Network Flows

UPCOPENCOURSEWARE number 34414

Topic 2: Paths, Trees and Cycles

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2.- Paths, Trees, and Cycles.
(Chap. 2 Ahuja,Magnanti,Orlin)

- Directed Graphs, Networks and Sub-graphs.
- *Walks and Paths.*
- Cycle.
- Connectivity.
- Cut.
- Trees and Forests.
- Cycles and Fundamental Cuts.
- Examples.

Directed Graphs, Networks and Subgraphs (I).

- **Directed Graph $G=(N,A)$:**
  "*It consist of a set N of nodes and a set A of arcs whose elements are ordered pairs of distinct nodes.*"
- **Directed Network:**
  "*Directed graph whose nodes and/or arcs have associated numerical values (costs, capacities, supplies and demands).*"
- **Subgraphs:**"*The graph $G'=(N',A')$ is a subgraph of $G=(N,A)$ if $N' \subseteq N$ and $A' \subseteq A$. G'* ***is the spanning subgraph*** *of G if $N' = N$ and $A' \subseteq A$.*"

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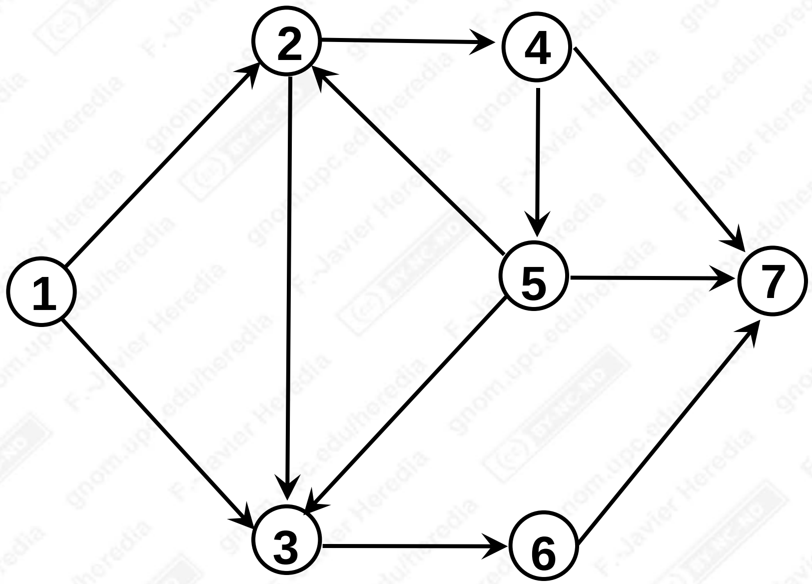
Directed Graphs, Networks, and subgraphs (II).

- **Example:**



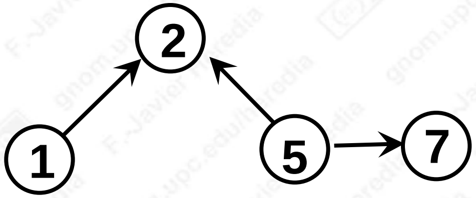
$$N=\{\ldots,\ldots,\ldots,\ldots,\ldots,\ldots\}$$

$$A=\{(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),$$
$$(\ldots,\ldots),(\ldots,\ldots),(\ldots,\ldots),\}$$

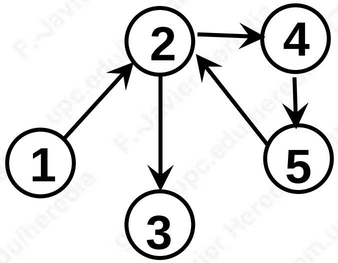
Walks and Paths

- ***Walk in a directed graph*** G**:**
  "subgraph of $G=(N,A)$ *consisting of a sequence of nodes* i_1- i_2-…- i_k…- i_{r-1}-i_r *such that:*

$$\forall 1 \le k \le r\text{-}1 : a_k=(i_k , i_{k+1})\in A \text{ or } a_k=(i_{k+1} , i_k)\in A\text{"}$$



a) $1\text{-}2\text{-}5\text{-}7$ (*not directed*)

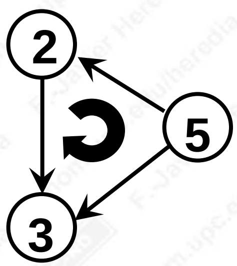


b) $1\text{-}2\text{-}4\text{-}5\text{-}2\text{-}3$ (*directed*)

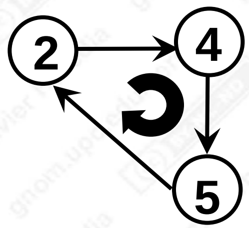
- ***Paths*****:**
  "A path is a walk without any repetition of nodes"
  - The walk in the graph *a) exhibits*:
    - *Forward Arcs* : (1,2), (5,7)
    - *Backward arcs*: (5,2)
  - **Directed path:** "*directed walk without any repetition of nodes (i.e. a directed path has no backward arcs)*"

Cycles

- **Cycle:**
  "*A cycle is a path* i_1- i_2-... -i_r *together with the arc* (i_r, i_1) *or* (i_1, i_r) : i_1- i_2-... -i_r-i_1"



- Cycle 2-5-3-2 (*undirected*):
  - Forward arcs: $(5,3)$.
  - Backward arcs: $(2,5)$, $(3,2)$.



- Cycle 2-4-5-2 (*directed*):
  - Forward arcs: $(2,4)$, $(4,5)$, $(5,2)$.
  - Backward arcs: $\emptyset$

- **Acyclic network:** "*graph without any directed cycle.*"

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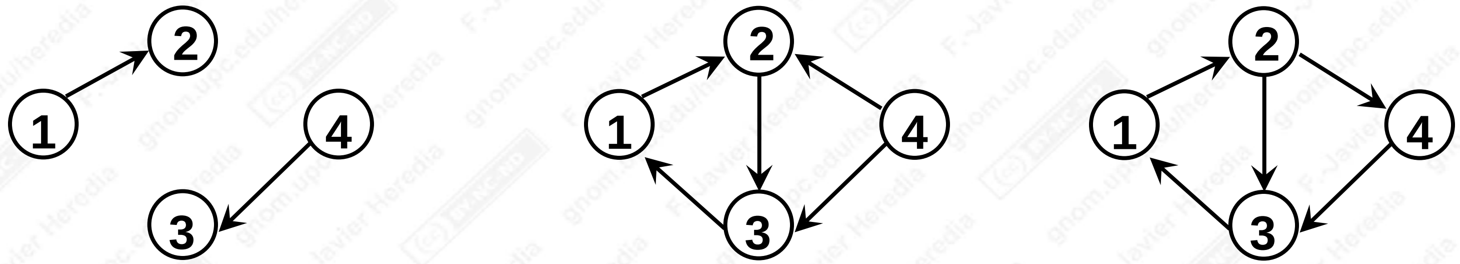
Connectivity

- **Connectivity:**
  "Two *nodes* i *and* j *are connected if the graph* G *contains at least one path from node* i *to node* j*. If every pair of nodes from the graph* G *are connected, then the graph is connected; otherwise it is disconnected"*
- **Strong connectivity:**
  "*A conn. graph is strongly connected if it contains at least one directed path from every node to every other node"*



a) Not connected *b)* Connected *c)* Strongly connected

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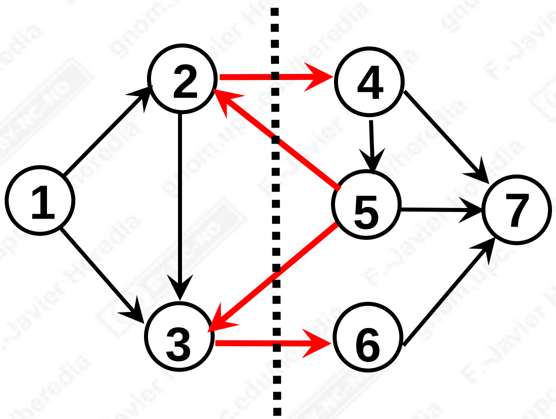
Cut

- **Cut:**
  "A cut is a *partition of the node set* N *into two parts of nodes,* S *and* $\underline{S}=N-S$"



- $S=\{1,2,3\}$; $\underline{S}=\{4,5,6,7\}$
- Cut: $[S, \underline{S}]=\{(2,4), (5,2), (5,3), (3,6)\}$

- **Cut** s**-**t **:** "*Cut* $[S, \underline{S}\,]$ *such that.* $s \in S$ *and* $t \in \underline{S}$ "

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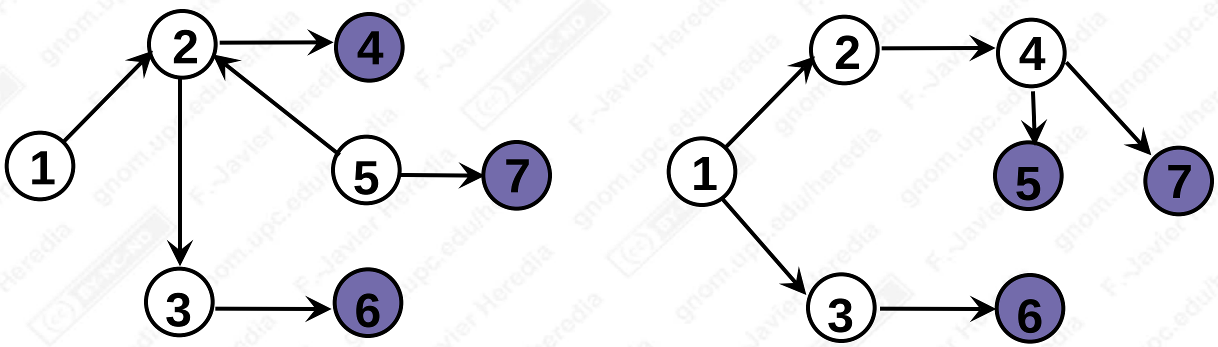
Trees and Forests (I)

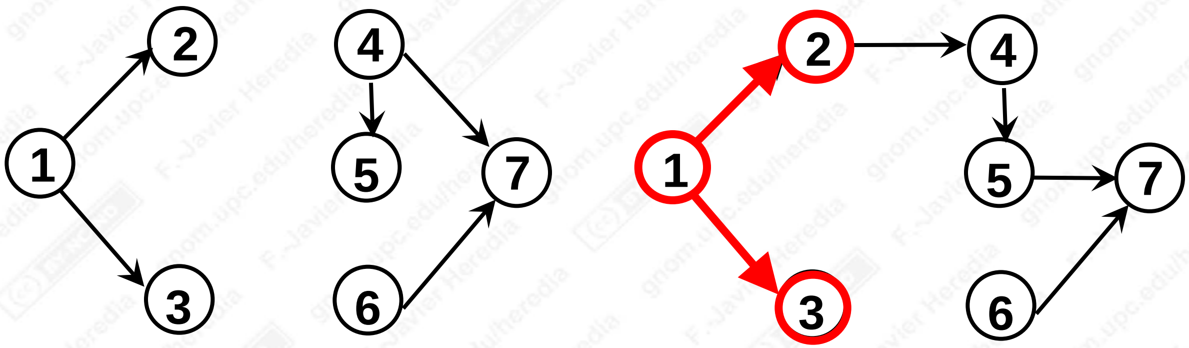
- **Tree:**
  "*A connected graph that contains no cycle.*"



- **Properties:**
  a) A tree on n nodes contains exactly n-1 arcs.
  b) A tree has at least two leaf nodes.
  c) Every two nodes of a tree are connected by a unique path.

Trees and Forests (II)

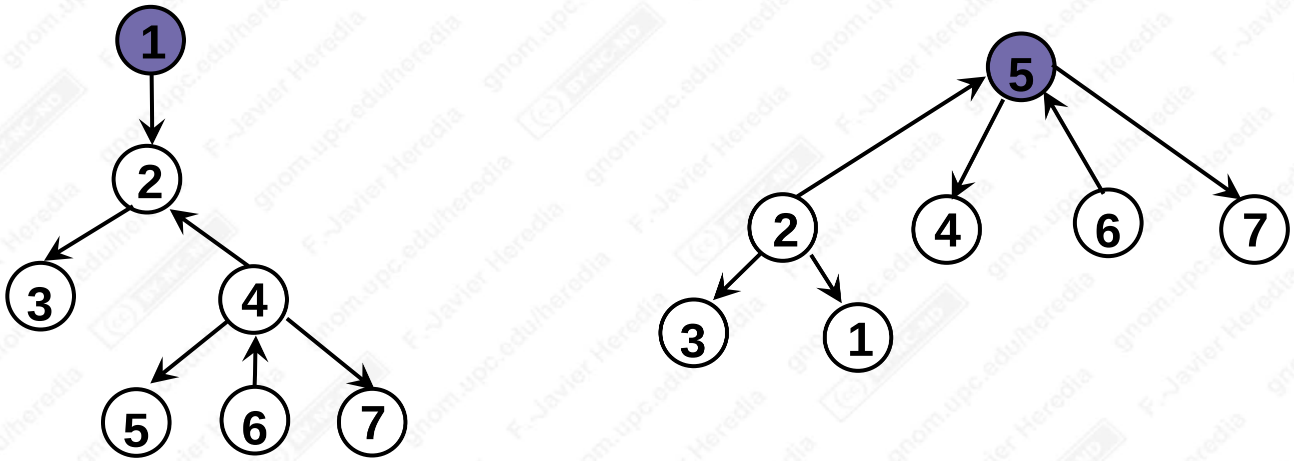
- **Forest:**
  "*A graph that contains no cycle*"
  "*A collection of trees*"



- **Subtree:**
  "*A connected subgraph of a tree*"

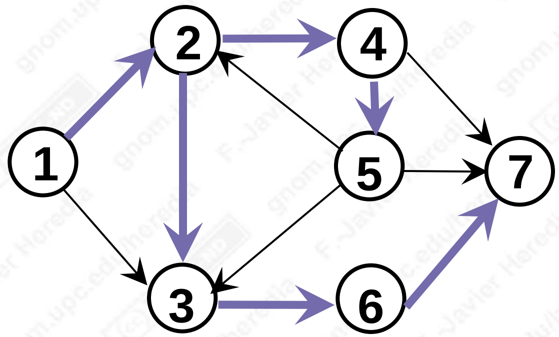
Trees and Forests (III)

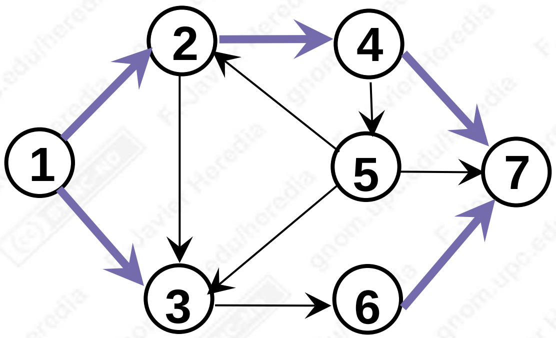
- **Rooted Tree:**
  "*Tree with a special designated node, called root. We regard a rooted tree as though it were hanging from its root.*"



Trees and Forests (IV)

- **Spanning Tree:**
  "A tree T *is a* ***spanning tree of*** G *if* T *is a spanning subgraph of* G*.*"





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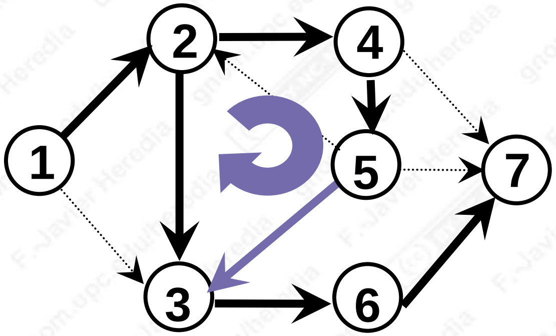
Cycles and fundamental cuts (I)

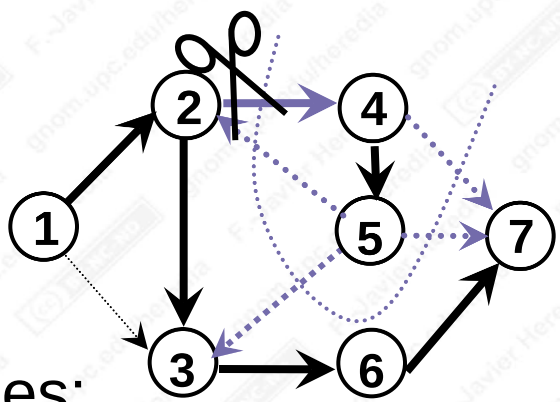
- **Fundamental cycle:**
  "*Let T be a spanning tree of the graph G. The addition of any non-tree arc to the spanning tree T creates exactly one cycle. We refer to any such cycle as a* ***fundamental cycle of G w.r.t. T***."



- Properties:
  a) There is no m-n+1 fundamental cycles.
  b) If we delete any arc in a fundamental cycle, we again obtain a spanning tree.

Cycles and fundamental cuts(II)

- **Fundamental cut:**
  "Let T *be a spanning tree of the graph* G*. The deletion of any tree arc of* T *produces a disconnected graph containing two subtrees* T_1 *and* T_2*. Arcs whose endpoints belong to different subtrees constitute a cut, which is known as a* ***fundamental cut of*** G ***respect to*** T*.*"

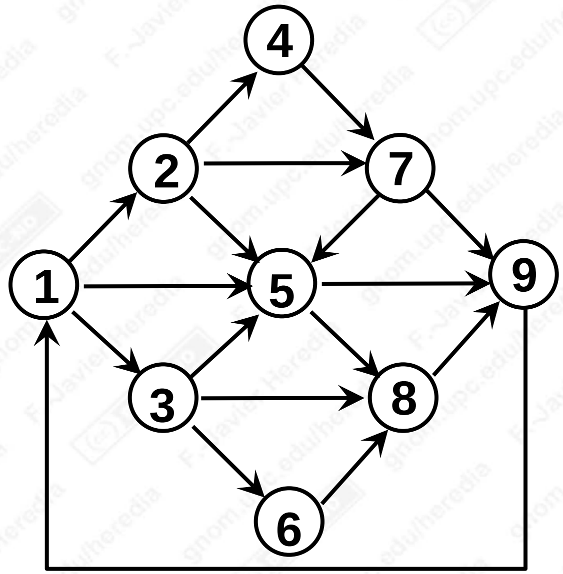


$T_1=\{(1,2),(2,3),(3,6),(6,7)\}$; $T_2=\{(4,5)\}$

$[T_1,T_2]=\{(2,4),(4,7),(5,2),(5,3),(5,7)\}$

- Properties:
  a) The graph G has n-1 fundamental cuts.
  b) Adding any arc in the fundamental cut to the two subtrees T_1 and T_2, we again obtain a spanning tree.

Example



1. Find a directed walk containing 6 arcs. Also find a walk containing 8 arcs.
2. Find a cycle containing 9 arcs and a directed cycle containing 7 arcs.
3. Find a spanning tree with 6 leaves and a cut with 6 arcs.
4. Are the graph strongly connected?
5. List all fundamental cycles w.r.t. the following spanning tree $T=\{(1,5),(1,3),(2,5),(4,7),(7,5),(7,9),(5,8),(6,8)\}$.
6. For the last spanning tree, list all fundamental cuts. Which of these are the s-t cuts $s=1$ and $t=9$?