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AN ELECTROELASTIC PROBLEM OF A SEMI-INFINITE BODY WITH D_∞ SYMMETRY SUBJECTED TO DISTRIBUTED SURFACE LOADING

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Abstract. Electroelastic field in an semi-infinite body with D_∞ symmetry subjected to a locally uniform electric potential on its surface is investigated. By extending a potential function method for transversely isotropic bodies, the electroelastic field inside the body is formulated. Furthermore, numerical calculation is performed to investigate the field qualitatively and quantitatively.

1 INTRODUCTION

The concepts of carbon neutrality have attracted considerable attention recently because of an increasing demand for a reduction in environmental loads. From the viewpoint of engineering production, wooden materials are one of the most promising candidates for achieving carbon neutrality.

To ensure the quality of wooden materials, nondestructive evaluation techniques need to be developed. In particular, the detection of local defects such as cracks, knots, and pith are of great importance for ensuring structural integrity. Wood has been known as a piezoelectric material since the middle of the 20th century, when Fukada succeeded in experimentally verifying the direct and converse piezoelectric effects of wood[1]. These effects are expected to be employed for nondestructive evaluation techniques[2-4].

From a *mesoscopic* viewpoint, woods are considered to belongs to point group D_∞ [1], which is characterized by an ∞*-fold rotation axis* and a *two-fold rotation axis* perpendicular to it[5]. The nonzero components of the piezoelectric constant are d_{14} and $d_{25}\left(=-d_{14}\right)$ only, allowing the ∞ -fold rotation axis be the third axis. In that case, the electric field perpendicular to the ∞ -fold rotation axis (third axis) induces shear strain in the plane perpendicular to the direction of the electric field.

The elastic problems of transversely isotropic bodies, which correspond to a special case in the absence of the piezoelectric effects in body with D_∞ symmetry, were extensively analyzed[6–8]. On the other hand, electroelastic problems of bodies with D_∞ symmetry were investigated experimentally[9-12]. However, for sound operation of nondestructive evaluation techniques, not only the input/output relationship but also the electroelastic field *inside* the material must be elucidated.

In this paper, therefore, we analyze the electroelastic field in a body with D_∞ symmetry. As an example, we treat a semi-infinite body subjected to a locally uniform electric potential on its surface. First, the displacement and electric field are expressed in terms of the potential functions. The governing equations for these functions are obtained by the equilibrium equations of stresses and the Gauss law. By solving the governing equations, the electroelastic field quantities are formulated. Moreover, by performing numerical calculation, the stress and electric field are investigated qualitatively and quantitatively, which helps us to understand the electroelascit field inside a body with D_∞ symmetry.

2 THEORETICAL ANALYSIS

2.1 Problem

We consider a semi-infinite piezoelectric body belonging to point group D_∞, as shown in Fig. 1, where the z axis is parallel to the ∞-fold rotation axis of the body. The surface of the body is subjected to a locally uniform distribution of electric potential and free from traction. The displacements and electric potential are assumed to be zero at infinity. Thus, the boundary conditions are given as

$$\left.\begin{array}{ll} x=0: & \sigma_{xx}=0,\sigma_{zx}=0,\sigma_{xy}=0,\Phi=\Phi_0\cdot H(|\delta|-y)H(|\delta|-z); \\ \sqrt{x^2+y^2+z^2}\to\infty: & u_x\to 0,u_y\to 0,u_z\to 0,\Phi\to 0 \end{array}\right\}. \tag{1}$$

In Eq. (1), σ_{ij}, u_i, and Φ denote the stress, displacement, and electric potential, respectively, $H(\cdot)$ denotes the Heaviside step function, and δ denotes the half-length of the square where the uniform electric potential is applied.

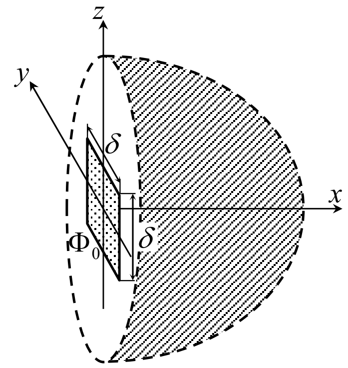

Figure 1: Analytical model

2.2 Governing equations

Let ε_{ij}, E_i, and D_i $(i,j=x,y)$ be the the strain, electric field, and electric displacement, respectively. The constitutive equations of the body are given as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ \text{sym.} & & & & & \dfrac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{14} & 0 & 0 \\ 0 & -e_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}, \tag{2}$$

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & 0 & 0 \\ & \eta_{11} & 0 \\ \text{sym.} & & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}, \tag{3}$$

where c_{ij} , η_{kl} , and e_{kj} denote the elastic stiffness constant, dielectric constant, and piezoelectric constant, respectively. The displacement-strain relations are given as

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x},\ \varepsilon_{yy} = \frac{\partial u_y}{\partial y},\ \varepsilon_{zz} = \frac{\partial u_z}{\partial z},\ 2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},\ 2\varepsilon_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z},\ 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{4}$$

The equilibrium equations of stresses and the Gauss law are given, respectively, by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0,\ \ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x} = 0,\ \ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0, \tag{5}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0. \tag{6}$$

Referring to a solution technique for elastic problems of transversely isotropic bodies[6], the displacement potential functions φ_i and ϑ_i are introduced as

$$u_x = \sum_{i=1}^{2}\left(\frac{\partial \varphi_i}{\partial x} + \frac{\partial \vartheta_i}{\partial y}\right),\ \ u_y = \sum_{i=1}^{2}\left(\frac{\partial \varphi_i}{\partial y} - \frac{\partial \vartheta_i}{\partial x}\right),\ \ u_z = \frac{\partial}{\partial z}\sum_{i=1}^{2} k_i \varphi_i, \tag{7}$$

where

$$k_i = \frac{c_{11}\mu_i - c_{44}}{c_{13} + c_{44}}, \tag{8}$$

μ_1 and μ_2 are the roots of a quadratic equation for μ :

$$c_{11}c_{44}\mu^2 - \left(c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}\right)\mu + c_{33}c_{44} = 0. \tag{9}$$

The components of the electric field are expressed by the electric potential function as

$$E_x = -\frac{\partial \Phi}{\partial x}, \quad E_y = -\frac{\partial \Phi}{\partial y}, \quad E_z = -\frac{\partial \Phi}{\partial z}. \tag{10}$$

Substituting Eqs. (4), (7), and (10) into Eqs. (2) and (3) and the results into Eqs. (5) and (6), we have

$$\left(\Delta_\mathrm{p} + \mu_i \frac{\partial^2}{\partial z^2}\right)\varphi_i = 0, \quad \left(\Delta_\mathrm{p} + \upsilon_i \frac{\partial^2}{\partial z^2}\right)\vartheta_i = 0 \quad \frac{e_{14}\mu_3}{c_{44}}\frac{\partial \Phi}{\partial z} = \sum_{i=1}^{2}\left(\Delta_\mathrm{p} + \mu_3 \frac{\partial^2}{\partial z^2}\right)\vartheta_i, \tag{11}$$

where υ_1 and υ_2 are the roots of a quadratic equation with respect to υ:

$$\upsilon^2 - \left[\mu_3\left(1 + k_\mathrm{couple}^2\right) + \eta\right]\upsilon + \mu_3\eta = 0 \tag{12}$$

and

$$\mu_3 = \frac{2c_{44}}{c_{11} - c_{12}}, \quad \eta = \frac{\eta_{33}}{\eta_{11}}, \quad k_\mathrm{couple}^2 = \frac{e_{14}^2}{c_{44}\eta_{11}}, \quad \Delta_\mathrm{p} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{13}$$

2.3 Electroelastic field quantities

By considering the symmetry of the electroelastic field and Eq. (1) and applying the Fourier transform techniques[13] to Eq. (11), the solutions to Eq. (11) are obtained as

$$\left.\begin{aligned}
\varphi_i &= \int_0^\infty \int_0^\infty A_i(\alpha,\beta)\exp(-\gamma_{\mu i}x)\sin(\alpha y)\sin(\beta z)\mathrm{d}\alpha\mathrm{d}\beta, \\
\vartheta_i &= \int_0^\infty \int_0^\infty C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\cos(\alpha y)\sin(\beta z)\mathrm{d}\alpha\mathrm{d}\beta, \\
\Phi &= \frac{c_{44}}{e_{14}}\frac{1}{\mu_3}\sum_{i=1}^{2}\left[\int_0^\infty \int_0^\infty (\mu_3 - \upsilon_i)\beta C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\cos(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right]
\end{aligned}\right\}, \tag{14}$$

where

$$\gamma_{\mu i} = \sqrt{\alpha^2 + \mu_i\beta^2}, \quad \gamma_{\upsilon i} = \sqrt{\alpha^2 + \upsilon_i\beta^2}; \tag{15}$$

$A_i(\alpha,\beta)$ and $C_i(\alpha,\beta)$ $(i = 1,2)$ are unknown constants to be determined by the boundary conditions described by Eq. (1). The distribution function for surface electric potential is expressed in the Fourier integral form[13] as

$$\left.\begin{aligned}
H(|\delta| - y)H(|\delta| - z) &= \int_0^\infty \int_0^\infty f^*(\alpha,\beta)\cos(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta, \\
f^*(\alpha,\beta) &= \frac{4\delta^2}{\pi^2}\frac{\sin(\alpha\delta)}{\alpha\delta}\frac{\sin(\beta\delta)}{\beta\delta}
\end{aligned}\right\}. \tag{16}$$

By substituting Eq. (14) into Eqs. (2)－(4), (7), and (10), the electroelastic field quantities are formulated, for example, as

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{c_{44}}{\mu_3}\sum_{i=1}^{2}\left\{\begin{aligned}&\int_0^\infty\int_0^\infty\left[\begin{aligned}&\left(2\alpha^2+(1+k_i)\mu_3\beta^2\right)A_i(\alpha,\beta)\exp(-\gamma_{\mu i}x)\\&+2\gamma_{\upsilon i}\alpha C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\end{aligned}\right]\\&\cdot\sin(\alpha y)\sin(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\end{aligned}\right\},\\
\sigma_{yz} &= \frac{c_{44}}{\mu_3}\sum_{i=1}^{2}\left\{\int_0^\infty\int_0^\infty\left[\begin{aligned}&\mu_3(1+k_i)\alpha\beta A_i(\alpha,\beta)\exp(-\gamma_{\mu i}x)\\&+\upsilon_i\gamma_{\upsilon i}\beta C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\end{aligned}\right]\cos(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right\},\\
\sigma_{zx} &= \frac{c_{44}}{\mu_3}\sum_{i=1}^{2}\left\{\int_0^\infty\int_0^\infty\left[\begin{aligned}&-\mu_3(1+k_i)\gamma_{\mu i}\beta A_i(\alpha,\beta)\exp(-\gamma_{\mu i}x)\\&-\upsilon_i\alpha\beta C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\end{aligned}\right]\sin(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right\},\\
\sigma_{xy} &= \frac{c_{44}}{\mu_3}\sum_{i=1}^{2}\left\{\int_0^\infty\int_0^\infty\left[\begin{aligned}&-2\gamma_{\mu i}\alpha A_i(\alpha,\beta)\exp(-\gamma_{\mu i}x)\\&-\left(2\alpha^2+\upsilon_i\beta^2\right)C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\end{aligned}\right]\cos(\alpha y)\sin(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right\},\\
E_x &= \frac{c_{44}}{e_{14}\mu_3}\sum_{i=1}^{2}\left[\int_0^\infty\int_0^\infty(\mu_3-\upsilon_i)\gamma_{\upsilon i}\beta C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\cos(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right],\\
E_y &= \frac{c_{44}}{e_{14}\mu_3}\sum_{i=1}^{2}\left[\int_0^\infty\int_0^\infty(\mu_3-\upsilon_i)\alpha\beta C_i(\alpha,\beta)\exp(-\gamma_{\upsilon i}x)\sin(\alpha y)\cos(\beta z)\mathrm{d}\alpha\mathrm{d}\beta\right]
\end{aligned}\right\}. \quad (17)$$

By substituting Eqs. (14), (16), and (17) into Eq. (1), $A_i(\alpha,\beta)$ and $C_i(\alpha,\beta)$ $(i=1,2)$ are obtained as

$$\begin{Bmatrix}A_1(\alpha,\beta)\\A_2(\alpha,\beta)\\C_1(\alpha,\beta)\\C_2(\alpha,\beta)\end{Bmatrix}=\frac{e_{14}\Phi_0}{c_{44}}\frac{f^*(\alpha,\beta)}{\beta}\frac{1}{\Delta(\alpha,\beta)}\begin{Bmatrix}A_1^*(\alpha,\beta)\\A_2^*(\alpha,\beta)\\C_1^*(\alpha,\beta)\\C_2^*(\alpha,\beta)\end{Bmatrix}, \quad (18)$$

where

$$\left.\begin{aligned}
\Delta(\alpha,\beta) &\equiv 2[2\mu_3-(\upsilon_1+\upsilon_2)](k_1-k_2)\alpha^2\gamma_{\mu1}\gamma_{\mu2}(\gamma_{\upsilon1}-\gamma_{\upsilon2})\\
&\quad+(\upsilon_1-\upsilon_2)\left\{\begin{aligned}&2(k_1-k_2)\alpha^2\left[\gamma_{\mu1}\gamma_{\mu2}(\gamma_{\upsilon1}+\gamma_{\upsilon2})-\mu_3\beta^2(\gamma_{\mu1}+\gamma_{\mu2})\right]\\&-\mu_3(1+k_1)(1+k_2)\left(2\alpha^2+\mu_3\beta^2\right)\beta^2(\gamma_{\mu1}-\gamma_{\mu2})\\&-4\alpha^4(k_1\gamma_{\mu1}-k_2\gamma_{\mu2})\end{aligned}\right\},\\
A_1^*(\alpha,\beta) &\equiv -\alpha\left\{\begin{aligned}&2(\upsilon_1-\upsilon_2)\alpha^2\left[2\alpha^2+(1+k_2)\mu_3\beta^2\right]\\&+2\left[2\alpha^2-\mu_3(1+k_2)\beta^2\right]\gamma_{\mu2}(\upsilon_2\gamma_{\upsilon1}-\upsilon_1\gamma_{\upsilon2})\\&-4\mu_3(1+k_2)\alpha^2\gamma_{\mu2}(\gamma_{\upsilon1}-\gamma_{\upsilon2})\end{aligned}\right\},\\
C_1^*(\alpha,\beta) &\equiv -\left\{\begin{aligned}&4(k_1-k_2)\mu_3\alpha^2\gamma_{\mu1}\gamma_{\mu2}\gamma_{\upsilon2}\\&+\left(2\alpha^2+\mu_3\beta^2\right)(\gamma_{\mu1}-\gamma_{\mu2})\left[2\alpha^2(\upsilon_2-\mu_3)-\mu_3^2(1+k_1)(1+k_2)\beta^2\right]\\&-2\mu_3(k_1-k_2)\upsilon_2\alpha^2\beta^2(\gamma_{\mu1}+\gamma_{\mu2})-4\alpha^4\mu_3(k_1\gamma_{\mu1}-k_2\gamma_{\mu2})\end{aligned}\right\}
\end{aligned}\right\}, \quad (19)$$

and $A_2^*(\alpha,\beta)$ and $C_2^*(\alpha,\beta)$ are obtained by interchanging subscripts "1" and "2" in $A_1^*(\alpha,\beta)$ and $C_1^*(\alpha,\beta)$, respectively.

3 NUMERICAL CALCULATION

To illustrate the numerical results, the following nondimensional quantities are introduced:

$$(\hat{x},\hat{y},\hat{z}) \equiv \frac{(x,y,z)}{\delta},\ \ (\hat{E}_x,\hat{E}_y) \equiv \frac{(E_x,E_y)}{\left(\dfrac{\Phi_0}{\delta}\right)},\ \ (\hat{\sigma}_{yz},\hat{\sigma}_{zx}) \equiv \frac{(\sigma_{yz},\sigma_{zx})}{\left(e_{14}\dfrac{\Phi_0}{\delta}\right)}. \tag{20}$$

Numerical parameters are chosen as

$$\frac{(c_{11},\ c_{12},\ c_{13},\ c_{33})}{c_{44}} = (1.2,\ 0.4,\ 0.6,\ 15),\ \ \eta = 1.5,\ \ k_{\text{couple}} = 0.1\,. \tag{21}$$

Figures 2－4 show the distributions of the electric fields and the resulting shear stresses in $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions, respectively. Figure 2 shows that the electric field $\hat{E}_x$ decreases monotonically toward zero with $\hat{x}$ and that the resulting shear stress $\hat{\sigma}_{yz}$ exhibits similar behavior. Figure 3 shows that the electric field $\hat{E}_x$ and shear stress $\hat{\sigma}_{yz}$ are maximum on the $\hat{x}$-axis and, roughly speaking, decrease toward zero with $\hat{y}$ and that, on the other hand, the electric field $\hat{E}_y$ and the resulting shear stress $\hat{\sigma}_{zx}$ are zero on the $\hat{x}$-axis, reach their maxima around the periphery of the surface electric potential, and decrease toward zero with $\hat{y}$. From Fig. 4 it is found that the distributions of the electric field $\hat{E}_x$ and shear stress $\hat{\sigma}_{yz}$ in $\hat{z}$ direction exhibit similar behavior to those in $\hat{y}$ direction.

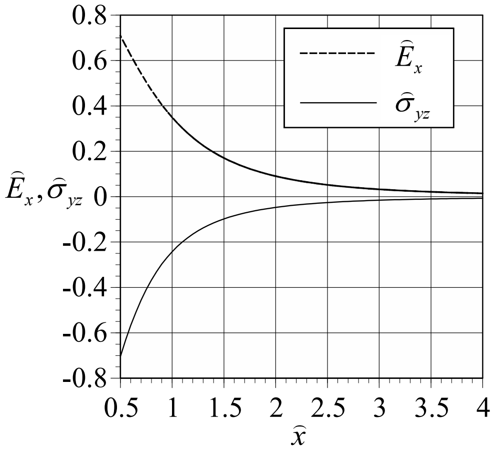

Figure 2: Distrubution of electric field and stress in $\hat{x}$ direction $(\hat{y}=0,\ \hat{z}=0)$

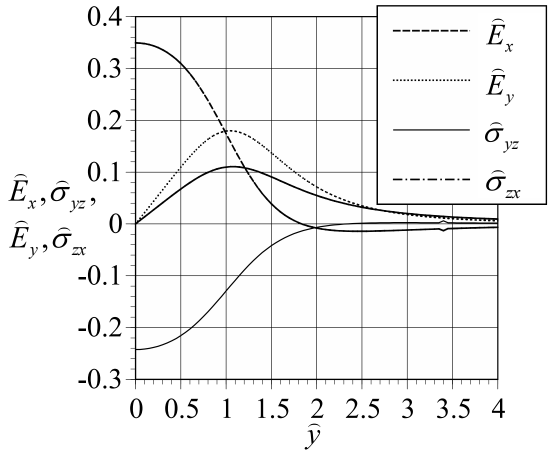

Figure 3: Distrubution of electric field and stress in $\hat{y}$ direction $(\hat{x} = 1, \hat{z} = 0)$

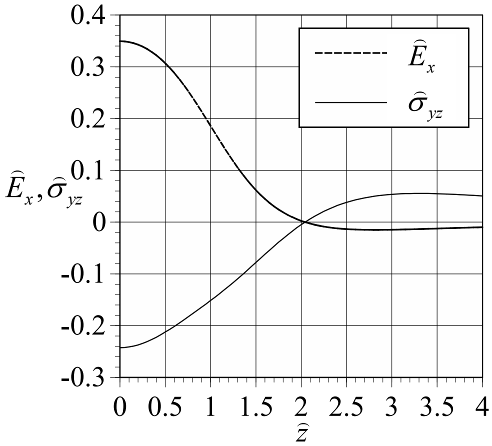

Figure 4: Distrubution of electric field and stress in $\hat{z}$ direction $(\hat{x} = 1, \hat{y} = 0)$

4 CONCLUSIONS

- The analytical solution of the electroelastic field in an semi-infinite body with D_∞ symmetry subjected locally uniform electric potential is formulated.
- For the analytical model above-mentioned, the electroelastic field inside the body, which is of great significance for sound operation of nondestructive evaluation techniques, is elucidated.

REFERENCES

[1] Fukada, E. Piezoelectricity of wood. *J. Phys Soc. Jpn* (1955) **10**:149-154.
[2] Smetana, J.A. and Kelso, P.W. Piezoelectric charge density measurements on the surface of Douglas-fir. *Wood Sci.* (1971) **3**:161-171.
[3] Knuffel, W and Pizzi, A. The piezoelectric effect in structural timber. *Holzforsch.* (1986)

40:157-162.
[4] Knuffel, W.E. The piezoelectric effect in structural timber - part II. The influence of natural defects. *Holzforsch.* (1988) **42**:247-252.
[5] Kim. S.K. *Group theoretical methods and applications to molecules and crystals*. Cambridge University Press, (1999).
[6] Elliott, H.A. Axial symmetric stress distributions in aeolotropic hexagonal crystals. *Math. Proc. Camb. Phil. Soc.* (1949) **45**:621-630.
[7] Chen, W.T. On some problems in transversely isotropic elastic materials. *J. Appl. Mech.* (1966) **33**:347-355.
[8] Atsumi, A. and Itou, S. Stresses in a transversely isotropic slab having a spherical cavity. *J. Appl. Mech.* (1973) **40**:752-758.
[9] Nakai, T., Igushi, N., and Ando, K. Piezoelectric behavior of wood under combined compression and vibration stresses I: Relation between piezoelectric voltage and microscopic deformation of a Sitka spruce (Picea sitchensis Carr.). *J. Wood Sci.* (1998) **44**:28-34.
[10] Nakai, T. and Ando, K. Piezoelectric behavior of wood under combined compression and vibration stresses II: Effect of the deformation of cross-sectional wall of tracheids on changes in piezoelectric voltage in linear-elastic region. *J. Wood Sci.* (1998) **44**:255-259.
[11] Nakai, T., Hamatake, M., and Nakao, T. Relationship between piezoelectric behavior and the stress – strain curve of wood under combined compression and vibration stresses. *J. Wood Sci.* (2004) **50**:97-99.
[12] Nakai, T., Yamamoto, H., Hamatake, M., and Nakao, T. Initial shapes of stress-strain curve of wood specimen subjected to repeated combined compression and vibration stresses and the piezoelectric behavior. *J. Wood Sci.* (2006) **52**:539-543.
[13] Sneddon, I.N. *The use of integral transforms.* McGraw-Hill, (1972).