

## AN ACCURATE SOLUTION OF SINGULAR THERMOPLASTIC PROBLEM OF PRESSURE – DEPENDENT PLASTICITY

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**Abstract.** The present paper concerns with a theoretical investigation into heat generation in the continued quasi-static plane strain compression of a thin strip between two rigid, parallel perfectly rough platens. The strip material obeys the double shearing model. The length of the platens is supposed to be much greater than the current strip thickness. The plastic work rate approaches infinity in the vicinity of perfectly rough friction surfaces. Since the plastic work rate is involved in the heat conduction equation, this greatly adds to the difficulties of solutions of this equation. In particular, commercial finite element packages are not capable of solving such boundary value problems. The present approximate solution is given in Lagrangian coordinates. In this case, the original initial/boundary value problem reduces to the standard second initial/boundary value problem for the nonhomogeneous heat conduction equation. Therefore, the Green's function is available in the literature. An example is provided to illustrate the general solution.

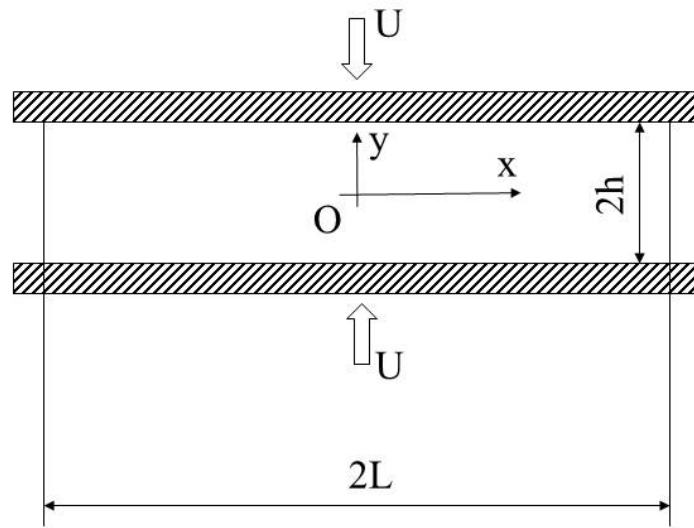
### 1 INTRODUCTION

The temperature of the workpiece rises during plastic deformation because of the heat generated by mechanical work. Under certain conditions, this temperature rise is of considerable importance. From the point of view of phenomenological plasticity theory, the first systematic study on temperature distributions in metal forming processes has been conducted in [1]. A recent review of the literature on this topic has been provided in [2]. Finite element approaches to coupled thermal flow in metal forming processes have been developed in [2 - 4]. Temperature and plastic strain are responsible for the generation of narrow fine grain layers in the vicinity of frictional interfaces [5]. Such layers may have a significant effect on the performance of machine parts [6 – 8]. The gradient in temperature is very high within the fine grain layers [9]. This greatly adds to the difficulties of solutions of corresponding boundary value problems. Moreover, in the case of the maximum friction law

finite element analyses usually fail to converge [10, 11]. Most likely, the reason for that is that exact solutions are singular in the vicinity of maximum friction surfaces for several material models [12 - 15]. In particular, the plastic work rate involved in the heat conduction equation approaches infinity in the vicinity of maximum friction surfaces. The present paper concerns with a theoretical investigation into heat generation in the continued quasi-static plane strain compression of a thin metal strip between two rigid, parallel perfectly rough dies. The maximum friction law is assumed at the die surface. The strip material obeys the double shearing model proposed in [16]. A semi-analytical solution of the aforementioned boundary value problem for the double shearing model without calculating the temperature field has been found in [17]. This solution is used in the present paper. In particular, the solution of the heat conduction equation is facilitated by using Lagrangian coordinates that are readily determined from the solution given in [17]. In these coordinates, the original initial/boundary value problem reduces to the standard second initial/boundary value problem for the nonhomogeneous heat conduction equation. Therefore, the Green's function is available in the literature, for example [18]. A similar method of solution has been used in [19] where the classical model of metal plasticity has been adopted.

## 2 STATEMENT OF THE PROBLEM

Consider a rigid plastic strip of initial thickness  $2h_0$  and initial width  $2L_0$ . The strip is compressed between two parallel platens. The speed of each platen is  $U$ . The current thickness and width of the strip are denoted by  $2h$  and  $2L$ , respectively. The Cartesian coordinate system  $(x, y)$  is chosen such that its  $x$  – and  $y$  – axes coincide with the axes of symmetry of the process (Fig. 1). Therefore, it is sufficient to consider the domain  $0 \leq x \leq L$



**Figure 1:** Configuration of the problem and Cartesian coordinate system

and  $0 \leq y \leq h$ . It is assumed that the initial temperature of the strip is constant,  $T_0$ . The current temperature is denoted by  $T$ . Let  $u_x$  and  $u_y$  be the velocity components referred to the

Cartesian coordinate system. Then, the exact velocity boundary conditions are

$$u_y = 0 \quad (1)$$

for  $y = 0$  and

$$u_y = -U \quad (2)$$

for  $y = h$ . In the case of  $h/L \ll 1$  the exact velocity boundary condition at  $x = 0$  is usually replaced with the following approximate condition [17, 20 - 23]

$$\int_0^h u_x dy = 0. \quad (3)$$

It is understood here that the velocity component  $u_x$  is calculated at  $x = 0$ . Let  $\sigma_{xx}$ ,  $\sigma_{yy}$  and  $\sigma_{xy}$  be the stress components referred to the Cartesian coordinate system. The exact stress boundary conditions are

$$\sigma_{xy} = 0 \quad (4)$$

for  $y = 0$  and the maximum friction law. The representation of the maximum friction law depends on the material model chosen. In the present paper, the double shearing model proposed in [16] is adopted. The system of equations comprising the constitutive equations and equilibrium equations is hyperbolic. Under plane strain conditions the characteristics of the stresses and the velocities coincide. Therefore, there are only two distinct characteristic directions at a point. The maximum friction law demands that a characteristic or an envelope of characteristics coincides with the friction surface. Let  $\psi$  be the angle between the direction of the algebraically greatest principal stress and the  $x$  – axis measured from the axis anticlockwise. Then, the angles between the characteristic directions and the  $x$  – axis are  $\psi + \pi/4 + \varphi/2$  and  $\psi - \pi/4 - \varphi/2$  [16]. Here  $\varphi$  is the angle of internal friction. The direction of flow dictates that  $\sigma_{xy} < 0$  at  $y = h$  (Fig. 1). Therefore,  $\psi < 0$  at  $y = h$  and the maximum friction law can be written as

$$\psi = -\frac{\pi}{4} - \frac{\varphi}{2} \quad (5)$$

for  $y = h$ . The exact stress boundary conditions at  $x = L$  are usually replaced with the following approximate condition [17, 20 - 23]

$$\int_0^h \sigma_{xx} dy = 0. \quad (6)$$

It is understood here that the stress component  $\sigma_{xx}$  is calculated at  $x = L$ . The magnitude of the shear stress at  $y = h$  is denoted by  $\tau_f$ . Therefore,

$$\tau_f = |\sigma_{xy}|. \quad (7)$$

It is understood here that the stress component  $\sigma_{xy}$  is calculated at  $y = h$ . The boundary

conditions on the temperature are

$$\frac{\partial T}{\partial y} = 0 \quad (8)$$

for  $y = 0$  and

$$\lambda \frac{\partial T}{\partial y} = \tau_f u_x + q_t \quad (9)$$

for  $y = h$ . It is understood here that the velocity component  $u_x$  is calculated at  $y = h$ . Also,  $q_t$  is the external heat flux through the friction surface from tool into the plastic strip and  $\lambda$  is the coefficient of thermal conductivity. In what follows,  $c_v$  and  $\rho$  will stand for the specific heat at constant volume and density, respectively. It is assumed that  $\lambda$ ,  $c_v$ ,  $\rho$  and  $q_t$  are constant. The initial condition is

$$T = T_0 \quad (10)$$

at  $h = h_0$ .

The Coulomb-Mohr yield criterion can be written as

$$q - p \sin \varphi = c \cos \varphi, \quad (11)$$

where  $c$  is the cohesion and

$$p = -\frac{(\sigma_{xx} + \sigma_{yy})}{2}, \quad q = \frac{1}{2} \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}. \quad (12)$$

The flow rule of the double shearing model is [16]

$$\begin{aligned} \xi_{xy} \cos 2\psi - \frac{1}{2}(\xi_{xx} - \xi_{yy}) \sin 2\psi + \sin \varphi \left( \omega_{xy} + \frac{d\psi}{dt} \right) &= 0, \\ \xi_{xx} + \xi_{yy} &= 0. \end{aligned} \quad (13)$$

Here  $\xi_{xx}$ ,  $\xi_{yy}$  and  $\xi_{xy}$  are the strain rate components in the Cartesian coordinates,  $\omega_{xy}$  is the only non-zero spin component in the Cartesian coordinates and  $d/dt$  denotes the convected derivative. The second equation in (13) is the equation of incompressibility. Therefore,

$$h_0 L_0 = hL. \quad (14)$$

The stress components in the Cartesian coordinates are represented as [16]

$$\sigma_{xx} = -p + q \cos 2\psi, \quad \sigma_{yy} = -p - q \cos 2\psi, \quad \sigma_{xy} = q \sin 2\psi. \quad (15)$$

### 3 GENERAL SOLUTION

The velocity components are given by [17]

$$\frac{u_x}{U} = \frac{x}{h} - \frac{2}{C} \cos 2\psi + B, \quad \frac{u_y}{U} = -\frac{y}{h}, \quad (16)$$

where  $B$  and  $C$  are constants of integration. It is evident that the solution (16) satisfies the boundary conditions (1) and (2). The dependence of  $\psi$  on  $y$  is given in implicit form as

$$\frac{y}{h} = \frac{\sin 2\psi + 2\psi \sin \varphi}{C}. \quad (17)$$

It is seen from (15) that this solution satisfies the boundary condition (4). The solution (17) and the boundary condition (5) combine to give

$$C = -\cos \varphi - \left( \frac{\pi}{2} + \varphi \right) \sin \varphi. \quad (18)$$

The solution for  $q$  is [17]

$$q = A \exp \left[ \frac{\sin \varphi}{\cos^2 \varphi} \left( \frac{Cx}{h} - \cos 2\psi \right) \right], \quad (19)$$

where  $A$  is a constant of integration. Using this solution  $p$  can be found from (11) and, then, the stress components from (15). In particular, the boundary condition (9) can be rewritten as

$$\lambda \frac{\partial T}{\partial y} = UA \exp \left[ \frac{\sin \varphi}{\cos^2 \varphi} \left( \frac{Cx}{h} + \sin \varphi \right) \right] \left( \frac{x}{h} + \frac{2 \sin \varphi}{C} + B \right) \cos \varphi + q, \quad (20)$$

for  $y = h$  or  $\psi = -\pi/4 - \varphi/2$ . The strain rate components are determined from (16) and (17) as

$$\xi_{xx} = \frac{U}{h}, \quad \xi_{yy} = -\frac{U}{h}, \quad \xi_{xy} = \frac{U \sin 2\psi}{h(\cos 2\psi + \sin \varphi)}. \quad (21)$$

It is seen from this solution and (5) that  $|\xi_{xy}| \rightarrow \infty$  as  $y \rightarrow h$ . Such behavior of the shear strain rate near the maximum friction surface is in agreement with the general theory [13]. The plastic work rate is defined as  $W = \sigma_{xx} \xi_{xx} + \sigma_{yy} \xi_{yy} + 2\sigma_{xy} \xi_{xy}$ . Substituting (15) and (21) into this equation yields

$$W = \frac{2Uq}{h} \frac{(1 + \cos 2\psi \sin \varphi)}{(\cos 2\psi + \sin \varphi)}. \quad (22)$$

It is known that the variation of temperature in the  $x$  – direction is negligible [23]. Therefore, it is usually assumed that  $T$  is independent of  $x$ . Then, the heat conduction equation can be written as

$$\frac{dT}{dt} = \frac{\lambda}{c_v \rho} \frac{\partial^2 T}{\partial y^2} + \frac{\beta}{c_v \rho} W. \quad (23)$$

Here the factor  $\beta$  determines the portion of plastic work converted into heat. The value of this factor is close to unity. Moreover, the temperature everywhere is directly proportional to  $\beta$ . Therefore, it is assumed that  $\beta = 1$ . It is seen from (5) and (22) that  $W \rightarrow \infty$  as  $y \rightarrow h$ . Therefore, equation (23) contains a singular term. In general, the assumption that the distribution of temperature is independent of  $x$  is not compatible with (20) and (22). Therefore, various approximations are used to get rid of the dependence of the right hand

sides of these equations on  $x$  [19, 23]. In what follows,  $W$  involved in (23) is replaced with its average value defined as

$$W_a = \frac{1}{L} \int_0^L W dx. \quad (24)$$

Substituting (19) and (22) into this equation and using (14) yields

$$W_a = \frac{2UA \cos^2 \varphi}{CL_0 \sin \varphi} w_1(\psi) w_2\left(\frac{h}{h_0}\right), \quad (25)$$

$$w_1(\psi) = \frac{(1 + \cos 2\psi \sin \varphi)}{(\cos 2\psi + \sin \varphi)} \exp\left(-\frac{\sin \varphi}{\cos^2 \varphi} \cos 2\psi\right), \quad w_2\left(\frac{h}{h_0}\right) = \left[ \exp\left(\frac{C \sin \varphi h_0^2 l_0}{\cos^2 \varphi h^2}\right) - 1 \right] \frac{h}{h_0}.$$

where  $l_0 = L_0/h_0$ . Then, equation (23) becomes

$$\frac{dT}{dt} = \frac{\lambda}{c_v \rho} \frac{\partial^2 T}{\partial y^2} + \frac{2UA \cos^2 \varphi}{c_v \rho CL_0 \sin \varphi} w_1(\psi) w_2\left(\frac{h}{h_0}\right). \quad (26)$$

The quantity  $\lambda \partial T / \partial y$  at  $y = h$  is replaced with

$$\lambda \frac{\partial T}{\partial y} = \lambda \left( \frac{\partial T}{\partial y} \right)_a \equiv \frac{\lambda}{L} \int_0^L \frac{\partial T}{\partial y} dx \quad (27)$$

Substituting the right hand side of (20) into this equation and using (14) yields

$$\lambda \left( \frac{\partial T}{\partial y} \right)_a = q_t + \frac{h^2}{h_0^2} \frac{UA \cos^3 \varphi}{l_0 C \sin \varphi} \exp(\tan^2 \varphi) \left\{ 1 + \left( \frac{l_0 C \sin \varphi}{\cos^2 \varphi} \frac{h_0^2}{h^2} - 1 \right) \exp\left(\frac{l_0 C \sin \varphi}{\cos^2 \varphi} \frac{h_0^2}{h^2}\right) + \left( \frac{2 \sin \varphi}{C} + B \right) \left[ \exp\left(\frac{l_0 C \sin \varphi}{\cos^2 \varphi} \frac{h_0^2}{h^2}\right) - 1 \right] \right\}. \quad (28)$$

Then, the boundary condition (20) becomes

$$\lambda \frac{\partial T}{\partial y} = \lambda \left( \frac{\partial T}{\partial y} \right)_a \quad (29)$$

for  $y = h$  or  $\psi = -\pi/4 - \varphi/2$ . Since  $dy/dt = u_y$  and  $dh/dt = -U$ , it follows from (16) that

$$\frac{dy}{dh} = \frac{y}{h}. \quad (30)$$

It is convenient to introduce a Lagrangian coordinate  $Y$  such that  $Yh_0 = y$  at  $h = h_0$ . The solution of equation (30) satisfying this initial condition is

$$y = hY. \quad (31)$$

Using this equation and taking into account that  $dh/dt = -U$  equation (26) in the Lagrangian coordinates can be written as

$$\frac{\partial \tau}{\partial p} = a \frac{\partial^2 \tau}{\partial Y^2} + \frac{b}{(p+1)^2} w_1(\psi) w_2\left(\frac{1}{1+p}\right). \quad (32)$$

where

$$a = \frac{\lambda}{c_v \rho h_0 U}, \quad b = \frac{2 A h_0 \cos^2 \varphi}{c_v \rho C T_0 l_0 \sin \varphi}, \quad \tau = \frac{T - T_0}{T_0}, \quad p = \frac{h_0}{h} - 1. \quad (33)$$

Using (31) and (33) the boundary condition (8) is transformed to

$$\frac{\partial \tau}{\partial Y} = 0 \quad (34)$$

for  $Y = 0$ . Analogously, the boundary condition (29) becomes

$$\frac{\partial \tau}{\partial Y} = \frac{h_0 q_t}{\lambda T_0 (p+1)} + \frac{b h_0 \cos \varphi}{2a (p+1)^3} \exp(\tan^2 \varphi) \left\langle 1 + \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 - 1 \right] \exp \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 \right] + \right. \\ \left. \left( \frac{2 \sin \varphi}{C} + B \right) \left\{ \exp \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 \right] - 1 \right\} \right\rangle. \quad (35)$$

for  $Y = 1$ . The initial condition (10) transforms to

$$\tau = 0 \quad (36)$$

at  $p = 0$ . In order to find  $B$  involved in (35), it is necessary to substitute (16) into (3). Using (17) and (18) results in

$$B = \frac{(\pi + 2\varphi + \sin 2\varphi)}{2[\cos \varphi + (\pi/2 + \varphi) \sin \varphi]^2}. \quad (37)$$

It remains to determine  $A$  involved in  $b$ . It follows from (11), (14), (15), (19), and (33) that

$$\sigma_{xx} = c \cot \varphi + A \exp \left\{ \frac{\sin \varphi}{\cos^2 \varphi} \left[ C l_0 (p+1)^2 - \cos 2\psi \right] \right\} \left( \cos 2\psi - \frac{1}{\sin \varphi} \right) \quad (38)$$

at  $x = L$ . Substituting (38) into (6), using (18) and replacing integration with respect to  $y$  with integration with respect to  $\psi$  by means of (17) and (31) yield

$$A = \frac{c \cos \varphi [\cos \varphi + (\pi/2 + \varphi) \sin \varphi]}{2} \times \\ \left\langle \int_0^{-\pi/4 - \varphi/2} \exp \left\{ \frac{\sin \varphi}{\cos^2 \varphi} \left[ C l_0 (p+1)^2 - \cos 2\psi \right] \right\} (\cos 2\psi \sin \varphi - 1) (\cos 2\psi + \sin \varphi) d\psi \right\rangle^{-1}. \quad (39)$$

Equation (32) together with the boundary and initial conditions (34) – (36) comprise the standard second initial/boundary value problem for the nonhomogeneous heat conduction equation [18]. However, a difficulty is that the function  $w_1(\psi)$  involved in (32) approaches infinity as  $\psi \rightarrow -\pi/4 - \varphi/2$  (or  $Y \rightarrow 1$ ). In order to overcome this difficulty, it is convenient to introduce the new function  $u$  by

$$u = \tau + f(Y)g(p) \quad (40)$$

where  $f(Y)$  is a function only  $Y$  and  $g(p)$  is a function only  $p$ . Substituting (40) into (32) yields

$$\frac{\partial u}{\partial p} = a \frac{\partial^2 u}{\partial Y^2} + f(Y) \frac{dg}{dp} - ag(p) \frac{d^2 f}{dY^2} + \frac{b}{(p+1)^2} w_1(\psi) w_2\left(\frac{1}{p+1}\right). \quad (41)$$

Assume that

$$g(p) = \frac{b}{a(p+1)^2} w_2\left(\frac{1}{p+1}\right) \quad \text{and} \quad \frac{d^2 f}{dY^2} = -w_1(\psi). \quad (42)$$

Then, equation (41) becomes

$$\frac{\partial u}{\partial p} = a \frac{\partial^2 u}{\partial Y^2} + f(Y) \frac{dg}{dp}. \quad (43)$$

It follows from (17) and (31) that

$$\frac{d\psi}{dY} = \frac{C}{2(\cos 2\psi + \sin \varphi)}. \quad (44)$$

Using (25), (33) and (44) equation (42) can be rewritten as

$$\begin{aligned} g(p) &= \frac{b}{a(p+1)^3} \left\{ \exp \left[ \frac{C \sin \varphi l_0}{\cos^2 \varphi} (p+1)^2 \right] - 1 \right\}, \\ \frac{d}{d\psi} \left( \frac{df}{dY} \right) &= -\frac{2(1 + \cos 2\psi \sin \varphi)}{C} \exp \left( -\frac{\sin \varphi}{\cos^2 \varphi} \cos 2\psi \right). \end{aligned} \quad (45)$$

Differentiating the first equation with respect to  $p$  gives

$$\frac{dg}{dp} = \frac{b}{a(p+1)^2} \left\{ \frac{3}{(p+1)^2} + \left[ \frac{2Cl_0 \sin \varphi}{\cos^2 \varphi} - \frac{3}{(p+1)^2} \right] \exp \left[ \frac{C \sin \varphi l_0}{\cos^2 \varphi} (p+1)^2 \right] \right\}. \quad (46)$$

The function  $f(Y)$  may be chosen such that  $df/dY = 0$  at  $Y = 0$ . Then, the solution of the second equation in (45) can be represented in the form

$$\frac{df}{dY} = -\frac{2}{C} \int_0^\psi (1 + \cos 2z \sin \varphi) \exp \left( -\frac{\sin \varphi}{\cos^2 \varphi} \cos 2z \right) dz. \quad (47)$$

Using (44) this solution can be rewritten as

$$\frac{df}{d\psi} = -\frac{4(\cos 2\psi + \sin \varphi)}{C^2} \int_0^\psi (1 + \cos 2z \sin \varphi) \exp \left( -\frac{\sin \varphi}{\cos^2 \varphi} \cos 2z \right) dz. \quad (48)$$

The function  $f(Y)$  may be chosen such that  $f = 0$  at  $Y = 0$ . Then, the solution of equation (48) is

$$f = -\frac{4}{C^2} \int_0^\psi (\cos 2\chi + \sin \varphi) \int_0^\chi (1 + \cos 2z \sin \varphi) \exp\left(-\frac{\sin \varphi}{\cos^2 \varphi} \cos 2z\right) dz d\chi. \quad (49)$$

Thus equations (17), (31) and (49) supply  $f$  as a function of  $Y$  in parametric form with  $\psi$  being the parameter varying in the range  $-\pi/4 - \varphi/2 \leq \psi \leq 0$ . Substituting this function and (46) into (43) determines the last term of this equation as a function of  $Y$  and  $p$ . Using (40) and the functions  $f(Y)$  and  $g(p)$  found the boundary conditions (34) and (35) are replaced with

$$\frac{\partial u}{\partial Y} = 0 \quad (50)$$

for  $Y = 0$  and

$$\begin{aligned} \frac{\partial u}{\partial Y} = & \frac{h_0 q_t}{\lambda T_0 (p+1)} + \frac{b h_0 \cos \varphi}{2a(p+1)^3} \exp(\tan^2 \varphi) \left\langle 1 + \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 - 1 \right] \exp \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 \right] + \right. \\ & \left. \left( \frac{2 \sin \varphi}{C} + B \right) \left\{ \exp \left[ \frac{l_0 C \sin \varphi}{\cos^2 \varphi} (p+1)^2 \right] - 1 \right\} \right. \\ & \left. + \frac{2b}{aC(p+1)^3} \left\{ \exp \left[ \frac{C \sin \varphi l_0}{\cos^2 \varphi} (p+1)^2 \right] - 1 \right\} \int_{-\pi/4 - \varphi/2}^0 (1 + \cos 2z \sin \varphi) \exp \left( -\frac{\sin \varphi}{\cos^2 \varphi} \cos 2z \right) dz \right. \end{aligned} \quad (51)$$

for  $Y = 1$ . The initial condition (36) becomes

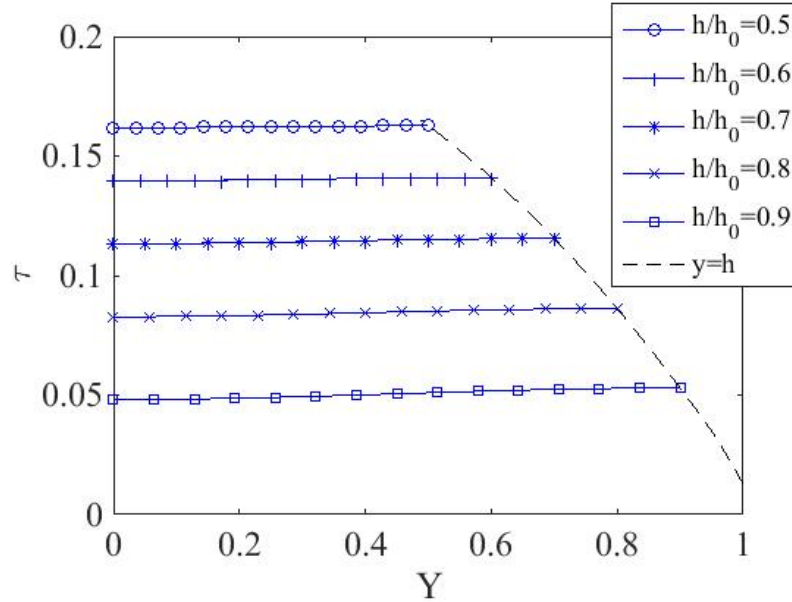
$$u = -\frac{4b}{aC^2} \left[ \exp \left( \frac{C \sin \varphi l_0}{\cos^2 \varphi} \right) - 1 \right] \int_0^\psi (\cos 2\chi + \sin \varphi) \int_0^\chi (1 + \cos 2z \sin \varphi) \exp \left( -\frac{\sin \varphi}{\cos^2 \varphi} \cos 2z \right) dz d\chi. \quad (52)$$

at  $p = 0$ . Here  $\psi$  should be eliminated by means of (17) and (31). The constants  $C$ ,  $B$  and  $A$  involved in (43), (51) and (52) should be eliminated by means of (18), (37) and (39).

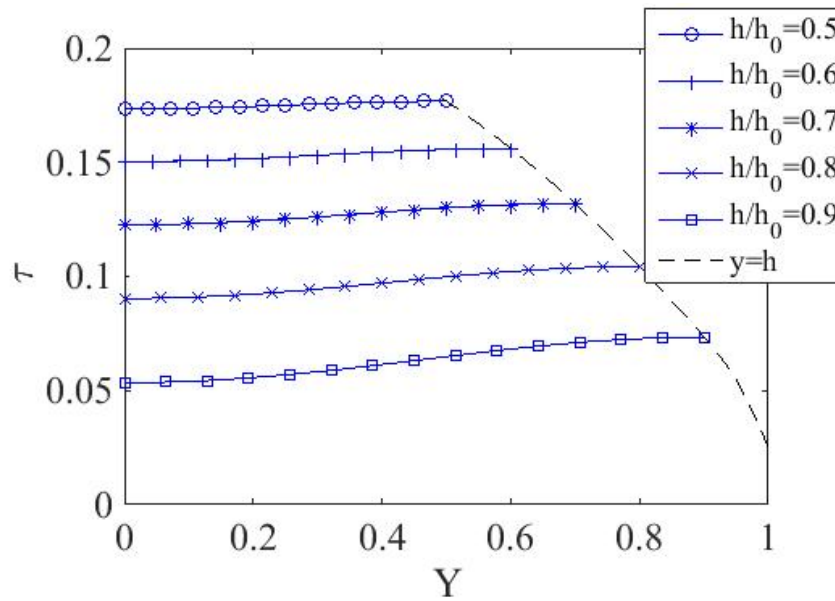
#### 4 ILLUSTRATIVE EXAMPLE

The numerical solution of equation (43) satisfying the initial condition (52) and the boundary conditions (50) and (51) has been found using the Green's function provided in [18]. It has been assumed that  $q_t = 0$ ,  $\varphi = 0.1$ ,  $l_0 = 5$ ,  $(c_v \rho) = 1.8 \cdot 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}$ ,  $T_0 = 1000^\circ \text{C}$ ,  $h_0 = 2 \text{ mm}$ , and  $c = 750 \text{ MPa}$ . The through thickness distribution of the dimensionless temperature determined by means of the numerical solution found and (40) is depicted in Fig. 2 at  $a = 30$  and several values of  $h/h_0$  and in Fig. 3 at  $a = 10$  and the same values of  $h/h_0$ . In the computations, 15 first terms in the series representing the Green's function have been used. For the set of parameters chosen, the accuracy of the Green's function calculated is higher than the standard MATLAB long format. All computations took a few seconds and provided the accuracy of  $10^{-5}$ . No difference in terms of the computation time and accuracy was observed when the Green's function was evaluated using 11 first terms in its series representation instead of 15 terms. This shows that the integration accuracy rather than the approximation of the Green's function should be improved to obtain a more accurate result.

It is worthy of note that the value of the dimensionless parameter  $l_0$  is rather large and this parameter is involved in the exponential function in (46). This might cause computational difficulties. However, it is seen from (33) and (39) that the product of  $b$  and the exponential function in (46) is of the order 1. Therefore, the numerical solution has been found with no difficulty.



**Figure 2:** Through thickness distribution of the dimensionless temperature at  $a = 30$  and several values of  $h/h_0$



**Figure 3:** Through thickness distribution of the dimensionless temperature at  $a = 10$  and several values of  $h/h_0$

## 5 CONCLUSIONS

The distribution of temperature in the continued quasi-static plane strain compression of a thin strip between two rigid, parallel perfectly rough dies has been found. The double shear model has been adopted. Using Lagrangian coordinates has enabled the original initial/boundary value problem to be reduced to the standard second initial/boundary value problem for the nonhomogeneous heat conduction equation. The Green's function for the latter is known [18].

The heat conduction equation contains a singular term (the plastic work rate approaches infinity near the friction surface). For this reason, commercial finite element packages are not capable of solving such boundary value problems. Probably, the generalized finite element method [24] can be used for this purpose. However, no specific code is currently available. In the present paper, the new unknown  $u$  has been introduced in equation (40) to transform the original heat conduction equation to (43). The latter contains no term that approaches infinity near the maximum friction surface.

## 6 ACKNOWLEDGMENTS

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