Network Flows

UPCOPENCOURSEWARE number 34414

Topic 10: Generalized Flows

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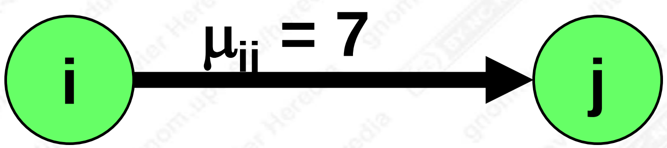
Generalized Flows

- Definitions.
- Applications.
- Properties and optimality conditions.
- Generalized Network Simplex Algorithm.
- Source material:
  - R.K. Ahuja, Th.L. Magnanti, J. Orlin “Network Flows”, chap. 15.
  - J. Orlin “Network Optimization” http://ocw.mit.edu/courses/sloan-school-of-management/15-082j-network-optimization-fall-2010/

Overview of Generalized Flows

- Suppose one unit of flow is sent in (i,j). We relax the assumption that one unit arrives at node j.

- If 1 unit is sent from i, μ_{ij} units arrive at j:



μ_{ij} is called the multiplier of (i,j)

LP Formulation of Generalized Flows

x_{ij} = amount of flow sent in (i,j)

μ_{ij} = multiplier of (i,j)

b(i) = supply at node i

c_{ij} = unit cost of flow in (i,j)

u_{ij} = upper bound on flow in (i,j)

$$\text{Minimize} \quad \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} \mu_{ji} x_{ji} = b(i) \ \forall i \in N$$

$$0 \le x_{ij} \le u_{ij} \quad \forall\ (i,j) \in A$$

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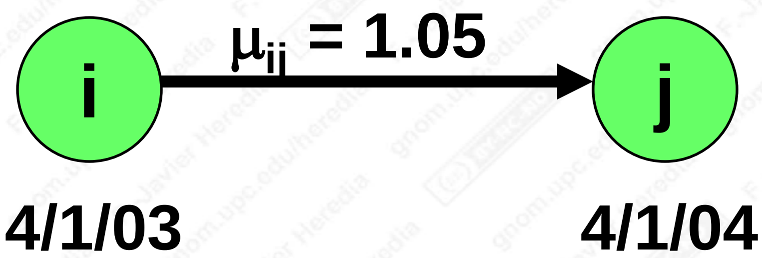
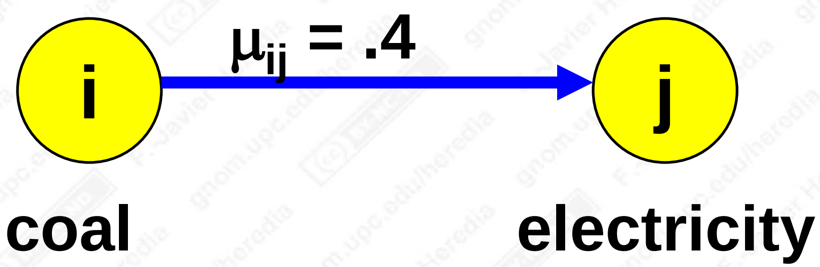
Conversions of physical entities



(i,j) represents a 1 year investment in an equity



(i,j) represents a conversion of coal into electricity

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Machine Scheduling

It takes 3 hours to make one unit of job i on machine j.

x_{ij} = units of product i made on machine j

μ_{ij} = number of hours to make product i on machine j

d(i) = number of units of product i that need to be made.

d(j) = total time available on machine j.

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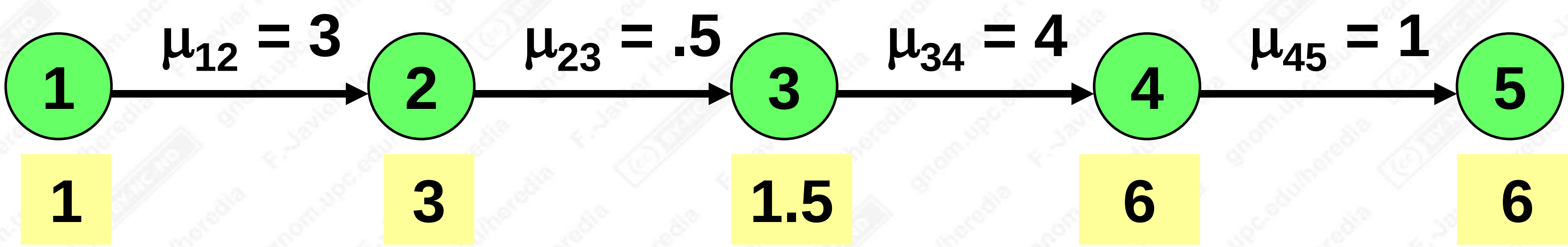
Flows Along Directed Paths

- Suppose that 1 **incremental** unit is sent from node 1, that flow is conserved in 2, 3, and 4, arrives at node 5.

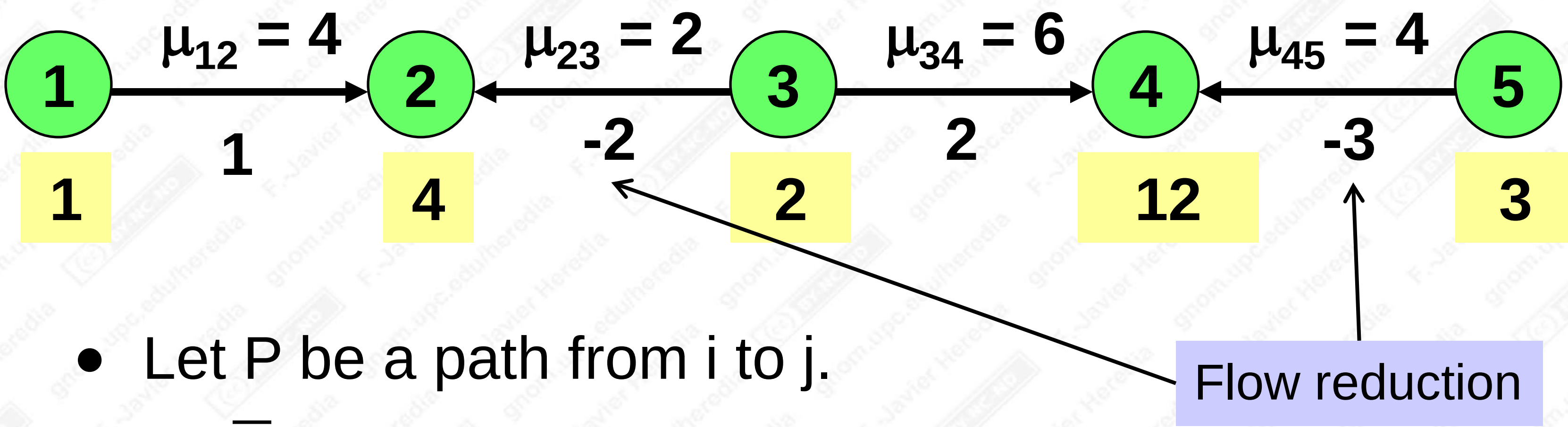


- For a directed path P from i to j, if one unit of flow is sent from i, then the amount arriving at j is:

$$\mu(P) = \prod_{(i,j)\in P} \mu_{ij}$$

Flows Along Non-directed Paths

- Suppose that 1 **incremental** unit is sent from node 1, that flow is conserved in 2, 3, and 4, arrives at node 5.



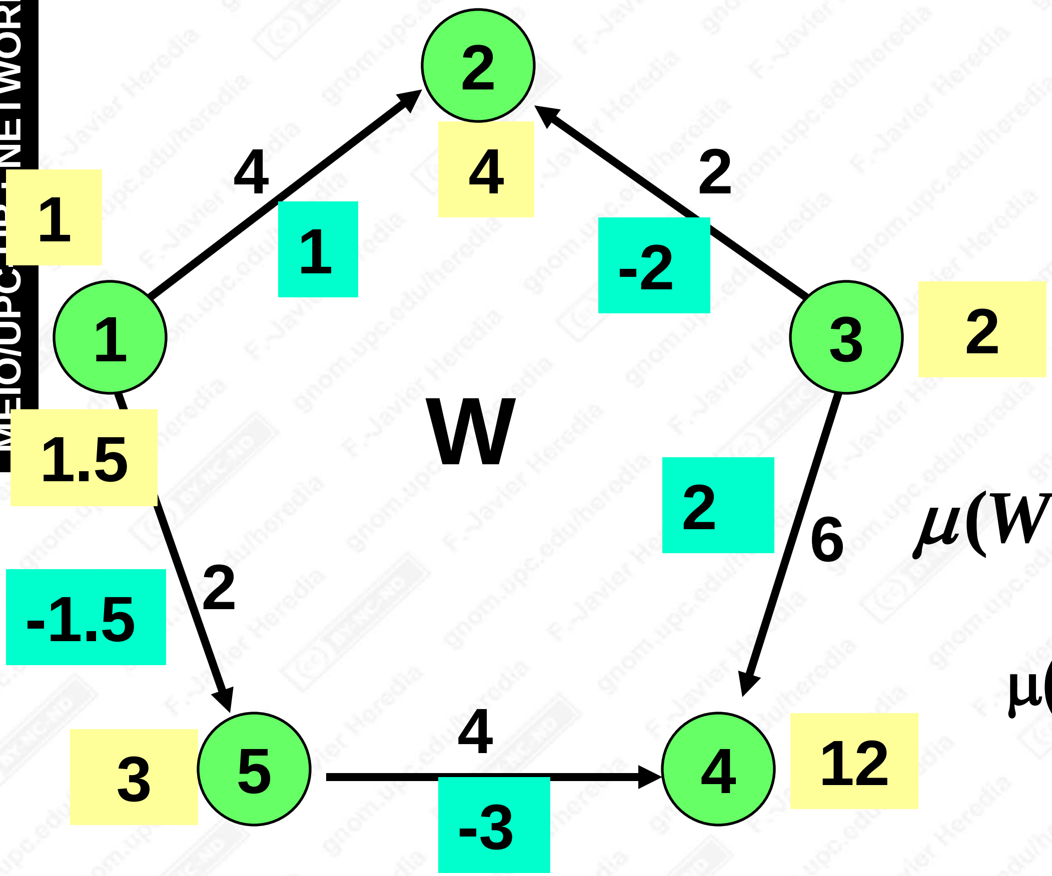
- Let P be a path from i to j.

  $\overline{P}$ = Forward arcs of P

  $\underline{P}$ = Backward arcs of P

- If one unit of flow is sent from i, then the amount arriving at j is: $\mu(P) = \prod_{(i,j)\in \overline{P}} \mu_{ij} / \prod_{(i,j)\in \underline{P}} \mu_{ij}$

Flows Along Cycles



- Suppose 1 unit is sent around W starting and ending at node 1.

$$\mu(W) = \prod_{(i,j)\in \overline{W}} \mu_{ij} / \prod_{(i,j)\in \underline{W}} \mu_{ij}$$

μ(W) = 1.5

- If μ(W) ≠ 1, then the amount of flow arriving at node 1 is different than the amount leaving node 1.
- If μ (W) = 1, W is called a **breakeven cycle**.

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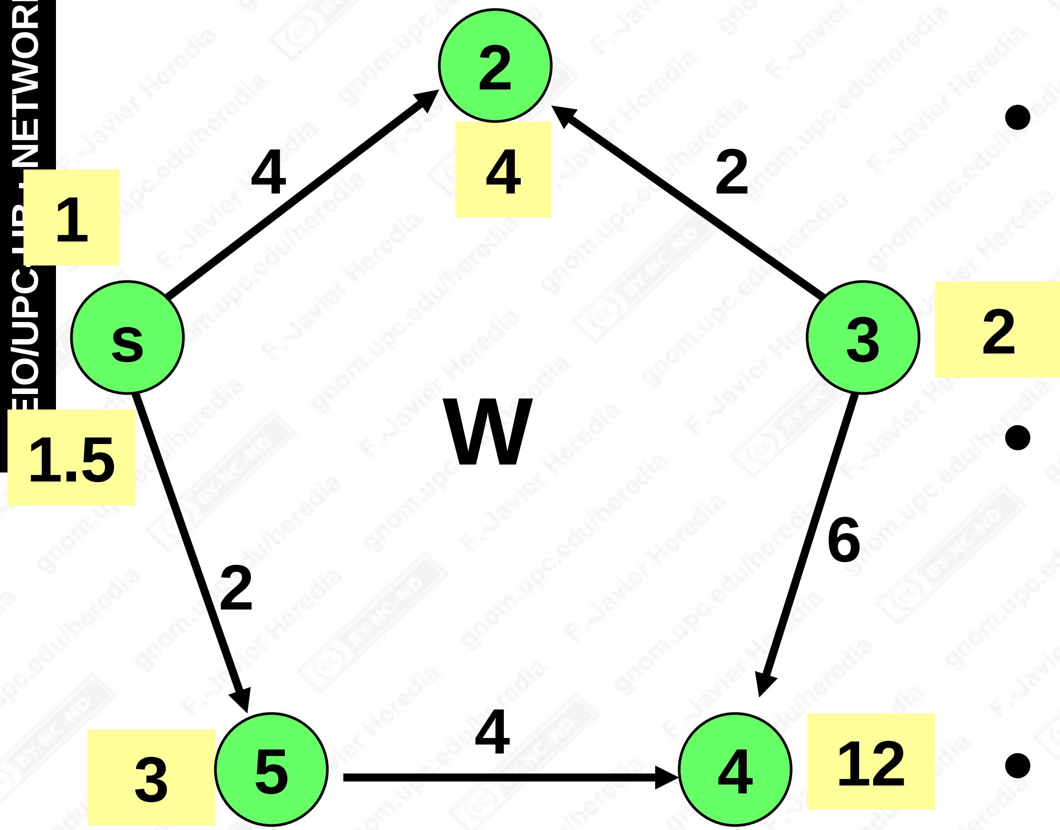
Flows Along Cycles



- Suppose θ units are send around W starting and ending at node s.
- The net amount arriving at node 1 is: $\theta[\mu(W)-1]$.
- To create a "supply" of "*a*" units at node s, send $\theta = a/[\mu(W)-1]$ units of flow.

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On the LP for Generalized Flows

$$\text{Minimize} \quad \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s. t.:} \quad \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(i,j)\in A} \mu_{ji} x_{ji} = b(i) \; i \in N$$

$$0 \le x_{ij} \le u_{ij} \quad \forall \; (i,j) \in A.$$

- The equality constraints have full row rank, which is n.
- A basis consists of n columns that are linearly independent.
- Equivalently, a basis has n columns such that no subset of these columns is dependent.

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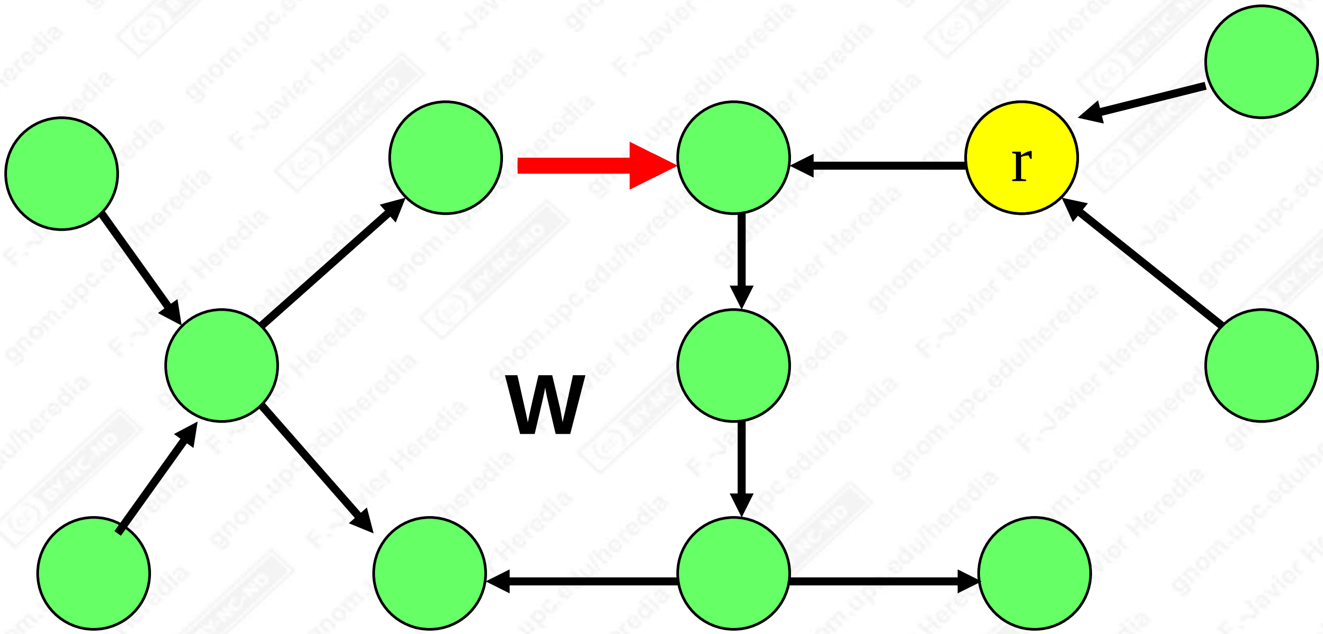
Augmented Trees



- An **augmented tree** T is a connected subset of k nodes and k arcs for some k i.e. T it is a spanning tree plus an **extra arc**.
- It usually has a **root**.
- T is called **good** if the cycle is **non-breakeven**.

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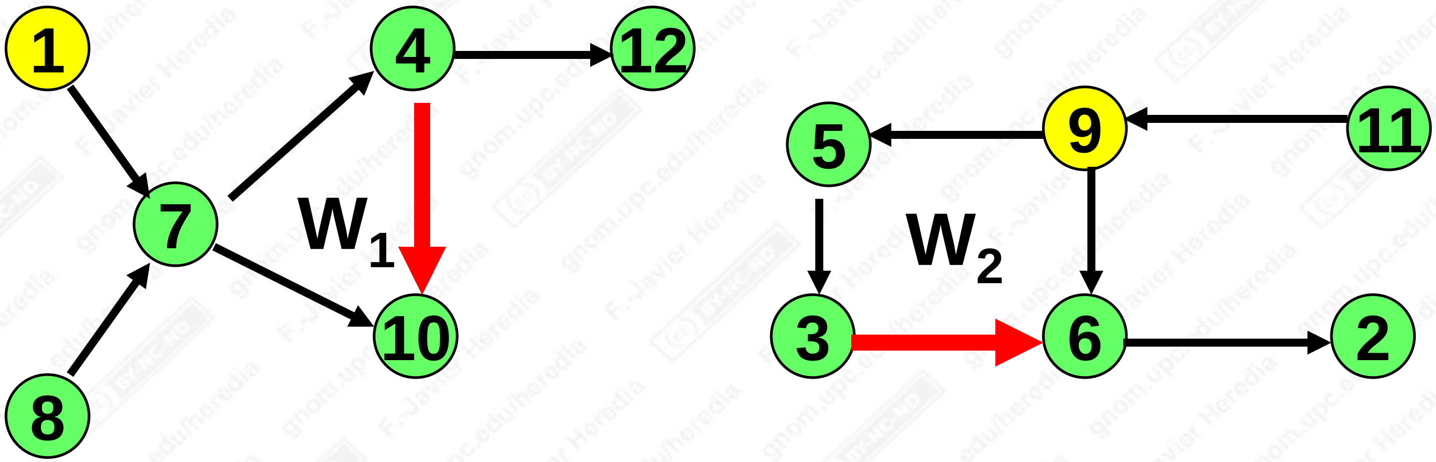
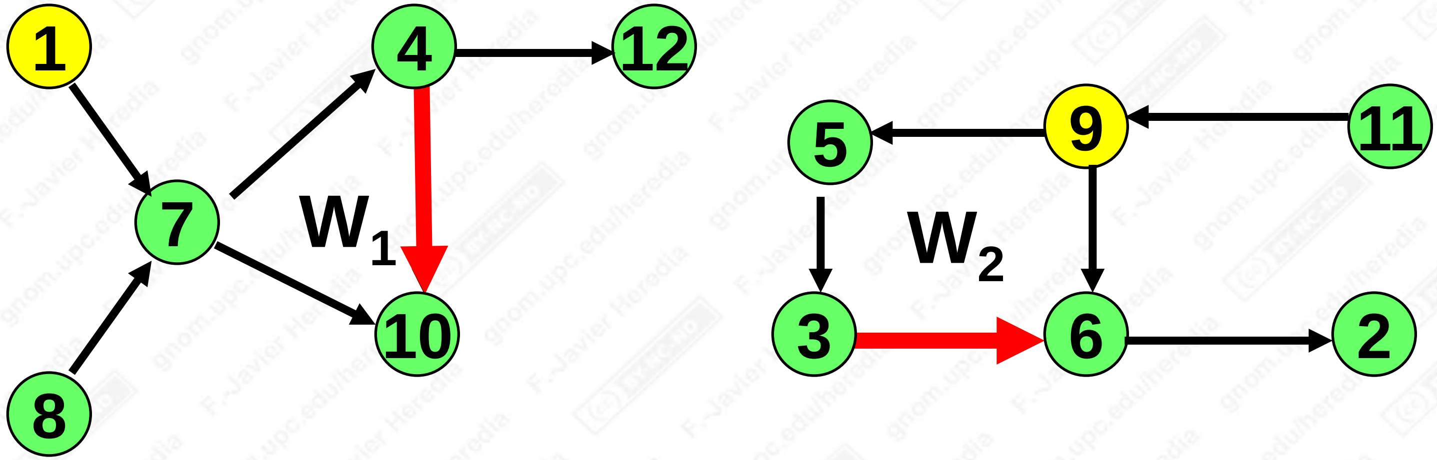
Augmented Forests



- An **augmented forest** is a collection of node disjoint augmented trees including all nodes.
- The augmented forest is **good** if each cycle is **non-breakeven**.

Augmented Forest Structure



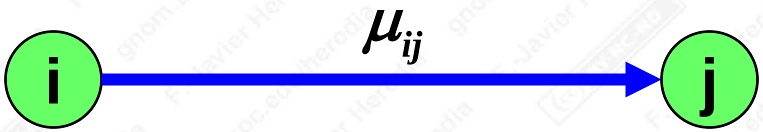
- **Def.:** (F, L, U) is an **augmented forest structure**
  - F are the arcs in the augmented forest.
  - L are the arcs at their lower bound: $x_{ij} = 0$ for $(i,j) \in L$
  - U are the arcs at their upper bound: $x_{ij} = u_{ij}$ for $(i,j) \in U$
- **Th. (good augmented forests and basis):**

A set B of arcs defines a basis of the generalized network flow problem if and only if B is a good augmented forest

proof: AMO, pag. 582-583

Node Potentials and Reduced Costs

- Let π(i) be the **node potential** for node i.
- The **reduced cost** of arc (i,j) is: $c_{ij}^{\pi} = c_{ij} - \pi(i) + \mu_{ij}\pi(j)$



Optimality Conditions (1/2)

Th. (Generalized Flow Sufficient Optimality Conditions)

A flow x is an optimal solution of the generalized network flow problem if it is feasible and for some vector π of node potentials, the pair (x*, π) satisfies the following optimality conditions:*

$$\text{(a)}\quad \text{If } 0 < x^*_{ij} < u_{ij},\ \text{ then } c^{\pi}_{ij} = 0$$

$$\text{(b)}\quad \text{If } x^*_{ij} = 0,\ \text{ then } c^{\pi}_{ij} \geq 0$$

$$\text{(c)}\quad \text{If } x^*_{ij} = u_{ij},\ \text{ then } c^{\pi}_{ij} \leq 0$$

proof: *AMO pag.: 576-577*

Optimality Conditions (2/2)

Property

(Augmented Forest Structure Optimality Conditions)

A feasible augmented forest structure (F, L, U) with the associated flow x is an optimal augmented forest structure if for some vector π of node potentials, the pair (x*, π) satisfies the following optimality conditions.*

$$\textbf{(a)}\quad \text{For all } (i,j) \in F,\ c_{ij}^{\pi} = 0$$

$$\textbf{(b)}\quad \text{For all } (i,j) \in L,\ c_{ij}^{\pi} \geq 0$$

$$\textbf{(c)}\quad \text{For all } (i,j) \in U,\ c_{ij}^{\pi} \leq 0$$

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Generalized Network Simplex

algorithm generalized network simplex

begin

determine the initial feasible augmented forest structure (F, L, U);

let x be the flow and π be the node potentials;

while some nonbasic arc violates its optimality condition **do**

begin

select an entering arc (k,l) violating its opt. condition;

add arc (k,l) to the forest and determine the leaving arc;

perform a forest update, and update x and π;

end;

end;

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The generalized simplex algorithm

1. Find an initial feasible augmented forest structure (F, L, U). (This is often an artificial solution, see AMO, pag 584).
2. Compute node potentials π reduced costs c^{π}_{ij}.
3. Find feasible flow x.
4. Simplex leaving arc rule.

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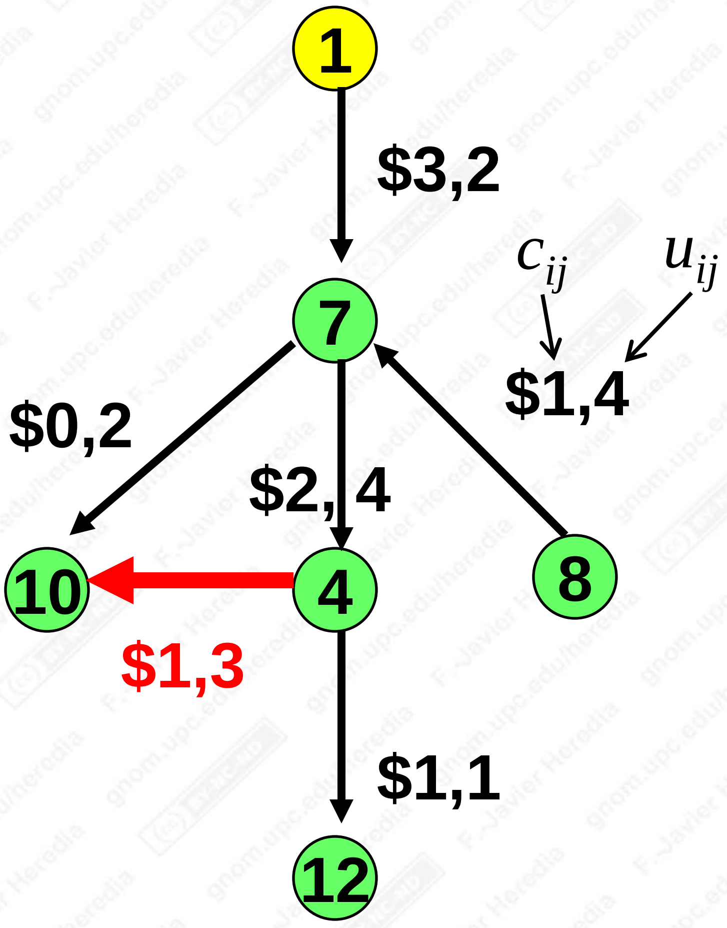
Computing Node Potentials for an Augmented Forest Structure

Compute node potentials

1. Set the potential of the root node to θ. We will determine θ later.
2. Determine the node potentials of all other nodes so that tree arcs have a reduced cost of 0.
3. Determine θ so that the extra arc also has a reduced cost of 0.



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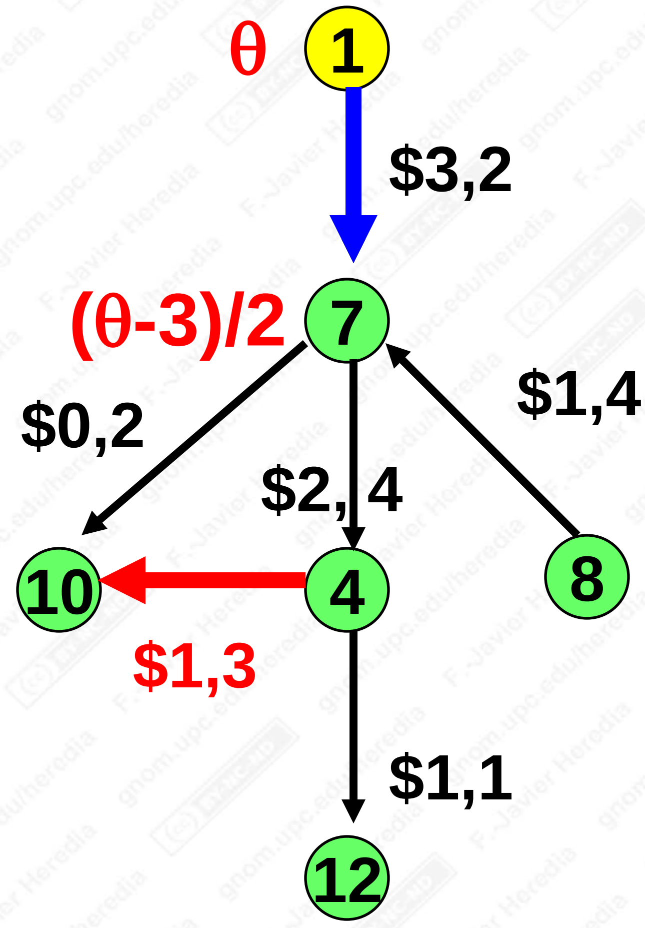
Computing Node Potentials

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c_{17}^{\pi} = c_{17} - \pi(1) + \mu_{17}\pi(7) = 0$$

$$3 - \theta + 2\pi(7) = 0$$

$$\pi(7) = (\theta - 3)/2$$



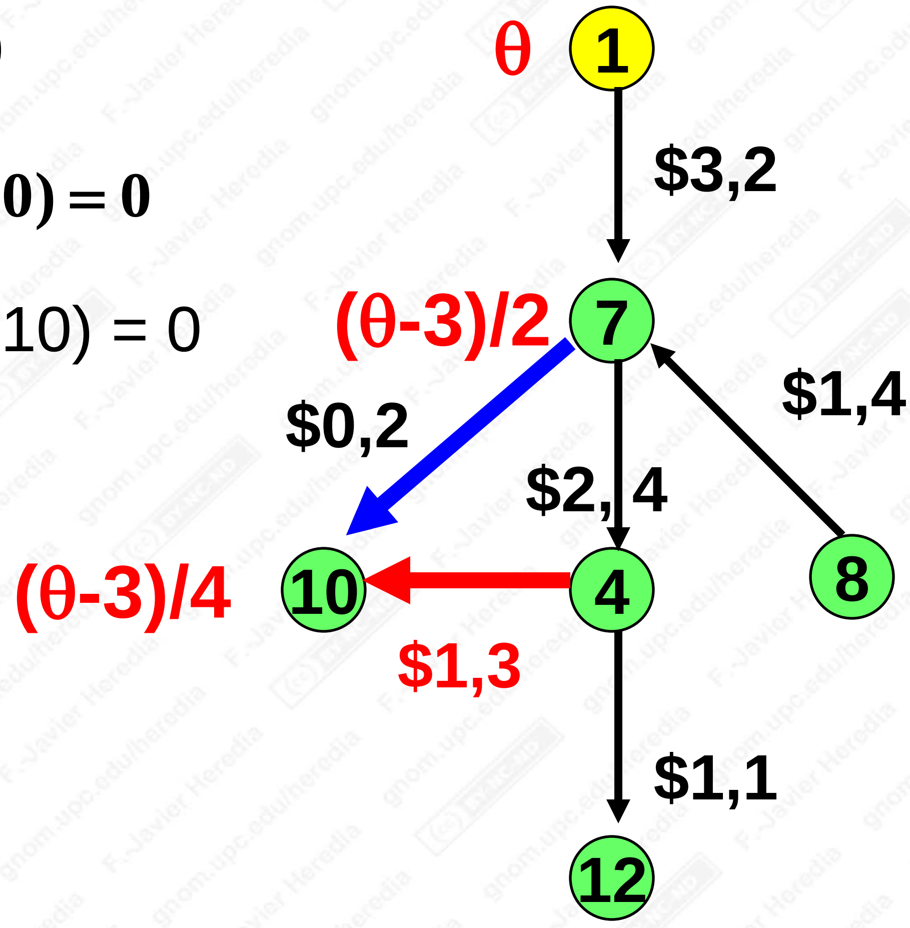
Computing Node Potentials

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c_{7,10}^{\pi} = c_{7,10} - \pi(7) + \mu_{7,10}\pi(10) = 0$$

$0 - (\theta - 3)/2 + 2\,\pi(10) = 0$

$\pi(10) = (\theta - 3)/4$



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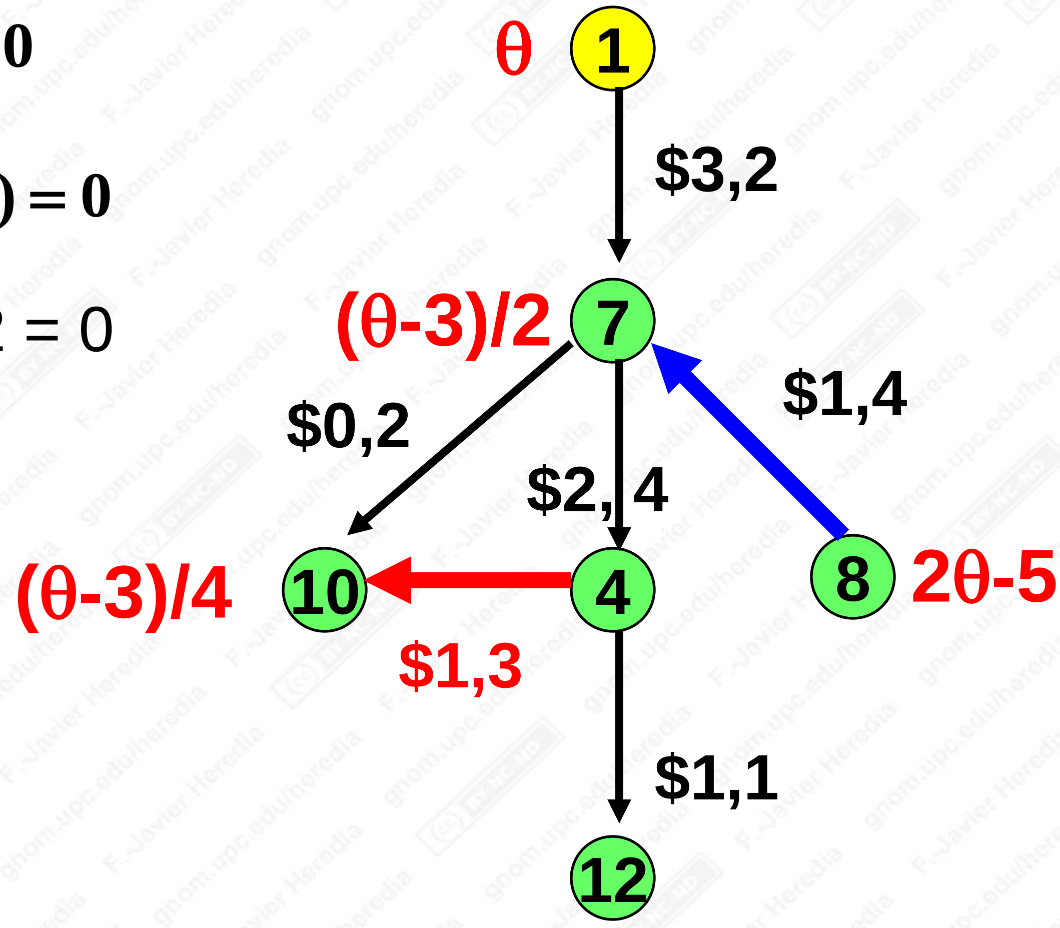
Computing Node Potentials

$$c^{\pi}_{ij} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c^{\pi}_{8,7} = c_{8,7} - \pi(8) + \mu_{8,7}\pi(7) = 0$$

$$1 - \pi(8) + 4(\theta - 3)/2 = 0$$

$$\pi(8) = 2\theta - 5$$



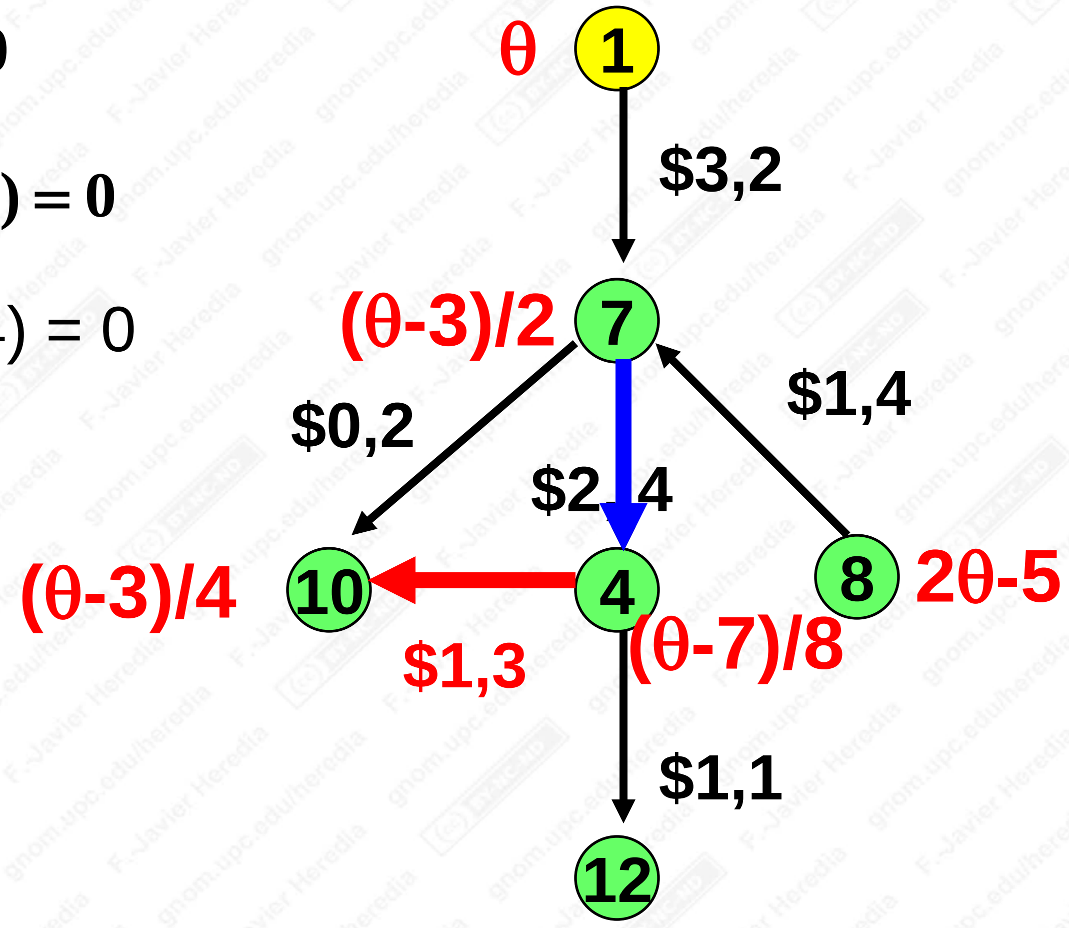
Computing Node Potentials

$$c^{\pi}_{ij} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c^{\pi}_{7,4} = c_{7,4} - \pi(7) + \mu_{7,4}\pi(4) = 0$$

$$2 - (\theta - 3)/2 + 4\,\pi(4) = 0$$

$$\pi(4) = (\theta - 7)/8$$



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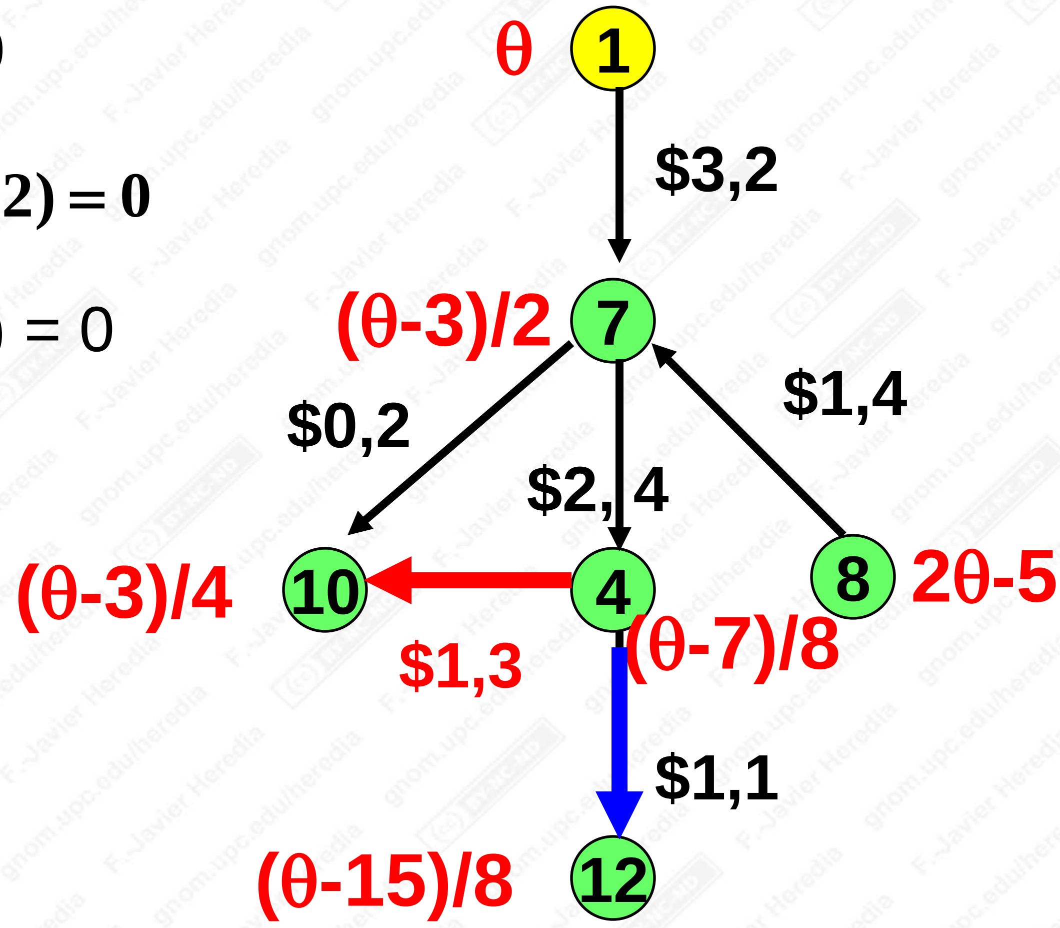
Computing Node Potentials

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c_{4,12}^{\pi} = c_{4,12} - \pi(4) + \mu_{4,12}\pi(12) = 0$$

$$1 - (\theta - 7)/8 + \pi(12) = 0$$

$$\pi(12) = (\theta - 15)/8$$



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NEXT: Look at the extra arc and compute θ

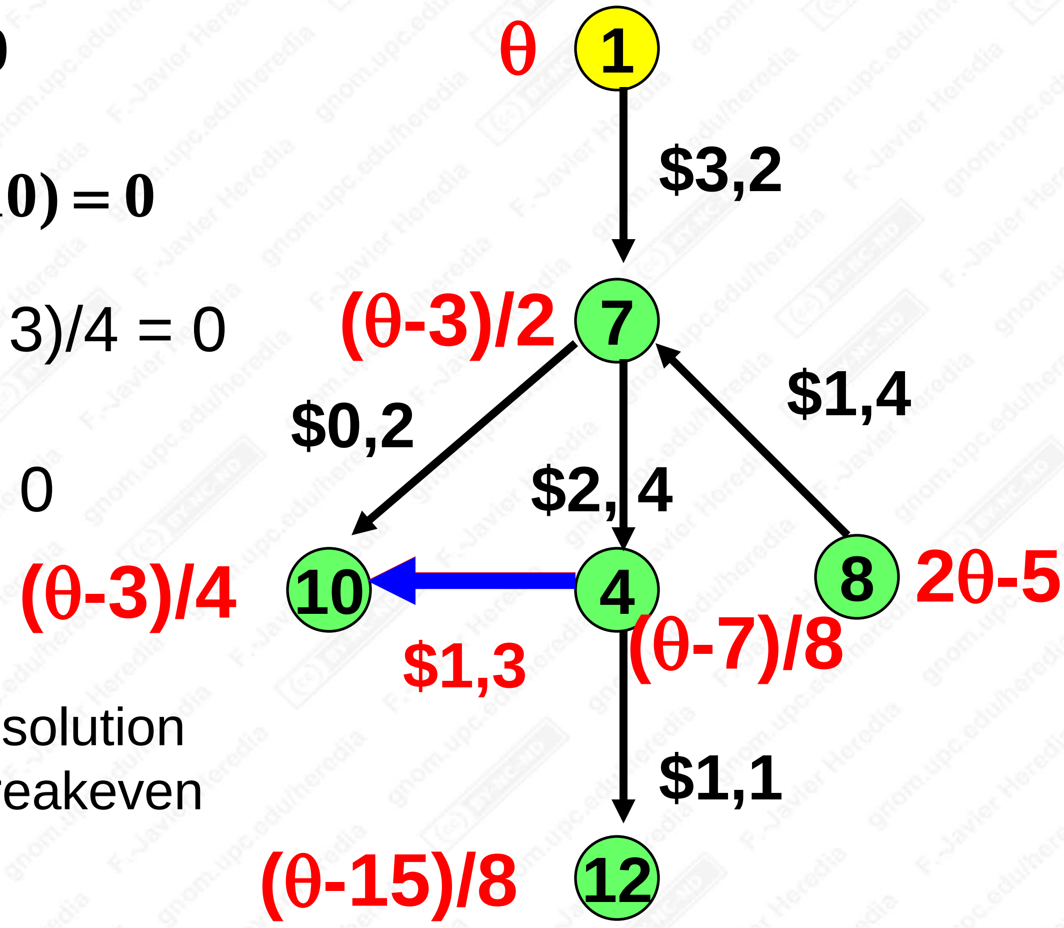
$$c^{\pi}_{ij} = c_{ij} - \pi(i) + \mu_{ij}\pi(j) = 0$$

$$c^{\pi}_{4,10} = c_{4,10} - \pi(4) + \mu_{4,10}\pi(10) = 0$$

$$1 - (\theta - 7)/8 + 3(\theta - 3)/4 = 0$$

$$8 - \theta + 7 + 6\theta - 18 = 0$$

$$\theta = 3/5$$



- This equation has a feasible solution whenever the cycle is non-breakeven (exercise 15.20).
- To compute the node potentials for a basis structure (F, L, U), compute the node potentials for each connected component of F.

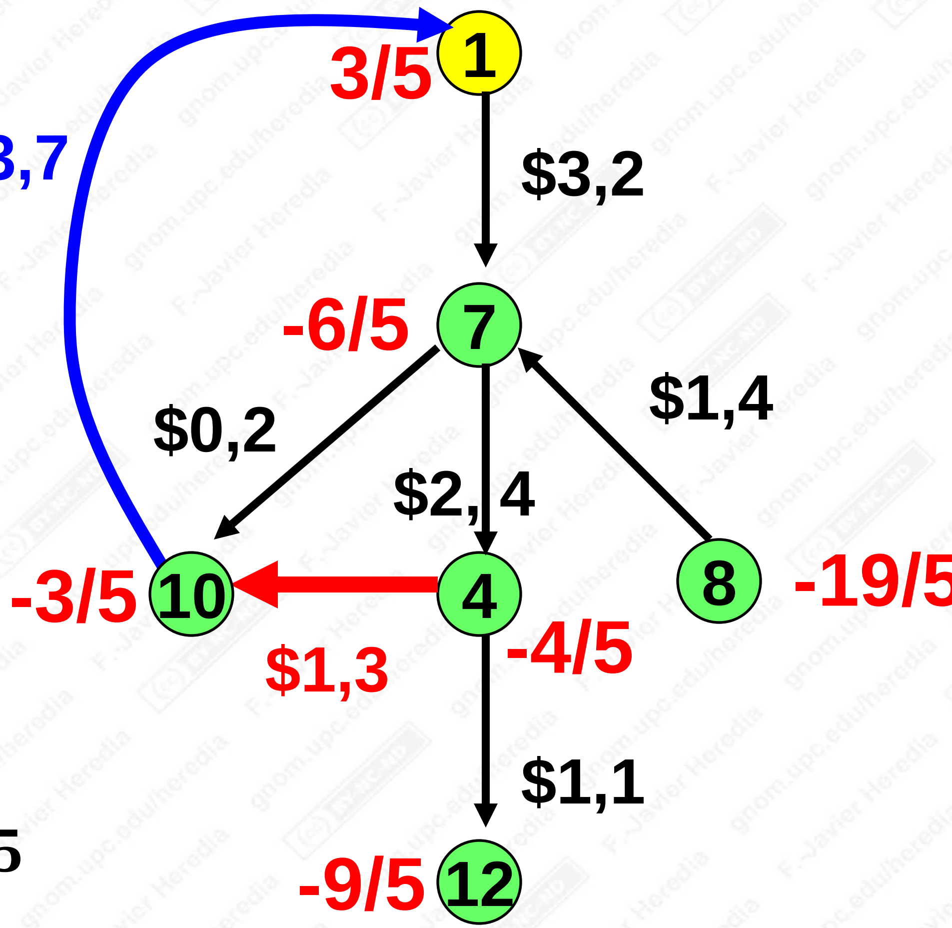
The reduced costs

- Compute reduced costs in the usual way.

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \mu_{ij}\pi(j)$$

$$c_{10,1}^{\pi} = c_{10,1} - \pi(10) + \mu_{10,1}\pi(1)$$

$$c_{10,1}^{\pi} = 3 + 3/5 + 7(3/5) = 39/5$$



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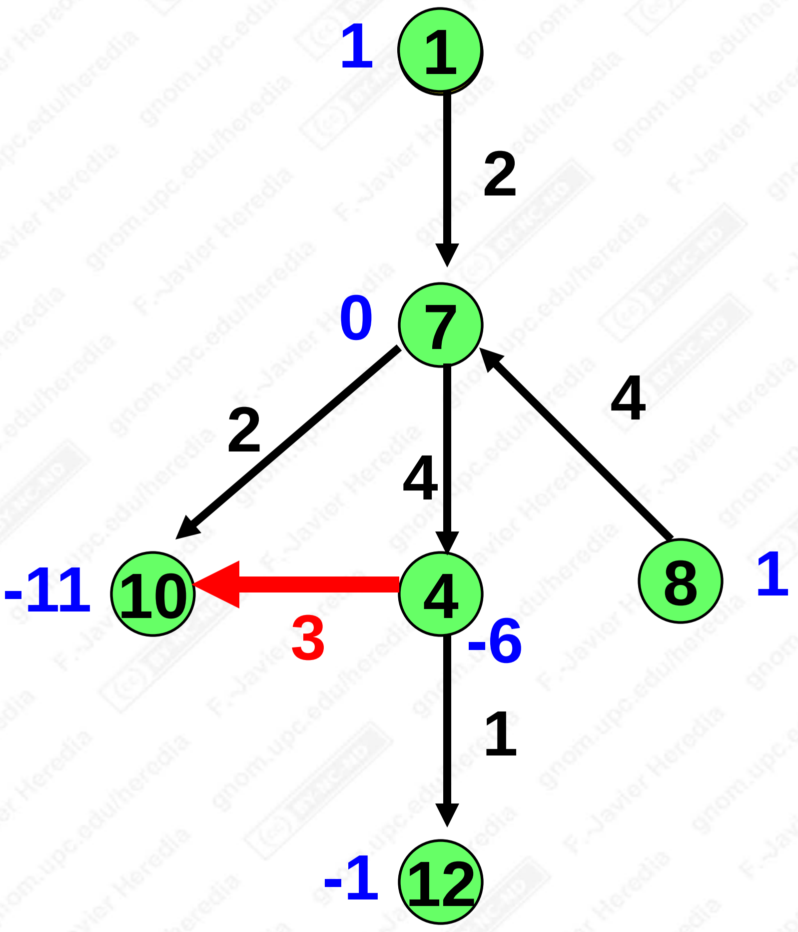
The Arc Flows

- The node numbers are supplies/demands.
  The arc numbers are the multipliers.

- To compute the arc flows, set the flow in the extra arc to θ and then compute the tree arcs in the usual way as a function of θ.



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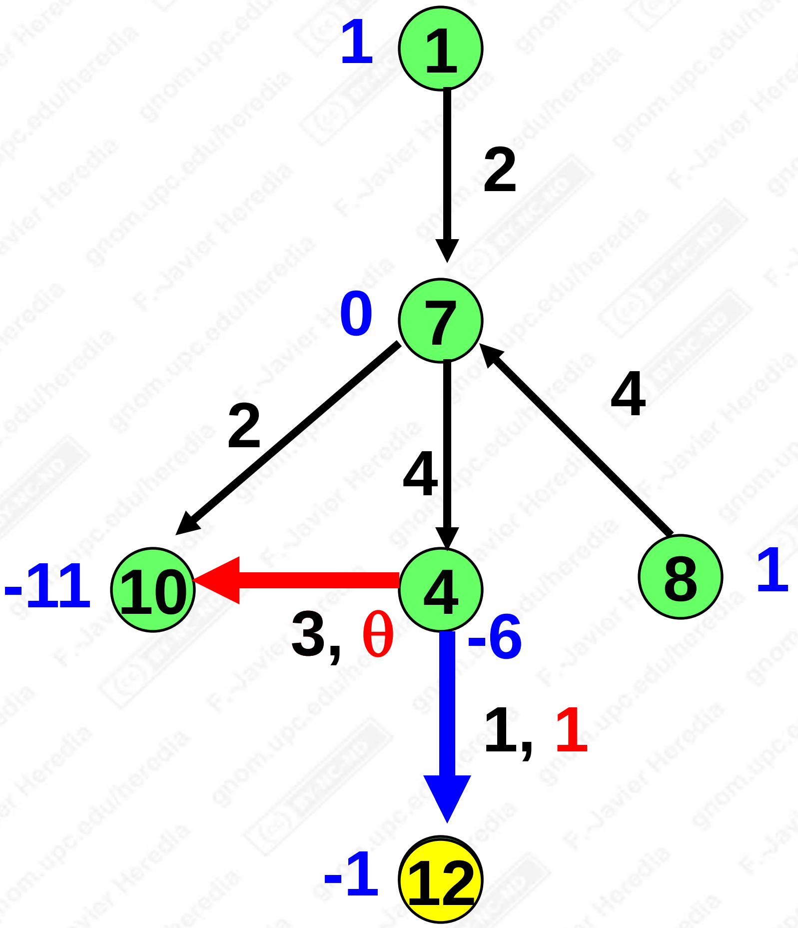
The supply of node 12 is -1

- Set the flow in the extra arc to θ.

- Compute the flow in (4,12)

$$x_{4,12} = 1$$



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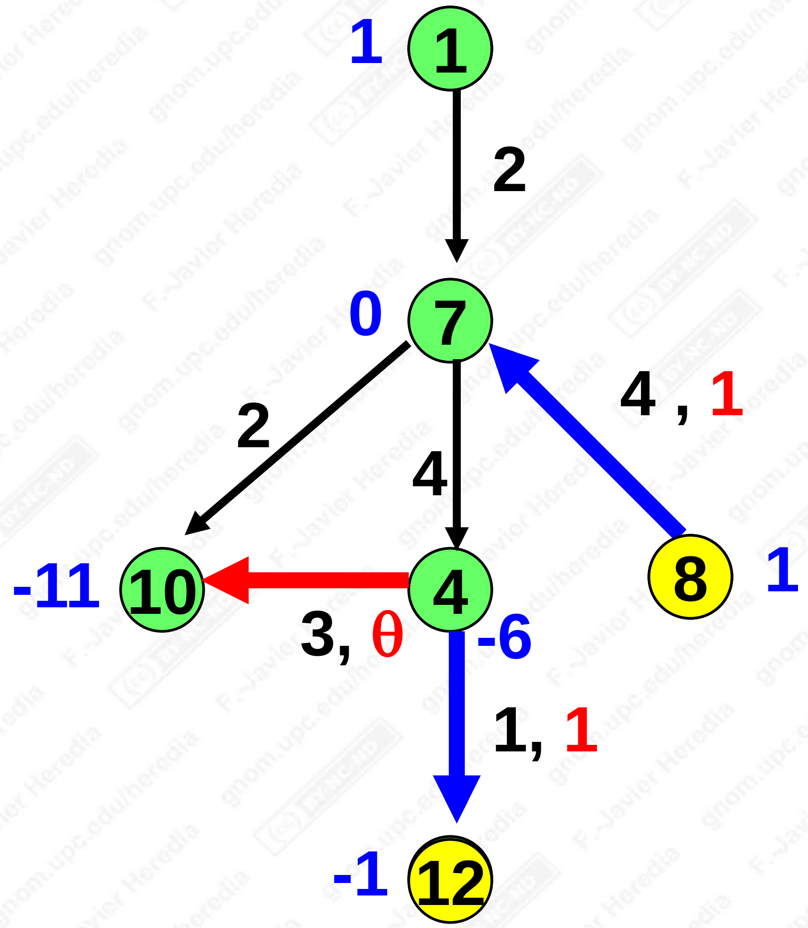
The supply of node 8 is 1

- Compute the flow in (8,7)

$x_{8,7} = 1$

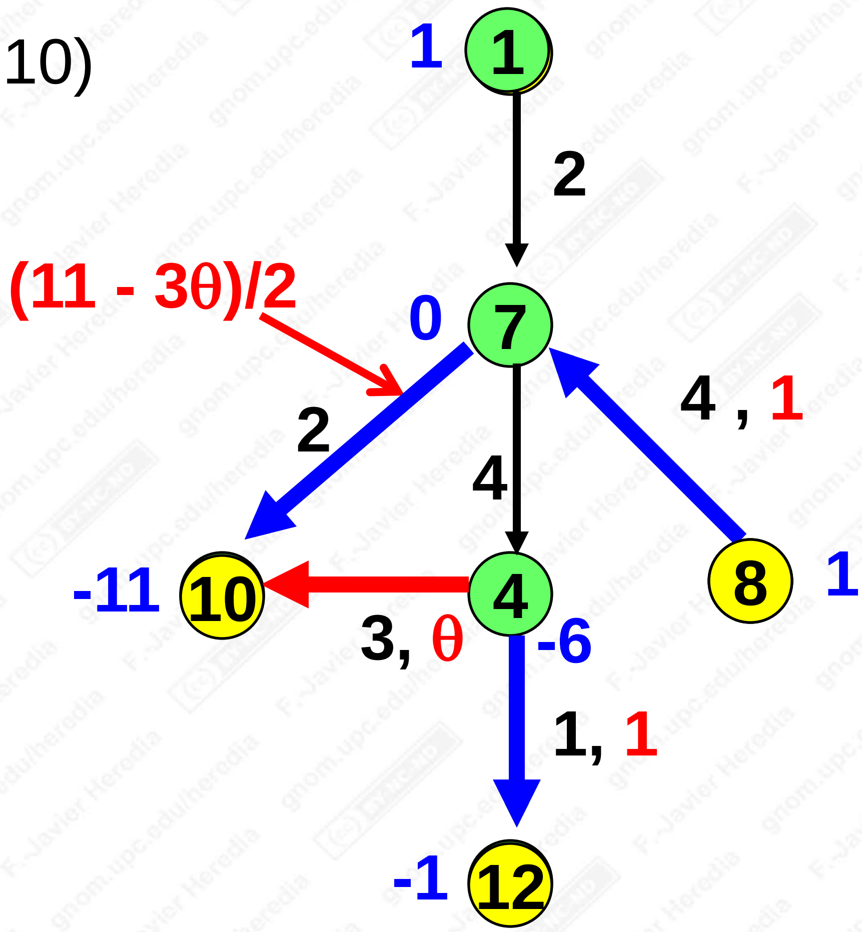


The supply of node 10 is -11

- Compute the flow in (7,10)

$$2\,x_{7,10} + 3\,x_{4,10} = 11$$

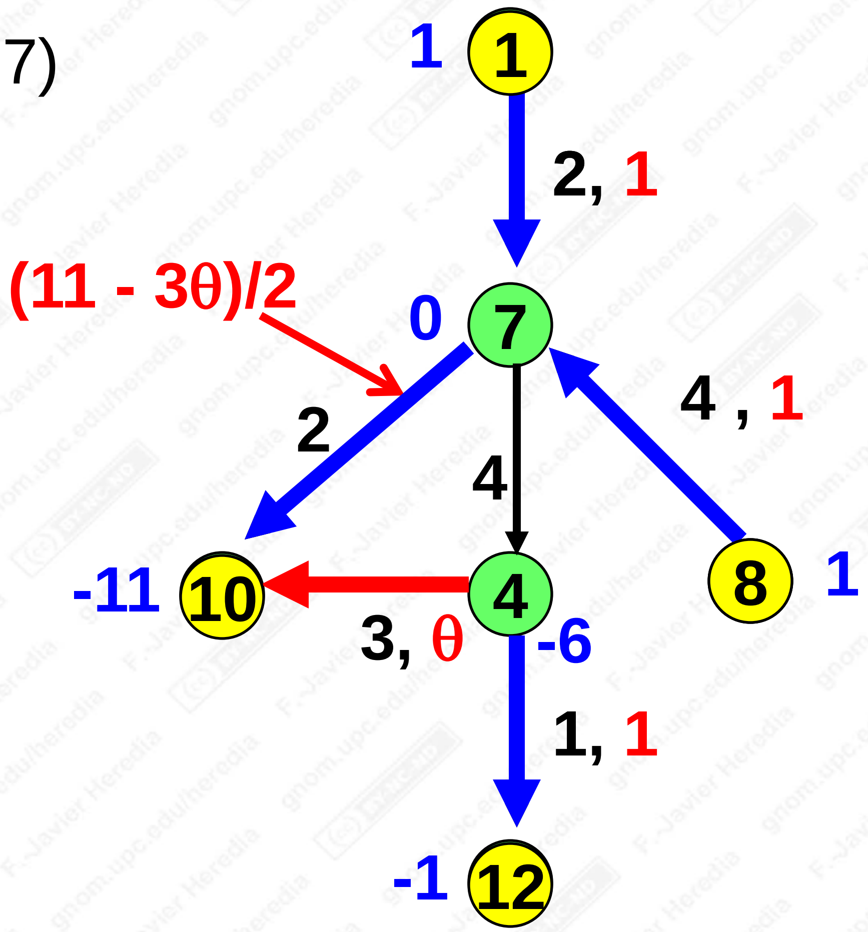
$$x_{7,10} = (11 - 3\theta)/2$$



The supply of node 1 is 1

- Compute the flow in (1,7)

$$x_{1,7} = 1$$



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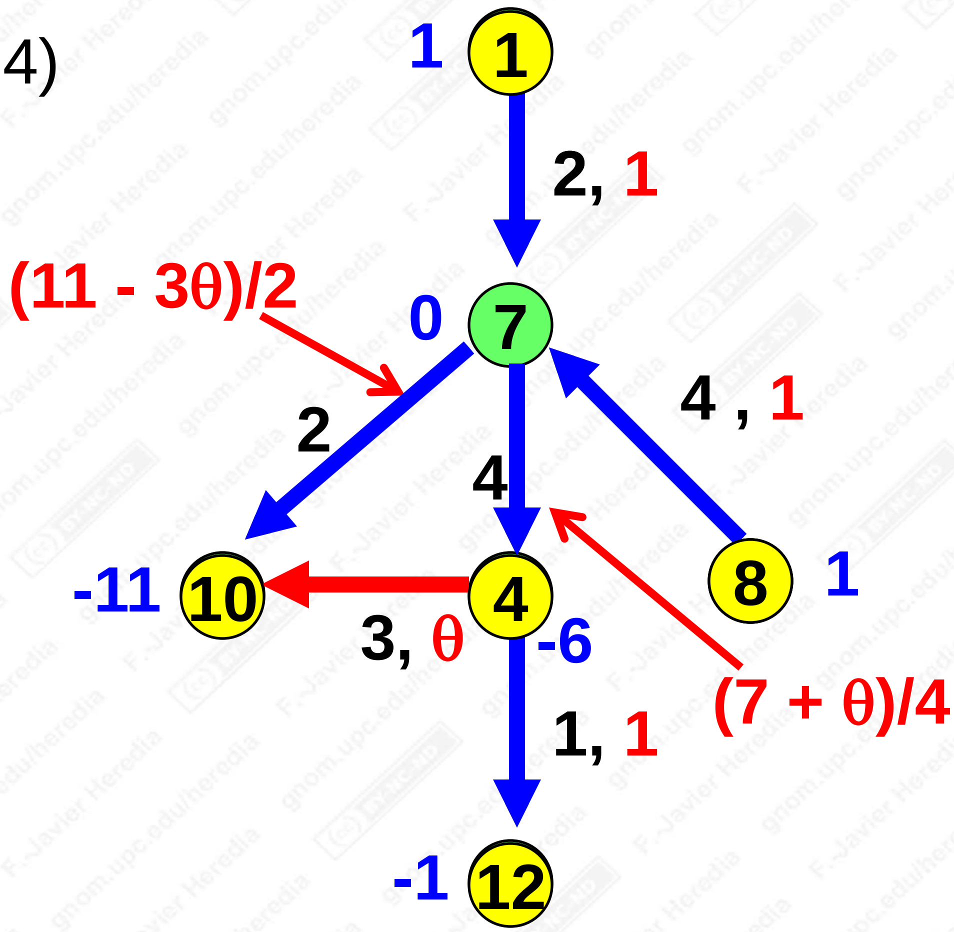
The supply of node 4 is -6

- Compute the flow in (7,4)

$$-4x_{7,4} + x_{4,12} + x_{4,10} = -6$$

$$x_{7,4} = (6 + 1 + \theta)/4$$
$$= (7 + \theta)/4$$



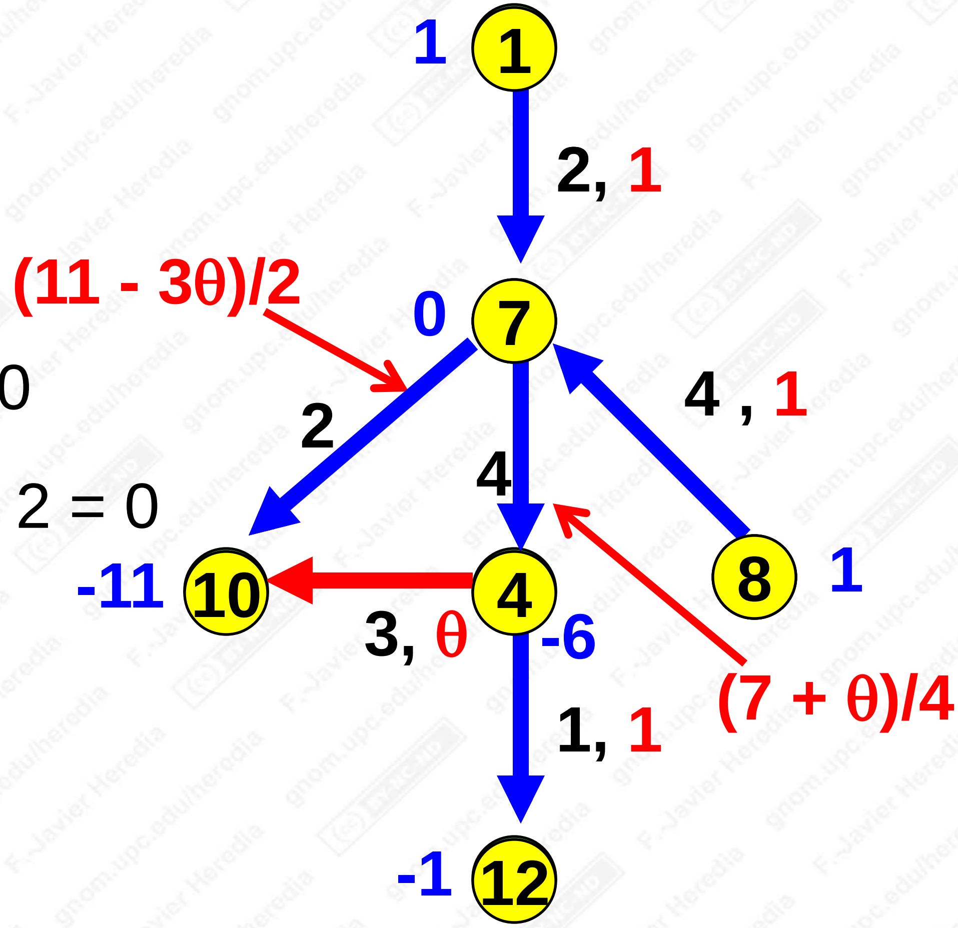
The supply of node 7 determines θ

$$x_{7,10} + x_{7,4} - 4x_{8,7} - 2\,x_{1,7} = 0$$

$$(11 - 3\theta)/2 + (7 + \theta)/4 - 4 - 2 = 0$$

$$(22 - 6\theta) + (7 + \theta) - 24 = 0$$

$$\theta = 1$$



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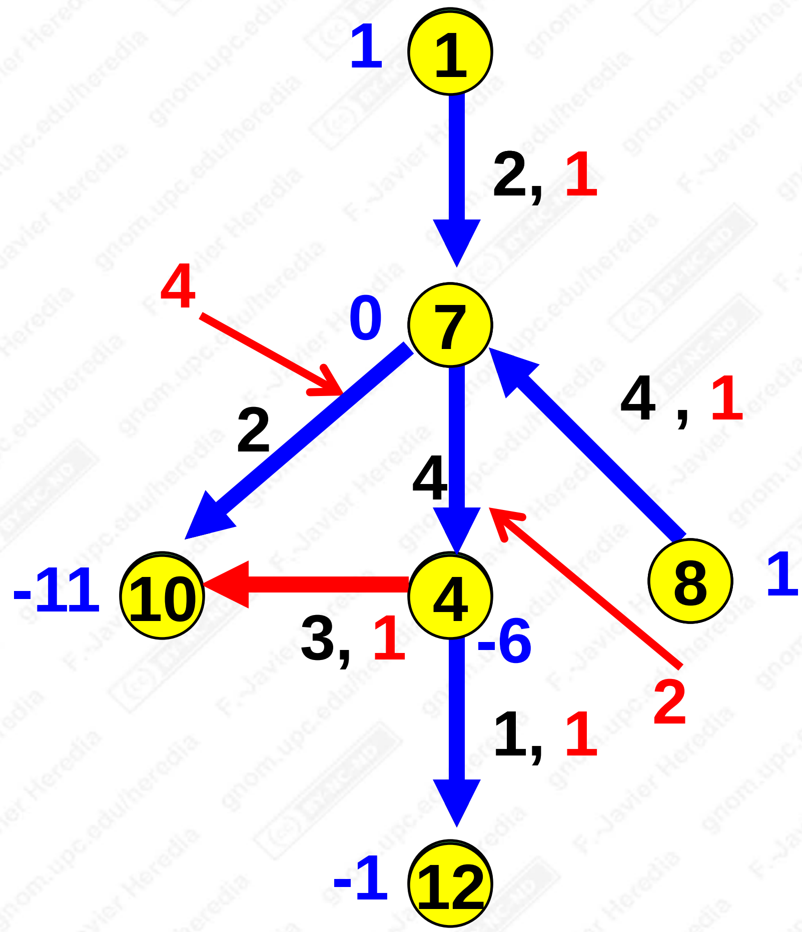
The basic flows

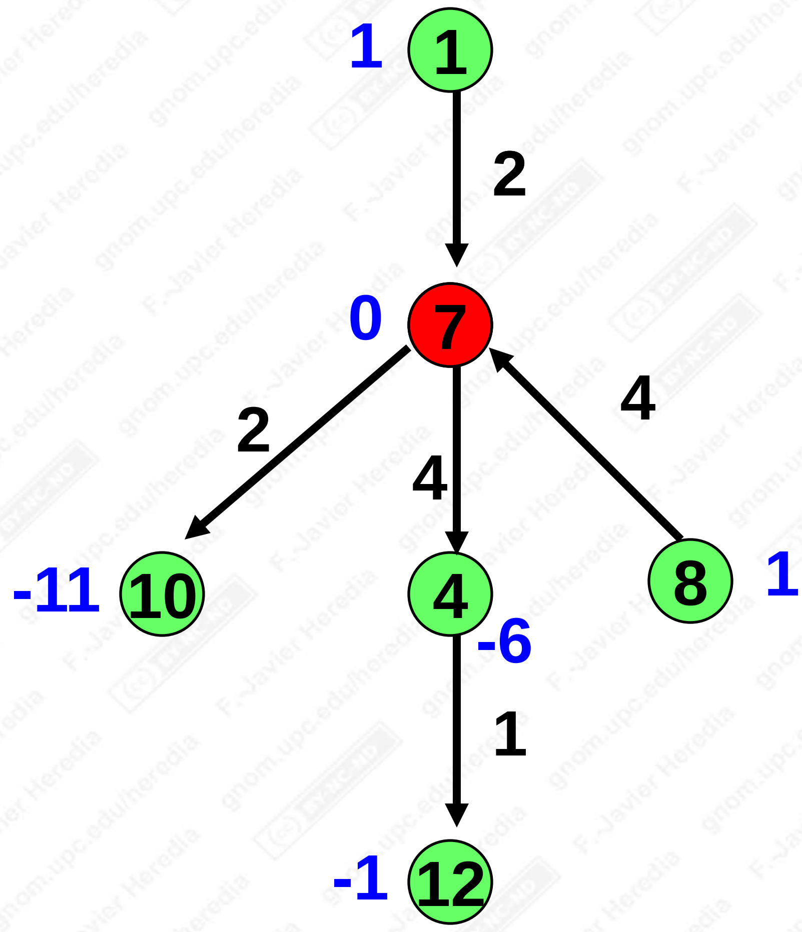
$\theta = 1$



- But how do we know that there will be a solution for θ? We next present an alternative approach that shows that there is a solution to the system of equations.

An alternative approach

- Choose a node of the cycle. Say node 7.
- Satisfy supply/demand constraints using tree arcs but ignoring node 7.
- Satisfy flow in node 7 by sending flow around the cycle.



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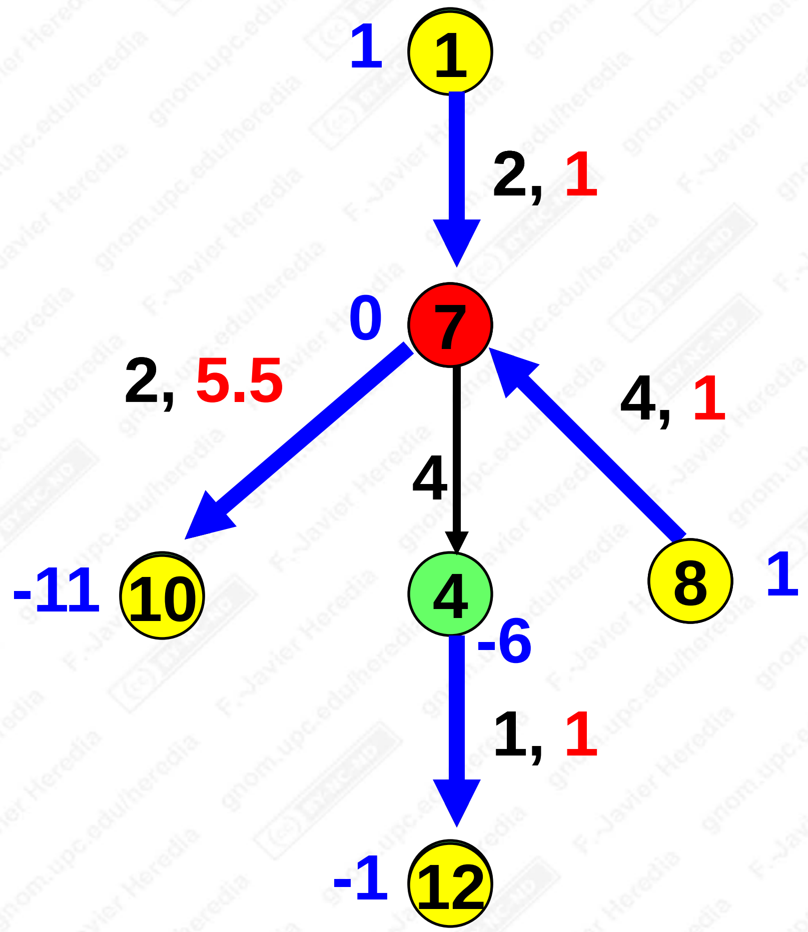
An alternative approach

$x_{4,12} = 1$

$x_{8,7} = 1$

$x_{7,10} = 5.5$

$x_{1,7} = 1$



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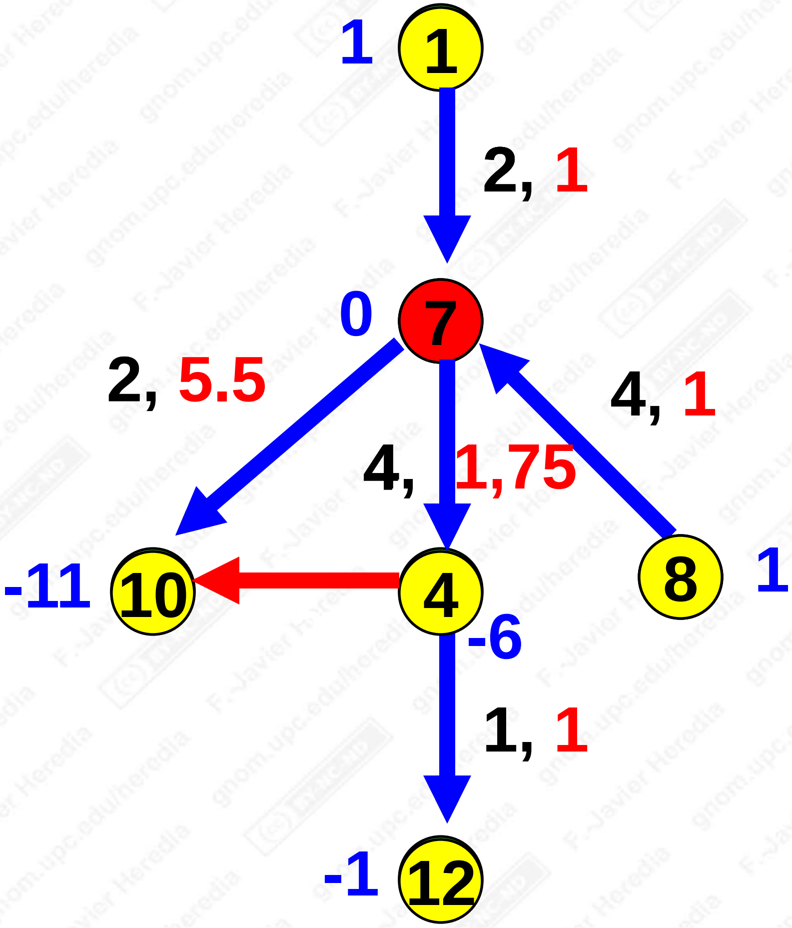
An alternative approach

$x_{4,12} - 4\ x_{7,4} = -6$

$1 - 4\ x_{7,4} = -6$

$x_{7,4} = 1.75$

- Now send flow around the cycle 7-4-10-7 to cancel the **excess flow of -1.25** units at node 7.



- Since the cycle is not breakeven, this is possible. Thus there is a feasible solution to this set of equations.

Sending flow around the cycle

- Send one unit out of 7 and satisfy conservation of flow at nodes 4 and 10:

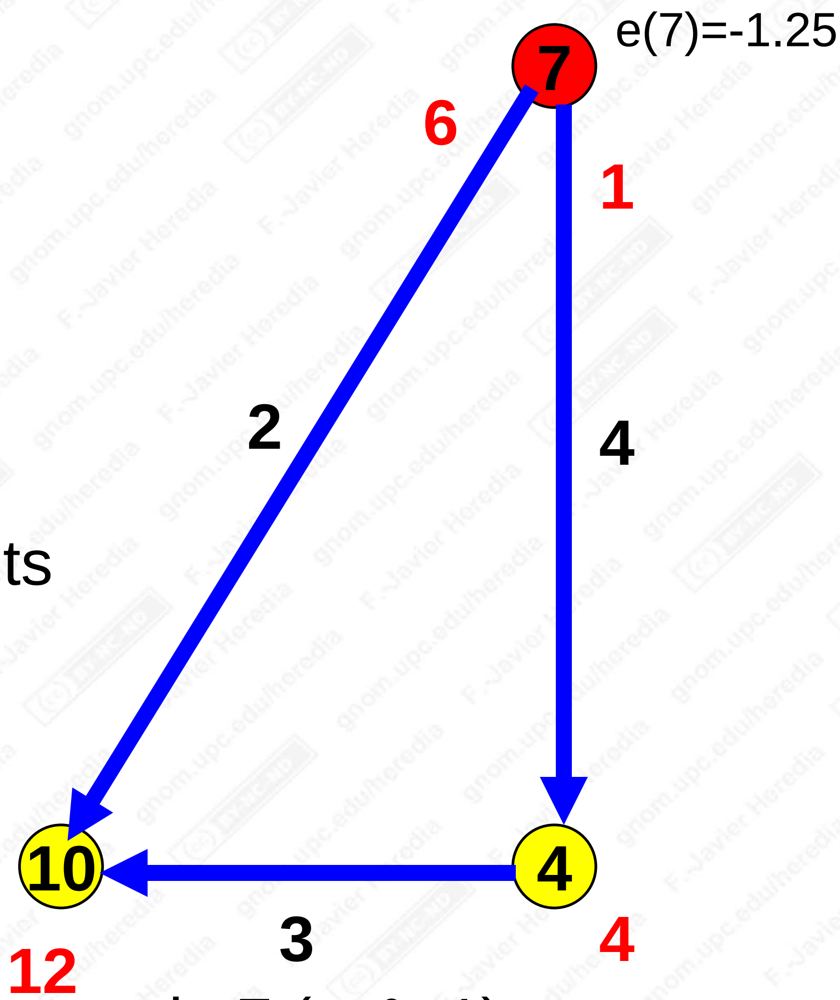
  $\mu(W) = 4*3/2 = 6$

- Currently, there is a **deficit** of 1.25 units at node 7. So, we need to send another 1.25 units to node 7.

- Sending

  $(5/4) /[\mu(W)- 1] = 1/4$

  units around the cycle balances node 7 ($\Rightarrow\theta=1$).



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What happens if U ≠ ∅ ?

- In computing flows, we assumed that all non-basic flows are 0. If U ≠ ∅, then we first compute the flows of arcs in U, and adjust the supplies and demands (or excess and deficits) accordingly, and then compute flows in arcs in F.

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Finding the leaving arc

- Suppose (i,j) enters the basis. Let y be the flow obtained in F + (i,j) by setting $y_{ij} = 1$, and determining flows in F so that there is conservation of flow everywhere.
- Let x* be the basic feasible flow for (F, L, U).
- Choose λ maximal so that x* + λy satisfies upper and lower bound constraints. Pivot out an arc (r, s) that has hit its upper or lower bound for this choice of λ.
- Time to determine leaving arc is O(n).

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A quick illustration of choosing λ

- Suppose x* = (2, 1, 3, 1, 0, 0, 5)
- Suppose y = (1, 2, -1, 0, 1, 0, 0)
- Suppose u = (4, 4, 4, 3, 6, 2, 5)

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} \le \begin{bmatrix}2\\1\\3\\1\\0\\0\\5\end{bmatrix} + \lambda \begin{bmatrix}1\\2\\-1\\0\\1\\0\\0\end{bmatrix} \le \begin{bmatrix}4\\4\\4\\3\\6\\2\\5\end{bmatrix} \qquad \begin{array}{l}\lambda \le 2\\ \lambda \le 1.5\\ \lambda \le 3\\ \\ \lambda \le 6\\ \\ \\ \end{array}$$

- So, we pick λ = 1.5. and variable 2 drops out of the basis.
- **Exercise.**

Summary

- The bases for generalized flow problems are good augmenting forests assuming that the graph is
  - connected and
  - has a non-breakeven cycle
- Complexity:
  - Theoretical: the number of iterations cannot be polinomial bounded (neither pseudopolinomial)
  - Practical: O(nm) (2 or 3 times slower than the network simplex)