VI International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2015
B. Schrefler, E. Oñate and M. Papadrakakis (Eds)

ROBUST OPTIMIZATION APPROACH FOR MIXED NUMERICAL/EXPERIMENTAL IDENTIFICATION OF ELASTIC PROPERTIES OF ORTHOTROPIC COMPOSITE PLATES

JANIS AUZINS*, EDUARDS SKUKIS†

* Institute of Mechanics
Riga Technical University
Ezermalas street 6k, Riga, LV-1014, Latvia
e-mail: auzinsjp@latnet.lv

† Institute of Materials and Structures
Riga Technical University
Azenes street 16/20, Riga, LV-1048, Latvia
e-mail: Eduards.Skukis@rtu.lv

Key words: Metamodeling, Elastic properties, Vibration tests, Finite element method, Uncertainty.

Abstract. This paper describes a method for determination of elastic parameters (elastic moduli and Poisson's ratio) of orthotropic composite plate-type structural elements using the results of natural frequency measurements. The identification of parameter values is provided by minimization of weighted squared difference (discrepancy) between physically measured frequencies and natural frequencies calculated by Finite Element Method. The metamodels for the frequency dependence on the elastic parameters and other geometrical and physical parameters of test specimens, including parameters with uncertainty ("noisy constants") are built using experimental designs optimized according to the Mean Squared Error space filling criterion and third-order polynomial approximations. The minimum of weighted squared difference is found using the multistart random search method. The expressions for standard deviations of identified parameters depending on deviations of "noisy constants" are derived using linearized metamodels. The expressions for identification errors allow the statement of the identification task as a robust minimization problem by simultaneous minimization of the discrepancy function and standard deviations of the identified values by varying the values of unknown elastic parameters and weighting coefficients for different frequencies. The partial scaling of natural frequencies is used for the reduction of the uncertainty impact on the identification error. This allows to reduce the identification error of elastic moduli about two times and Poisson's ratio about 20 times in comparison with the results obtained by using dimensioned frequencies.

1 INTRODUCTION

The method of identification of elastic parameters (Young's moduli, shear moduli, Poisson's ratio) using eigenfrequency measurements of specimens is quite old. Currently there exists an extensive amount of literature on the identification of elastic properties of layered composite materials using physical measurements and numerical calculations of natural frequencies, mostly using Finite Element Method (FEM) [1,2,3]. The traditional numerical-experimental identification procedure is based on the minimization of discrepancy between numerical and experimental results. There have been papers published suggesting that normal maximum likelihood is superior to weighted least squares for the determination of material elastic properties using the vibration method [4,5]. At the same time it has been proven that the three main estimation methods (normal maximum likelihood, weighted least squares and ridge regression) all have the same asymptotic covariance and that there is no gain in efficiency among them [6].

During the first years of using discrepancy minimization method, the main problem was the minimization of the discrepancy functional. FEM software cannot be used in identification, since the minimization of the differences between physical measurements and numerical results requires many thousands of FEM calculations that would take years to run. Therefore the metamodeling (also called surrogates) methodology is used [2,3]. The methodology consists in the creation of experimental designs for computer experiments with FEM software, carrying out computer experiments (100-300 experimental runs), building the approximate model for the dependence of natural frequencies of a given structural element on their geometrical and elastic parameters and finding the values of elastic parameters that minimize the difference between measured and calculated frequency values. The last step in this methodology is the verification of the result – repeated calculation of FEM results and validation – comparing identified elastic parameter values with values obtained by a different method (bending tests, for example).

Today, using modern numerical experimental designs and nonparametric approximation methods, the discrepancy minimization is not a hard task. However, the estimation of the variance of obtained identified parameters is a pressing problem. The errors of identification depend on errors introduced by material production, cutting testing specimens, physical measurement errors and errors caused by disregarding significant factors in the finite element model. A large amount of literature is devoted to analysis of the accuracy of FEM, but the influence of errors of physical experiments, caused by parameter variance during material production, specimen preparation and errors of registration and measurement of natural frequencies, is significantly less studied.

This paper will propose a methodology that allows both to minimize the difference between physically measured and numerically calculated natural frequency values, and to evaluate the robustness of the identification results and organize the identification process in a way that would allow obtaining the best possible identification precision.

2 VIBRATION-BASED IDENTIFICATION OF ELASTIC PARAMETERS

Identifiable elastic parameters usually are Young's moduli, shear moduli, Poisson's ratio. The classical idea is to find those values of elastic parameters of the mathematical FEM model which will give minimal discrepancy between calculated and physically measured natural frequencies.

We will designate the column vector of *n* physically measured natural frequencies as $\boldsymbol{f}^{\mathrm{EXP}}$

$$\boldsymbol{f}^{\mathrm{EXP}} = [f_1^{\mathrm{EXP}}, f_2^{\mathrm{EXP}}, \dots, f_L^{\mathrm{EXP}}]^{\mathrm{T}}, \tag{1}$$

column vector of numerically calculated (mostly using FEM) natural frequencies as $\boldsymbol{f}^{\mathrm{FEM}}$

$$\boldsymbol{f}^{\mathrm{FEM}} = [f_1^{\mathrm{FEM}}, f_2^{\mathrm{FEM}}, \dots, f_L^{\mathrm{FEM}}]^{\mathrm{T}} \tag{2}$$

and column vector of n physically identifiable parameters of elasticity as $\boldsymbol{E}$

$$\boldsymbol{E} = [E_1, E_2, \dots, E_n]^{\mathrm{T}}. \tag{3}$$

A superscript $^{\mathrm{T}}$ denotes the matrix transpose operation. The number n of identified parameters can be different, including elastic modulus and Poisson's ratio for different composite layers. The discrepancy Φ between measured and calculated natural frequencies is calculated as weighted sum of L squared differences:

$$\Phi = \sum_{i=1}^{L} w_i \left(f_i^{\mathrm{FEM}} - f_i^{\mathrm{EXP}}\right)^2, \tag{4}$$

where w_i – nonnegative weighting coefficient for i-th frequency.
A frequently used weighting method for discrepancy measure is the squared relative error:

$$\Phi = \sum_{i=1}^{L} \left(\frac{f_i^{\mathrm{FEM}} - f_i^{\mathrm{EXP}}}{f_i^{\mathrm{EXP}}}\right)^2. \tag{5}$$

The discrepancy minimization approach means that input parameters which give minimal value of functional Φ will be considered as identified values for unknown parameters $\boldsymbol{E}$:

$$\boldsymbol{E}^* = \underset{\boldsymbol{E}}{\arg\min}\, \Phi(\boldsymbol{E}). \tag{6}$$

When using FEM software, the minimization requires physical experiments as well as numerical experiments; therefore this approach is sometimes called Mixed Numerical-Experimental Technique (MNET) [2]. The traditional MNET steps are:

Step 1. Preparation of specimen samples, providing frequency measurements by resonance measurements or Fourier analysis of free oscillations registered after initial excitation.

Step 2. Design of numerical experiments for FEM software. The variable input factors for eigenfrequency calculations are identifiable elastic parameters. Mostly the Latin Hypercube (LH) type designs are used. Here we use LHs and non-LH type designs optimized according to the Mean Square Error space-filling criterion, introduced in [7]. The values for other input parameters (geometrical, mass, density, layer configuration and others) must correspond to specimens used in physical experiments. The number of runs for numerical experiments depends on the number m of identifiable parameters. For relatively simple plate-type specimens the calculations are fast enough to execute 100-300 trial runs in 15 minutes of computing time on PC with Intel quad-core i7 processor.

Step 3. Carrying out numerical experiments, registering and grouping the eigenfrequencies according to vibration modes.

Step 4. Building the metamodels (surrogate model) for the dependency of calculated eigenfrequencies $\boldsymbol{f}^{\mathrm{FEM}}$ on the input parameters $\boldsymbol{E}$.

$$\boldsymbol{f}^{\mathrm{FEM}} \approx \hat{\boldsymbol{f}}(\boldsymbol{E}), \tag{7}$$

where the "hat" above a function symbol signifies approximation. The software EDAOpt [8], created at Institute of Mechanics of Riga Technical University, was used for design of computer experiments, metamodel building and minimization of discrepancy functional.

Quality of approximation is estimated using leave-one-out cross-validation [9]. The relative cross-validation error *CVE* is calculated relatively to standard deviation from the mean value of responses. In cases where analytical thick plate models [9] are used instead of FEM, the prediction error was calculated in more than 10^6 test points and the mean error agrees very well with the cross-validation estimate.

Practice shows that almost in all cases third order polynomials give the approximation of frequencies with relative cross-validation error less than 0.02%. For some more complicated specimens the best accuracy can be obtained using nonparametric approximation methods: kriging, locally weighted polynomials.

Step 5. Finding the values of identifiable parameters by minimization of approximated discrepancy functional

$$\boldsymbol{E}^* = \underset{\boldsymbol{E}}{\arg\min} \sum_{i=1}^{n} w_i \left(\hat{f}_i(\boldsymbol{E}) - f_i^{\mathrm{EXP}}\right)^2 \tag{8}$$

The software EDAOpt uses a modified multi-start simulated annealing method [10] and always gives global minimums of the discrepancy functional. It must be noted that the polynomial metamodels are relatively simple: for evaluation of the objective function Eq. (8) even several millions of evaluations need only a few minutes of computing time.

Step 6. Traditionally, the next step is the recalculation of the metamodel in the sub-area near the found values of identified parameters, and analysis of the significance of different elasticity parameters for natural frequencies [2,3]. However, this analysis gives insufficient information for the estimation of accuracy of identified values. Therefore the present work proposes a method for accuracy estimation.

3 STANDARD DEVIATIONS OF IDENTIFICATION

3.1 Sources of composite material parameter deviations

The errors of elastic parameter identification using the vibration method depend on errors introduced by material production, cutting of testing specimens, physical measurement errors and errors of the finite element model. Practice shows that the mode measurements using the Polytec PSV-400-3D Laser Scanning Vibrometer (http://www.polytec.com/) have very high accuracy. The repeated measurements for a given specimen give almost the same results. At the same time the measurements of 4-6 different specimens from the experimental sample allow to estimate the standard deviations from the mean as about 0.5% up to 2%. This means that the variance of parameters introduced by production (elasticity, density, thickness uniformity, misalignment of the reinforcing fibre etc.) and errors introduced by the preparation of the sample (geometrical errors, errors of density and weight estimation, microdamages created by sample cutting) have determining influence on the variance of identified elastic parameters. Here we will use the term „noisy constants" for parameters that are taken as constants in FEM calculations, but that contain uncertainty for the actual physical specimens.

3.2 Discrepancy minimization approach in the presence of noisy constants

Let us define the following variables:

$\boldsymbol{E}$ – column vector of size n of elastic parameters whose numerical values must be identified. In the simplest case of orthotropic material plate, this vector consists of 4 components ($n = 4$): longitudinal modulus E_x, transverse modulus E_y, shear modulus G_{xy} and Poisson's ratio ν_{xy}.

$$\boldsymbol{E} = \left[E_x, E_y, G_{xy}, \nu_{xy}\right]^{\mathrm{T}}, \tag{9}$$

$\boldsymbol{P}$ – column vector of size m of *noisy constants* – parameters that are taken into account as constants both in FEM and in approximated analytical calculations of eigenfrequencies, but physically are parameters with uncertainty. In the simplest case of rectangular plate, the vector $\boldsymbol{P}$ consists of four components (m = 4): plate length a, plate width b, plate total thickness h and the average material density ρ:

$$\boldsymbol{P} = [a, b, h, \rho]^{\mathrm{T}}. \tag{10}$$

$\boldsymbol{f}^{\mathrm{EXP}}$ – column vector of measured L lowest natural frequencies f_i^{EXP}, $i = 1,2,\ldots, L$.

$\hat{\boldsymbol{f}}$- column vector of the approximated model for the dependence of natural frequencies on material elastic parameters $\boldsymbol{E}$ and specimen parameters $\boldsymbol{P}$. This functional dependence can be obtained by metamodeling (approximation of FEM results) or using approximate analytical expressions:

$$\hat{\boldsymbol{f}} = \hat{\boldsymbol{f}}(\boldsymbol{E}, \boldsymbol{P}) \tag{11}$$

$\boldsymbol{W}$ – diagonal matrix of weighting coefficients:

$$\boldsymbol{W} = \begin{bmatrix} w_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_L \end{bmatrix} \tag{12}$$

The discrepancy function that measures the deviation between measured and numerically calculated frequencies in matrix form:

$$\Phi(\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{W}) = (\boldsymbol{f}(\boldsymbol{E}, \boldsymbol{P}) - \boldsymbol{f}^{\mathrm{EXP}})^{\mathrm{T}} \boldsymbol{W} (\boldsymbol{f}(\boldsymbol{E}, \boldsymbol{P}) - \boldsymbol{f}^{\mathrm{EXP}}) \tag{13}$$

Most frequently used weighting coefficients vary inversely as the square of measured frequency $w_i = \left(1/f_i^{\mathrm{EXP}}\right)^2$, so that the discrepancy function is the sum of squared relative differences between measured and calculated frequencies [3,10 and others].

The discrepancy minimization approach means that for given values of specimen parameters $\boldsymbol{P}$ and weighting coefficients $\boldsymbol{W}$, input parameters $\boldsymbol{E}$ which give minimal value of functional Φ will be considered as identified values for parameters $\boldsymbol{E}^*$:

$$\boldsymbol{E}^* = \underset{\boldsymbol{E}}{\arg\min}\, \Phi(\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{W}) \tag{14}$$

Because the simple third-order polynomial metamodels of FEM results give very small prediction error and the number of minimization variables is relatively small, the minimization of discrepancy function is easy and needs 1-3 seconds of PC processor time using the multistart random search method [8]. The main problem is not the discrepancy minimization, but the estimation and improvement of the accuracy of identification.

3.3 Calculation of the standard deviations of identified values

Let us assume that the sources of identification errors are:

1) Uncertainty of specimen's parameters $\boldsymbol{P}$. We assume that these uncertainties may be described by unbiased independent random errors conforming to normal probability density distribution with zero means and given standard deviations:

$$\boldsymbol{P} = \overline{\boldsymbol{P}} + \boldsymbol{\varepsilon}^P, \ \ \varepsilon_i^P = N(0, \sigma_i^P), \ i = 1,2,3, m \tag{15}$$

2) Additional errors of measured frequencies, caused by measurement errors and uncertainty of parameters which are not included in vector $\boldsymbol{P}$, for example dimensional inaccuracies like deviations of rectangularity, curvature, deviations of orthotropic axes and others. For simplicity, we consider these errors as unbiased independent random errors conforming to normal probability density distribution with zero means and given standard deviations:

$$\boldsymbol{f}^{\mathrm{EXP}} = \overline{\boldsymbol{f}}^{\mathrm{EXP}} + \boldsymbol{\varepsilon}^F, \ \ \varepsilon_i^F = N(0, \sigma_i^F), \ i = 1,2, \dots, L \tag{16}$$

3) Deviations of elasticity parameters themselves. We consider these deviations as unbiased independent random errors conforming to normal probability density distribution with zero means and given standard deviations:

$$\boldsymbol{E} = \overline{\boldsymbol{E}} + \boldsymbol{\varepsilon}^E, \ \ \varepsilon_i^E = N(0, \sigma_i^E) \tag{17}$$

4) Other sources of identification errors include the lack of correspondence between the mathematical model and the physical experiment. This may occur due to neglecting temperature effects, internal and external friction (air resistance and fastening effects), the use of inadequate FEM elements (homogeneous shell elements instead of layered structural shell elements for multilayer plate, etc.). These errors are inherently more systematic than random errors and can be discovered by analysis of means and standard deviations of the difference between measured and calculated values. Comparisons with the elastic parameter measurements obtained by other methods, for example bending tests are always useful.

5) Outliers - large errors made during physical and mathematical experimentation. The main source of this type of errors is the incorrect grouping of oscillation modes. All FEM software packages give the eigenfrequencies and corresponding modes in order of increasing frequency. The automatic mode recognition will be explained below.

After the minimization of the weighted discrepancy function Φ, the column vector of identified values $\boldsymbol{E}^*$ is found. For the analysis of the identification errors we will use the linearized model of the frequency dependence on elastic parameters $\boldsymbol{E}$ and specimen parameters $\boldsymbol{P}$ in the form

$$\boldsymbol{f}(\boldsymbol{E}, \boldsymbol{P}) = \boldsymbol{AE} + \boldsymbol{BP} + \boldsymbol{C} \tag{18}$$

where $\boldsymbol{A}$ – constant matrix $L\times n$, $\boldsymbol{B}$ – constant matrix $L\times m$, $\boldsymbol{C}$ – constant column vector of size L. Matrices $\boldsymbol{A} = \frac{\partial \hat{f}}{\partial \boldsymbol{E}}(\boldsymbol{E}^*, \overline{\boldsymbol{P}})$, $\boldsymbol{B} = \frac{\partial \hat{f}}{\partial \boldsymbol{P}}(\boldsymbol{E}^*, \overline{\boldsymbol{P}})$ and column vector $\boldsymbol{C} = \hat{f}(\boldsymbol{E}^*, \overline{\boldsymbol{P}}) - \boldsymbol{AE}^* - \boldsymbol{B}\overline{\boldsymbol{P}}\boldsymbol{C}$ can be calculated analytically, if the metamodel is polynomial or by use of numerical differentiation for other types of metamodels, including kriging.

Then the discrepancy function assumes the form:

$$\boldsymbol{\Phi}(\boldsymbol{E},\boldsymbol{P},\boldsymbol{W}) = (\boldsymbol{f}(\boldsymbol{E},\boldsymbol{P}) - \boldsymbol{f}^{\text{EXP}})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{f}(\boldsymbol{E},\boldsymbol{P}) - \boldsymbol{f}^{\text{EXP}}) = \tag{19}$$
$$= (\boldsymbol{AE} + \boldsymbol{BP} + \boldsymbol{C} - \boldsymbol{f}^{\text{EXP}})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{AE} + \boldsymbol{BP} + \boldsymbol{C} - \boldsymbol{f}^{\text{EXP}})$$

This is a second-order polynomial regarding components of vector $\boldsymbol{E}$, therefore the minimum of the discrepancy function can be found by equating partial derivatives to zero:

$$\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{E}}(\boldsymbol{E}^*,\boldsymbol{P},\boldsymbol{W}) = 0 \tag{20}$$

This gives a system of linear algebraic equations regarding the vector of parameters to be identified $\boldsymbol{E^*}$

$$\boldsymbol{A^T W A E^*} + \boldsymbol{A^T K C} + \boldsymbol{A^T W B P} - \boldsymbol{A^T W f}^{\text{EXP}} = 0 \tag{21}$$

Introducing the designation for system matrix $\boldsymbol{R} = \boldsymbol{A^T K A}$, we obtain the expression for vector of identified values:

$$\boldsymbol{E}^* = \boldsymbol{R}^{-1}(\boldsymbol{A^T W f}^{\text{EXP}} - \boldsymbol{A^T W C} - \boldsymbol{A^T W B P}) \tag{22}$$

The size of the symmetrical matrix $\boldsymbol{R}$ is equal to the number of parameters to be identified. The determinant and condition number of matrix $\boldsymbol{R}$ is very important. Small value of the determinant signifies an ill-conditioned identification problem, which can occur by using small number of frequencies and by including simultaneously non-identifiable parameters (like density, ply thickness and elastic moduli) in the vector $\boldsymbol{E}$.
We introduce symbols for matrices:

$$\boldsymbol{Q} = \boldsymbol{R}^{-1}(\boldsymbol{A^T W B}) \tag{23}$$

and

$$\boldsymbol{S} = \boldsymbol{R}^{-1}(\boldsymbol{A^T W}) \tag{24}$$

Taking into account assumptions about normally dispersed parameter errors and additional frequency errors, the standard deviation of *k*-th identifiable parameter can be calculated according to the statistical law for linear sum of random variables [11].

$$\text{STD}(E_k^*) = \sqrt{\sum_{i=1}^{m}(Q_{ik}\sigma_i^P)^2 + \sum_{i=1}^{L}(S_{ik}\sigma_i^F)^2 + \sigma_k^E}\ , \quad k = 1,\ldots, n, \tag{25}$$

It also is not difficult to take in account the correlation between frequency and parameter errors, if the covariance matrices can be estimated.

4 PARAMETER IDENTIFICATION WITH ROBUSTNESS CONTROL

The identification consists of two-stage optimization. During the first stage, the metamodel is built for the dependence of natural frequencies on the elastic parameters $\boldsymbol{E}$ and noisy constants $\boldsymbol{P}$, and the unconstrained minimization of the discrepancy function is provided using fixed weighting coefficients $\boldsymbol{W}$ for frequencies. The metamodel is created on the basis of computer experiments, which are carried out according to space-filling Latin hypercube experimental designs [7]. Usually third-order polynomial approximations are used. The range of variation of noisy constants in the computer experiment should not exceed the real uncertainty level more than twice, because the following analysis of robustness assumes small parameter deviations. During the creation of the metamodel, accurate mode recognition must be provided to exclude large errors of responses (outliers). In most cases the modal assurance criterion (MAC) based on Pearson correlation coefficient between two mode shapes allows fast and accurate mode grouping.

During the second stage, the robustness of the identification is checked. The calculation of the standard deviations of identified parameter values is based on the assumed values of deviations of noisy constants and natural frequency measurements. In reality these values are known only very approximately or not at all. In any case, the estimation of the deviations of parameters, used in physical measurements and FEM calculations, should be obtained with replicated measurements using a sufficiently large number of specimens. If the deviations of noisy constants (and the probability density functions for them) would be known with high precision, then the step of robust optimization could include the minimization of the standard deviation of identification by variation of weighting coefficients. Since in reality the deviations of noisy constants are known rather approximately, then in practice robust minimization in this case means simply the choice – to use or not to use some measured frequency in the discrepancy function. But the main significance of robustness analysis is the ability to compare the precision of different methods (using different modes, non-dimensionalization and scaling of frequencies) with approximately identical deviations of measured parameters.

After the analysis of the standard errors of identification and the choice of a different frequency set for identification, the first stage must be repeated. This process (discrepancy minimization – STD minimization) usually converges quickly.

5 FREQUENCY APPROXIMATION AND NONDIMENSIONALIZATION

The frequently used simple approximate analytical formula for natural frequencies of orthotropic plates with completely free boundary conditions is given by Dickinson [9]. He applied characteristic beam functions in Rayleigh's method to obtain an approximate formula for the flexural vibration of specially orthotropic plates. (Specially orthotropic means that the orthotropic axes are parallel to rectangular plate edges).

Using the four elasticity parameters (E_x, E_y, G_{xy}, ν_{xy}), three dimensional parameters a, b, h and average density parameter ρ, the formula assumes the form:

$$f_{ij} = \frac{\pi}{2\sqrt{12}\sqrt{1-\nu_x\nu_x E_y/E_x}} \cdot \frac{h}{\sqrt{\rho}} \left(\frac{G_i^4 E_x}{a^4} + \frac{G_j^4 E_y}{b^4} + \frac{2H_i H_j \nu_x E_y}{a^2 b^2} + \frac{4J_i J_j G_{xy}(1-\nu_x\nu_x E_y/E_x)}{a^2 b^2} \right)^{1/2} \quad (26)$$

where f_{ij} is the natural frequency, measured in Hz, i and j are the number of halfwaves of mode shape in x and y direction respectively. The dimensionless parameters G, H, and J are functions of the indices i and j and can be taken from literature [9]. Comparison with the FEM calculations shows that the error of analytical approximation is about 4% for the case of homogenous plate. The analytical expressions of natural frequencies therefore cannot be used for accurate identification of elastic properties, but they can help to analyze the nature and sources of identification errors.

As can be seen from equation (26), all frequencies have the coefficient $h/\sqrt{\rho}$ and the multiplication of all three elasticity moduli with any constant coefficient c causes the increasing of frequency $\sqrt{c}$ times. From this follows that it is not possible to simultaneously identify plate material density or thickness and elasticity parameters. This effect is called isospectral family – systems with different geometrical and elasticity parameters but equal all natural frequencies [12].

5.1 Partial nondimensionalization of natural frequencies.

Let us assume that independent standard deviations of frequencies are proportional to mean values of frequencies. As can be seen, in approximated analytical expressions (26) the plate thickness gives a multiplier $\sqrt{h^3}$ for all frequencies, if the density is calculated as division of mass by volume $\rho = M/abh$. Therefore it would be very desirable to use not the frequency values, but the ratios between them, for example dividing all first L frequency values by the sum of frequencies:

$$\tilde{f}_i = \frac{f_i}{\sum_{j=1}^{L} f_j}, i = 1,2, \dots, L \tag{27}$$

This nondimensionalization can be carried out for physically measured and numerically calculated frequency values. Obviously, the nondimensional frequencies of this type (at least for analytical model Eq.(26)) do not depend on the plate thickness and are not affected by thickness errors. The drawback is that the elastic moduli cannot be determined uniquely – the frequencies $\tilde{\boldsymbol{f}}$ build an isospectral family. As can be seen from expression (26), multiplying the values of moduli E_x, E_y, G_{xy} by the same coefficient will give exactly the same values of scaled frequencies $\tilde{\boldsymbol{f}}$. Numerical experiments with FEM metamodels showed that the use of scaled frequencies gives very accurate identification of Poison's ratio and the values of elastic moduli are determined with an accuracy of common unknown coefficient. This means that the values of nondimensional ratios E_y/E_x and G_{xy}/E_x can be determined very accurately. Therefore for practical use in identification, the lowest frequency was used without scaling:

$$\tilde{f}_1 = f_1, \ \tilde{f}_i = \frac{f_i}{\sum_{j=1}^{L} f_j}, i = 2,3, \dots, L \tag{28}$$

The lowest frequency for the Dickinson's analytical frequency model [9] is the torsional mode $f_{2,2}$

$$f_1 = f_{2,2} = \frac{1.10279 h \sqrt{G_{xy}}}{ab\sqrt{\rho}} = \frac{1.10279\sqrt{h^3}\sqrt{G_{xy}}}{\sqrt{Mab}} \quad . \tag{29}$$

6 EXAMPLE OF IDENTIFICATION

The designs of computer experiments optimized according to the MSE space filling criterion [7], the polynomial and kriging metamodels and the optimization software EDAOpt [8] have been used for the identification of composite material properties for more than 10 years [3]. In the present study we demonstrate the use of the proposed method for the example problem and frequency measurements of P. Pedersen and P.S. Frederiksen [14], that have been used by many authors [4,10,13]. They measured the first ten natural frequencies of a thin glass/epoxy composite laminate with a stacking sequence of $[0,-40,40,90,40,0,90,-40]_s$. We seek the four in-plane ply-elastic constants of a thin composite laminate: E_x, E_y, G_{xy} and ν_{xy}. The rectangular plate had the dimensions (length a, width b and thickness h) given in Table 1.

Normal probability density distributions were used for uncertainties of noisy constants a, b, h, ρ. The plate was attached by two strings which were assumed to be modeled appropriately by free boundary conditions. The measured frequencies are provided in Table 2.

Table 1: Plate properties: length (a), width (b), thickness (h) and density (ρ) and their standard deviations

Parameter	a (mm)	b (mm)	h (mm)	ρ (kg/m^3)
Mean value	209	192	2.59	2120
Standard deviation	0.25	0.25	0.01	10.6

Table 2: Measured natural frequencies [12] and assumed standard deviations [4]

Frequency	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
Value (Hz)	172.5	250.2	300.6	437.9	443.6	760.3	766.2	797.4	872.6	963.4
STD %	0.5	0.52	0.54	0.58	0.585	0.686	0.687	0.697	0.721	0.75

The 301-point 8-factor sequential MSE-optimized design [7] was used for metamodel building. The 8 factors were four elastic parameters E_x, E_y, G_{xy}, ν_{xy} and four noisy constants a, b, h, ρ.

For FEM calculations the software ANSYS was used with elements SHELL 281. Third-order polynomial approximations were used for metamodeling. The oscillation modes 4-5 and 6-7 were mixed up (interchanged) for many points of experimental design in the output file of ANSYS. The relative cross-validation errors for frequencies 4,5, and 6, 7 with mixed modes were 3.9% and 10.2% respectively. After automatic mode recognition the cross-validation errors decreased to 0.03%, except for modes 2,3 and 9,10 which had CVE of about 0.2%. It must be noted that standard deviation from the mean value of frequency responses are about 25 times smaller than the mean value. Therefore if the relative error would be calculated in relation to the mean value, the numbers would be correspondingly 25 times smaller, that is, approximately 0.008%. This means that the approximation error is at least 100 times smaller than the error of physical measurements.

Using partially nondimensionalized frequencies the mean difference between measured and calculated frequencies for identified parameter values were about ±0.2%.

Table 3 shows the comparison of identification results obtained by different authors. As can be seen, the estimated standard deviation of Poison's ratio when using dimensioned frequencies is 59%. This means that using discrepancy minimization with dimensioned Hz frequencies it is not possible to identify the Poison's ratio if the frequency measurement standard deviation exceeds 0.5%. At the same time, using the partially scaled frequency approach, the estimated standard deviation for Poison's ratio is only 2.6%, which means that the accuracy of identification is very high.

In the publications of other authors regarding the identification of this plate, the evaluation of the standard deviation of identified parameters is given only in [4]. The present results of identification within 1-2 standard deviations overlap with the results given in [4], but the results obtained with the partial scaling method have 2 times smaller standard deviation for elasticity moduli and 5 times smaller STD for Poison's ratio. Therefore identification with the robust optimization approach is much more precise if it is assumed that the FEM model gives sufficiently accurate frequency calculations without systematic error.

Table 3: Comparison of identification results obtained by different authors

Method	E_x STD CV%	E_y STD CV%	G_{xy} STD CV%	ν_{xy} STD CV%
Present Hz	62.81 4.41 7.01	20.97 2.88 13.7	9.47 2.39 25.2	0.246 0.14 59
Present scaled	65.40 1.15 1.75	22.63 0.42 1.84	8.39 0.21 2.51	0.186 0.005 2.6
[14]	61.3 ----- ------	21.4 ------ -----	9.8 ----- -----	0.280 ----- -----
[4]	60.8 1.85 3.05	21.3 1.16 5.46	9.87 0.59 5.96	0.27 0.034 12.2
[13]	56.5 ----- ------	20.8 ------ -----	11.8 ----- -----	0.349 ----- -----
[10]	57.2 ----- -----	21.4 ------ -----	11.3 ----- -----	0.300 ----- -----

7 CONCLUSIONS

- The use of the natural frequency metamodel with „noisy constants" included in the set of input variables and weighted discrepancy minimization method allows obtaining the estimated values for standard deviations of identified parameters.
- The use of calculated standard deviations of identified parameters allows the formulation of the identification problem in the form of robust minimization – simultaneous minimization of the discrepancy function and the standard deviations of determined values by variation of unknown values of elastic parameters and weighting coefficients.
- The third-order polynomial approximations for the dependence of natural frequencies (calculated by FEM) on the elastic parameters and „noisy constants" gives the prediction error of metamodel, which is approximately 100 to 200 times less than the frequency errors caused by uncertainty of geometrical and physical parameters of the composite material specimens.
- The use of partially scaled frequency values allows to reduce the standard deviations of determined values of elastic moduli about two times and the STD value of Poisson's ratio about 20 times in comparison with the use of dimensioned frequency values.
- The relative standard deviations of all identified values of elastic parameters are typically about 3-4 times larger than the relative standard deviations of physically measured natural frequencies. The accuracy of elastic moduli identification was about 4%, Poisson's ratio – about 14% for 95% confidence level. The method proved that the Poisson's ratio for plate type specimens can be determined using only out-of-plane bending modes with about 5% error for 95% confidence level, when frequency standard deviations are about 1%.

Acknowledgements

This work has been supported by the European Social Fund within the Project No. 2013/0025/1DP/1.1.1.2.0/13/APIA/VIAA/019 "New "Smart" Nanocomposite Materials for Roads, Bridges, Buildings and Transport Vehicles".

REFERENCES

[1] Larrson, D. Using modal analysis for estimation of anisotropic material constants. *Journal of Engineering Mechanics*, Vol. **123**, No. **3**, 222-229, (1997).

[2] Lauwagiea, T., Solb, H.W., Roebbenc, G. Determination of the in-plane elastic properties of the different layers of laminated plates by means of vibration testing and model updating. *Journal of Sound and Vibration*, **274**, 529–546, (2004).

[3] Rikards, R. and Auzins, J. Response Surface Method for Solution of Structural Identification Problems, *Inverse Problems in Science and Engineering*, Vol. **12**, No. 1, , pp. 59-70. (2004).

[4] Gogu, C., Haftka, R., Le Riche, R., Molimard, J. Effect of approximation fidelity on vibration-based elastic constants identification. *Structural and Multidisciplinary Optimization*, Vol. **42**, Issue **2**, (August 2010), 293-304.

[5] Smarslok, B.P., Haftka, R.T., Ifju, P.G. Correlation model for composite elastic properties using only measurements from a single specimen. *Probabilistic Engineering Mechanics*, **29**, (2012), pp. 64-69.

[6] Withers, C.S. and Nadarajah, A. Normal maximum likelihood, weighted least squares, and ridge regression estimates. *Probability and mathematical statistics*, Vol. **32**, fasc. 1 (2012), pp. 11–24.

[7] Auzins, J. Direct Optimization of Experimental Designs, 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, AIAA Paper 2004-4578, CD-ROM Number 17, (2004).

[8] Auzins, J., Janushevskis, A., Janushevskis, J. and Skukis, E. Software EDAOPT For Experimental Design, Analysis And Multiobjective Robust Optimization. OPT-i International Conference on Engineering and Applied Sciences Optimization, M. Papadrakakis, M.G. Karlaftis, N.D. Lagaros (eds.) Kos Island, Greece, 4-6 June 2014. CD-edition, ISSN: 2241-9098, ISBN: 978-960-99994-5-8, (2014).

[9] Dickinson, S.M. The buckling and frequency of flexural vibration of rectangular isotropic and orthotropic plates using Rayleigh's method, *Journal of Sound and Vibration*, Vol. **61**, No. **1**, 1-8, 1978.

[10] Mota Soares, C.M., Moreira de Freitas, M., Araújo, A.L., Pedersen, P. Identification of material properties of composite plate specimens, *Composite Structures*, Vol. **25**, 277-285, (1993).

[11] DeVore, J.L. *Probability and Statistics for Engineering and Science*, 8th edition, (Duxbury Press, 2011).

[12] Gladwell, G.M.L. *Inverse Problems in Vibration*. Second Edition, Kluwer Academic Publishers, 2005.

[13] Rikards, R., Chate, A., Steinchen, W., Kessler, A., Bledzki, A.K. Method for identification of elastic properties of laminates based on experiment design. *Composites: Part B*, **30**, pp. 279–289. (1999).

[14] Pedersen, P., Frederiksen, P.S. Identification of orthotropic material moduli by a combined experimental/numerical method. *Measurement*, Vol. **10**, Issue **3**, (July–September 1992), 113–118.