

A CONSTITUTIVE MODEL OF COUPLED THERMOELASTICITY WITH PLASTICITY

R. BARRETTA^{*}, F. MAROTTI DE SCIARRA[†] AND M. SALERNO⁺

^{*} University of Naples Federico II, Department of Structures for Engineering and Architecture
Via Claudio 21 80121 Naples, Italy
e-mail: rabaret@unina.it

[†] University of Naples Federico II, Department of Structures for Engineering and Architecture
Via Claudio 21 80121 Naples, Italy
e-mail: marotti@unina.it

⁺ University of Naples Federico II, Department of Structures for Engineering and Architecture
Via Forno Vecchio 36 80134 Naples, Italy
e-mail: maria.salerno@unina.it

Key words: Thermoelasticity without energy dissipation, Thermoplasticity, Constitutive model, Thermodynamic potentials.

Abstract. A consistent set of the thermodynamic functions in the framework of the Green and Naghdi (GN) coupled thermoelasticity with plasticity is addressed. The relations between these functions are provided and the constitutive relations are obtained.

1 INTRODUCTION

It is well-known that the classical linear theory of heat conduction, based on Fourier's law for the thermal flux, predicts that a thermal effects at a point of a body is felt instantly at other points of the body. Therefore in past years several alternative theories of heat conduction have been proposed. In this paper we consider the non-classical theory of thermoelasticity developed by Green and Naghdi [1,2] which incorporates the approach based on Fourier's law (referred to as type I), the theory without energy dissipation (type II) and a theory which allows finite wave propagation as well as energy dissipation (type III). Contributions on the Green and Naghdi (GN) approach can be found, among others, in Bargmann and Steinmann [3] and references therein.

An analytical treatment of coupled thermoelastic problems is complex so that the development of alternative methods of analysis turns out to be important. Accordingly variational formulations of such problems can be of great interest from a theoretical point of view and from a computational standpoint since they are the foundation to develop mixed finite elements.

In the present paper we treat the non-classical thermoelastic problem following the GN model of type II. The purpose of the contribution is to provide a constitutive model of thermoelasticity coupled with plasticity suitable to formulate variational formulations (see Romano et al. [4-6] and Marotti de Sciarra [7-9] for nonlocal problems), also for functionally graded materials [10-13], in order to consistently derive a finite element [14-15] or boundary element [16] approach and the related algorithmic procedure.

The GN coupled thermoelastic model with plasticity is formulated within the framework of generalized standard material (Halphen and Nguyen [17]).

Moreover we adopt the hypotheses of small strains, additive plasticity and rate-dependent plasticity (Bartels and Roubíček [18]), whereas we get the assumption that the elastoplastic behaviour of the material does not influence the thermoelastic constitutive properties, i.e. the thermal behaviour in all the elastic ranges. (Bertram and Krawietz [19]).

The relations between the thermodynamic functions and their alternative formulations are obtained. A consistent set of the eight thermodynamic functions has been derived into the framework of convex analysis and conjugate functions.

In the considered GN model, the thermodynamic functions depend on three state variables where the dual set of constitutive state variables are given by strain, stresses, temperature, entropy, gradient of thermal displacement and entropy flux.

Using a systematic procedure based on Legendre transforms, the thermodynamic potentials are expressed in terms of different combinations of the abovementioned state variables obtaining a set of eight alternative functions. A characteristic feature of the proposed approach is that the derivatives of the thermodynamic potentials and of their alternative forms provide different expressions of the constitutive relations which turn out to be all equivalent each others.

2 CLASSICAL THERMODYNAMIC FRAMEWORK

Let us consider the classical thermodynamic framework of the thermo-mechanical processes for an elastoplastic body. The constitutive model follows from the first principle of thermodynamics, expressed in pointwise form of the internal energy U :

$$\dot{U} = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} + \dot{Q} + r \quad (1)$$

where $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ denote the strain and the stress tensors, $Q = -\text{div } \mathbf{q}$ is the heat supply from conduction, being \mathbf{q} the heat flux, and r is the heat supply from irradiation.

The second principle of thermodynamics is assumed in form of Clausius-Duhem inequality:

$$\dot{\eta} - \left(\frac{r}{\theta} - \frac{\text{div } \mathbf{q}}{\theta} \right) \geq 0 \quad (2)$$

where $\dot{\eta}$ is the internal entropy production rate per unit volume and θ is the absolute temperature ($\theta > 0$).

The thermodynamic eqs. (1) and (2) yield the non-negative total dissipation:

$$D = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - \dot{U} + \dot{\eta}\theta - \nabla\theta \cdot \frac{\mathbf{q}}{\theta} \geq 0 \quad (3)$$

The inequality (3) is employed to derive some restrictions on the constitutive equations.

The thermo-elastoplastic solid is assumed to have a saddle internal energy (convex in terms of elastic strain and internal variables and concave in terms of the elastic entropy), differentiable with respect to the arguments, as

$$U = U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e) \quad (4)$$

where the plastic behaviour with hardening is introduced by a suitable set of kinematic internal variables $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ that describe the kinematic and isotropic hardening. The dual static internal variables are $(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$.

Moreover, following a consolidated procedure, the total strain $\boldsymbol{\varepsilon}$ is assumed to be the sum of an elastic strain $\boldsymbol{\varepsilon}_e$ and of a plastic strain $\boldsymbol{\varepsilon}_p$ (Coleman and Owen [20])

Expanding the derivative of $U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e)$ and substituting in (3), we can write the overall dissipation:

$$D = (\boldsymbol{\sigma} - \partial_{\boldsymbol{\varepsilon}_e} U) \cdot \dot{\boldsymbol{\varepsilon}}_e + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \partial_{\alpha_1} U \cdot \dot{\alpha}_1 - \partial_{\alpha_2} U \cdot \dot{\alpha}_2 - \partial_{\eta_e} U \cdot \dot{\eta}_e + \dot{\eta}\theta - \nabla\theta \cdot \frac{\mathbf{q}}{\theta} \geq 0 \quad (5)$$

that leads to

$$\begin{aligned} \boldsymbol{\sigma} &= \partial_{\boldsymbol{\varepsilon}_e} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e) \\ \boldsymbol{\chi}_1 &= \partial_{\alpha_1} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e) \quad \boldsymbol{\chi}_2 = \partial_{\alpha_2} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e) \\ \theta &= \partial_{\eta_e} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e) \end{aligned} \quad (6)$$

where $\partial_{\boldsymbol{\varepsilon}_e}$ denotes the partial derivative of U with respect to $\boldsymbol{\varepsilon}_e$ and similarly the other ones.

Then the inequality (3) reduces to following form

$$D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \boldsymbol{\chi}_1 \cdot \dot{\alpha}_1 - \boldsymbol{\chi}_2 \cdot \dot{\alpha}_2 - \theta \cdot \dot{\eta}_e + \dot{\eta}\theta - \nabla\theta \cdot \frac{\mathbf{q}}{\theta} \geq 0$$

Accordingly, introducing the following additive decomposition of the total entropy η in elastic and plastic parts, it turns out to be:

$$\eta = \eta_e + \eta_p \quad (7)$$

so then it follows

$$D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \boldsymbol{\chi}_1 \cdot \dot{\alpha}_1 - \boldsymbol{\chi}_2 \cdot \dot{\alpha}_2 + \theta \cdot \dot{\eta}_p - \nabla\theta \cdot \frac{\mathbf{q}}{\theta} \geq 0 \quad (8)$$

where $\dot{\eta}_p$ is the rate plastic entropy $\dot{\eta}_p = \dot{\eta} - \dot{\eta}_e$.

The plastic entropy is related to dissipative plastic structural changes and does not affect the internal energy (Simo and Miehe [21]).

3 NON-CLASSICAL DISSIPATIONLESS THERMOPLASTICITY

In this paper we analyse the GN thermoelastic framework of type II coupled with plastic behaviour with linear hardening.

The GN model introduces a scalar variable α , called the thermal displacement, which is related to the temperature ϑ by the relation:

$$\alpha(\mathbf{x}, t) = \int_0^t \vartheta(\mathbf{x}, \tau) dt + \alpha_0 \quad (9)$$

where \mathbf{x} is a point pertaining to the thermo-elastoplastic body defined on a regular bounded domain Ω of an Euclidean space, $\vartheta = \theta - \theta_r$ represents the temperature variation from the uniform reference temperature θ_r and α_0 is the initial value of α at the time $t=0$. As a consequence the time derivative of the thermal displacement field is the temperature variation, i.e. $\dot{\alpha} = \dot{\vartheta}$. The thermal displacement gradient $\nabla\alpha$ is denoted by $\mathbf{g} = \nabla\alpha$.

The internal energy assumes the following form:

$$U = U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g}) \quad (10)$$

Expanding the derivative of $U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g})$ and substituting in (3), we can write the overall dissipation:

$$D = (\boldsymbol{\sigma} - \partial_{\boldsymbol{\varepsilon}_e} U) \cdot \dot{\boldsymbol{\varepsilon}}_e + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \partial_{\alpha_1} U \cdot \dot{\alpha}_1 - \partial_{\alpha_2} U \cdot \dot{\alpha}_2 - \partial_{\eta_e} U \cdot \dot{\eta}_e - \partial_{\mathbf{g}} U \cdot \dot{\mathbf{g}} + \dot{\eta}\theta - \nabla\theta \cdot \frac{\mathbf{q}}{\theta} \geq 0 \quad (11)$$

that leads to

$$\begin{aligned} \boldsymbol{\sigma} &= \partial_{\boldsymbol{\varepsilon}_e} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g}) \\ \chi_1 &= \partial_{\alpha_1} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g}) \quad \chi_2 = \partial_{\alpha_2} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g}) \\ \theta &= \partial_{\eta_e} U(\boldsymbol{\varepsilon}_e, \alpha_1, \alpha_2, \eta_e, \mathbf{g}) \end{aligned} \quad (12)$$

Recalling that $\dot{\mathbf{g}} = \nabla\dot{\alpha} = \nabla\dot{\theta}$, introducing the entropy flux vector $\mathbf{p} = \frac{\mathbf{q}}{\theta}$ and considering the relations (12), the total dissipation (11) reduces to this form

$$D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \chi_1 \cdot \dot{\alpha}_1 - \chi_2 \cdot \dot{\alpha}_2 + \theta \cdot \dot{\eta}_p - (\partial_{\mathbf{g}} U - \mathbf{p}) \cdot \nabla\theta \geq 0 \quad (13)$$

that leads to

$$\mathbf{p} = -\partial_{\mathbf{g}} U \quad (14)$$

Finally the total dissipation in the GN model turns to be:

$$D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \chi_1 \cdot \dot{\alpha}_1 - \chi_2 \cdot \dot{\alpha}_2 + \theta \cdot \dot{\eta}_p \geq 0 \quad (15)$$

being null the dissipation due to heat conduction. It is apparent that the above dissipation coincides to the classical mechanical dissipation $D_m = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}_p - \chi_1 \cdot \dot{\alpha}_1 - \chi_2 \cdot \dot{\alpha}_2$ [22] and the thermal dissipation $D_{th} = \theta \cdot \dot{\eta}_p$.

4 ENERGETIC FUNCTIONS FOR COUPLED THERMOELASTICITY WITH PLASTICITY

In this paper the GN thermodynamic framework is formulated by introducing the following generalized vectors of the kinematic and dual static internal variables:

- total strain $\underline{\underline{\boldsymbol{\varepsilon}}} = (\boldsymbol{\varepsilon}, \mathbf{0}, \mathbf{0})$,
- elastic strain $\underline{\underline{\boldsymbol{\varepsilon}}}_e = (\boldsymbol{\varepsilon}_e, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$,
- plastic strain $\underline{\underline{\boldsymbol{\varepsilon}}}_p = (\boldsymbol{\varepsilon}_p, -\boldsymbol{\alpha}_1, -\boldsymbol{\alpha}_2)$,
- stress $\underline{\underline{\boldsymbol{\sigma}}} = (\boldsymbol{\sigma}, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$,

The scalar product between dual quantities $\underline{\underline{\boldsymbol{\varepsilon}}}$ and $\underline{\underline{\boldsymbol{\sigma}}}$ has the mechanical meaning of the internal virtual work:

$$\underline{\underline{\boldsymbol{\sigma}}}^* \underline{\underline{\boldsymbol{\varepsilon}}}_e = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_e + \boldsymbol{\chi}_1 \cdot \boldsymbol{\alpha}_1 + \boldsymbol{\chi}_2 \cdot \boldsymbol{\alpha}_2 \quad (16)$$

Let us now derive the complete set of the energy functions for the considered GN model and the related constitutive relations in the framework provided by convex/saddle functions (Houlsby and Puzrin [23]) and, for different models, see also [24-28] and [29-31].

Classically, the analysis of a thermodynamic process starts from the Helmholtz free energy Φ , which it is a function of strain $\underline{\underline{\boldsymbol{\varepsilon}}}$, temperature θ and thermal gradient vector \mathbf{g} , that is $\Phi(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{g})$. The Helmholtz free energy Φ is assumed to be convex in the strain $\underline{\underline{\boldsymbol{\varepsilon}}}_e$ and in the thermal gradient vector \mathbf{g} and concave in the temperature θ at any point \mathbf{x} . The dual variables of the triplet $(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{g})$ are the stress $\underline{\underline{\boldsymbol{\sigma}}}$, the elastic entropy η_e and the entropy flux vector \mathbf{p} .

Now we derive the three conjugates of the free energy Φ with respect to one state variable.

A saddle function φ_1^* can be associated with the Helmholtz free energy Φ , by considering the conjugate of Φ with respect to the strain tensor in the form:

$$\varphi_1^*(\underline{\underline{\boldsymbol{\sigma}}}, \theta, \mathbf{g}) = \sup_{\underline{\underline{\boldsymbol{\varepsilon}}}_e} \{ \underline{\underline{\boldsymbol{\sigma}}}^* \underline{\underline{\boldsymbol{\varepsilon}}}_e - \Phi(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{g}) \} \quad (17)$$

The function φ_1^* turns out to be convex in $(\underline{\underline{\boldsymbol{\sigma}}}, \theta)$ and concave in \mathbf{g} .

The conjugate of the function Φ with respect to θ is the concave function φ_2^* given by:

$$\varphi_2^*(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \eta_e, \mathbf{g}) = \inf_{\theta} \{ \eta_e \theta - \Phi(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{g}) \} \quad (18)$$

where η_e is the entropy.

A saddle function φ_3^* can be associated with the Helmholtz free energy Φ , by considering the conjugate of Φ with respect to the thermal gradient vector \mathbf{g} , in the form:

$$\varphi_3^*(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{p}) = \sup_{\mathbf{g}} \{ \mathbf{p}^* \mathbf{g} - \Phi(\underline{\underline{\boldsymbol{\varepsilon}}}_e, \theta, \mathbf{g}) \} \quad (19)$$

and it is convex in (θ, \mathbf{p}) and concave in $\underline{\underline{\boldsymbol{\varepsilon}}}_e$. The variable \mathbf{p} denotes the entropy flux vector and is related to the heat flux \mathbf{q} by the relation $\mathbf{p} = \mathbf{q}/\vartheta$.

Let us now derive the conjugate functions of the Helmholtz free energy with respect to the pairs of variables $(\underline{\boldsymbol{\varepsilon}}_e, \theta)$, (θ, \mathbf{g}) and $(\underline{\boldsymbol{\varepsilon}}_e, \mathbf{g})$.

The conjugate of Φ with respect to $(\underline{\boldsymbol{\varepsilon}}_e, \theta)$ is the saddle function φ_{12}^* , convex in $\underline{\boldsymbol{\sigma}}$ and concave in (η_e, \mathbf{g}) ; given by:

$$\varphi_{12}^*(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) = \inf_{\theta} \sup_{\underline{\boldsymbol{\varepsilon}}_e} \{ \underline{\boldsymbol{\sigma}} * \underline{\boldsymbol{\varepsilon}}_e + \eta_e \theta - \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \} \quad (20)$$

The saddle function φ_{23}^* is the conjugate of Φ with respect to the pair (θ, \mathbf{g}) :

$$\varphi_{23}^*(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) = \inf_{\theta} \sup_{\mathbf{g}} \{ \eta_e \theta + \mathbf{p} * \mathbf{g} - \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \} \quad (21)$$

and results concave in $(\underline{\boldsymbol{\varepsilon}}_e, \eta_e)$ and convex in \mathbf{p} .

The conjugate of the Helmholtz free energy Φ with respect to the pair $(\underline{\boldsymbol{\varepsilon}}_e, \mathbf{g})$ is the convex function φ_{13}^* in $(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p})$ given by:

$$\varphi_{13}^*(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) = \sup_{\underline{\boldsymbol{\varepsilon}}_e, \mathbf{g}} \{ \underline{\boldsymbol{\sigma}} * \underline{\boldsymbol{\varepsilon}}_e + \mathbf{p} * \mathbf{g} - \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \} \quad (22)$$

The saddle function φ^* is the conjugate of the Helmholtz free energy Φ with respect to the triplet $(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$ and is defined as:

$$\varphi^*(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) = \inf_{\theta} \sup_{\underline{\boldsymbol{\varepsilon}}_e, \mathbf{g}} \{ \underline{\boldsymbol{\sigma}} * \underline{\boldsymbol{\varepsilon}}_e + \eta_e \theta + \mathbf{p} * \mathbf{g} - \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \} \quad (23)$$

The conjugate function φ^* turns out to be convex in $(\underline{\boldsymbol{\sigma}}, \mathbf{p})$ and concave in η_e .

It is worth noting that further relations connecting the above conjugate potentials can be derived following the above mentioned procedure but they are not explicitly provided for sake of conciseness.

5 CONSTITUTIVE RELATIONS

Let us now show that the constitutive relations for the GN thermoelasticity can involve, as a maximum, eight different thermodynamic potentials.

To this end the following equivalent relations in terms of the functions Φ , φ_1^* , φ_2^* , φ_{12}^* and φ^* , φ_{23}^* , φ_{13}^* , φ_3^* are provided:

$(\underline{\boldsymbol{\sigma}}, -\eta_e, -\mathbf{p}) = d\Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$	(24)
$(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) = d\varphi^*(\underline{\boldsymbol{\sigma}}, -\eta_e, -\mathbf{p})$	
$(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) = d\varphi_1^*(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g})$	
$(-\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) = d\varphi_{23}^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, -\mathbf{p})$	
$(-\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) = d\varphi_2^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, \mathbf{g})$	
$(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) = d\varphi_{13}^*(\underline{\boldsymbol{\sigma}}, \theta, -\mathbf{p})$	
$(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) = d\varphi_{12}^*(\underline{\boldsymbol{\sigma}}, -\eta_e, \mathbf{g})$	
$(-\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) = d\varphi_3^*(\underline{\boldsymbol{\varepsilon}}_e, \theta, -\mathbf{p})$	

The relations (24) among conjugate functions are equivalent to the following Fenchel's equalities, in terms of the functions Φ , φ_1^* , φ_2^* , φ_{12}^* and φ^* , φ_{23}^* , φ_{13}^* , φ_3^* :

$$\begin{aligned}
\Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) + \varphi^*(\underline{\boldsymbol{\sigma}}, -\eta_e, -\mathbf{p}) &= \underline{\boldsymbol{\sigma}}^* \underline{\boldsymbol{\varepsilon}}_e - \eta_e \theta - \mathbf{p}^* \mathbf{g} \\
-\varphi_{23}^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, -\mathbf{p}) + \varphi_1^*(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) &= \underline{\boldsymbol{\sigma}}^* \underline{\boldsymbol{\varepsilon}}_e + \eta_e \theta + \mathbf{p}^* \mathbf{g} \\
\varphi_{13}^*(\underline{\boldsymbol{\sigma}}, \theta, -\mathbf{p}) - \varphi_2^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, \mathbf{g}) &= \underline{\boldsymbol{\sigma}}^* \underline{\boldsymbol{\varepsilon}}_e + \eta_e \theta - \mathbf{p}^* \mathbf{g} \\
-\varphi_3^*(\underline{\boldsymbol{\varepsilon}}_e, \theta, -\mathbf{p}) - \varphi_{12}^*(\underline{\boldsymbol{\sigma}}, -\eta_e, \mathbf{g}) &= -\underline{\boldsymbol{\sigma}}^* \underline{\boldsymbol{\varepsilon}}_e + \eta_e \theta - \mathbf{p}^* \mathbf{g}
\end{aligned} \tag{25}$$

On the basis of the above relations (24) eight different thermodynamic functions, with different combinations of the state variables, can be defined.

As a consequence, the differential relations (24) show that the constitutive relations for the considered GN model can be equivalently expressed in terms of such eight thermodynamic functions.

The energy definitions and the constitutive relations for the considered GN thermoelastic model without dissipation can be obtained by giving the mechanical meanings to the introduced energetic functions as reported in Table 1.

Table 1: Relations among the convex/concave functions and the thermodynamic energy functions.

Helmholtz free energy:	$\Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$
Internal energy:	$U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) = -\varphi_2^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, \mathbf{g})$
Enthalpy:	$H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) = -\varphi_{12}^*(\underline{\boldsymbol{\sigma}}, -\eta_e, \mathbf{g})$
Gibbs free energy:	$G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) = -\varphi_1^*(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g})$

In addition, further four thermodynamic potentials can be defined with different combinations of the state variables from the above energy functions.

These thermodynamic potentials are named alternative since they differ from the GN thermodynamic potentials reported in Table 1 by the exchange of the displacement gradient \mathbf{g} with the entropy heat flux \mathbf{p} .

The alternative thermodynamic potentials are reported in Table 2.

Table 2: Relations among the convex/concave functions and the alternative formulations of the thermodynamic energy functions.

Alternative form of Helmholtz free energy:	$\hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) = -\varphi_3^*(\underline{\boldsymbol{\varepsilon}}_e, \theta, -\mathbf{p})$
Alternative form of the internal energy:	$\hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) = -\varphi_{23}^*(\underline{\boldsymbol{\varepsilon}}_e, -\eta_e, -\mathbf{p})$
Alternative form of the enthalpy:	$\hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) = -\varphi^*(\underline{\boldsymbol{\sigma}}, -\eta_e, -\mathbf{p})$
Alternative form of Gibbs free energy:	$\hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) = -\varphi_{13}^*(\underline{\boldsymbol{\sigma}}, \theta, -\mathbf{p})$

The constitutive relations for the considered GN model can be consistently deduced from the differential relations reported in (24) in terms of the thermodynamic energy functions and

of their alternative formulations. In particular, the GN constitutive relations in terms of the four thermodynamic potentials are provided in Table 3.

Table 3: The thermodynamic potentials and the related constitutive relations.

Helmholtz free energy $\Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$	Internal energy $U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g})$
$\underline{\boldsymbol{\sigma}} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \begin{cases} \boldsymbol{\sigma} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \\ \chi_1 = \partial_{\alpha_1} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \\ \chi_2 = \partial_{\alpha_2} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g}) \end{cases}$ $-\eta_e = \partial_{\theta} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$ $-\mathbf{p} = \partial_{\mathbf{g}} \Phi(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{g})$	$\underline{\boldsymbol{\sigma}} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) \begin{cases} \boldsymbol{\sigma} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) \\ \chi_1 = \partial_{\alpha_1} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) \\ \chi_2 = \partial_{\alpha_2} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g}) \end{cases}$ $\theta = \partial_{\eta_e} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g})$ $-\mathbf{p} = \partial_{\mathbf{g}} U(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{g})$
Enthalpy $H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g})$	Gibbs free energy $G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g})$
$-\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) \begin{cases} -\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) \\ -\alpha_1 = \partial_{\chi_1} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) \\ -\alpha_2 = \partial_{\chi_2} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g}) \end{cases}$ $\theta = \partial_{\eta_e} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g})$ $-\mathbf{p} = \partial_{\mathbf{g}} H(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{g})$	$-\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) \begin{cases} -\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) \\ -\alpha_1 = \partial_{\chi_1} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) \\ -\alpha_2 = \partial_{\chi_2} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g}) \end{cases}$ $-\eta_e = \partial_{\theta} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g})$ $-\mathbf{p} = \partial_{\mathbf{g}} G(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{g})$

Moreover the GN constitutive relations in terms of the four alternative forms of the thermodynamic potentials are given in Table 4.

Table 4: Alternative formulations of the thermodynamic potentials and the related constitutive relations.

Alternative form of Helmholtz free energy $\hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p})$	Alternative form of the internal energy $\hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p})$
$\underline{\boldsymbol{\sigma}} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) \begin{cases} \boldsymbol{\sigma} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) \\ \chi_1 = \partial_{\alpha_1} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) \\ \chi_2 = \partial_{\alpha_2} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p}) \end{cases}$ $-\eta_e = \partial_{\theta} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p})$ $\mathbf{g} = \partial_{\mathbf{p}} \hat{\Phi}(\underline{\boldsymbol{\varepsilon}}_e, \theta, \mathbf{p})$	$\underline{\boldsymbol{\sigma}} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) \begin{cases} \boldsymbol{\sigma} = \partial_{\underline{\boldsymbol{\varepsilon}}_e} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) \\ \chi_1 = \partial_{\alpha_1} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) \\ \chi_2 = \partial_{\alpha_2} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p}) \end{cases}$ $\theta = \partial_{\eta_e} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p})$ $\mathbf{g} = \partial_{\mathbf{p}} \hat{U}(\underline{\boldsymbol{\varepsilon}}_e, \eta_e, \mathbf{p})$
Alternative form of the enthalpy $\hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p})$	Alternative form of Gibbs free energy $\hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p})$
$-\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) \begin{cases} -\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) \\ -\alpha_1 = \partial_{\chi_1} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) \\ -\alpha_2 = \partial_{\chi_2} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p}) \end{cases}$ $\theta = \partial_{\eta_e} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p})$ $\mathbf{g} = \partial_{\mathbf{p}} \hat{H}(\underline{\boldsymbol{\sigma}}, \eta_e, \mathbf{p})$	$-\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) \begin{cases} -\underline{\boldsymbol{\varepsilon}}_e = \partial_{\underline{\boldsymbol{\sigma}}} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) \\ -\alpha_1 = \partial_{\chi_1} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) \\ -\alpha_2 = \partial_{\chi_2} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p}) \end{cases}$ $-\eta_e = \partial_{\theta} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p})$ $\mathbf{g} = \partial_{\mathbf{p}} \hat{G}(\underline{\boldsymbol{\sigma}}, \theta, \mathbf{p})$

It is worth noting that the constitutive relations reported in Table 3 and 4 in terms of the

complete set of the thermodynamic functions turn out to be all equivalent each other.

The correspondences between the thermodynamic potentials provide a generalization to the present GN thermoelastic framework of the classical Legendre transform concerning the sum of the elastic and complementarity energies. In fact, assuming that the state variables $(\underline{\varepsilon}_e, \theta, \mathbf{g})$ and $(\underline{\sigma}, \eta_e, \mathbf{p})$ fulfil the constitutive relations reported in Tables 3 and 4, the Fenchel's equalities (24) can be rewritten in the following form:

$$\begin{aligned} \Phi(\underline{\varepsilon}_e, \theta, \mathbf{g}) - \hat{H}(\underline{\sigma}, \eta_e, \mathbf{p}) &= \underline{\sigma} * \underline{\varepsilon}_e - \eta_e \theta - \mathbf{p} * \mathbf{g} \\ -G(\underline{\sigma}, \theta, \mathbf{g}) + \hat{U}(\underline{\varepsilon}_e, \eta_e, \mathbf{p}) &= \underline{\sigma} * \underline{\varepsilon}_e + \eta_e \theta + \mathbf{p} * \mathbf{g} \\ U(\underline{\varepsilon}_e, \eta_e, \mathbf{g}) - \hat{G}(\underline{\sigma}, \theta, \mathbf{p}) &= \underline{\sigma} * \underline{\varepsilon}_e + \eta_e \theta - \mathbf{p} * \mathbf{g} \\ H(\underline{\sigma}, \eta_e, \mathbf{g}) + \hat{\Phi}(\underline{\varepsilon}_e, \theta, \mathbf{p}) &= -\underline{\sigma} * \underline{\varepsilon}_e + \eta_e \theta - \mathbf{p} * \mathbf{g} \end{aligned} \quad (26)$$

In particular we underline the mechanical meaning of the second relation of (26):

- the difference between the alternative form of the internal energy and the Gibbs free energy provides the sum of mechanical virtual work and thermal virtual work between elastic entropy and temperature and between the thermal displacement gradient and the entropy flux vector.

5 CONCLUSION

Using a systematic procedure based on convex/concave functions and Legendre transforms, the consistent set of the thermodynamic functions in the framework of GN dissipationless thermo-elasto-plasticity is addressed. Starting from Helmholtz free energy, the thermodynamic potentials, i.e. the internal energy, the enthalpy and the Gibbs free energy, are derived. Moreover it is shown that four more thermodynamic potentials, named alternative, can be provided. The relations between these functions are provided and the constitutive relations are obtained.

REFERENCES

- [1] Green, A.E., Naghdi, P.M. A re-examination of the basic postulates of thermomechanics. *Proc. R. Soc. London* (1991) **432**: 171-194.
- [2] Green, A.E., Naghdi, P.M. Thermoelasticity without energy dissipation. *J. Elasticity* (1993) **31**: 189-208.
- [3] Bargmann, S., Steinmann, P. Theoretical and computational aspects of non-classical thermoelasticity. *Comput. Meth. Appl. Mech. Eng.* (2006) **196**: 516-527.
- [4] Romano, G., Barretta, R. Covariant hypo-elasticity. *Eur. J. Mech. A, Solids* (2011) **6**: 1012-1023.
- [5] Romano G., Barretta R. On Euler's stretching formula in continuum mechanics. *Acta Mech.* (2013) **224**: 211-230.
- [6] Romano G., Barretta R. Geometric constitutive theory and frame invariance. *Int. J. Non-Linear Mech.* (2013) **51**: 75-86.
- [7] Marotti de Sciarra, F. Hardening plasticity with nonlocal strain damage. *Int. J. Plasticity*

- (2008) **34**: 114-138.
- [8] Marotti de Sciarra, F. Novel variational formulations for nonlocal plasticity. *Int. J. Plasticity* (2009) **25**: 302-331.
- [9] Marotti de Sciarra, F. A nonlocal model with strain-based damage. *Int. J. Solids Structures* (2009) **46**: 4107-4122.
- [10] Fraldi M., Carannante F., Nunziante L., Analytical solutions for n-phase Functionally Graded Material Cylinders under de Saint Venant load conditions: homogenization and effects of Poisson ratios on the overall stiffness. *Composites: Part B: Engineering* (2013) **45**: 1310-1324.
- [11] Fraldi M., Nunziante, L., Gesualdo, A., Guarracino F., On the bounding of limit multipliers for combined loading, *Proceedings Royal Society A* (2010) **466**: 493-514.
- [12] Fraldi, M., Carannante, F., Nunziante, L. Axis-Symmetrical Solutions for n-ply Functionally Graded Material Cylinders under Strain no-decaying conditions. *Mechanics of Advanced Materials and Structures* (2007) **14**: 151-174.
- [13] Fraldi M., Cowin S. C. Inhomogeneous elastostatic problem solutions constructed from stress-associated homogeneous solutions. *J. Mech. Phys. Solids* (2004) **52**: 2207-2233.
- [14] Marotti de Sciarra, F. A finite element for nonlocal elastic analysis, *IV International Conference on Computational Methods for Coupled Problems in Science and Engineering*, Coupled Problems (2011) 496–505.
- [15] Marotti de Sciarra, F. On non-local and non-homogeneous elastic continua, *Int. J. Solids Structures* (2009) **46**: 651-676.
- [16] Salerno M., Terravecchia S., Zito L. Frictionless contact-detachment analysis: iterative linear complementarity and quadratic programming approaches. *Comput Mech* (in print), DOI: 10.1007/s00466-012-0752-4, 2012.
- [17] Halphen, B., Nguyen, Q.S. Sur les matériaux standards généralisés. *J. Mécanique* (1975) **14**: 39–63.
- [18] Bartels S., Roubíček, T. Thermoviscoplasticity at small strain. *Z. Angew. Math. Mech.* (2008) **9**: 735–754.
- [19] Bertram, A., Krawietz, A. On the introduction of thermoplasticity. *Acta Mech.* (2012) **223**: 2257–2268.
- [20] Coleman B.D., Owen D. On thermodynamics and elastic-plastic materials. *Arch. Ration. Mech. Anal.* (1975) **59**: 25-51.
- [21] Simo J.C., Miehe C. Associative coupled thermoplasticity at finite strains: formulation, numerical analysis and implementation. *Comput. Meth. Appl. Mech. Eng.* (1992) **98**: 41-104.
- [22] Marotti de Sciarra F. Nonlocal and gradient rate plasticity. *Int. J. Solids Structures* (2004) **26**: 7329–7349.
- [23] Houlsby G.T., Puzrin, A.M. A thermomechanical framework for constitutive models for rate-independent dissipative materials. *Int. J. Plast.* (2000) **16**: 1017–1047.
- [24] Caccese E., Guarracino, F. On the “relativistic” description of motion of soliton-like defects in elastic media, *Chaos, Solitons and Fractals* (2006), **27**: 868–880.
- [25] Caccese E., Guarracino, F. ‘Relativistic’ characterisation of the energy of soliton-like defects in continuous media, *Chaos, Solitons and Fractals* (2007), **33**: 1103–1116.
- [26] Fraldi M., Cutolo A., Esposito L., Guarracino F., The role of viscoelasticity and stress gradients on the outcome of conductive keratoplasty, *Biomech Model Mechanobiol*,

- (2011) **10**: 397–412.
- [27] Fraldi M., Guarracino F., An improved formulation for the assessment of the capacity load of circular rings and cylindrical shells under external pressure. Part 1. Analytical derivation, *Thin-Walled Structures*, (2011) **49**, 1054–1061.
- [28] Fraldi, M., Freeman, R., Slater, S., Walker A.C., Guarracino F., An improved formulation for the assessment of the capacity load of circular rings and cylindrical shells under external pressure. Part 2. A comparative study with design codes prescriptions, experimental results and numerical simulations, *Thin-Walled Structures*, (2011) **49**, 1062–1070.
- [29] Guarracino, F. On the extension of Crotti's theorem to the theory of finite elastic displacements *Int. J. Solids Structures* (1994) **31**: 1911-1922.
- [30] Fraldi M., Guarracino, F. On a general property of a class of homogenized porous media, *Mech. Res. Comm.* (2001) **28**: 213-221.
- [31] Guarracino F., Considerations on the Numerical Analysis of Initial Post-Buckling Behaviour in Plates and Beams, *Thin-Walled Structures* (2007) **45**: 845-848,