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ON THE EFFECT OF THE ANISOTROPIC DRY FRICTION AND THE DEFORMED STATE OF TIRES ON THE SHIMMY INITIATION

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Abstract. The shimmy theories based on the Keldysh assumption can be easily implemented analytically and are still quite efficient for the preliminary analysis of the stability of steady state rolling regimes with no slip and spin of tires. On the other hand, such a shimmy theory uses the non-holonomic rolling model, therefore it is inconsistent with unsteady rolling regimes characterized by the non-vanishing sliding and spin. The qualitatively different model accounting the dry friction effect on the stability of motion is constructed on the groundwork of the coupled dry friction theory. This model has shown its' applicability to some practical problems of engineering design even if a wheel is assumed to be rigid. Here the improved model accounting for the tire deforming, the complex contact pressure distribution, and the anisotropy of the dry friction coefficient in case of the combined kinematics is presented.

1 INTRODUCTION

The phenomenon of the shimmy of wheels of aircraft landing gears and automotive vehicles is well-known but not completely studied. Indeed, the traditionally used model of a shimmy proposed by M. V. Keldysh [1] is based on the following assumptions: (a) the sliding in each point of the road-wheel contact area vanishes and (b) the motion instability is induced essentially by the elastic forces appearing in deformed tires. Such an approach result the very simple model with reduced number of degrees of freedom, does not require complex numerical methods and remains still useful in the engineering practice [2]. In the same time the instable motion of various wheels is observed at the stages of non-steady rolling with significant longitudinal sliding in the contact spot; moreover the spinning can appear in case of the disturbed motion [3]. It is clear that the Keldysh assumptions are inconsistent with such a regime, and the dry friction effects on the motion stability cannot be studied using the traditional shimmy theory [2]. On the other hand the combined kinematics of relative motion in the contact area requires the qualitative improvement of dry friction theories as it was shown by P. Constnsou [4] and Th. Erissmann [5]; the classical Coulomb dry friction law is unable to describe a wide range of phenomena due to the coupling of sliding and spinning.

The rigid wheel shimmy is one of such phenomena; it is induced only by the coupling of the dry friction forces and torque [6, 7]. This effect was found using the new dry friction model so-called “poly-component” or “multi-component” [8, 9] and being the essential improvement of previous theories of Contensou [4] and Erismann [5]. The multi-component theory uses the local formulation of the Coulomb law for each small element of the contact spot where the corresponding summary sliding velocity is resulted by the longitudinal slip and the spinning. Therefore the resultant dry friction force vector and moment are obtained as a result of integration over the contact area; both are depending as well on the slip velocity as on the angular spin velocity. The exact integral formulation for the resultant force vector and couple were obtained in [8] assuming the contact area be small and consequently circular and the contact pressure distribution be Hertzian. These formulae are too complex to use them in the engineering analysis of dynamics of systems with the dry friction; it was shown nevertheless that their linear-fractional approximation is quite sufficient in most dynamical problems [9]. The Pade approximations were later replaced by the new type of the approximate formulae [10]. Such a results were obtained for some others particular cases of the contact pressure distributions [11].

The quasi-rigid wheel theory [9, 10] was applied to investigate the shimmy of landing gears [3, 12-14]. It seems to be a good first approximation, nevertheless this simplest approximate model has almost nothing to do with the real objects of the engineering analysis. Indeed, the Hertzian contact pressure distribution is consistent only with very small deformations of a typical pneumatic tire [15], the contact area is not circular, moreover the dry friction is not isotropic due to the tire tread effect. Thus, the shimmy models [3, 6, 7, 12-14] require some significant improvements as well as the combined dry friction theory [8-11].

The elliptic contact area was considered in [16] under the assumption of the generalized Hertzian contact pressure distribution. In the recently published paper [17] the polynomial approximation of the complex contact pressure distribution in the pneumatic tire obtained from the finite element simulation [15] was introduced into the model assuming the contact spot be circular. On the other hand the dry friction anisotropy was introduced as a generalized Coulomb law by many authors (e. g. see [18-22]), the tensor coefficient of the dry friction was introduced, and the effect of the anisotropy on the dynamics of the material point on the plane was investigated in details [20]. It has to be noted nevertheless that all the mentioned models generalize the classical Coulomb law in case of the simple motion and have nothing to do with the combined kinematics. Here the basics of the improved theory of coupled dry friction are presented, the dry friction anisotropy is introduced, and the formulae for the resultant vector and couple of the dry friction are constructed. Such a model seems to be a good second approximation for a wheel under combined rolling, sliding, and spin and can be applied to study the shimmy initiation conditions.

2 MODEL OF THE COMBINED ROLLING, SLIDING, AND SPIN OF THE RIGID BODY BASED ON THE COUPLED DRY FRICTION THEORY

2.1 On the anisotropic dry friction

In case of the frictional anisotropy induced by the structure of interacting bodies and/or texture of the contacted surface the Amonton-Coulomb dry friction law can be written in the following formulation [18-21]:

$$\mathbf{F} = -|N|\frac{\mathbf{f}\cdot\mathbf{v}}{|\mathbf{v}|} \quad (\mathbf{v}\neq 0). \tag{1}$$

Here the second rank tensor $\mathbf{f} = f_{\alpha\beta}\mathbf{e}^{\alpha}\mathbf{e}^{\beta}$ is the dry friction coefficient; $\mathbf{e}^{\alpha}, \alpha = 1,2$ are base vectors of some frame Ox^1x^2 on the plane of interaction of the contacting bodies, $\mathbf{v} = v^{\alpha}\mathbf{e}_{\alpha}$ is the vector of the relative sliding velocity, and the symbol «•» denotes the scalar product. In general $f_{\alpha\beta} \neq f_{\beta\alpha}$ (e. g. see [18, 22]).

Let us consider the dry friction tensor $\mathbf{f}$ be positively defined:

$$\mathbf{v}_1^{\mathrm{T}}\cdot\mathbf{f}_S\cdot\mathbf{v}_1 > 0 \quad \forall\mathbf{v}_1(\mathbf{q}):\ \mathbf{q}\in\Omega\subset R^n, \quad \mathbf{f}_S = \tfrac{1}{2}\left(\mathbf{f}+\mathbf{f}^{\mathrm{T}}\right), \tag{2}$$

where Ω is the configuration space of the considered mechanical system, and $\mathbf{q}$ is the vector of generalized coordinates. Thus, the linear mapping of the set of unit vectors $\mathbf{v}_1$ into the set of dry friction force vectors $\mathbf{F}$ (1) is defined by the linear operator $\mathbf{f}$ (2). As usually, the unit vector $\mathbf{v}_1 = \mathbf{v}/|\mathbf{v}|$ is the relative sliding director.

The general cohesion condition can be represented in the formulation [20]:

$$I_2(\mathbf{f})\neq 0:\quad I_2^{-2}(\mathbf{f})\left\{I_1^2(\mathbf{f})|\mathbf{F}|^2 + \left[I_1(\mathbf{f})\mathbf{f}_S + \mathbf{f}^{\mathrm{T}}\cdot\mathbf{f}\right]:\mathbf{F}\otimes\mathbf{F}\right\} = |N|^2, \tag{3}$$

where $I_1(\mathbf{f})$ and $I_2(\mathbf{f})$ are invariants of dry friction tensor $\mathbf{f}$, and the symbol «$\otimes$» denotes the tensor product. The physical components $f^*_{\alpha\beta}$ of the tensor $\mathbf{f}$ can be defined experimentally on the groundwork of specific tests. For instance the concept proposed by A. Zmitrowiecz [20] can be used; the dry friction forces are measured directly for two non-collinear sliding directions. In general these ones has not to coincide with the base vectors $\mathbf{e}_{\alpha}$ of the main frame Ox^1x^2.

2.2 The local model of the anisotropic dry friction. Differential formulation

In general, the plane-parallel relative motion, i. e. the simultaneous sliding and spinning, of the rigid bodies with the finite contact spot S requires the qualitative improvement of the Amonton-Coulomb dry friction law [5-11]. The first generalized formulation accounting for the spin in the formulae for the dry friction force was proposed by Th. Erissmann [5] and P. Contensou [4]; it was shown that the resultant vector of dry friction forces depends significantly on the spin. V. Zhuravlev [7-9] and A. Kireenkov [9-11] have proposed the further improvement of this concept; finally, the multi-component dry friction theory was formulated for a general case of combined slip, spin, and rolling [7]. The aim of this theory consists in the differential formulation of the Coulomb law as a local model of the friction interaction in each point of the contact area S :

$$\forall M \in S \quad \boldsymbol{\tau} = -|\sigma_\nu| \frac{\mathbf{f} \cdot \mathbf{v}_\Sigma}{|\mathbf{v}_\Sigma|} \quad (\mathbf{v}_\Sigma \neq 0), \tag{4}$$

$$\mathbf{v}_\Sigma = \mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau. \tag{5}$$

Here $\mathbf{v}_\Sigma$ denotes the summary velocity of the relative slip in the arbitrary point $M \in S$, $\mathbf{v}_0(\mathbf{q})$ is the longitudinal absolute velocity, $\boldsymbol{\omega}_\tau(\mathbf{q})$ is the angular velocity of rolling, , $\boldsymbol{\omega}_\nu(\mathbf{q})$ is the angular velocity of spinning, $R(M)$ is the curvature radius of the rolling body calculated in the point M, $\mathbf{r}_\tau(M)$ is the vector radius of the point $M \in S$ in the plane of contact, $\mathbf{e}_3$ denotes the normal unit vector of the contact plane, $\boldsymbol{\tau}$ is the frictional tangential stress in the contact area S, and σ_ν denotes the normal contact pressure. Thus, the cohesion condition (3) can be formulated locally in the point $M \in S$ as follows:

$$I_2^{-2}(\mathbf{f})\left\{I_1^2(\mathbf{f})|\boldsymbol{\tau}|^2 + \left[I_1(\mathbf{f})\mathbf{f}_S + \mathbf{f}^{\mathrm{T}} \cdot \mathbf{f}\right] : \boldsymbol{\tau} \otimes \boldsymbol{\tau}\right\} = |\sigma_\nu|^2. \tag{6}$$

Let us consider the combined kinematics, i. e. the simultaneous slip, spin, and rolling (5). Therefore, taking into account the dry friction anisotropy, we obtain the following formula for the tangential stress (4):

$$\boldsymbol{\tau}_1 = -|\sigma_\nu| \left|\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right|^{-1} \mathbf{f} \cdot \left(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right), \tag{7}$$

where the normal pressure accounting for the rolling effect is represented by the linear approximation [23]:

$$\sigma_\nu = \sigma_0\left[1 + \left(\mathbf{r}_\tau \times \mathbf{h} \cdot \boldsymbol{\omega}_\tau / |\boldsymbol{\omega}_\tau|\right) \cdot \mathbf{e}_3\right]. \tag{8}$$

Here $\sigma_0 = \sigma_\nu(\boldsymbol{\omega}_\tau = 0)$, and $\mathbf{h} = h_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta$ is the “rolling friction tensor” for the anisotropic elastic body; we assume it being homogeneous and positively defined:

$$\mathbf{h} \neq \mathbf{h}(M); \quad \forall \boldsymbol{\omega}_\tau = \boldsymbol{\omega}_\tau(\mathbf{q}) \quad \boldsymbol{\omega}_\tau^{\mathrm{T}} \cdot \mathbf{h} \cdot \boldsymbol{\omega}_\tau > 0. \tag{9}$$

The static normal contact stress σ_0 is determined by the solution of the appropriate contact problem of elasticity theory in quasi-static statement. Thus, the model (7)-(9) allows one to use the static solution as a first approximation to model the coupled rolling and sliding friction, and the complex modeling of the dynamics of contact interaction is not required.

Now, accounting the rolling effect on the contact pressure on the basis of the formula (8) and using the formula (7) for the tangential contact stress that accounts by-turn both sliding (this term is denoted as $\boldsymbol{\tau}_1$) and spinning (this one is denoted as $\boldsymbol{\tau}_2$), we obtain finally the local model of the anisotropic dry friction in case of the combined kinematics:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2;$$

$$\begin{aligned}
\boldsymbol{\tau}_1 &= -|\sigma_0|\left[1+\left(\mathbf{r}_\tau \times \frac{\mathbf{h}\cdot\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}\right)\cdot \mathbf{e}_3\right]\frac{\mathbf{f}\cdot\left(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3\right)}{\left|\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right|};\\
\boldsymbol{\tau}_2 &= -|\sigma_0|\left[1+\left(\mathbf{r}_\tau \times \frac{\mathbf{h}\cdot\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}\right)\cdot \mathbf{e}_3\right]\frac{\mathbf{f}\cdot\left(\boldsymbol{\omega}_\tau \times \mathbf{r}_\tau\right)}{\left|\mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right|}.
\end{aligned} \tag{10}$$

2.2 The global model of the anisotropic dry friction under combined kinematics

The dynamic interaction of the slightly deformed rigid body with the rough support plane is defined by the normal reaction $\mathbf{N}$, the resultant vector of tangent forces $\mathbf{T}$, the anti-rolling couple $\mathbf{M}_\tau$, and the dry friction torque $\mathbf{M}_\nu$. These quantities are obtained by integration of the normal contact stress (8) as well as the summary tangential stress (10) over the contact area S. Taking into account the dry friction anisotropy, we can represent the appropriate integral relationships formulated in [16] as follows:

$$\mathbf{N} = \int_S \sigma_\nu \mathbf{e}_3 dS = \int_S \sigma_0 \left[\mathbf{e}_3 + \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|}\right] dS; \tag{11}$$

$$\mathbf{M}_\tau = \int_S \sigma_\nu \mathbf{r}_\tau \times \mathbf{e}_3 dS = \int_S \sigma_0 \mathbf{r}_\tau \times \left[\mathbf{e}_3 + \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|}\right] dS; \tag{12}$$

$$\mathbf{T} = -\int_S \boldsymbol{\tau} dS = -\int_S \sigma_0 \left[1 + \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{f}\cdot\left(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau\times\mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right)}{\left|\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right|} dS; \tag{13}$$

$$\mathbf{M}_\nu = -\int_S \mathbf{r}_\tau \times \boldsymbol{\tau} dS = -\int_S \sigma_0 \left[1 + \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|}\cdot \mathbf{e}_3\right] \frac{\mathbf{r}_\tau \times \left[\mathbf{f}\cdot\left(\mathbf{v}_0 - R\boldsymbol{\omega}_\tau\times\mathbf{e}_3 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right)\right]}{\left|\mathbf{v}_0 + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau\right|} dS. \tag{14}$$

In the formula (11), the resultant force $\mathbf{N}_0$ of the static contact pressure σ_0 and it's variation $\mathbf{N}_1$ induced by the rolling effect can be written as follows:

$$\mathbf{N}_0 = N_0 \mathbf{e}_3, \quad N_0 = \int_S \sigma_0 dS; \tag{15}$$

$$\mathbf{N}_1 = \int_S \sigma_0 \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|} dS = N_1 \mathbf{e}_3, \quad N_1 = -\mathbf{S}_\sigma \cdot \mathbf{h} \cdot \frac{\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|} \tag{16}$$

We have the similar formulae for the anti-rolling couple where the “static” anti-rolling couple (that vanishes not in case of the rolling asymmetry of the body) and the “dynamic” one are defined by the formulae (17) and (18) respectively:

$$\mathbf{M}_\tau^0 = \int_S \sigma_0 \mathbf{r}_\tau \times \mathbf{e}_3 dS = \mathbf{S}_\sigma; \tag{17}$$

$$\mathbf{M}_\tau^1 = \int_S \sigma_0 \mathbf{r}_\tau \times \frac{\mathbf{r}_\tau \times \left(\mathbf{h}\cdot\boldsymbol{\omega}_\tau\right)}{|\boldsymbol{\omega}_\tau|} dS = -\mathbf{J}_\sigma \cdot \mathbf{h} \cdot \frac{\boldsymbol{\omega}_\tau}{|\boldsymbol{\omega}_\tau|}. \tag{18}$$

Here the first moment vector $\mathbf{S}_\sigma$ and the inertia moment tensor $\mathbf{J}_\sigma$ of the plane area of contact S with the static contact pressure distribution σ_0 are introduced:

$$\mathbf{S}_\sigma = \int_S \sigma_0 \mathbf{r}_\tau \times \mathbf{e}_3 dS = \int_S \sigma_0 \epsilon_{\alpha\beta} \xi^\beta \mathbf{e}^\alpha dS; \tag{19}$$

$$\mathbf{J}_\sigma = \int_S (\wedge \mathbf{r}_\tau) \otimes (\mathbf{r}_\tau \wedge) dS = \int_S \epsilon_{\alpha\beta} \xi^\beta \epsilon_{\gamma\delta} \xi^\delta \mathbf{e}^\alpha \otimes \mathbf{e}^\beta dS, \tag{20}$$

$$\epsilon_{\alpha\beta} \equiv \epsilon_{\alpha\beta 3} = (\mathbf{r}_\alpha \times \mathbf{r}_\beta) \cdot \mathbf{e}_3, \quad \alpha, \beta, \gamma, \delta = 1,2.$$

The homogeneity of the tensor (9) and the formulae (15) - (20) lead to the vanishing as well normal reaction $\mathbf{N}_1$ as the rolling initiation moment $\mathbf{M}_\tau^0$ in the frame $O\xi^1\xi^2$ attached to the center of the figure S, therefore we have the following formulae for the normal reaction and anti-rolling couple:

$$\mathbf{N} = N_0 \mathbf{e}_3; \quad \mathbf{M}_\tau = -\mathbf{J}_\sigma \cdot \mathbf{h} \cdot \left(\boldsymbol{\omega}_\tau / |\boldsymbol{\omega}_\tau| \right). \tag{21}$$

The resultant vector of the anisotropic dry friction (13) under combined kinematics can be now expressed through the following terms:

$$\mathbf{T} = \sum_{k=1}^{4} \mathbf{T}_k,$$

$$\mathbf{T}_1 = -\int_S \sigma_0 \frac{\mathbf{f} \cdot \mathbf{v}_S}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{22}$$

$$\mathbf{T}_2 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h} \cdot \boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} \cdot \mathbf{e}_3 \right] \frac{\mathbf{f} \cdot \mathbf{v}_S}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{23}$$

$$\mathbf{T}_3 = -\int_S \sigma_0 \frac{\mathbf{f} \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{24}$$

$$\mathbf{T}_4 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h} \cdot \boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} \cdot \mathbf{e}_3 \right] \frac{\mathbf{f} \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS, \tag{25}$$

Here (22) is the static dry friction force, (23) is the supplementary quasi-static dry friction force resulted by the rolling effect, (24) is the supplementary dry friction force due to the spin. and the term (25) denotes the variation of the dry friction force due to the coupling of the rolling and spinning of the body.

The torque of the anisotropic dry friction under combined kinematics (14) is also represented as a sum of four terms:

$$\mathbf{M}_\nu = \sum_{k=1}^{4} \mathbf{M}_k,$$

$$\mathbf{M}_1 = -\int_S \sigma_0 \frac{\mathbf{r}_\tau \times (\mathbf{f} \cdot \mathbf{v}_S)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{26}$$

$$\mathbf{M}_2 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h} \cdot \boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} \cdot \mathbf{e}_3 \right] \frac{\mathbf{r}_\tau \times (\mathbf{f} \cdot \mathbf{v}_S)}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{27}$$

$$\mathbf{M}_3 = -\int_S \sigma_0 \frac{\mathbf{r}_\tau \times [\mathbf{f} \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)]}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS; \tag{28}$$

$$\mathbf{M}_4 = -\int_S \sigma_0 \left[\frac{\mathbf{r}_\tau \times (\mathbf{h} \cdot \boldsymbol{\omega}_\tau)}{|\boldsymbol{\omega}_\tau|} \cdot \mathbf{e}_3 \right] \frac{\mathbf{r}_\tau \times [\mathbf{f} \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau)]}{|\mathbf{v}_S + \boldsymbol{\omega}_\nu \times \mathbf{r}_\tau|} dS, \tag{29}$$

Here (26) is the dry friction torque under the pure sliding that is resulted by the coupling of the slip and spin, the formula (27) defines the supplementary quasi-static dry friction torque resulted by the rolling effect on the slip, the "proper static" dry friction torque is defined by the formula (28) and it's variation due to the coupling of the rolling and spinning is given by the term (29). The vector $\mathbf{v}_S$ denotes the summary velocity in the point $M \in S$:

$$\mathbf{v}_S = \mathbf{v}_0 - R\boldsymbol{\omega}_\tau \times \mathbf{e}_3; \quad |\mathbf{v}_\Sigma|^2 = v_S^2 + 2\mathbf{v}_S \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau) + (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau) \cdot (\boldsymbol{\omega}_\nu \times \mathbf{r}_\tau). \tag{30}$$

Thus, the invariant formulation for the coupled dry friction theory is obtained. The constructed model describes the coupling effects under combined sliding, spinning, and rolling of the deformed rigid body with the finite area of contact with the support plane. The invariant relationships (11)-(14) can be rewritten in the coordinate form by the appropriate choice of the main frame.

3 APPROXIMATE MODEL OF THE ROLLING WHEEL

Let us consider the orthotropic dry friction given by the tensor

$$\mathbf{f} = f \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad f \neq 0, \quad \kappa \neq 0. \tag{31}$$

Here f and κf are the principal components of the tensor $\mathbf{f}$. Let us introduce the frame Ox^1x^2 attached to the centroid of the contact spot S; the corresponding base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are collinear to the principal directions of the tensor $\mathbf{f}$. The static contact pressure symmetry, $\sigma_0(x^1, x^2) = \sigma_0(\pm x^1, \pm x^2)$, as well as the rolling friction isotropy are assumed.

Let us consider the motion defined by the longitudinal velocity $\mathbf{v}_0 = V_0 \mathbf{e}'_1$ along the axis OX^1 of the global rest frame (for more details, see [16]), the rolling angular velocity $\boldsymbol{\omega}_\tau = -\omega_\tau \mathbf{e}_2$, and the spinning velocity ω_ν. Such a situation corresponds to the rolling of the wheel with the thread characterized by the friction factors f along the tread and κf across it. Let us also consider here only circular area of contact with the radius R.

The quantities (22)-(29) can now be simplified as it was implemented in [10, 11, 16, 17].

However the integral relationships are too complex to apply them to the analysis of dynamics of real systems while their linear-fractional approximations are adequately accurate and simple in the same time [7, 9-11, 16]. Let us sketch below only the main results; for more details concerning the construction of the approximate formulae, see [16, 17].

The resultant force vector can be represented as $\mathbf{T} = T_{\|}\mathbf{e}_1 + T_{\perp}\mathbf{e}_2$, so that $T_{\|}$ is the longitudinal and $T_{\perp}$ is the lateral friction force. It was shown [7-11, 16] that the last one is due to the coupling effects. As a result, we have the following formulae:

$$T_{\|} = F_0 \frac{v_s}{\sqrt{v_s^2 + au^2}}; \tag{32}$$

$$T_{\perp} = h\kappa F_0 \frac{uv_s}{\sqrt{\left(u^2 + bv_s^2\right)\left(v_s^2 + au^2\right)}}; \tag{33}$$

$$M_{\nu} = M_0 \frac{u}{\sqrt{u^2 + mv_s^2}}. \tag{34}$$

Here $u = \omega_{\nu} R$, $F_0 = fN_0$ is the longitudinal resting friction force, the resting friction torque can be computed as follows:

$$M_0 = \pi\left(1+\kappa\right) f \int_0^1 \sigma_0\left(r\right) r^2 dr. \tag{35}$$

Here the dimensionless radial coordinate is introcuced. For the factors a, b, m we have following formulae analogous to these published in [17].

4 ACCOUNTING FOR THE CONTACT PRESSURE DISTRIBUTION

The contact pressure distribution close to the real one as well as the contact spot diameter can be obtained on the basis of the finite element model of a tire; several levels of accuracy of the model can be considered. The first and simplest one consists in the numerical simulation of the quasi-static nonlinear deforming of a tire using the plane elasticity problem statement for the tire cross-section [15]. The contact pressure distribution computed for several levels of the vertical load or the radial deformation of the tire are then interpolated analytically.

The polynomial interpolation of the finite element solution σ_{0i} is constructed as follows:

$$\sigma_0\left(r\right) \approx \sigma_0^k p_k\left(r\right), \quad r \in \left[-1,1\right], \quad k = 0, 2 \ldots N \in \mathbb{N}, \tag{36}$$

where r is the dimensionless coordinate in the contact spot and $p_k\left(r\right)$ are Legendre polynomials. The contact pressure distribution is assumed to be symmetric, $\sigma_0(r) = \sigma_0(-r)$. The factors $\sigma_0^k \in \mathbb{R}$ are obtained from the solution of the quadratic programming problem:

$$G_{km}\sigma_0^k\sigma_0^m \to \min, \quad G_{km} = 2\delta_{km}/\left(2k+1\right) \tag{37}$$

with the following restrictions:

$$\sigma_0^k p_k(r_i) \le \sigma_{0i}, \quad \sigma_0^k p_k(1) = 0, \quad \sigma_0^k \int_{-1}^{1} p_k(r)\,dr = \sum_{i=2}^{n} (\sigma_{0i} + \sigma_{0i-1})(r_i + r_{i-1}), \tag{38}$$

where the first one corresponds to the lower approximation of the FEM solution, the second one corresponds to the condition $\sigma_0|_{\partial S} = 0$, and the third one guarantees the equivalence of the normal reaction following from the interpolation and the load applied to the FE model. The typical contact pressure and its approximation ($N = 10$) is shown on the Figure 1.

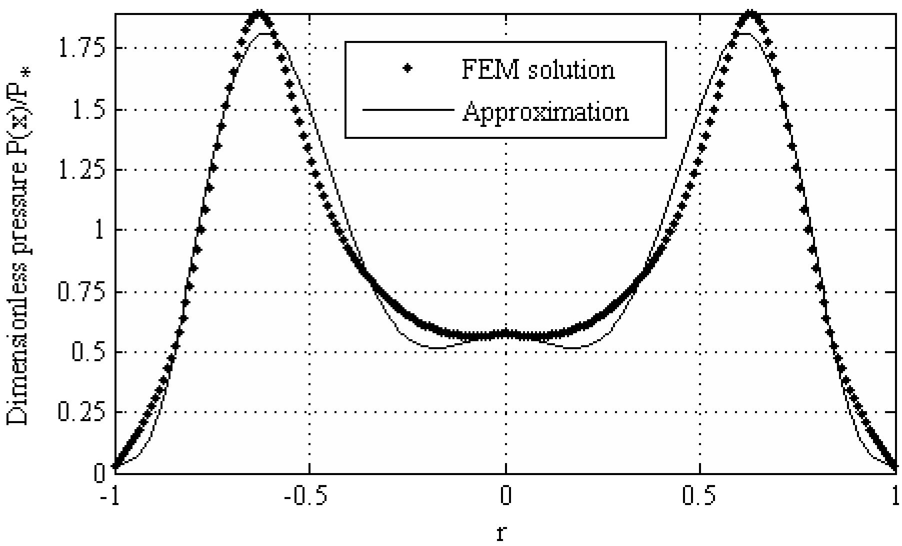

Figure 1: Contact pressure distributions for a typical tire [17]

The high level of vertical loads results the O-shaped area of contact where the contact pressure vanishes near the center of a spot [15]. In such a situation, the higher-order approximation may be required. On the other hand, the strongly deformed state of a tire leads to the accounting for the tangential deformation and the elastic forces in the pneumatics, therefore this model become combined with the Keldysh concept.

CONCLUSIONS

The theory of the coupled dry friction under combined kinematics is improved by accounting for the anisotropy of the dry friction coefficient represented in a form of the second-rank tensor. The general invariant formulation for the resultant vector of dry friction forces as well as for the dry friction couple and the rolling friction torque are obtained in case of the simultaneous sliding, spinning, and rolling of the rigid body. The contact pressure for a tire can be obtained from the numerical simulation on the groundwork of the finite element modeling, thus, the polynomial interpolation of such a solution is introduced, and the coefficients of the approximate model of the dry friction arte obtained. The constructed model being applied to the rolling wheel can be interpreted as the second approximation improving the model of the shimmy of quasi-rigid wheel.

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