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THE PERTURBATION METHOD IN THE PROBLEM ON A NEARLY CIRCULAR INCLUSION IN AN ELASTIC BODY

MIKHAIL A. GREKOV*, ALEKSANDRA B. VAKAEVA*

*St. Petersburg State University (SPbU)
7/9, Universitetskaya nab., 199034, St. Petersburg, Russia
e-mail: m.grekov@spbu.ru, a.vakaeva@spbu.ru, web page: http://spbu.ru

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Abstract. The two-dimensional boundary value problem on a nearly circular inclusion in an infinity elastic solid is solved. It is supposed that the uniform stress state takes place at infinity. Contact of the inclusion with the matrix satisfies to the ideal conditions of cohesion. To solve this problem, Muskhelishvili's method of complex potentials is used. Following the boundary perturbation method, this potentials are sought in terms of power series in a small parameter. In each-order approximation, the problem is reduced to the solving two independent Riemann – Hilbert's boundary problems. It is constructed an algorithm for funding any-order approximation in terms of elementary functions. Based on the first-order approximation numerical results for hoop stresses at the interface are presented under uniaxial tension at infinity.

1 INTRODUCTION

Stress concentration caused by different defects (such as holes and inclusions) existing in materials and structures is one of the reasons of devices failure. Apparently, it is not possible to obtain an exact analytical solution of an elastic boundary value problem for an arbitrary defect in an infinite plane. The real hole or inclusion has a shape which can't be usually described by a conformal image using, for example, for elliptic holes [1, 2]. So-called circular defects have practically relief surface slightly deviated from a circle and a circular shape of them is nothing but idealization. In the works [3, 4] the perturbation method was used to solve the problem of an elastic infinity plane with a nearly circular hole at the macro- and nanolevel. The results obtained in [3] for the nearly circular holes, by means of the boundary perturbation technique allow us to solve a more complex problem of determining the stress-strain state inside and outside of an elastic inclusion having a different shape.

In the present work, the approach developed in [3] for the analysis of the infinity elastic body with a nearly circular hole is used to study stress-strain state of an elastic plane with a nearly circular inclusion. We consider the 2-D problem on an inclusion in an elastic solid under remote tension. To solve the problem, we use Goursat – Kolosov complex potentials, Muskhelishvili representations and universal boundary perturbation technique applied recently to some problems of elasticity (see, for example [3]–[8]). First, based on Muskhelishvili's technique [9], we seek complex potentials in terms of power series in a small parameter. Then, in each-order approximation, the problem is reduced to the solving two independent Riemann – Hilbert's boundary problems. In contrast to the work [10], where only the first-order approximation has been derived, we construct an algorithm for finding any-order approximation expressed in elementary functions. For the periodic shape of the inclusions determined by the cosine function, the first-order formulas of approximation are derived in a closed form. At the end of the work, we give the most essential numerical results and their analysis for some shapes of the inclusion.

2 STATEMENT OF THE PROBLEM

We consider an infinite elastic body with an inclusion the shape of which is weakly deviated from a circle. Under arbitrary remote loading, the body is in plane strain. So, it leads to the 2-D boundary value problem for the elastic plane of complex variable $z = x_1 + ix_2$ (i is the imaginary unit) with a nearly circular inclusion. Suppose that the matrix corresponds to the domain Ω_1 and the inclusion — to the domain Ω_2, and the interface between the matrix and inclusion Γ is determined by the relation

$$z \equiv \zeta = \rho e^{i\theta} = (1 + \varepsilon f(s))\, s. \tag{1}$$

Here $s = e^{i\theta}$, $f(s)$ is the continuous differentiable function satisfying $|f| \leq 1$, ε is the small parameter which is equal to the maximum deviation of the inclusion boundary from the circular one, $\varepsilon > 0$, $\varepsilon \ll 1$. The elastic properties of each domain Ω_k $(k = 1, 2)$ are determined by the Poisson coefficient ν_k and the shear modulus μ_k.

It is suppose that contact of the inclusion with the matrix satisfies the ideal conditions of cohesion

$$\Delta\sigma_n(\zeta) = \sigma_n^+ - \sigma_n^- = 0, \quad \Delta u(\zeta) = u^+ - u^- = 0, \tag{2}$$

and stresses σ_{ij} $(i, j = 1, 2)$ and the rotation angle ω are specified at infinity as

$$\lim_{z\to\infty} \sigma_{ij} = \sigma_{ij}^{\infty}, \quad \lim_{z\to\infty} \omega = 0. \tag{3}$$

Here, $\sigma_n(\zeta) = \sigma_{nn} + i\sigma_{nt}$, $u = u_1 + iu_2$; σ_{nn}, σ_{nt} are the normal and tangential stress tensor components at the interface, correspondently; u_1, u_2 — components of the displacement vector in the Cartesian coordinates x_1, x_2. In equation (2), $\sigma_n^{\pm} = \lim\limits_{z\to\zeta\in\Gamma} \sigma_n(z)$, $u^{\pm} = \lim\limits_{z\to\zeta\in\Gamma} u(z)$. The superscript "−" corresponds to $z \in \Omega_1$ and "+", to $z \in \Omega_2$.

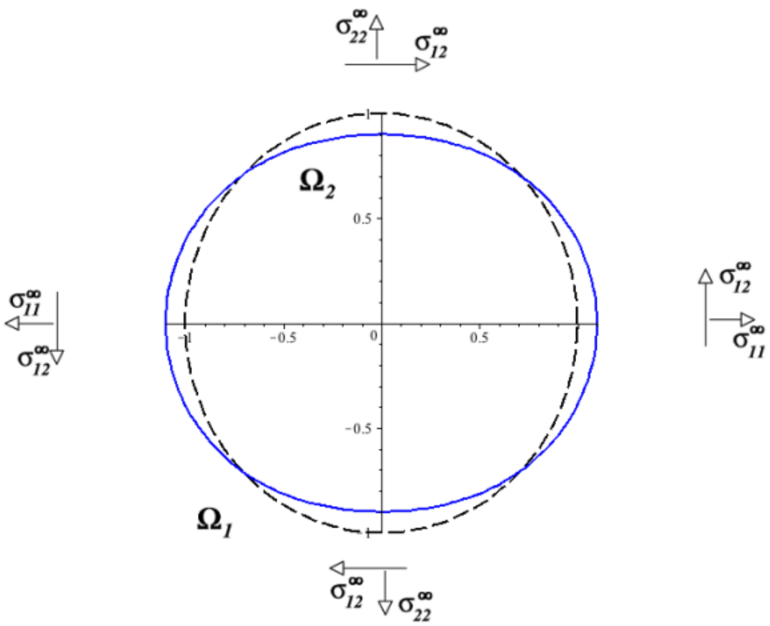

Figure 1: A nearly circular inclusion (firm line) in an infinite elastic plane under arbitrary remote loading ($\varepsilon = 0,1$).

The boundary of the inclusion determined by equation (1) is shown in Fig. 1 for the function $f(\theta) = \cos 2\theta$ when parameter $\varepsilon = 0,1$. This function is used in the work to get numerical results.

3 BASIC RELATIONS

According to [5], the stresses and the displacement in each domain Ω_k ($k = 1, 2$) are expresses in terms of two holomorphic functions $\Phi_k(z)$ and $\Psi_k(z)$

$$G(z, \eta_k) = \eta_k \Phi_k(z) + \overline{\Phi_k(z)} + \left[z\overline{\Phi'_k(z)} + \overline{\Psi_k(z)} \right] e^{-2i\alpha}, \quad z \in \Omega_k.$$

Here

$$G(z, \eta_k) = \begin{cases} \sigma_n, & \eta_k = 1, \\ -2\mu_k \dfrac{du}{dz}, & \eta_k = -\varkappa_k, \end{cases}$$

where $\varkappa_k = (3 - \nu_k)/(1 + \nu_k)$ for a plane stress state and $\varkappa_k = 3 - 4\nu_k$ for a plane deformation, σ_n is the traction at the area with the normal $\mathbf{n}$, α is the angle between the direction $\mathbf{t}$ of the area and the x_1 axis.

Following [9], introduce new functions $\Upsilon_k(z)$ holomorphic in the domain $\widetilde{\Omega}_k = \{z : \bar{z}^{-1} \in \Omega_k\}$ with the boundary $\widetilde{\Gamma}$ which is symmetrical to the interface Γ relative to the unit circle,

$$\Upsilon(z) = -\overline{\Phi(\bar{z}^{-1})} + z^{-1}\overline{\Phi'(\bar{z}^{-1})} + z^{-2}\overline{\Psi(\bar{z}^{-1})}, \quad \bar{z}^{-1} \in \Omega_k. \tag{4}$$

We determine the unknown functions $\Phi_k(z)$ and $\Upsilon_k(z)$ from the boundary conditions (2). Passing to the limit for $z \to \zeta \in \Gamma, z \in \Omega_k$ [5] and taking into account the relation (4), the following boundary equations for complex potentials Φ_k and Υ_k can be written as

$$G(\zeta, \eta_k) = \eta_k \Phi_k(\zeta) + \overline{\Phi_k(\zeta)} + \frac{\rho' - i\rho}{\rho' + i\rho}\left[\frac{1}{\overline{\overline{\zeta}^2}}\left(\overline{\Phi_k(\zeta)} + \Upsilon_k\left(\frac{1}{\bar{\zeta}}\right)\right) + \left(\zeta - \frac{1}{\bar{\bar{\zeta}}}\right)\overline{\Phi_k'(\zeta)}\right]\bar{s}^2, \tag{5}$$

where $\Phi_k(\zeta) = \lim\limits_{z\to\zeta} \Phi_k(z)$ when $z \in \Omega_k$ and $\Upsilon_k(\zeta) = \lim\limits_{z\to\zeta} \Upsilon_k(z)$ when $z \in \widetilde{\Omega}_k$.

4 BOUNDARY PERTURBATION TECHNIQUE

In equation (5) $\zeta \in \Gamma$, but $\overline{\zeta}^{-1} \in \widetilde{\Gamma}$. If $\varepsilon = 0$, then $\Gamma = \widetilde{\Gamma}$ that corresponds to the appropriate boundary value problem for the circular inclusion. In the general case for $0 < \varepsilon \ll 1$, the curves Γ and $\widetilde{\Gamma}$ represent small perturbations of the unit circle. Consequently, to find unknown functions $\Phi_k(z)$, $\Upsilon_k(z)$ and the solution of the problem, we can use boundary perturbation procedure. Following [3], [4], [11]–[13], we represent functions Φ_k and Υ_k as power series in the small parameter ε

$$\Phi_k(z) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\Phi_{kn}(z), \quad \Upsilon_k(z) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\Upsilon_{kn}(z). \tag{6}$$

Expand the boundary values of functions Φ_{kn}, Υ_{kn} at Γ and $\widetilde{\Gamma}$ into Taylor series in the vicinity of unit circle ($|z| = 1$)

$$\Phi_{kn}(\zeta) = \sum_{m=0}^{\infty} \frac{(\varepsilon f(s)s)^m}{m!}\Phi_{kn}^{(m)}(s), \quad \Upsilon_{kn}\left(\frac{1}{\bar{\bar{\zeta}}}\right) = \sum_{m=0}^{\infty} \frac{(\varepsilon f(\bar{s})\bar{s})^m}{m!}\frac{d^m}{d\bar{s}^m}\Upsilon_{kn}\left(\frac{1}{\bar{s}}\right). \tag{7}$$

We also derive the expressions for all functions in (5) as power series in the small parameter ε [3].

Taking into account the definition of the function G (5), the equations (2) can be transformed into the following

$$m_1 G^+(\zeta, \eta_2) - m_2 G^-(\zeta, \eta_1) = 0, \tag{8}$$

where $m_k = 1, k = 1, 2$ for $\eta_1 = \eta_2 = 1$ and $m_k = \mu_k$ for $\eta_k = -\varkappa_k$.

Substituting (5) into equation (8) and taking into account series (6), (7), we equate the sum of coefficients of the same power ε^n $(n = 0, 1, \ldots)$ to zero. Then we arrive at the

Rimann — Gilbert boundary value problems on the jump of holomorphic functions $\Sigma_n(z)$ and $V_n(z)$ for n-order approximation

$$\begin{aligned} \Sigma_n^+(s) - \Sigma_n^-(s) &= q_n(s), \quad |s| = 1, \\ V_n^+(s) - V_n^-(s) &= r_n(s), \quad |s| = 1. \end{aligned} \tag{9}$$

Here $\Sigma_n^{\pm} = \lim\limits_{|z|\to 1\mp 0} \Sigma_n(z)$, $V_n^{\pm} = \lim\limits_{|z|\to 1\mp 0} V_n(z)$; q_n, r_n – are the known functions depending on all previous approximations and the conditions at infinity (3). The piecewise holomorphic functions $\Sigma_n(z)$, $V_n(z)$ are defined as

$$\Sigma_n(z) = \begin{cases} \Upsilon_{1n}(z) + \Phi_{2n}(z), & |z| < 1, \\ \Upsilon_{2n}(z) + \Phi_{1n}(z), & |z| > 1, \end{cases} \tag{10}$$

$$V_n(z) = \begin{cases} \mu_2 \Upsilon_{1n}(z) - \mu_1 \varkappa_2 \Phi_{2n}(z), & |z| < 1, \\ \mu_1 \Upsilon_{2n}(z) - \mu_2 \varkappa_1 \Phi_{1n}(z), & |z| > 1. \end{cases} \tag{11}$$

According to [5], solutions of the problems (9) can be written in terms of Cauchy type integrals

$$\begin{aligned} \Sigma_n(z) &= I_1(z) + b_0 + S_n(z) + D_1, \quad z \in \Omega_1 \cup \Omega_2, \\ V_n(z) &= I_2(z) + \mu_1 b_0 - \mu_2 \varkappa_1 D_1 + \mu_2 S_n(z), \quad z \in \Omega_1 \cup \Omega_2, \end{aligned}$$

where

$$I_1(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{q_n(\zeta)}{\zeta - z} d\zeta, \quad I_2(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{r_n(\zeta)}{\zeta - z} d\zeta,$$

and $S_0 = \overline{D}_2 z^{-2}$, $S_n = 0$ $(n = 1, 2, \ldots)$, $4D_1 = \sigma_{11}^\infty + \sigma_{22}^\infty + i8\mu_1\omega^\infty/(\varkappa_1 + 1)$, $2D_2 = = \sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty$.

The constant b_0 is found from equation [5]

$$(\mu_1 - \mu_2) b_0 - (\mu_2 + \mu_1 \varkappa_2) \overline{b_0} = \mu_2 (1 + \varkappa_1) D_1.$$

The expressions for the complex potentials of n-order approximations are derived from (10), (11)

$$\Phi_{kn}(z) = \frac{\mu_k \Sigma_n(z) - V_n(z)}{\mu_k + \mu_l \varkappa_k}, \quad \Upsilon_{kn}(\bar{z}^{-1}) = \frac{\mu_k \varkappa_l \Sigma_n(\bar{z}^{-1}) - V_n(\bar{z}^{-1})}{\mu_l + \mu_k \varkappa_l}, \tag{12}$$

where $z \in \Omega_k$, $l = 3 - k$, $k = 1, 2$.

5 FIRST-ORDER APPROXIMATION

The complex potentials in the zero-order approximation which correspond to the solution of the appropriate boundary value problem for the circular inclusion, are determined [5] as

$$\Upsilon_{10}\left(\frac{1}{\bar z}\right) = \frac{\mu_1(\varkappa_2+1)b_0 + (\mu_1\varkappa_2 - \mu_2\varkappa_1)D_1}{\mu_2+\mu_1\varkappa_2} + \overline{D_2}\bar z^2,$$

$$\Phi_{10} = D_1 + \frac{\mu_1-\mu_2}{\mu_1+\mu_2\varkappa_1}\frac{\overline{D_2}}{z^2}, \quad |z|>1,$$

$$\Upsilon_{20}\left(\frac{1}{\bar z}\right) = \frac{\mu_2(\varkappa_1+1)\overline{D_2}\bar z^2}{\mu_1+\mu_2\varkappa_1} + b_0,$$

$$\Phi_{20}(z) = \frac{(\mu_2-\mu_1)b_0 + \mu_2(\varkappa_1+1)D_1}{\mu_2+\mu_1\varkappa_2}, \quad |z|<1.$$

By solving corresponding Riemann – Hilbert's boundary problems and taking into account (12), we obtain the complex potentials of the first-order approximation for the nearly circular inclusion the shape of which is determined by the function $f(s) = (s^2+s^{-2})/2 = \cos 2\theta$:

$$\Phi_{11}(z) = \frac{1}{1+M\varkappa_1}(z^{-2}(D_1(1-\beta)+\overline{D_1}+b_0\gamma - M(D_1(2-\beta)+\gamma b_0)) + z^{-4}(\overline{D_2}(\xi+2) - M\overline{D_2}(2+\varkappa_1\xi))), \quad z\in\Omega_1, \tag{13}$$

$$\Upsilon_{11} = \frac{1}{M+\varkappa_2}(\varkappa_2\xi(\overline{D_2}+D_2) + M(\xi D_2 + 2\overline{D_2}) - 2\overline{D_2}(1-\xi) - M\varkappa_1\xi\overline{D_2}) - 3z^2(\varkappa_2(2(1-\gamma)b_0 + (D_1+\overline{D_1})(\beta-1)) + b_0(2-\gamma) + \beta\overline{D_1} - M(D_1(2-\beta)+\gamma b_0)) + 5\xi D_2 z^4(\varkappa_2+M)), \quad z\in\widetilde\Omega_1, \tag{14}$$

$$\Phi_{21} = \frac{1}{M+\varkappa_2}(z^{-2}(b_0(2-\gamma) + \beta\overline{D_1} - M(D_1+\overline{D_1}(\beta-1)+b_0(2-\gamma))) + \overline{D_2}z^{-4}(3\xi M - 2M + 2(1-\xi) - \varkappa_1\xi M)), \quad z\in\Omega_2,$$

$$\Upsilon_{21} = \frac{1}{1+M\varkappa_1}(M(\varkappa_1\xi D_2 + \xi D_2 + 2\overline{D_2}) - 2\overline{D_2}(1-\xi) - 3z^2(M\varkappa_1(2(1-\gamma)b_0 + (D_1+\overline{D_1})(\beta-1)) + b_0(2-\gamma) + \beta\overline{D_1} - M(D_1(2-\beta)+\gamma b_0)) + 5M\xi D_2 z^4(\varkappa_1+1)), \quad z\in\widetilde\Omega_2.$$

Here, we introduce the following notations

$$M = \frac{\mu_2}{\mu_1}, \quad \xi = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2 \varkappa_1}, \quad \beta = \frac{\mu_2(\varkappa_1 + 1)}{\mu_2 + \mu_1 \varkappa_2}, \quad \gamma = \frac{\mu_1(\varkappa_2 + 1)}{\mu_2 + \mu_1 \varkappa_2}.$$

For the boundary value problem on an elastic plane with a nearly circular hole, we assume $\mu_2 = 0$ in equations (13), (14) . Then for the case $\omega^\infty = 0$, complex potentials are written in the form:

$$\Phi_{11} = 3\overline{D_2} z^{-4} + 2D_1 z^{-2}, \quad \Upsilon_{11} = \overline{D_2} + D_2 + 6D_1 z^2 + 5D_2 z^4.$$

This solution coincides with the solution obtained in [3]. In this case $\Phi_{21}(z) = 0$, $\Upsilon_{21}(z) = 0$.

In the zero-order approximation, we obtain the expressions for the hoop stresses σ_{tt} at the circular interface for $k = 1, 2$ in the form

$$\sigma_{tt}^{k0}(s) = \Re\left[\Phi_{k0}(s) + 2\overline{\Phi_{k0}}(s) + \Upsilon_{k0}\left(\bar{s}^{-1}\right)\right].$$

In the first-order approximation

$$\sigma_{tt}^{k}(s) = \sigma_{tt}^{k0}(s) + \varepsilon \sigma_{tt}^{k1}(s),$$

where $\sigma_{tt}^{k1}(s) = \Re[F_1]$,

$$F_1 = \left(\Phi_{k1}(s) + 2\overline{\Phi_{k1}(s)} + \Upsilon_{k1}\left(\bar{s}^{-1}\right) + s f(s)\Phi'_{k0}(s) + \bar{s} f(\bar{s})\left(2\overline{\Phi'_{k0}(s)} + \frac{d\Upsilon_{k0}\left(\bar{s}^{-1}\right)}{d\bar{s}}\right)\right) +$$
$$+2\left(\left(i f'_\theta(s) - f(s)\right)\left(\overline{\Phi_{k0}(s)} + \Upsilon_{k0}\left(\bar{s}^{-1}\right)\right) + \bar{s} f(s)\overline{\Phi'_{k0}(s)}\right).$$

6 NUMERICAL RESULTS AND DISCUSSION

Selected numerical results obtained for the hoop stress σ_{tt} at the interface are shown in Fig. 2 in the case of the uniaxial tension σ_{22}^∞ along axis x_2, i. e., for $\sigma_{11}^\infty = \sigma_{12}^\infty = 0$, $\sigma_{22}^\infty > 0$ when $\varepsilon = 0$ and $\varepsilon = 0,1$. The graphs are plotted for two values of the bimaterial parameter $M = \mu_2/\mu_1$.

Analyzing the dependences shown in Fig. 2, we come to the following conclusion:

- For a softer nearly circular inclusion (when $M < 1$) stress concentration in the inclusion and in the matrix is greater than in the case of circular inclusion corresponding to the zero-order approximation. At the same time, stress concentration in the matrix is greater than in the inclusion (Fig. 2a). It is worth noting that in the case of $M < 1$, the zone of compression in the matrix is wider for the nearly circular inclusion than for the ideal circular one.

- If the inclusion more rigid than the matrix ($M > 1$), the results are opposite (Fig. 2b).

- Comparison with similar results obtained for the nearly circular hole ($\mu_2 = 0$) in [3] shows that, for any value of parameter M, the presence of the inclusion reduces the stress concentration in the matrix.

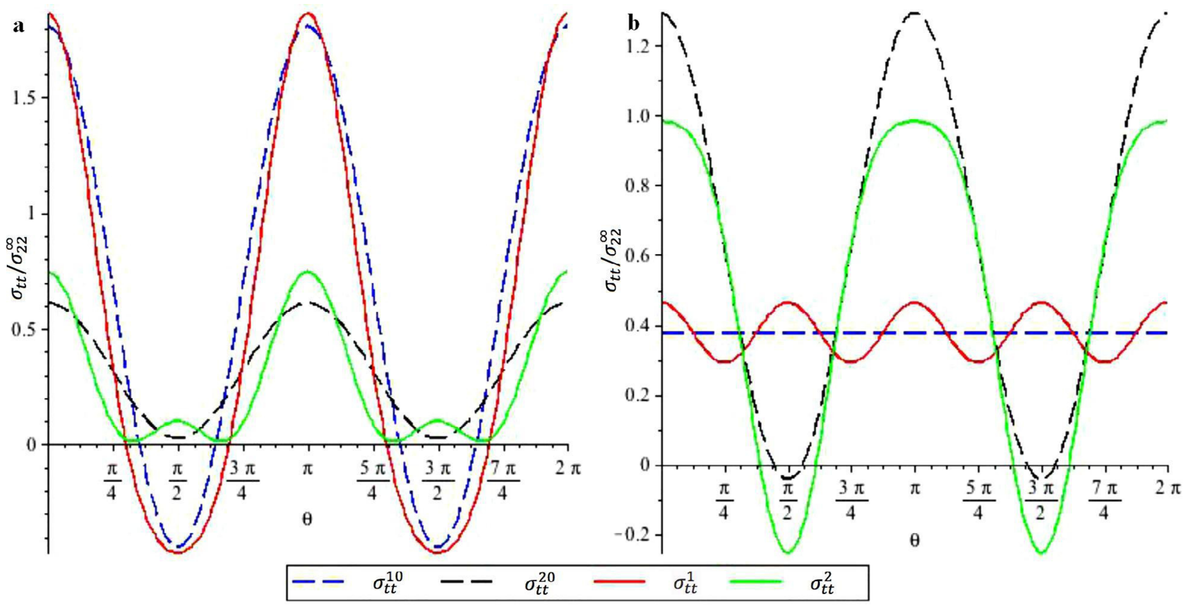

Figure 2: Dependence of hoop stresses σ_{tt} at the boundary of the inclusion, given by the function $f(\theta) = \cos 2\theta$, upon the polar angle θ under uniaxial tension for $\varepsilon = 0,1$ and $M = 1/3; 10/3$ (a, b).

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