^I(x)$ is either $\phi_1^{\emptyset}(x) = \phi_1(x)$ or $\phi_1^{\{0\}}(x) \equiv \phi_1^*(x)$, the characteristic polynomial of $G_1 - \{0\}$. Therefore,

$$\begin{aligned} \phi_H(x) &= \det\big(\phi_1(x)\boldsymbol{I}_2 - \phi_1^*(x)\boldsymbol{A}_2\big) = \det\left(\phi_1^*(x)\left[\frac{\phi_1(x)}{\phi_1^*(x)}\boldsymbol{I}_2 - \boldsymbol{A}_2\right]\right) \\ &= \big(\phi_1^*(x)\big)^{n_2}\phi_2\left(\frac{\phi_1(x)}{\phi_1^*(x)}\right), \end{aligned}$$

  as obtained in [2].

- $r = 2$: In this case, Eq. (8) becomes

$$\begin{aligned} \phi_H(x) &= \det\Big(\phi_1(x)\boldsymbol{I}_2 - \Big(\phi_1^{\{0\}}(x) + \phi_1^{\{1\}}(x)\Big)\,\boldsymbol{A}_2 + \phi_1^{\{0,1\}}(x)\,\boldsymbol{A}_2^2\Big) \\ &= \det\Big(\phi_1^{\{0,1\}}(x)\big(\mu_+(x)\boldsymbol{I}_2 - \boldsymbol{A}_2\big)\,\big(\mu_-(x)\boldsymbol{I}_2 - \boldsymbol{A}_2\big)\Big) \\ &= \big(\phi_1^{\{0,1\}}(x)\big)^{n_2}\,\phi_2\big(\mu_+(x)\big)\,\phi_2\big(\mu_-(x)\big), \end{aligned}$$

  where

$$\mu_\pm(x) = \frac{\phi_1^{\{0\}}(x)+\phi_1^{\{1\}}(x)\pm\sqrt{\Big(\phi_1^{\{0\}}(x)+\phi_1^{\{1\}}(x)\Big)^2 - 4\,\phi_1(x)\,\phi_1^{\{0,1\}}(x)}}{2\,\phi_1^{\{0,1\}}(x)}.$$

- $r = n_1$: In this case, the generalized hierarchical product becomes the Cartesian product, $H = G_2 \sqcap G_1(V_1) = G_2 \square G_1$, and Eq. (8) gives

$$\begin{aligned} \phi_H(x) &= \det \sum_{|I|\le n_1} (-\boldsymbol{A}_2)^{|I|} {\phi_1}^{I}(x) \\ &= \det \Big(\phi_1(x)\boldsymbol{I}_2 - \big(\phi_1^{\{0\}}(x) + \cdots + \phi_1^{\{n_1-1\}}(x)\big)\boldsymbol{A}_2 + \cdots \\ &\qquad + (-1)^{n_1-1} n_1 x \boldsymbol{A}_2^{n_1-1} + (-1)^{n_1}\boldsymbol{A}_2^{n_1}\Big). \end{aligned}$$

Moreover, in the last case, using the same reasoning that allowed us to get Eq. (7), we obtain an expression for the characteristic polynomial of the Cartesian product of two graphs.

Lemma 4.4 *Given two graphs G_1, G_2, with respective adjacency matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, the characteristic polynomial of their Cartesian product $G_2 \square G_1$ is*

$$\phi_H(x) = \det\big(\phi_1(x\boldsymbol{I}_2 - \boldsymbol{A}_2)\big) = \det\big(\phi_2(x\boldsymbol{I}_1 - \boldsymbol{A}_1)\big).$$

To illustrate the application of both Theorem 4.2 and Theorem 4.3, we now compute the characteristic polynomial of the hierarchical product of $H = C_4 \sqcap K_5(U_1)$, the 4-cycle $G_2 = C_4$ and the complete graph $G_1 = K_5$ with $U_1 = \{0, 1, 2\}$. Recall that the spectrum of the former is $\mathrm{sp}(C_4) = \{2, 0^2, -2\}$, where the superscript stands for the eigenvalue multiplicity.

Using mathematical software, we get

$$\phi_H(x) = (x-3)\,(x+2)\,(x^2-5x-2)\,(x-1)^2\,(x+3)^2\,(x-4)^2\,(x+1)^{10}.$$

Now, in this case, the 'condensed matrix' is

$$\boldsymbol{A}(\lambda) = \begin{pmatrix} \lambda & 1 & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

For each $\lambda \in \mathrm{ev}\, C_4$, the characteristic polynomial of $\boldsymbol{A}(\lambda)$ is

$$\begin{aligned} \phi_2(x) &= (x+1)\,(x^2-5x-2)\,(x-1)^2, \\ \phi_0(x) &= (x-4)\,(x+1)^4, \\ \phi_{-2}(x) &= (x-3)\,(x+2)\,(x+1)\,(x+3)^2, \end{aligned} \tag{9}$$

and $\phi_H(x) = \phi_2(x)\,\phi_0(x)^2\,\phi_{-2}(x)$.

Taking into account that the characteristic polynomial of the complete graph K_n is $\phi(x) = (x-n+1)(x+1)^{n-1}$ and the fact that removing any vertex of K_n gives K_{n-1}, Theorem 4.2 yields

$$\phi_\lambda(x) = (x-4)\,(x+1)^4 - 3\,(x-3)\,(x+1)^3\,\lambda + 3\,(x-2)\,(x+1)^2\,\lambda^2 - (x+1)(x-1)\,\lambda^3,$$

and for $\lambda = 2, 0, -2$ we have (9), as expected.

Let $\boldsymbol{C}$ be the adjacency matrix of the 4-cycle. If we work with the block matrices as in Theorem 4.3, the characteristic polynomial is

$$\phi_H(x) = \det\big((x-4)(x+1)^4\boldsymbol{I}_4 - x-3)(x+1)^3\boldsymbol{C} + 3(x-2)(x+1)^2\boldsymbol{C}^2 - (x+1)(x-1)\boldsymbol{C}^3\big)$$

$$= \begin{vmatrix} (x^3-2x^2-x-16)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) & 6(x-2)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) \\ -(x+1)(3x^3-3x^2-11x-13) & (x^3-2x^2-x-16)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) & 6(x-2)(x+1)^2 \\ 6(x-2)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) & (x^3-2x^2-x-16)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) \\ -(x+1)(3x^3-3x^2-11x-13) & 6(x-2)(x+1)^2 & -(x+1)(3x^3-3x^2-11x-13) & (x^3-2x^2-x-16)(x+1)^2 \end{vmatrix}.$$

Then, computing the determinant, we get

$$\phi_H(x) = (x-3)(x+2)(x^2-5x-2)(x-1)^2(x+3)^2(x-4)^2(x+1)^{10},$$

as claimed. Note that, in this example, we have been able to simplify the expressions (6) and (8) because of the property mentioned above of the complete graph.

5 Hamiltonian cycles

It is well known that the Cartesian product $G = G_1 \square G_2$ of the Hamiltonian graphs G_1, G_2 is also Hamiltonian; see, for instance, [4]. As commented above, such a product corresponds to our hierarchical product $G_2 \sqcap G_1(U_1)$ when $U_1 = V_1$. Here we show that the existence of a Hamiltonian cycle is also granted under a much less restricted condition on the subset U_1.

Proposition 5.1 *If the graphs $G_i = (V_i, E_i)$, $i = 1, 2$, are Hamiltonian and the graph induced by the vertices in $U_1 \subset V_1$ has a path P_3 contained in the Hamiltonian cycle of G_1, then the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$ is Hamiltonian.*

Proof. The Hamiltonian cycle of H is constructed by appropriately joining n_2 Hamiltonian quasi-cycles of subgraphs isomorphic to G_1 and three Hamiltonian quasi-cycles of subgraphs isomorphic to G_2 (a quasi-cycle is a cycle with some edges removed), as it is shown in Fig. 2. □

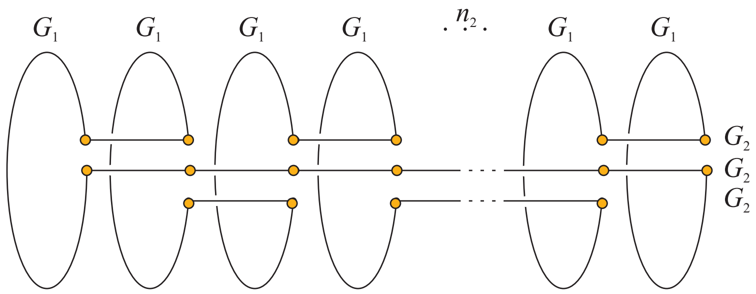

Figure 2: A Hamiltonian cycle in $G_2 \sqcap G_1(U_1)$ going through three copies of G_2 and n_2 copies of G_1.

In fact, if n_2 is even we also have the following result whose proof is based on the construction depicted in Fig. 3.

Proposition 5.2 *If the graphs $G_i = (V_i, E_i)$, $i = 1, 2$, are Hamiltonian, $n_2 = |V_2|$ is even and the graph induced by the vertices in $U_1 \subset V_1$ has an edge in the Hamiltonian cycle of G_1, then the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$ is Hamiltonian.*

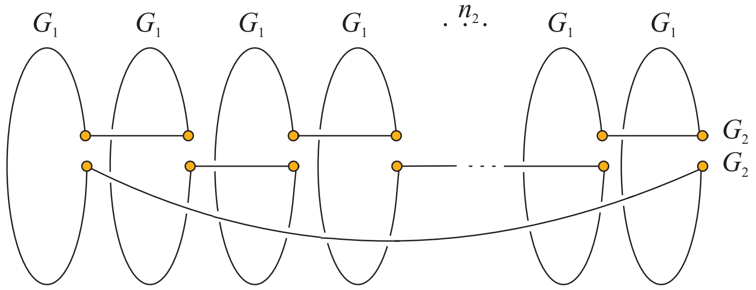

Figure 3: A Hamiltonian cycle in $G_2 \sqcap G_1(U_1)$ going through two copies of G_2 and n_2 copies of G_1 when n_2 is even.

6 Vertex- and edge-coloring

This section deals with vertex- and edge-coloring of the hierarchical product and the generalized hierarchical product of graphs.

As usual, we denote by $\chi(G)$ and $\chi'(G)$ the chromatic number and the chromatic index, respectively, of a graph G. For the Cartesian product, Sabidussi [16] proved that

$$\chi(G_2 \,\square\, G_1) = \max\{\chi(G_2), \chi(G_1)\}.$$

As it is shown in the following result, this is also the case for the chromatic number of the generalized hierarchical product $G_2 \sqcap G_1(U_1)$, for every $U_1 \subset V_1$, and, in particular, for the hierarchical product $G_2 \sqcap G_1$ (where $U_1 = \{0\}$).

Proposition 6.1 *Given two graphs G_1 and G_2 and a subset $U_1 \subset V_1$, the chromatic number of its generalized hierarchical product is*

$$\chi\big(G_2 \sqcap G_1(U_1)\big) = \max\{\chi(G_2), \chi(G_1)\}.$$

Proof. We already know that $G_2 \sqcap G_1(U_1)$ contains a subgraph isomorphic to G_2 and a subgraph isomorphic to G_1. Moreover, $G_2 \sqcap G_1(U_1)$ is a subgraph of $G_2 \,\square\, G_1$. This implies that

$$\max\{\chi(G_2), \chi(G_1)\} \leq \chi\big(G_2 \sqcap G_1(U_1)\big) \leq \chi(G_2 \,\square\, G_1) = \max\{\chi(G_2), \chi(G_1)\}.$$

□

According to Vizing's theorem [19], the chromatic index of a graph G satisfies

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ is the maximum degree of G. A graph G is said to be of *class 1* if its chromatic index equals its maximum degree, and of *class 2* in the other case.

Mahmoodian [12] showed that, if one of the two factors is of class 1, then their Cartesian product also is. Namely,

$$\chi'(G_1) = \Delta(G_1) \text{ or } \chi'(G_2) = \Delta(G_2) \Rightarrow \chi'(G_2 \,\square\, G_1) = \Delta(G_2 \,\square\, G_1) = \Delta(G_2) + \Delta(G_1).$$

In the next two results we use the following notation for the subgraphs isomorphic to G_1 and G_2 in $H = G_2 \sqcap G_1(U_1)$. The n_2 copies of G_1 in H are denoted by $G_{1i} = H\langle ix\rangle$, $i = 0, 1, \ldots, n_2 - 1$, and the $|U_1|$ copies of G_2 in H are denoted by $G_{2i} = H\langle xi\rangle$, $i = 0, 1, \ldots, r - 1$ (see Lemma 2.3).

For the particular case of the hierarchical product, we have the following result.

Proposition 6.2 *The chromatic index of the hierarchical product of the graphs G_1 and G_2 satisfies*

$$\chi'(G_2 \sqcap G_1) = \max\{\Delta(G_2) + d_0, \chi'(G_1)\},$$

where $d_0 = \partial_{G_1}(0)$ denotes the degree of the root vertex of G_1.

Proof. First, notice that

$$m = \max\{\Delta(G_2) + d_0, \chi'(G_1)\} \le \chi'(G_2 \sqcap G_1).$$

To show the reverse inequality, we need to give a proper edge-coloring of $G_2 \sqcap G_1$ with m colors.

Note first that for every $m \ge \chi'(G_1)$, there exists a proper m-edge-coloring of G_1 with the $d_0(\ge 1)$ edges incident to vertex 0 having some prescribed colors.

Since, by Vizing's theorem, $\chi'(G_2) - 1 \le \Delta(G_2)$, we have

$$m \ge \Delta(G_2) + d_0 \ge \chi'(G_2) - 1 + d_0 \ge \chi'(G_2).$$

Therefore, we can have a proper edge-coloring of the subgraph G_{20} using m colors.

With respect to each subgraph G_{1i}, as $m \ge \chi'(G_1)$, we can also have a proper edge-coloring of G_{1i} with m colors. However, to avoid conflicts with the colors of the edges of G_{20} incident to vertex $i0$, we cannot use $\partial_{G_2}(i) \le \Delta(G_2)$ of the m available colors and this gives the following number of available colors:

$$m - \partial_{G_2}(i) \ge m - \Delta(G_2) \ge d_0,$$

which are enough to color the edges of G_{1i} incident to $i0$. □

For the generalized hierarchical product of graphs, we can give the following bounds.

Proposition 6.3 *The chromatic index of $H = G_2 \sqcap G_1(U_1)$ satisfies*

$$\max\{\Delta(G_2) + \Delta_{U_1}(G_1), \chi'(G_1)\} \le \chi'(H) \le \max\{\chi'(G_2) + \Delta_{G_1(U_1)}, \chi'(G_1)\},$$

where $\Delta_{U_1}(G_1) \equiv \Delta_{G_1(U_1)}$ and $\Delta_{V_1}(G_1) \equiv \Delta_{G_1}$.

Proof. To properly color the edges of $H = G_2 \sqcap G_1(U_1)$ we have to color the n_2 copies of G_1. Thus, we need at least $\chi'(G_1)$ colors. Moreover, in H there is at least one vertex of degree $\Delta(G_2) + \Delta_{U_1}(G_1)$. This implies the lower bound,

$$\max\{\Delta(G_2) + \Delta_{U_1}(G_1), \chi'(G_1)\} \le \chi'(H).$$

To show that the upper bound also holds, we color the edges of H in the following way. We fix the same edge-coloring for all the copies of G_1. Some of the $\chi'(G_1)$ colors already used can also be employed to color the copies of G_2. In fact, for a fixed $i \in U_1$, all the vertices of G_{2i} have the same set of forbidden colors, i.e., the colors used in G_{1j} to color the edges incident to vertex ji, which are independent of j. Thus, to color G_{2i}, we have $\chi'(G_1) - \partial_{G_1}(i)$ available colors. If $\chi'(G_1) \ge \chi'(G_2) + \partial_{G_1}(i)$, we are done. Otherwise, we need to add to our set of colors

$$\chi'(G_2) - \big(\chi'(G_1) - \partial_{G_1}(i)\big) = \chi'(G_2) + \partial_{G_1}(i) - \chi'(G_1)$$

new colors. That is, we will use in total the number of colors

$$\chi'(G_2) + \partial_{G_1}(i) - \chi'(G_1) + \chi'(G_1) = \chi'(G_2) + \partial_{G_1}(i).$$

Taking the maximum over all the vertices in U_1, we get

$$\chi'(H) \le \max\{\chi'(G_2) + \Delta_{U_1}(G_1), \chi'(G_1)\}.$$

□

Corollary 6.4 *If either G_1 is of class 1 and U_1 contains a vertex of degree $\Delta(G_1)$, or G_2 is of class 1, then the chromatic index of $H = G_2 \sqcap G_1(U_1)$ satisfies*

$$\chi'(H) = \max\{\Delta(G_2) + \Delta_{U_1}(G_1), \chi'(G_1)\}.$$

7 Connectivity

In the current section we give some results on the vertex-connectivity of the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$. Observe that, as in the case of the Cartesian product $G_2 \Box G_1$, H is connected if and only if G_2 and G_1 are. In fact, for such an extreme case (where $U_1 = V_1$), only recently an exact value of its connectivity has been given [18]. Namely,

$$\kappa(G_2 \Box G_1) = \min\{\kappa_1|V_2|, \kappa_2|V_1|, \delta_1 + \delta_2\},$$

where κ_i and δ_i denote, respectively, the connectivity and minimum degree of G_i, $i = 1, 2$.

To study the general case, where $U_1 \subsetneq V_1$, we need to introduce the following new connectivity parameter: For a graph $G = (V, E)$ and a vertex subset $U \subsetneq V$, let $\kappa(U|\overline{U})$ be the minimum cardinality of a vertex subset S such that in $G - S$ there exist some vertex $u \in \overline{U}$ and there is no path from u to any vertex of U. In particular, taking $S = U \neq V$, we get $\kappa(U|\overline{U}) \leq |U|$.

Proposition 7.1 *Using the above notation, the connectivity κ_H of the generalized hierarchical product $H = G_2 \sqcap G_1(U_1)$, $U_1 \subsetneq V_1$, satisfies*

$$\kappa_H \leq \min\{\kappa_1|V_2|, \kappa(U_1|\overline{U}_1), \delta_H\},$$

where $\delta_H = \min\{\delta_{G_1(\overline{U}_1)}, \delta_{G_1(U_1)} + \delta_{G_2}\}$.

Proof. The fact that $\kappa_H \leq \delta_H$ for any H is trivial. Moreover, $\kappa_H \leq \kappa_1|V_2|$, because $H = G_2 \sqcap G_1(U_1)$ is a subgraph of $G_2 \Box G_1(U_1)$ with the same vertex set. Finally, we have seen in the section on the metric parameters that any path between vertices (x_2, y_2) and (y_2, y_1), with $x_2 \neq y_2$ and $x_1 \notin U_1$, requires the presence of a x_1-y_1 path through U_1 in G_1, which does not exist if $\kappa(U_1|\overline{U}_1)$ vertices have been removed from the copy G_{1x_2}. Therefore, we also have $\kappa_H \leq \kappa(U_1|\overline{U}_1)$, and this complete the proof. □

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