VI International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2015
B. Schrefler, E. Oñate and M. Papadrakakis (Eds)

MODELING AND NUMERICAL CALCULATION OF PISTON-LIKE OIL DISPLACEMENT FOR DOUBLY-PERIODIC SYSTEMS OF OIL FIELDS DEVELOPMENT

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Key words: Mathematical modeling, Piston-like oil displacement, Weierstrass elliptic functions, Singular integral equation.

Summary. Prediction of the motion of the oil-water contact boundary has great importance in the problems of design of oilfield development by waterflooding: knowledge of the nature of coupled motion of oil and water, displacing oil in the reservoir allows us to optimize the system of oil field development. The simplest model of coupled filtering of oil and water is the model of "multicolored" liquids, which assumes that oil and water have the same or similar physical properties (density and viscosity).

In this paper we consider a more complex "piston-like" model of oil-water displacement, which takes into account differences in viscosity and density of the two fluids. Oil reservoir assumed to be homogeneous and infinite, fixed thickness, with constant values of porosity and permeability coefficients. It is assumed that the reservoir is developed by a group of a finite number of production and injection wells recurrent in two directions (doubly-periodic cluster). Filtration of liquids is described by Darcy's law. It is assumed, that both fluids are weakly compressible and the pressure in the reservoir satisfies the quasi-stationary diffusion equation.

Piston-like displacement model leads to the discontinuity of the tangential component of the velocity vector at the boundary of oil-water contact. Use of the theory of elliptic functions in conjunction with the generalized Cauchy integrals reduces the problem of finding the current boundaries of oil-water contact to the system of singular integral equations for the tangential and normal components of the velocity vector and the Cauchy problem for the integration of the differential equations of motion of the boundary of oil-water contact.

An algorithm for the numerical solution of this problem is developed. The monitoring of oil-water boundary motion for different schemes of waterflooding (linear row, four-point, five-point, seven-point, nine-point, etc.) is carried out.

1 INTRODUCTION

Among the methods of oil fields development the waterflooding method [1, 2] became widespread. The main objective of waterflooding is to maintain by flooding the reservoir pressure, inevitably falling in the primary field development. Simulation of the flooding process, analysis of its qualitative and quantitative characteristics for different schemes of flooding are the purpose of the present study. In this paper, the model of the piston-like oil displacement by water [2], which takes into account the difference in physical properties (density and viscosity) displaced and displacing fluids.

The task of monitoring of the line flooding motion (the line separating the displacement of the water and oil) was first considered by Muskat [3] and subsequently aroused great interest among researchers. It was noted by Leibenson [4], where the viscosity of the displacing fluid was neglected. In the paper of Danilov and Kats [5], based on the potential theory, the original problem of monitoring of the line flooding motion has been reduced to a nonlinear integro-differential equations. Danilov's method was used by Fazlyev [6] for the some scheme of the areal flooding.

2 MATHEMATICAL MODEL

Consider the plane filtration flow of a viscous compressible fluid with viscosity μ and compressibility β in an infinite horizontal reservoir with permeability k, porosity m and thickness h. For the quasi-stationary state of filtration flow the pressure in the reservoir $p(x,y,t)$ satisfies the diffusivity equation and the velocity components $V_x(x,y,t)$ and $V_y(x,y,t)$ are calculated by the Darcy law [1-3]

$$\begin{gathered} \frac{\partial p}{\partial t} = \chi\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right), \\ V_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}, V_y = -\frac{k}{\mu}\frac{\partial p}{\partial y} \end{gathered} \tag{1}$$

where $\chi = k/m\mu\beta$ - the coefficient of diffusivity.

Simulated reservoir is developed by a doubly periodic system of production and injection wells. The whole set of production and injection wells can be represented as an infinite number of repetitions of the wells in two directions. Such repetition can be described by a doubly periodic lattice, which in the complex plane $z=x+iy$ is defined by two complex periods ω_1 and ω_2. The whole set of lattice points in the complex plane is defined as $\omega=m\omega_1+n\omega_2$ ($m,n=0,\pm1,\pm2, \ldots$), the value of $\Delta = \mathrm{Im}(\bar{\omega}_1\omega_2)$ corresponds to the area of a parallelogram lattice (Figure 1).

Solution of the equation (1) in the case of doubly periodic system of production and injection wells (doubly periodic claster) has been obtained in [7-9]. The distribution of the velocity field was presented by the Weierstrass zeta function and written as follows:

$$\begin{aligned} \bar{V}(z,\bar{z}) &= V_x(x,y) - iV_y(x,y) = \\ &= -\sum_{k=1}^{n} \frac{Q_k}{2\pi h}\left(\zeta(z-z_k) + \alpha(z-z_k) - \beta(\bar{z}-\bar{z}_k)\right), \end{aligned} \tag{2}$$

where $\zeta(z)=\frac{1}{z}+\sum_{i,j=-\infty}^{\infty}{}'(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2})$ - Weierstrass zeta function, z_k - location of the k-th well in the cluster, Q_k - flow rate of the k-th well ($Q_k<0$ for the injection well and $Q_k>0$ for the production well), $\beta=\pi/\Delta$ and $\alpha=(\beta\overline{\omega}_1-2\zeta(\omega_1/2))/\omega_1$.

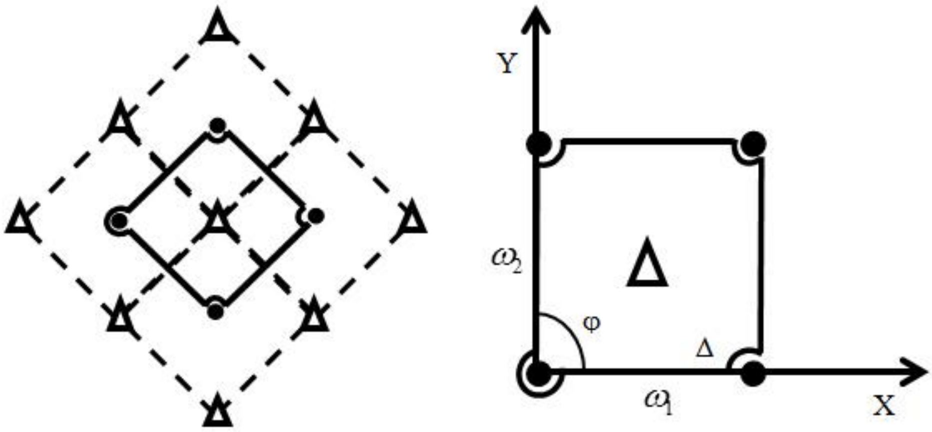


Figure 1: A five-point scheme of flooding (production wells are marked by black circles, injection – by white triangles) and doubly periodic lattice with its basic elements

In the case of the piston-like flooding (moving boundary problem in the theory of filtration) the boundary conditions on the flooding line L (figure 2) for the tangential V_t and normal V_n components of the filtration velocity and for the pressure p are the follows (the index o refers to particles of oil, and the index w - to the particles of water):

$$V_t^{(o)}\mu_{(o)}=V_t^{(w)}\mu_{(w)};\quad V_n^{(o)}=V_n^{(w)};\quad p^{(o)}=p^{(w)}. \tag{3}$$

Discontinuous on the line L, but doubly periodic function $\overline{V}(z,\overline{z})=V_x(x,y)-iV_y(x,y)$ in the complex plane $z=x+iy$ will be sought in the form of the Cauchy-type integral [7-9]

$$\overline{V}(z,\overline{z})=F(z,\overline{z})+\frac{1}{2\pi i}\oint_L\zeta(\tau-z)\gamma(\tau)d\tau, \tag{4}$$

where $F(z,\overline{z})$ is given by equation (2).

The left and right boundary values of this function (4) at a point $z(s)$ on the line L can be written in the form of Sokhotski–Plemel formulas [10]

$$\begin{cases}\overline{V}^{(w)}(z(s))=F(z(s))+\dfrac{1}{2\pi i}\oint_L\zeta(z(\sigma)-z(s))\gamma(z(\sigma))\dfrac{dz}{d\sigma}d\sigma+\dfrac{\gamma(z(s))}{2};\\ \overline{V}^{(o)}(z(s))=F(z(s))+\dfrac{1}{2\pi i}\oint_L\zeta(z(\sigma)-z(s))\gamma(z(\sigma))\dfrac{dz}{d\sigma}d\sigma-\dfrac{\gamma(z(s))}{2}.\end{cases} \tag{5}$$

Applying the Sokhotskiy-Plemel formulas (5), the unknown function *γ(s)* and the complex velocity $\overline{V}(z(s))$ at a point *z(s)* of the line *L* can be expressed as

$$\begin{aligned}\gamma(s)&=\overline{V}^{(w)}(s)-\overline{V}^{(o)}(s),\\ \overline{V}(s)&=(\overline{V}^{(w)}(s)+\overline{V}^{(o)}(s))/2.\end{aligned} \tag{6}$$

Taking into account the boundary conditions on the moving boundary (3), the relations (6) can be rewritten as

$$\begin{aligned}\gamma(s)&=(1-\mu_{(w)}/\mu_{(o)})V_t^{(w)}(s)e^{-i\alpha},\\ \overline{V}(s)&=((1+\mu_{(w)}/\mu_{(o)})V_t^{(w)}(s)/2+iV_n^{(w)}(s))e^{-i\alpha},\end{aligned} \tag{7}$$

where α – the angle between the tangent line *L* and the axis *X,* i.e. $e^{i\alpha}=dz/ds$ (figure 2).

Denoting the viscosity ratio as $\kappa=\mu_{(w)}/\mu_{(o)}$, the normal and tangential velocity components of the water as $T(s)=V_t^{(w)}(s)$ and $N(s)=V_n^{(w)}(s)$, the equations (7) allow us to obtain the following singular integral equation for the unknown functions *T(s)* and *N(s)* on the line *L:*

$$\frac{1+\kappa}{2}T(s)+iN(s)=[F(s)+\frac{1-\kappa}{2\pi i}\oint_L\zeta(z(\sigma)-z(s))T(\sigma)d\sigma]\frac{dz}{ds}\cdot \tag{8}$$

The integral equation (8) must be supplemented by a differential equation that determines the time evolution of the line *L*. This equation has the form [1-3]

$$\begin{aligned}&m\frac{\partial\overline{z}(s,t)}{\partial t}=\overline{V}(z(s,t)),\\ &z(s,0)=z_0+r_we^{i\theta}.\end{aligned} \tag{9}$$

where z_0 - the center of the injection well with radius r_w, through which water is pumped into the reservoir. The initial condition (9) indicates the starting position of the point $z(s,0)=z_0+r_we^{i\theta}$ in the beginning of flooding, the corresponding angle θ is determined on the contour of the injection well.

Using once more the second Sokhotskiy-Plemel formula (7) and the equation (8), the equation (9) can be rewritten as

$$\begin{aligned}&m\frac{\partial\overline{z(s,t)}}{\partial t}\frac{\partial z}{\partial s}=\frac{(1+\kappa)}{2}T(s,t)+iN(s,t),\\ &z(s,0)=z_0+r_we^{i\theta}.\end{aligned} \tag{10}$$

where functions *T(s)* and *N(s)* for a given time *t* are obtained by solving the singular integral equation (8).

3 NUMERICAL SOLUTION OF THE MOVING BOUNDARY PROBLEM

Consider the algorithm for the numerical solution of integro-differential equations (8) and (10). For the numerical solution of singular integral equation (8) we divide the contour *L* by discrete set of points on the elements $[z_i, z_{i+1}]$, $(i = 0, 1, \ldots N-1)$ (Figure 2). Due to the closure of line L, the first and the last points of the partition are the same, i.e., $z_N=z_0$. Each of the points z_k corresponds to the length of the arc s_k. Let us choose and fix the point $z_k=z(s_k)$ on the contour L. Separating in equation (6) the real and imaginary parts, we obtain the following equations for the unknown values of $T_k=T(s_k)$ and $N_k=N(s_k)$ at the points $z_k=z(s_k)$:

$$\begin{cases} \dfrac{1+\kappa}{2}T(s_k) = \mathrm{Re}\left\{[F(s_k)+\dfrac{1-\kappa}{2\pi i}\oint\limits_L \zeta(z(\sigma)-z_k)T(\sigma)d\sigma]z_k^{'}\right\}; \\ N(s_k) = \mathrm{Im}\left\{[F(s_k)+\dfrac{1-\kappa}{2\pi i}\oint\limits_L \zeta(z(\sigma)-z_k)T(\sigma)d\sigma]z_k^{'}\right\}. \end{cases} \tag{11}$$

Omitting the rather cumbersome intermediate calculations, we write the approximation of the integral term in equation (11) as follows:

$$\oint\limits_L \zeta(z(\sigma)-z_k)T(\sigma)d\sigma = \\ = \frac{1}{2z_k'}\left[2T_k \ln\left(\frac{\Delta s_k}{\Delta s_{k-1}}\right) + T_{k+1} - T_{k-1} + \sum_{i=k+1}^{N+k-1}\zeta(z_i-z_k)T_i(\Delta s_i+\Delta s_{i-1})z_k'\right]. \tag{12}$$

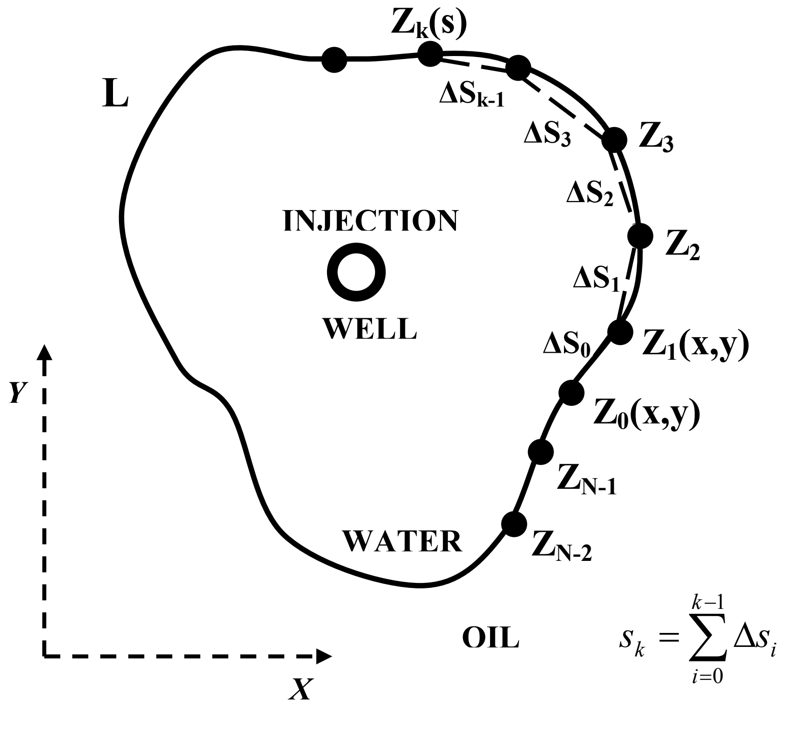


Figure 2: Parameterization scheme of line L and its discretization by points Z_k $(k = 0 \ldots N\text{-}`1)$.

Equations (11) show us that the real part of the singular integral equation (8) matches the finding of values of the unknown function $T_k=T(s_k)$ at a given time t_n on a discrete set of points $z_k=z(s_k)$. The imaginary part of this equation is, in fact, the formula for calculating the values of the function $N_k=N(s_k)$ at a given time t_n at the same discrete set of points.

The obtained values of T_k and N_k are then used to calculate the displacements of points $z_k=z(s_k)$ in the time interval $[t_n, t_{n+1}]$. These displacements are determined by the numerical solution of the Cauchy problem (10) with the Runge-Kutta method, modified in view of the complex nature of the differential equation (10). In the calculations was chosen the dimensionless time *τ,* associated with the original time *t* as $\tau = 10^{-4} Qt / 2\pi mh\omega_1^2$.

4 RESULTS OF CALCULATIONS

To solve this problem there was developed software system to track the evolution of the flooding front (line *L*) in time, as well as to quantify the effectiveness of a particular scheme of flooding (water breakthrough time to production well and waterflood sweep efficiency K_{wse}). In numerical calculations there was considered frontal row, four-point, five-point, seven-point and nine-point scheme of flooding. To determine the current position of the flooding front it was involved 180 tracers - points coming out at the initial time of the injection well.

It is known [11] that the viscosity ratio κ has a negative impact on the ultimate recovery: the decrease in the parameter κ leads to the decrease in the volume of recoverable oil due to the growing instability of oil displacement by water. In addition, when viscosity ratio $\kappa<1$, the Saffman-Taylor instability [12] occurs, which is often called as "viscous fingering". The viscous fingering effect was observed in the course of our numerical calculations. The figure 3 shows the formation of viscous fingers for five-point scheme of flooding at $\kappa=1/5$ and $\tau=0.045$ values. For comparison, the figure 3 is supplemented by the figure from [13] (figure 6a in the cited paper) obtained experimentally for similar geometry placement of wells.

Additionally, the figures 4 and 5 show us the flooding area for seven-point and nine-point schemes of flooding. For comparison the left parts of these pictures show the flooding area with viscosity ratio $\kappa=1$ and the right parts – with $\kappa=1/4$.

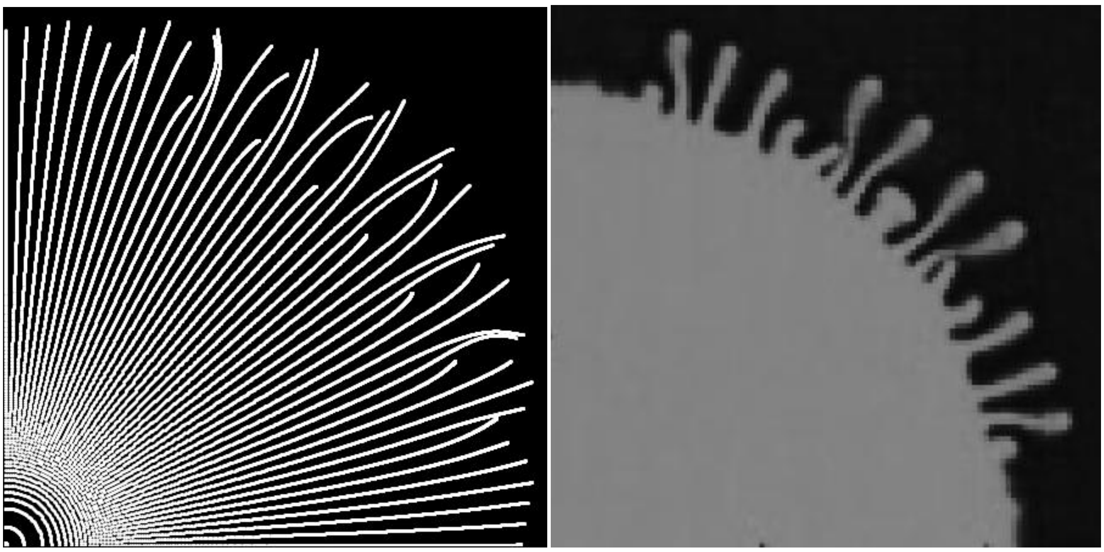
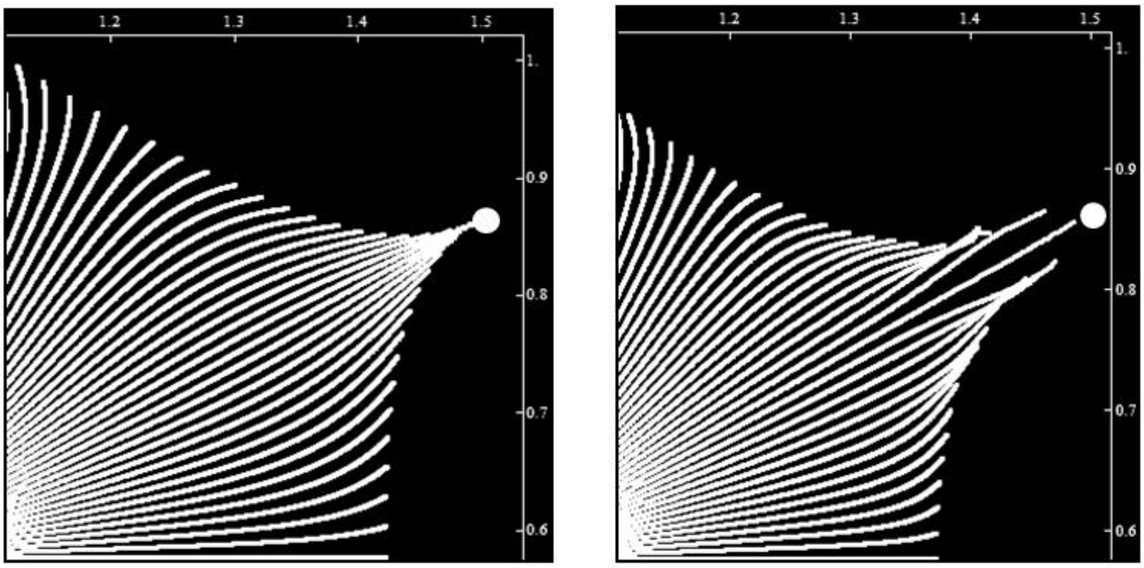
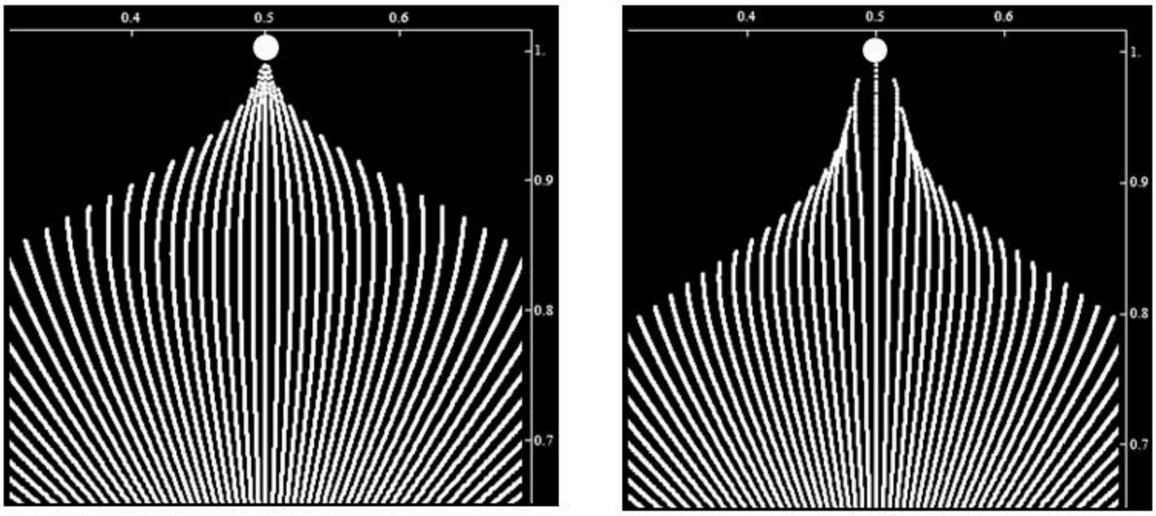


Figure 3: Saffman-Taylor instability of the front of flooding for the five-point scheme (on the left – the numerical calculation for the mobility κ=1/5 = 1/5 and τ = 0.0450, on the right - the data from [13]).



$\kappa = 1, \tau = 0.0516$ $\kappa = 1/4, \tau = 0.0439$

Figure 4: The flooding area for seven-point scheme of flooding.



$\kappa = 1, \tau = 0.0277$ $\kappa = 1/4, \tau = 0.0232$

Figure 5: The flooding area for nine-point scheme of flooding.

Further, the Table 1 shows the values of the water breakthrough time into the producing wells τ and the Table 2 - the waterflood sweep efficiency K_{wse}, calculated for four of waterflood patterns with different values of κ. Abbreviation VF (viscous fingers) indicates the appearance of the "viscous fingering" for the selected scheme of the flooding. Therefore, due to violation of the smoothness of the flooding front the count of the waterflood sweep efficiency K_{wse} for a given value of κ is not possible. For greater clarity, the columns of Table 2 are supplemented with values taken from the book by F. Craig [11] for the case of κ=1.

Table 1: The values of dimentionless time τ for different values of the mobility $\varkappa$

Scheme of flooding	Values of the dimensionless breakthrough time τ			
	κ=1	κ=1/2	κ=1/3	κ=1/4
Five-point	1152	1017	960	VF
Front row	1820	1500	VF	VF
Seven-point	516	472	452	439
Nine-point	277	252	240	232

Table 2: The values of waterflooding coverages K_{cov} for different values of the mobility $\varkappa$

Scheme of flooding	Values of the waterflood sweep efficiency K_{wse}				
	κ=1 [8]	κ=1	κ=1/2	κ=1/3	κ=1/4
Five-point	70%	72,5%	63,5%	60%	VF
Front row	58%	57,2%	47%	VF	VF
Seven-point	73%	75,2%	68,5%	65,3%	63,3%
Nine-point	55%	52,7%	48%	45,5%	44%

ACKNOWLEDGEMENTS

This work was performed as the basic part of the state problem of the Russian Ministry of Science and Education for Samara State Technical University (code project 1785). Authors are also thankful the Russian Foundation for Basic Research for financial support under grant 14-01-97041-r-Volga region-a.

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