APPLICATION OF HIGH-ORDER ELEMENTS FOR COUPLED ANALYSIS IN GEOMECHANICS

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Key words: Coupled Problems, Geomechanics, and High-order elements.

Abstract. In this paper high-order triangular elements are implemented in the framework of the Arbitrary Lagrangian-Eulerian method for the analysis of large strain consolidation problems in geomechanics. The theory of consolidation, as well as details of the high-order elements, including cubic (10-noded), quartic (15-noded), quantic (21-noded) and sextic (28-noded) elements are discussed. The accuracy and the efficiency of high-order elements in the analysis of consolidation problems are demonstrated conducting a small deformation analysis of the soil under a strip footing as well as a large deformation analysis of a vertical cut subjected to a surcharge loading. Based on the numerical results, it is shown that high-order elements not only improve the accuracy of solution but can also significantly decrease the required computational time. It is also demonstrated that assuming identical order for displacement shape functions and the pore water pressure shape functions does not affect the stability of the time-marching analysis of consolidation nor the accuracy of the numerical predictions.

1 INTRODUCTION

The theory of linear consolidation was first proposed by Biot in 1941. However, it took several years before researchers applied the finite element method to solve Biot's consolidation equations, e.g., Sandu and Wilson (1969). The works of Christian and Boehemer (1970) and Kraus (1978) are the good examples of employing the finite element technique to solve elastic consolidation problems. Later, Small et al. (1976) proposed the first extension to Biot's theory to accommodate elastoplastic behaviour of the soil. Works of Carter et al. (1977) and Carter et al. (1979) are further extension of the theory of consolidation for elastic and elastoplastic soil subjected to large strains.

The finite element method has proven to be very effective for dealing with consolidation problems in geomechanics. However, simultaneously minimising the computational time while maximising the accuracy of numerical solution, is still a research subject of some interest. One of the ways to achieve this goal is to apply high-order elements in which increasing the order of the nodal polynomial functions may result in more accurate results in less computational time. The 15-noded triangular elements have already been applied by a few researchers to improve the solution of some well-known geotechnical problems.

Examples include study of the collapse load of an incompressible soil under strip and circular footings by Sloan and Randolph (1982), investigation of the collapse load of soil with a non-associated flow rule with small strains by De Borst and Vermeer (1984), and the consolidation problem of composite soft clay by Horpibulsuk et al. (2012). Recently, Kardani et al. (2013) compared the efficiency of high-order triangular elements such as quartic and quadratic elements by investigating the large deformation of an undrained soil under a footing. Later Kardani et al. (2014) compared the performance of the same set of high-order elements in analysing large strain coupled problems. When dealing with large deformation problem, the Arbitrary-Lagrangian-Eulerian (ALE) formulation presented by Nazem et al. (2008) has been applied to refine the mesh in order to prevent the occurrence of mesh distortion during the analysis.

In this work, the performance of 28-noded triangular elements in analysing the coupled problems of geomechanics is compared with the outcomes of 10-, 15- and 21-noded elements. The comparison will be shown by investigating the bearing capacity of the soil under a strip footing assuming small deformations as well as the stability of a vertical cutting in an undrained soil subjected to large deformations.

2 GOVERNING EQUATIONS

In geotechnical problems the deformations of the solid phase are usually coupled with the pore fluid pressures. In order to analyse these consolidation problems, the governing equations are obtained by coupling the conservation of mass and the equilibrium resulting the governing finite element equations (see Nazem *et al.* 2008):

$$\begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{H}} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{F}}_{ext} \\ \dot{\mathbf{Q}}_{ext} \end{Bmatrix} \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{L}$ represents the coupling matrix, $\mathbf{H}$ denotes the flow matrix, $\mathbf{u}$ and $\mathbf{p}$ are respectively the vectors of nodal displacements and pore water pressures, and $\mathbf{F}_{ext}$ and $\mathbf{Q}_{ext}$ represent the external force vector and the fluid supply vector, respectively. A superimposed dot in Equation (1) represents the time derivative of a variable. It is noted that the Arbitrary Lagrangian-Eulerian method proposed by Nazem *et al.* (2006) is employed for solving the large deformation problems in this study. The implicit backward Euler method is used to numerically integrate the coupled differential equation in (1). For further details, see Nazem *et al.* (2008).

3 HIGH-ORDER ELEMENTS

In this paper, high-order triangular elements including the 10-, 15, 21- and 28-noded elements are applied to discretise the problem domain. In triangular elements, the order of the polynomial shape function, p, is directly related to the number of nodes, m, according to (Dunavant, 1985)

$$m = \frac{1}{2}(p+1)(p+2) \tag{1}$$

In dealing with the coupled problem, it is generally accepted that the order of the pore water pressure shape functions should be one degree lower than the order of the displacement shape functions. For the 6-noded elements this can easily be achieved by only considering pore pressure degrees of freedom at the corner nodes. However, with higher order elements such consideration is not feasible. To overcome the problem in this study, the same order of shape function is assumed for both displacement and pore water pressure. Later, through some numerical examples, it is demonstrated that this assumption is satisfactory. Table 1 contains the characteristic information for each type of high-order element considered, including number of nodes on each side of the element, the number of internal nodes, the polynomial order and the minimum number of quadrature points.

Table 1: Characteristic information for high-order triangular elements

Element type	No. of nodes per side	No. of internal nodes	Order of shape function	Plane strain		Axi-symmetric	
				Order of Integrand	Gauss points	Order of Integrand	Gauss points
10-noded	4	1	3	6	6	7	12
15-noded	5	3	4	8	12	9	16
21-noded	6	6	5	10	16	11	25
28-noded	7	10	6	12	25	13	37

4 NUMERICAL EXAMPLES

In this study, the accuracy and efficiency of high-order elements in tackling coupled problems is investigated by analysing two geotechnical problems. In the first example, a soil layer under a strip footing is analysed to find its undrained bearing response while assuming small deformations. In the second example, the loading of a vertical cutting is studied assuming large deformations. In both examples, the study is focused on the possibility of achieving a prescribed accuracy with fewer degrees of freedom while increasing the order of the elements.

It is noted that the high-order elements have been implemented in SNAC, a finite element program developed for analysing geotechnical problems at the University of Newcastle.

4.1 Elastoplastic analysis of a soil layer under a strip footing

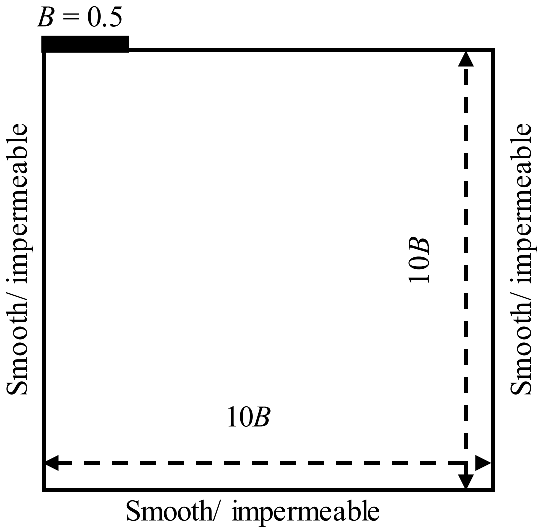
In this example, the efficiency of high-order element in dealing with geotechnical coupled problems is shown by small deformation analysis of the undrained bearing response of a soil layer under a rigid strip footing. Small (1977) showed that for a weightless soil the drained and undrained strength parameter must satisfy

$$\frac{c_u}{c'} = \frac{2\sqrt{N_\phi}}{1+N_\phi} \qquad \textit{where} \qquad N_\phi = \frac{1+\sin\phi'}{1-\sin\phi'} \tag{1}$$

in which ϕ' and c' are respectively the drained friction angle and cohesion, and c_u denotes the undrained shear strength of the soil. In a coupled analysis, the excess pore water pressure will not have sufficient time to dissipate provided that the loading rate is relatively fast rate (see Small 1977). Based on Prandtl's plasticity solution, the undrained bearing capacity of soil under a strip footing is obtained by

$$q_u = N_c c_u \tag{1}$$

where $N_c=2+\pi$ represents the bearing capacity factor. The Mohr-Coulomb material model is used to predict the soil behaviour in this example. To avoid an artificial increase in shear strength of soil due to the suppression of any dilation under undrained (constant volume) conditions, it is important to assume that dilation angle is zero. The problem domain, boundary conditions and material properties are illustrated in Figure 1. Note that only the right half of the problem domain is considered in the analysis due to symmetry. Also, plane strain conditions are assumed.



Soil Properties

Drained Young's modulus, $E' = 500\ kPa$
Drained Poisson's ratio, $\nu' = 0.3$
Drained shear strength, $c' = 0.01E'$
Drained friction angle, $\phi' = 20°$
Dilation angle, $\psi' = 0°$
Permeability, $k = 5\times10^{-9}\ m/s$
Unit weight of the soil, $\gamma = 0\ kN/m^3$
Unit weight of the water, $\gamma_w = 10\ kN/m^3$

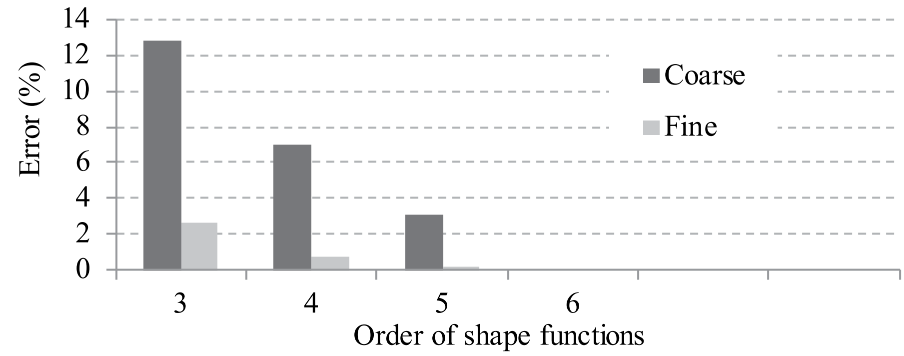
Figure 1: An undrained layer of soil under a strip footing

To provide a comparison of the efficiency of the high-order elements, two fixed grids were used to discretise the problem domain. For a meaningful comparison it is important to use more or less an identical number of degrees-of-freedom regardless of the element type. First, the problem domain is discretised by an uniform 60x60 grid which provides a relatively coarse mesh with 3721 nodal points, noting that 60 is the least common multiple of 3, 4, 5 and 6 (number of segments on one side of 10-, 15-, 21-, and 28-node elements, respectively). The second grid is relatively fine, including 120x120 uniform divisions and 14641 nodal points. Both uniform grids are used to form triangular meshes by applying 10-, 15-, 21- and 28-node elements.

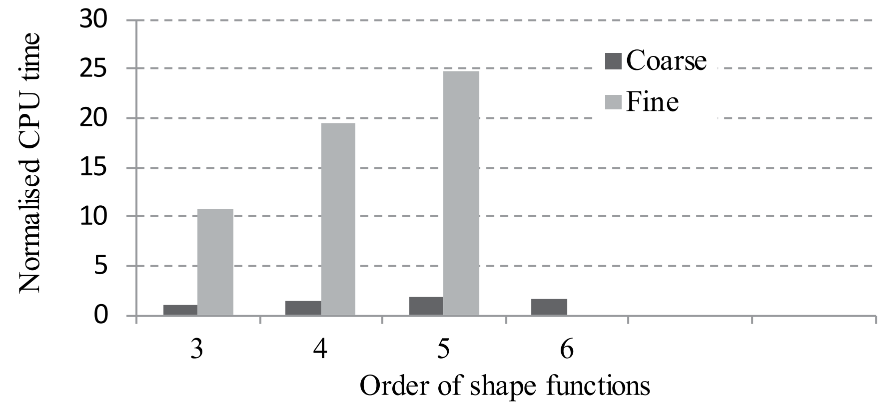
Table 2 presents the number of elements and integration points in each mesh as well as the numerical results including the error in predicted N_c and the CPU time normalised by the CPU time of the fastest analysis. The plots of error versus the order of the shape functions, as well as the plots of error versus the normalised CPU time, are presented in Figures 2a and 2b, respectively. According to Table 2, the coarse mesh of 28-noded elements provides the most accurate estimation of the undrained bearing capacity of the soil and hence there is no need to repeat the analysis using the fine mesh of 28-noded elements.

Table 2: Finite element meshes and numerical result

Element Type	Mesh	Number of elements	Number of Gauss points	N_c	Error in N_c (%)	Normalised CPU time
10-noded	Coarse	800	4800	5.80	12.84	1
	Fine	3200	19200	5.279	2.7	10.85
15-noded	Coarse	450	5400	5.50	7.0	1.4
	Fine	1800	21600	5.186	0.77	19.43
21-noded	Coarse	288	4608	5.301	3.13	1.84
	Fine	1152	18432	5.141	0.18	24.8
28-noded	Coarse	200	5000	5.140001	1.9E-7	1.6
	Fine	800	20000	-	-	-



a. Error versus order of shape functions



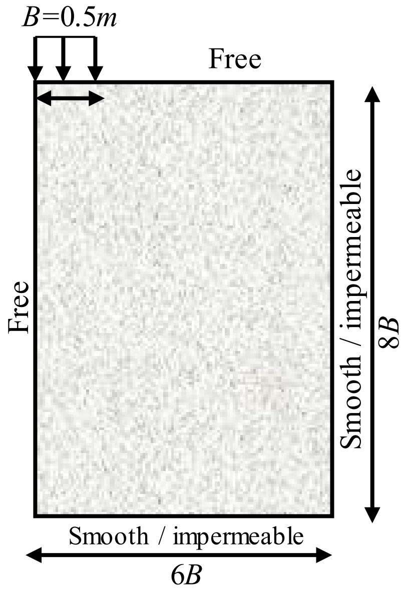
b. Normalised CPU time versus order of shape functions

Figure 2: Numerical results

Based on the result presented in Table 2 and Figure 2, the coarse mesh with cubic elements represents the fastest analysis, but the bearing capacity of the soil is considerably overestimated. On the other hand, the analysis with 28-noded elements with a CPU time just 1.66 times the fastest analysis, achieves the most accurate result. Also in this example, by comparing the analysis results of 21-noded and 28-noded elements, it is concluded that the 28-noded elements are more efficient than 21-noded elements as they not only improve the final solution but decrease the computational time.

4.2 Vertical cut

In this example, the efficiency of high-order element in dealing with coupled geotechnical problem is studied by large deformation analysing of a weightless soil under a uniform vertical load applied adjacent to a vertical cutting.

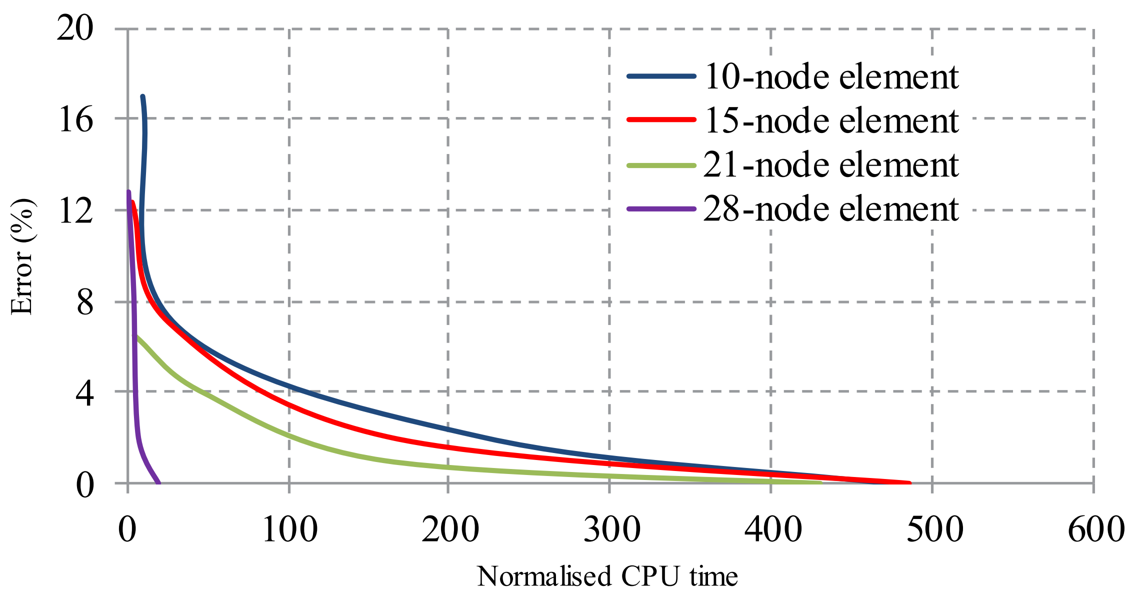


Soil Properties

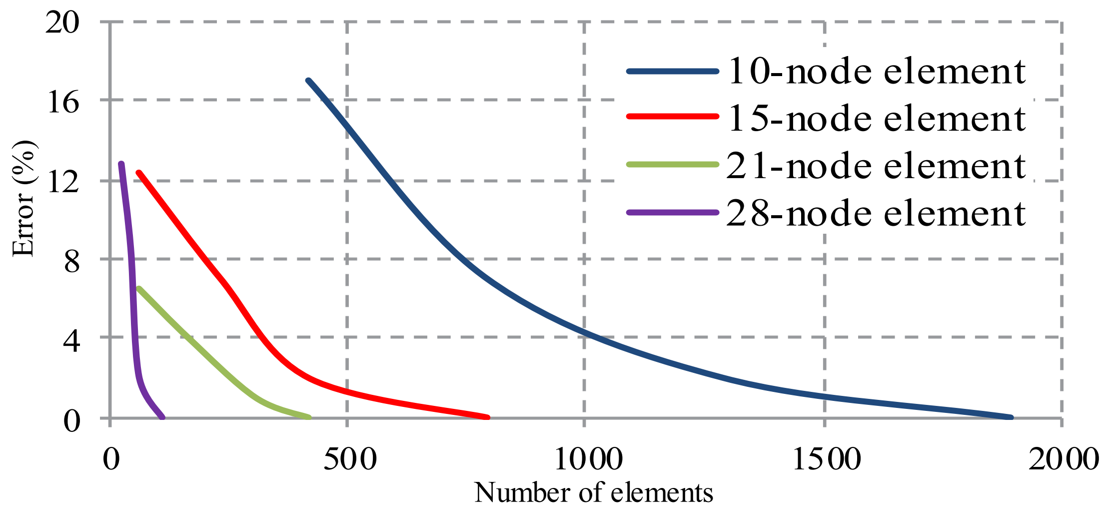
Drained Young's modulus, $E' = 100\,kPa$
Drained Poisson's ratio, $\nu' = 0.3$
Drained shear strength, $c' = 5.0\,kPa$
Drained friction angle, $\phi' = 20°$
Dilation angle, $\psi' = 0°$
Permeability, $k = 5 \times 10^{-9}\ m/s$
Unit weight of the soil, $\gamma = 0\ kN/m^3$
Unit weight of the water, $\gamma_w = 10\,kN/m^3$

Figure 3: A vertical slope under vertical prescribed displacement

To model the undrained soil behaviour, the drained material properties are considered and a coupled pore water pressure-displacement analysis is carried out with a rather fast loading rate. The problem domain, material properties and boundary condition are shown in Figure 3. Based on the theoretical solution, the maximum vertical pressure which can be applied on an undrained soil is equal $2c_u$. The analyses were conducted using a trial and error strategy. For each type of high-order elements, the first analysis employed a very coarse mesh and the error was calculated. The analysis was then repeated by gradually increasing the density of the mesh, specifically in the area under the load, with the aim of approaching the exact solution. The characteristics of the discretisation as well as the error and CPU time of each analysis are summarised in Table 3. For each analysis, the errors versus number of elements as well as the error versus normalised CPU time are plotted in Figure 4. According to Figure 4a, by increasing the number of elements the error decreases, whereas the computational time increases. According to the Figure 4b, the performance of the high-order element improves as



a. Error versus number of elements in the vertical cut.



b. Error versus normalised computational time

Figure 4: Analysis results of vertical cut

the order of element is increased. The 28-noded element achieved the same accuracy of the other type of elements with the lowest computational time. This given accuracy can be achieved in the fastest time using the 28-noded element.

5 CONCLUSION

In this study, high-order finite elements are applied for coupled problems in a finite element framework, using the Arbitrary Lagrangian-Eulerian method. The accuracy and efficiency of these elements were examined by analysing two typical geotechnical problems including the undrained bearing capacity of soil under a rigid footing and the behaviour of undrained soil under a vertical pressure applied adjacent to a cutting.

For the coupled problems investigated in this study, applying high-order elements was proven to be very effective in improving the computational time. This means that increasing the order of elements leads to a decrease in the number of degrees of freedom required to achieve a given accuracy and therefore a significant drop in the computational time. Also, it was demonstrated that considering the same order of shape functions for both displacement and pore water pressure does not affect the accuracy of the numerical results. In dealing with the coupled problems studied in this paper, the 28-noded triangular elements prominently outperformed the other high-order elements.

Table 3: Descritisations in vertical cut problem

Element type	Number of Elements	Degrees of freedom	Gauss points	Error (%)	Normalised CPU time
10-node	416	5790	2496	17	9.8
	792	10938	4752	7	29.97
	1289	17697	7734	2.0	222.58
	1894	25878	11364	0	471.8
15-node	59	1533	708	12.35	3.2
	231	5733	2772	7	26.98
	416	10215	4992	2	164.78
	792	19335	9504	0	485.59
21-node	59	2367	944	6.5	4.58
	164	6333	2624	4	46.98
	305	11733	4880	1	159.79
	416	15888	6656	0	430.396
28-node	22	1317	550	12.8	1.0
	42	2415	1050	8.5	4.2
	59	3360	1457	2.1	7.0
	108	6051	2700	0	19.6

REFERENCES

[1] Sandhu, R.S., and Wilson, E.L. Finite element analysis of seepage in elastic media. *J. of Eng. Mechanics* (1969) **95**: 641-652.
[2] Christian, J.T., and Boehmer, J.W. Plain strain consolidation by finite elements. *J. of the Soil Mechanics and Foundation Division of ASCE* (1970) 1435-1457.
[3] Small, J.C., Booker, J.R., and Davis, E.H. Elasto-plastic consolidation of soil. *Int. J. of Solids and Structures* (1976) **12**: 47-76.

[4] Carter, J.P., Small, J.C., and Booker, J.R. A theory of finite elastic consolidation. *Int. J. of Solids and Structures* (1977) **13**: 467-478.
[5] Krause, G. Finite element schemes for porous media. *J. of the Eng. Mechanics Division, ASCE* (1978) **104**: 605-620.
[6] Carter, J.P., Booker, J.R., and Small, J.C. The analysis of finite elasto-plastic consolidation. *Int. J. for Num. and Analy. Meth. in Geomechanics* (1979) **3**: 107-129.
[7] Sloan, S.W., and Randolph, M.F. Numerical Prediction of collapse loads using the finite element methods. *Int. J. of Num. and Analy. Meth. in Geomechanics* (1982) **6**: 47-76.
[8] De Borst, R., and Vermeer, P.A. Possibilities and limitations of finite elements for limit analysis. Géotechnique (1984) **34**: 199-210.
[9] Nazem, M., Sheng, D., and Carter J.P. Stress integration and mesh refinement in numerical solutions to large deformations in geomechanics, *Int. J. Numer. Meth. Eng.* (2006) **65:** 1002-1027.
[10] Nazem, M., Sheng, D., Carter, J.P., and Sloan, S.W. Arbitrary Lagrangian-Eulerian method for large-strain consolidation problems. *Int. J. for Num. and Analy. Meth. in Geomechanics* (2008) **32**: 1023-1050.
[11] Horpibulsuk, S., Chinkulkijniwat, A., Cholaphatsron, A., Suebsuk, J., and Liu, M.D. Finite element analysis of consolidation behaviour of composite soft ground. *GeoCongress* (2012) 940-947
[12] Kardani, M., Nazem, M., Carter, J.P., and Abbo, A.J., The efficiency of high-order elements in large deformation problems of geomechanics. *Int. J. for Geomechanics* (2014) 10.1061/(ASCE)GM.1943-5622.0000457 , 04014101.
[13] Kardani, M., Nazem, M., Carter, J.P., and Abbo, A.J., Large strain coupled analysis of geotechnical problems using high-order elements. *Australian Geomechanics* (2014) **49**:167-174