Vector Calculus

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Preface

This problem book evolved from problem lectures I gave in UPC (Universitat Politècnica de Catalunya). There are two hundred problems completely solved in detail. Some are public domain, some are recreations and some others have been told by colleagues. If any merit can be given to the collection it could be the selection and grouping of the material, mainly following Polya's dictum that 'problems grow as mushrooms'.

These problems could interest students of mathematics, physics and engineering. As the title shows the main emphasis is on calculus and not so much on analysis. Nevertheless bibliography on some delicated questions is presented here and there. For a quick overview of the style I would like to mention some of my favourite problems: **3**, **6**, **14**, **54** and its companion **57**, **81**, the surprising result in problem **114**, **145**, **160**, **173**, **178**. Physics and engineering students may like to browse last chapter on electromagnetism.

I only assume a mild responsability as to possible errors because as late Prof. Wieszlaw Slenck said "correcting is an infinite non convergent process". The existence of errors can be seen as a stimulus for the student to be careful.

In any case if the reader wants to point out an error, he can do so writing to miquel.dv@gmail.com, but he needn't use this address only for that reason and if he liked some part(s) of the book he can show it through the same channel.

Needless to say that my English is IE (International English), not SE (Shakespeare English); I apologize about that.

Acknowledgments

The author would like to thank Profs. Jordi Saludes for the confidence, and Juan Jose Morales who took care of the course later on. Thanks as well to Prof. Natalia Sadovskaia who pointed out errors and contributed some problems.

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Some notations

As they are mainly standard we mention only a few:

$a := b$ defines a in terms of a known b.

$\boxed{\mathbb{T}}$:= Beginning of a theoretical section.

$\square$:= End of a theoretical section or of a problem.

iff:= if and only if

Matrix (m, n) :=m rows, n columns.

S^1 :=circumference with center at $\mathbf{0}$ and radius 1.
S^1_R :=circumference with center at $\mathbf{0}$ and radius R.

S^2 :=sphere with center at $\mathbf{0}$ and radius 1.
S^2_R :=sphere with center at $\mathbf{0}$ and radius R.

Contents

Chapter 1

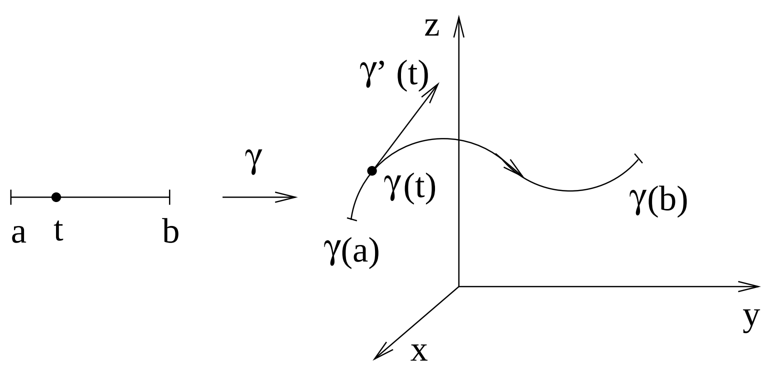
Curves

1.1 Parametrized curves

$\boxed{\mathbb{T}}$ A *parametrized curve* in $\mathbb{R}^n$ is a differentiable map

$$\boxed{\gamma : [a,b] \longrightarrow \mathbb{R}^n}$$

of class $\mathcal{C}^1$ (:= with continuous first derivative) at least.

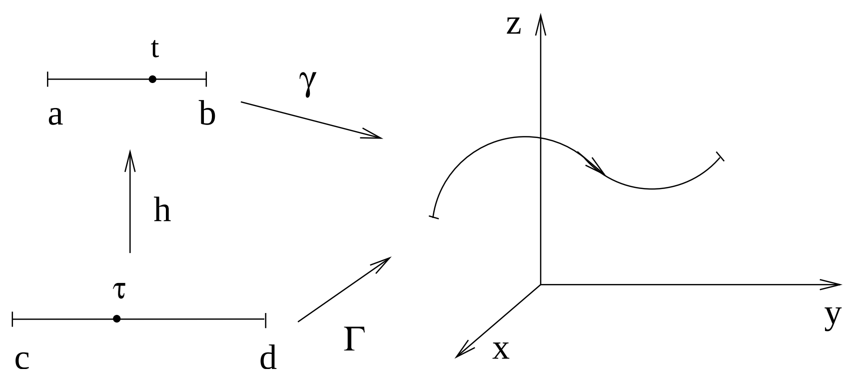


- The vector $\gamma'(t)$ is the *velocity vector* at the instant t and its norm $|\gamma'(t)|$ is called the *celerity*.
- An instant $t \in [a,b]$ is *regular* if $\gamma'(t) \neq 0$ (the tangent vector is not null), and it is *singular* if $\gamma'(t) = 0$.
  A *parametrization* is regular if all instants are regular. The velocity vector is then never null.

- Let $\gamma(t), t \in [a,b]$ be a parametrized curve in $\mathbb{R}^n$ and make the change of variable $t = h(\tau)$ to obtain another parametrization $\Gamma(\tau) = \gamma(h(\tau)), \tau \in [c,d]$. We call Γ a *reparametrization* of γ and we shall say that both parametrizations are *equivalent*. The function

$$h : [c,d] \quad \rightarrow \quad [a,b]$$

that does the change of variable must be bijective and both h and h^{-1} are to be of class $\mathcal{C}^1$ at least (another name for a change of variables is *diffeomorphism*).



As equivalent parametrizations do the same job we define the curve C to be the whole of equivalent parametrizations. The common trace of all those parametrizations is the *geometric curve* of C, a subset of $\mathbb{R}^n$.

We can use as well *piecewise* $\mathcal{C}^1$ parametrizations; those are continuous functions

$$\gamma : [a,b] \rightarrow \mathbb{R}^n$$

with a partition of $[a,b]$ in n subintervals $\{a = t_0 < t_1 < \cdots < t_n = b\}$ such that γ is $\mathcal{C}^1$ in each subinterval $[t_{i-1}, t_i], i = 1, \ldots, n$. A polygonal line may be a handy example (see problem 3 a), but see as well 3 b)).

□

Problem 1. Parametrizations.

Find parametrizations of the following paths:

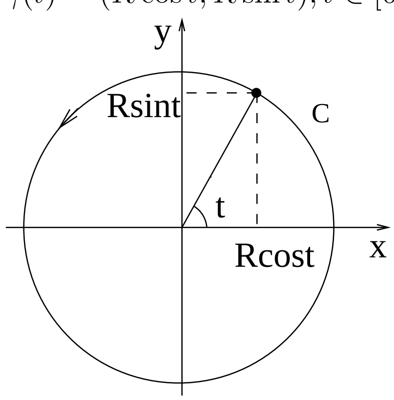
a) A complete wind in the positive sense (anti clockwise) around a circumference with center at $\mathbf{0}$ and radius R.

b) A spiral that turns in the positive sense and opens.

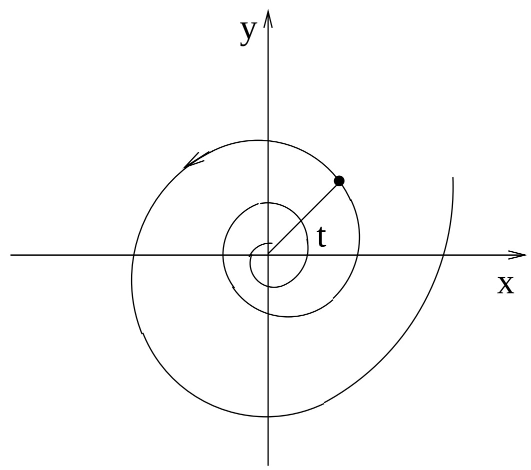
c) A complete turn of a helix (the composition of a uniform circular movement and a uniform translation movement perpendicular to the plane of the circular motion).

Solution:

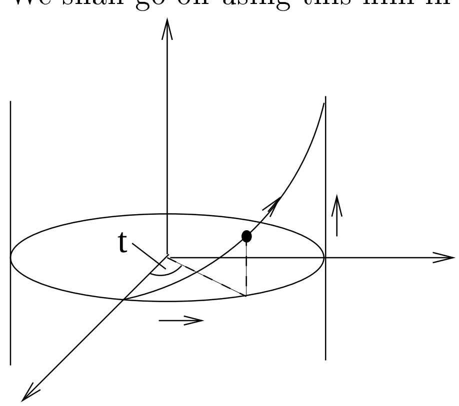
a) $\gamma(t) = (R\cos t, R\sin t), t \in [0, 2\pi]$.



b) $\gamma(t) = f(t)(\cos t, \sin t), t \in \mathbb{R}$ $f(t)$ being a function such that $\mid f(t) \mid$ increases. In this case the domain of the parameter is the whole of $\mathbb{R}$.



c) $\gamma(t) = (a\cos t, a\sin t, bt), a, b > 0, t \in [0, 2\pi]$. We can visualize precisely a helix turn if we draw the diagonal of a film tranparency and wrap it to have a cylinder.
We shall go on using this film in the study of the cycloid.



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Problem 2. Equivalent parametrizacions. Opposite curve.

Parametrize:

a) Two complete turns around the unit circumference in the positive sense.

b) One turn around the unit circumference in the negative sense.

c) Two complete turns around the unit circumference in the negative sense.

d) A particle goes from $(0,0)$ to $(1,1)$.

e) A particle goes along the segment $(0,0)$ to $(1,1)$, returns to $(0,0)$ along the same path, and goes to $(1,1)$ again following the same route. Find the obvious piecewise $\mathcal{C}^1$ parametrization and a not so obvious $\mathcal{C}^1$ parametrization as well.

Identify some equivalent parametrizations.

Solution:

a) We can double the time elapsed

$$\gamma_1(t) = (\cos t, \sin t), t \in [0, 4\pi],$$

or double the celerity:

$$\gamma_2(t) = (\cos 2t, \sin 2t), t \in [0, 2\pi]$$

Those parametrizations are equivalent because if we make in $\gamma_1(t)$ the substitution $t = 2u$ we obtain $\gamma_2(u)$. The diffeomorphism we have used is

$$\begin{array}{rccc} h: & [0, 2\pi] & \longrightarrow & [0, 4\pi] \\ & u & \mapsto & t = 2u \end{array}$$

On another hand none of these two parametrizations is equivalent to $\gamma(t) = (\cos t, \sin t)$ because γ is one to one in $(0, 2\pi)$ but γ_1 and γ_2 wind twice around the unit circle and are not one to one. Being a reparametrization bijective it doesn't change the injectivity, so γ_1 and γ_2 cannot be equivalent to γ.

b)

i) It is geometrically clear that $\gamma(t) = (\cos t, -\sin t), t \in [0, 2\pi]$ is a solution to our problem.

ii) In general to describe a given parametrized curve $\gamma(t), t \in [a, b]$ traversed in the opposite sense we put:

$$\gamma_1(t) = \gamma(-t), t \in [-b, -a]$$

which is a parametrized curve named the *opposite curve* of γ, notated γ^-. Applying this method to the present problem we have:

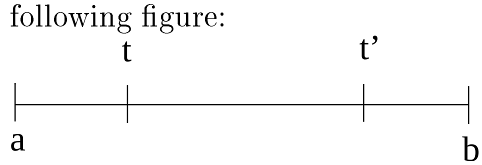
$$\gamma^-(t) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t), t \in [-2\pi, 0]$$

But $\cos t$ and $\sin t$ are 2π-periodic functions, so a parametrization equivalent to γ^- is

$$\Gamma^-(t) = (\cos t, -\sin t), t \in [0, 2\pi],$$

the parametrization we guessed at the start.

iii) Yet another way to obtain the opposite curve follows from the following figure:



where $t' = b - (t - a) = a + b - t$. We see that as t goes from a to b, t' goes from b to a and the opposite curve is:

$$\gamma^-(t) = \gamma(a + b - t), t \in [a, b]$$

Using this method we obtain

$$\gamma^-(t) = (\cos(2\pi - t), \sin(2\pi - t)) = (\cos t, -\sin t), t \in [0, 2\pi]$$

c) Using the parametrization γ_2 of a), the method of b) ii) and the 2π-periodicity we have:

$$\gamma_2^-(t) = (\cos(-2t), \sin(-2t)) = (\cos 2t, -\sin 2t), t \in [0, 2\pi]$$

We may as well use the parametrization γ_1 in a) :

$$\gamma_1^-(t) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t), t \in [0, 4\pi]$$

Or, if we prefer so, we can use the method in b) iii) and the parametrization γ_2, leading to:

$$\begin{aligned} \gamma_2^-(t) &= (\cos 2(2\pi - t), \sin 2(2\pi - t)) = \\ &= (\cos 2t, -\sin 2t), t \in [0, 2\pi] \end{aligned}$$

d) We can connect the points through a segment

$$\gamma_1(t) = (t,t), t \in [0,1],$$

an arc of a parabola

$$\gamma_2(t) = (t,t^2), t \in [0,1],$$

segments in the axis directions

$$\gamma_3(t) = \begin{cases} (t,0) & \text{if } 0 \le t \le 1 \\ (1,t-1) & \text{if } 1 \le t \le 2 \end{cases},$$

among an infinity of options.

e) It's easy to write a piecewise $\mathcal{C}^1$ parametrization. First

$$\gamma_1(t) = \begin{pmatrix} t \\ t \end{pmatrix}, t \in [0,1] \text{ parametrizes the segment } [\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

while, using the method of b) iii), we see that

$$\gamma_2(t) = \begin{pmatrix} 1-t \\ 1-t \end{pmatrix}, t \in [0,1] \text{ parametrizes the segment } [\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}]$$

Now define $\gamma_3 = \gamma_1$ and then, to have a single interval for the parameter, adapt γ_2 to the interval $[1,2]$ and γ_3 to the interval $[2,3]$:

$$\Gamma_2(t) = \begin{pmatrix} 2-t \\ 2-t \end{pmatrix}, t \in [1,2]$$

$$\Gamma_3(t) = \begin{pmatrix} t-2 \\ t-2 \end{pmatrix}, t \in [2,3]$$

Then

$$\gamma(t) = \begin{cases} (t,t) & \text{if } t \in [0,1] \\ (2-t,2-t) & \text{if } t \in [1,2] \\ (t-2,t-2) & \text{if } t \in [2,3] \end{cases}$$

is a parametrization of the whole path. It is not diferentiable at $t = 1,2$ but it is continuous in $[0,3]$ and of class $\mathcal{C}^1$ in each subinterval so it is

piecewise $\mathcal{C}^1$.
A $\mathcal{C}^1$ parametrization is

$$\gamma(t) = (\sin^2 t, \sin^2 t), t \in [0, 3\pi/2].$$

but the tangent vector $\gamma'(t) = (2\sin t\cos t, 2\sin t\cos t)$ vanishes for the values of the parameter $t = 0, \pi$ corresponding to the point $(0,0)$ and for the values $t = \pi/2, 3\pi/2$ corresponding to the point $(1,1)$. The velocity vector vanishes there and the $\mathcal{C}^1$ parametrization is not regular at those points.

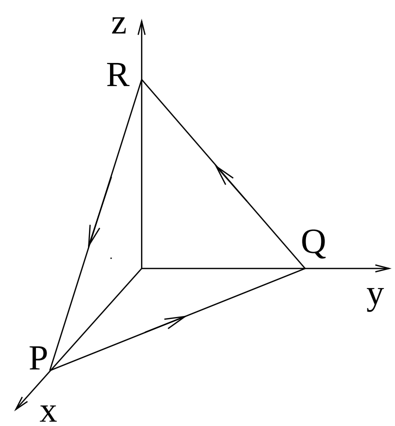
□

Problem 3: Differentiable parametrization of a path with sharp points.

a) Give a piecewise $\mathcal{C}^1$ parametrization of the triangle with vertices $P = (1,0,0), Q = (0,1,0), R = (0,0,1)$, traversed in the sense P, Q, R.

b) Obtain a $\mathcal{C}^1$ parametrization as well.

Solution:



a) As in the preceding problem we first parametrize the three segments PQ, QR, RP:

$$PQ : \gamma_1(t) = (1-t)\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1-t\\t\\0\end{pmatrix}, t \in [0,1]$$

$$QR : \gamma_2(t) = (1-t)\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1-\ t\\t\end{pmatrix}, t \in [0,1]$$

$$RP : \gamma_3(t) = (1-t)\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}t\\0\\1-t\end{pmatrix}, t \in [0,1]$$

We get a single interval reparametrizing γ_2 and γ_3 and keeping γ_1:

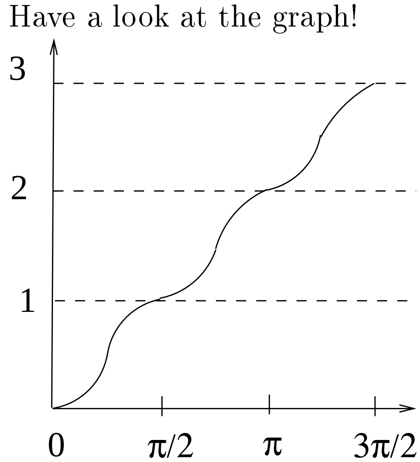
$$\gamma(u) = \begin{cases} (1-u, u, 0) & \text{if } 0 \le u \le 1 \\ (0, 2-u, u-1) & \text{if } 1 \le u \le 2 \\ (u-2, 0, 3-u) & \text{if } 2 \le u \le 3 \end{cases}$$

But this parametrization is not diferenciable at $u = 1, 2$. For instance $\gamma'_-(1) = (-1, 1, 0)$ and $\gamma'_+(1) = (0, -1, 1)$: left and right derivatives are different at $u = 1$; the same thing happens at $u = 2$. So the parametrization is only piecewise $\mathcal{C}^1$.

b) The useful idea to construct a $\mathcal{C}^1$ parametrization is to enter the sharp points with velocity zero (see the end of the preceding problem). The function

$$u(v) = \begin{cases} \sin^2 v & \text{if } 0 \le v \le \pi/2 \\ 1 + \sin^2(v - \pi/2) & \text{if } \pi/2 \le v \le \pi \\ 2 + \sin^2(v - \pi) & \text{if } \pi \le v \le 3\pi/2 \end{cases}$$

has vanishing lateral derivatives at $v_0 = 0, v_1 = \pi/2, v_2 = \pi, v_3 = 3\pi/2$. Have a look at the graph!



Then the parametrization of the given path

$$\Gamma(v) = \gamma(u(v)), v \in [0, 3\pi/2]$$

is of class $\mathcal{C}^1$ in $[0, 3\pi/2]$ because at v_0 and at v_3 we have

$$\Gamma'_-(\pi/2) = \gamma'_-(1) \cdot u'_-(\pi/2) = 0$$
$$\Gamma'_+(\pi/2) = \gamma'_+(1) \cdot u'_-(\pi/2) = 0$$

so Γ' is continuous at $v_1 = \pi/2$, corresponding to the point Q. Similarly one sees that Γ' is continuous at $v_2 = \pi$, corresponding to the point R. This parametrization is not equivalent to γ ($u(v)$ is *not* a change of variable because, despite being bijective, its derivative vanishes at four points) but it is a $\mathcal{C}^1$ parametrization of the given path.

□

Problem 4: Circumference.

Let S^1_R be the circumference with center $(0, 0)$ and radius R; parametrize:

a) S^1_R using an angular coordinate.

b) S^1_R as the graph of a function.

c) S^1_R projecting the axis Ox on S^1_R from the north pole.

Solution:

a) We know this one:

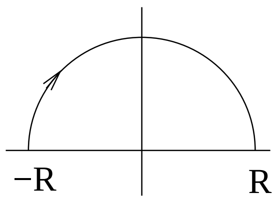
$$\gamma(\theta) = (R\cos\theta, R\sin\theta), \theta \in [0, 2\pi]$$

This parametrized curve winds in the positive sense once around S^1_R (the point $(R, 0)$ is accessed twice) and is of class C^∞.

b) Isolating y in the equation $x^2 + y^2 = R^2$ we obtain the function $y = f(x) = \sqrt{R^2 - x^2}$; a parametrization of the graph of f is

$$\gamma_1(x) = (x, \sqrt{R^2 - x^2}), x \in [-R, R]$$

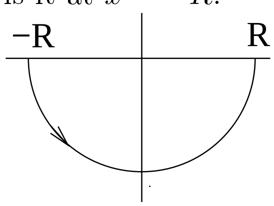
The image of this parametrized curve traverses once the upper semicircumference in the negative sense; it is not differentiable at $x = R$ nor is it at $x = -R$.



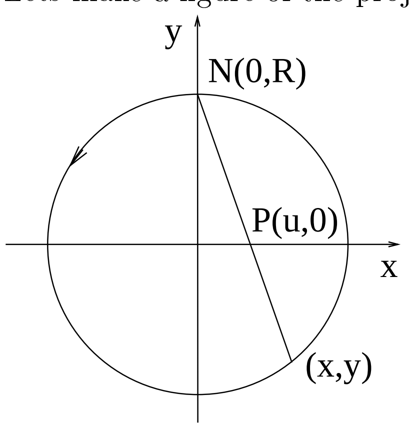
A parametrization of the lower semicircumference is

$$\gamma_2(x) = (x, -\sqrt{R^2 - x^2}), x \in [-R, R]$$

The image of this parametrized curve traverses once the lower semicircumference in the positive sense; it is not differentiable at $x = R$ nor is it at $x = -R$.



c) Lets make a figure of the projection



The equation of the line NP is:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ R \end{pmatrix} + t \begin{pmatrix} u \\ -R \end{pmatrix}$$

We compute the intersection point of NP with S^1_R :

$$\left.\begin{array}{ccc} X & = & tu \\ Y & = & R - tR \\ X^2 + Y^2 & = & R^2 \end{array}\right\} \Rightarrow t = \frac{2R^2}{u^2 + R^2} \Rightarrow \left\{\begin{array}{ccc} x & = & \frac{2R^2 u}{u^2+R^2} \\ y & = & \frac{R(u^2-R^2)}{u^2+R^2} \end{array}\right.$$

So the looked for parametrization is:

$$\gamma(u) = R(\frac{2uR}{u^2 + R^2}, \frac{u^2 - R^2}{u^2 + R^2}), u \in \mathbb{R}$$

The intersection point can also be computed from the equations $y = -\frac{R}{u}x + R$ and $x^2 + y^2 = R^2$:

$$x^2 + \frac{R^2}{u^2}x^2 - 2\frac{R^2}{u}x + R^2 = R^2$$

leading to the same solution, of course!
This parametrization winds once in the positive sense around S^1_R except the north pole; parameters with $\mid u \mid \leq R$ go to points in the lower semicircumference and those with $\mid u \mid \geq R$ go to points in the upper semicircumference (except N). In the usual case of the unit circumference S^1 the parametrization is:

$$\gamma(u) = (\frac{2u}{u^2 + 1}, \frac{u^2 - 1}{u^2 + 1})u \in \mathbb{R}$$

Similar parametrizations are obtained projecting the Ox axis from the south pole $(0, -R)$. Or we can project the Oy axis from $(R, 0)$ or from $(-R, 0)$. It is also possible to project from the north pole a straight line through the south pole, etc.

□

Problem 5: Ellipse.

Parametrize the ellipse $E = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0\}$

a) Using an angular coordinate.

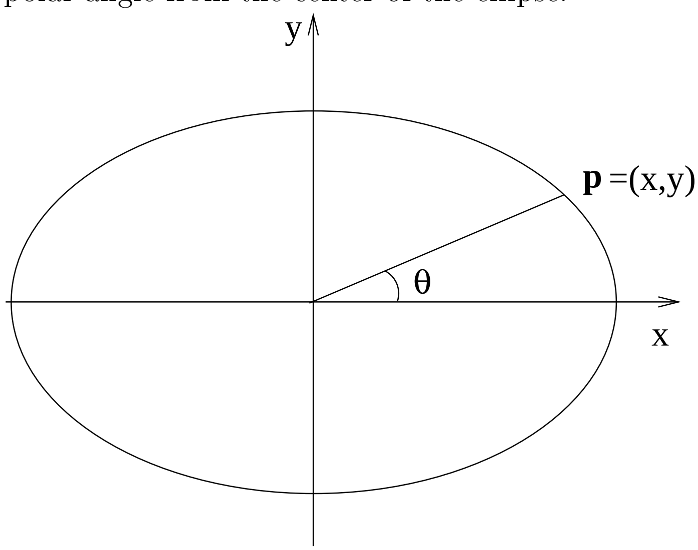
b) Projecting a coordinate axis on E from a vertex of the ellipse.

Solution:

a) By analogy with the circumference we may suspect that

$$\gamma(\theta) = (a\cos\theta, b\sin\theta), \theta \in [0, 2\pi]$$

could be the sought for parametrization of E. If we substitute $x = a\cos\theta, y = b\sin\theta$ into the equation defining E we obtain the identity $1 = 1$ showing that $\gamma(\theta) \in E, \forall\theta \in [0, 2\pi]$, so $\gamma(\theta)$ is on E. The point is whether we traverse the *whole* ellipse or not; to answer this question let us study the geometrical meaning of θ. It cannot possibly be the polar angle from the center of the ellipse:



because then

$$x = |\mathbf{p}|\cos\theta, y = |\mathbf{p}|\sin\theta$$

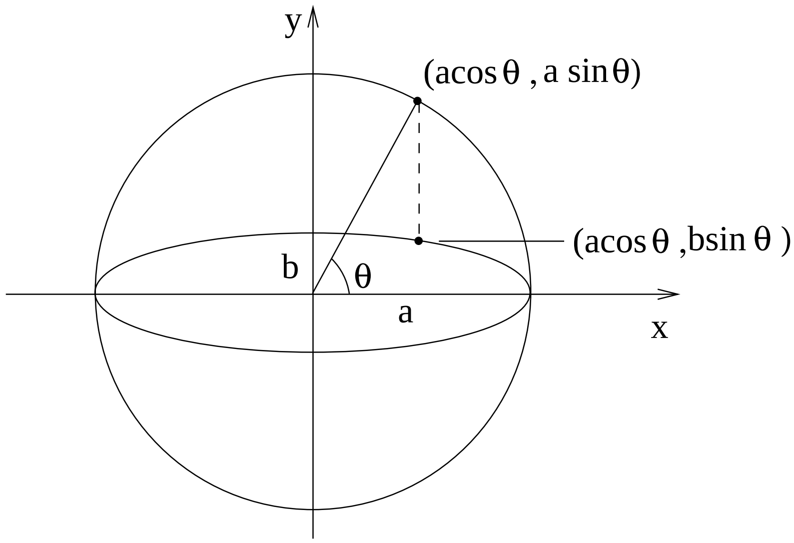
and substituting into the equation of E we have

$$\frac{|\mathbf{p}|^2}{a^2}\cos^2\theta + \frac{|\mathbf{p}|^2}{b^2}\sin^2\theta = 1 \Rightarrow |\mathbf{p}| = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

leading to the parametrization

$$\varphi(\theta) = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}(\cos\theta, \sin\theta),$$

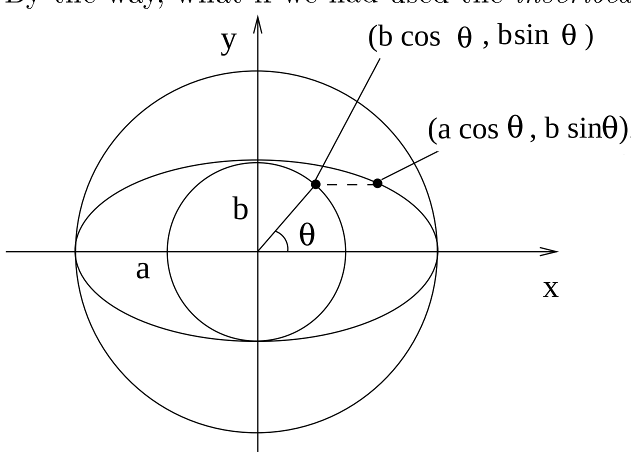
which is not our favourite one. Nevertheless the analogy with the circumference strongly favours θ being a polar angle from the origin; the conclusion is that θ is the polar angle of points *not* on the ellipse. Let's try with points of the circumscribed circumference; the following picture may emerge:



Those points are $(a\cos\theta, a\sin\theta), \theta \in [0, 2\pi]$; the first coordinate of the projection on E is $a\cos\theta$ and we compute the second coordinate using E's equation:

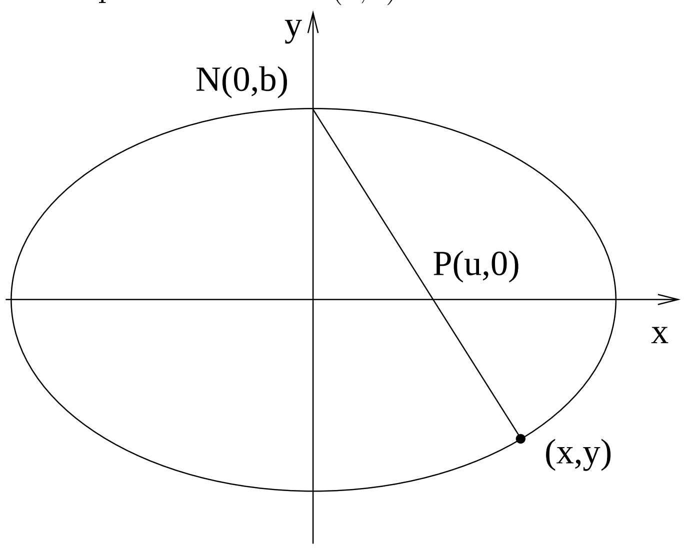
$$\frac{a^2\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1$$

Then $y^2 = b^2\sin^2\theta \Rightarrow \mid y \mid = b \mid \sin\theta \mid$ and taking $y = b\sin\theta$ all signs are right. This is the geometric sense of θ and we now see that our parametrization covers the whole of E in a 'time-interval' of 2π.
By the way, what if we had used the *inscribed* circumference?



If we project horizontally, the point $(b\cos\theta, b\sin\theta)$ of the inscribed circumference goes onto the point $(a\cos\theta, b\sin\theta)$ on the ellipse and we have another geometric construction of the same parametrization.

b) Lets imitate c) of the preceding problem and project the Ox axis on the ellipse E from $N = (0, b)$:



The straight line through $N = (0, b)$ and $P = (u, 0)$ cuts E at

$$\left.\begin{array}{rcl} X & = & tu \\ Y & = & (1-t)b \\ \frac{x^2}{a^2}+\frac{y^2}{b^2} & = & 1 \end{array}\right\} \Rightarrow t = \frac{2a^2}{u^2+a^2} \Rightarrow \left\{\begin{array}{rcl} x & = & \frac{2ua^2}{u^2+a^2} \\ y & = & \frac{b(u^2-a^2)}{u^2+a^2} \end{array}\right.$$

and we obtain the parametrization of the ellipse

$$\gamma(u) = (\frac{2ua^2}{u^2+a^2}, \frac{b(u^2-a^2)}{u^2+a^2}), u \in \mathbb{R}$$

We can compare with the corresponding parametrization of the circumference:

$$\gamma(u) = (\frac{2uR^2}{u^2+R^2}, \frac{R(u^2-R^2)}{u^2+R^2}), u \in \mathbb{R}$$

□

Problem 6: Moving along the ellipse $\gamma(\theta) = (a\cos\theta, b\sin\theta)$.

The point $(a\cos\theta, a\sin\theta)$ moves uniformly along the circumference S_a^1 but the corresponding point on the ellipse $(a\cos\theta, b\sin\theta)$ moves with a nonconstant celerity $|\gamma'(\theta)| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$.

a) At what points of the ellipse has the celerity its maximums (minimums)?

b) Is the angular velocity constant along the ellipse ?

c) Is the areal velocity constant?

Solution:

This problem has such an astronomical flavour that it is sound thinking about θ as time running.

a) We find the points of extremum of a function equating to 0 its derivative. The celerity is a differentiable function except when $|\gamma'(\theta)| = 0 \Leftrightarrow a^2\sin^2\theta + b^2\cos^2\theta = 0 \Leftrightarrow \sin\theta = \cos\theta = 0$ but this is impossible. So $|\gamma'(\theta)| > 0$, the celerity is a differentiable function and to find the extremums we write

$$\frac{d}{d\theta}|\gamma'(\theta)| = \frac{1}{|\gamma'(\theta)|}(a^2 - b^2)\sin\theta\cos\theta = 0$$

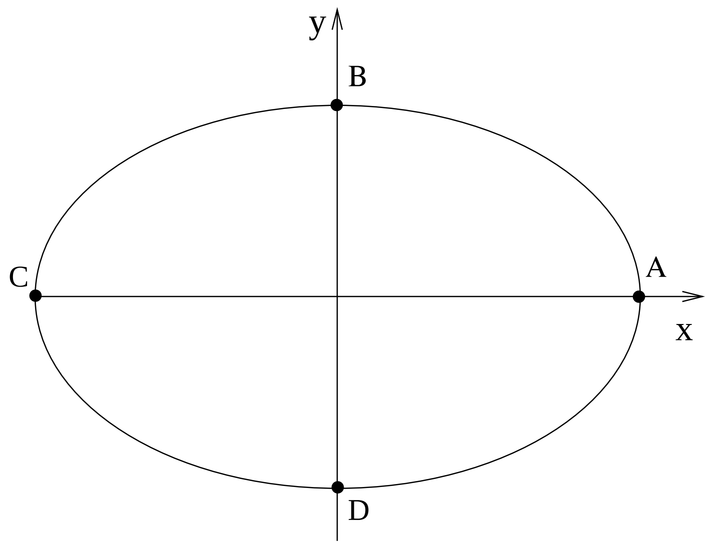
Assuming $a \neq b$ this is equivalent to

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = 0$$

with solutions in $[0, 2\pi)$

$$\theta = 0, \pi/2, \pi, 3\pi/2$$

that correspond to the vertices of the ellipse, A, B, C, D respectively:



Assuming $a > b$ the sign of the derivative is the same as that of $\sin 2\theta$; we see that $|\gamma'(\theta)|$ increases in $(0, \pi/2)$ and decreases in $(\pi/2, \pi)$. We conclude that the celerity has a maximum at $\theta = \pi/2$, that is at the point B. Similarly we can see it has another maximum at D and minimums at A, C.

b) The angle φ the radius vector of the point $(a\cos\theta, b\sin\theta)$ makes with the OX axis is:

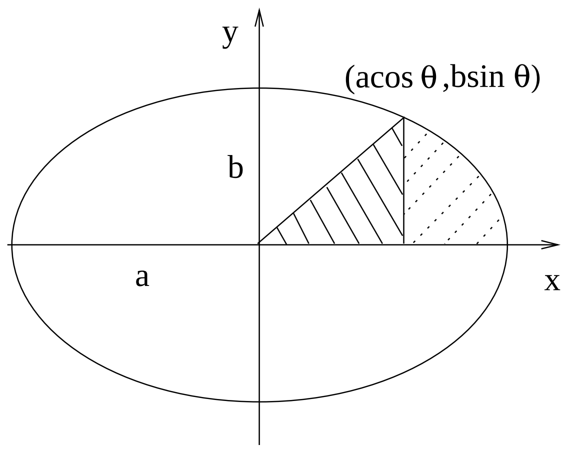
$$\tan\varphi = \frac{b\sin\theta}{a\cos\theta} \Leftrightarrow \varphi = \arctan(\frac{b}{a}\tan\theta)$$

(we must exclude the values of the parameter $\theta = \pi/2, 3\pi/2$). The angular velocity is the derivative of this angle

$$\omega = \frac{d\varphi}{d\theta} = \frac{a}{a^2\cos^2\theta + b^2\sin^2\theta}$$

which is not a constant.

c) Lets first compute $A(\theta)$, the area of the region swept by the radius vector of the point $(a\cos\theta, b\sin\theta)$ when the parameter varies between 0 and θ:



The upper semiellipse is the graph of the function

$$y(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

So the area seeked is

$$A(\theta) = \frac{ab\sin\theta\cos\theta}{2} + \int_{a\cos\theta}^{a} b\sqrt{1 - \frac{x^2}{a^2}}dx$$

To evaluate the integral we make the change of variable

$$\left\{\begin{array}{rcll} x &=& a\cos u & \\ dx &=& -a\sin u\,du & \\ x &=& a\cos\theta, & u=\theta \\ x &=& a, & u=0 \end{array}\right\}$$

For $\theta \in [0,\pi]$ we have

$$\begin{aligned} \int_{a\cos\theta}^{a} b\sqrt{1 - \frac{x^2}{a^2}}dx &= \int_{\theta}^{0} b\sqrt{1-\cos^2 u}(-a\sin u)\,du = \\ &= ab\int_0^{\theta}\sin^2 u\,du = ab\int_0^{\theta}\frac{1-\cos 2u}{2}\,du = \\ &= ab(\frac{\theta}{2} - \frac{\sin 2\theta}{4}) \end{aligned}$$

the area is

$$A(\theta) = \frac{ab\sin\theta\cos\theta}{2} + ab(\frac{\theta}{2} - \frac{\sin 2\theta}{4}) = ab\frac{\theta}{2}$$

and the areal velocity is

$$\frac{d}{d\theta}A(\theta) = \frac{ab}{2},$$

a constant. Notice the radius vector emanates from the origin; should it emanate from a focus, a similar calculation shows that

$$\begin{aligned} A(\theta) &= ab(\frac{\theta}{2} - \frac{\sin 2\theta}{4}) + \frac{(a\cos\theta - \sqrt{a^2-b^2})b\sin\theta}{2} = \\ &= ab\frac{\theta}{2} - \frac{b\sin\theta\sqrt{a^2-b^2}}{2} \end{aligned}$$

and $\frac{d}{d\theta}A(\theta)$ is not constant.
Incidentally this shows that this parametrization does not describe planetary motion, since the second Kepler's law stipulates that the areal velocity is a constant (if the radius vector is taken from a focus, where the Sun is).

□

Problem 7: Hyperbola.

We want to parametrize the hyperbola $H = \{(x, y) : x^2 - y^2 = 1\}$.

a) Prove that for every point in the right branch, $(x, y) \in H_+ = H \cap \{x > 0\}$, there is one and only one $u \in \mathbb{R}$ such that

$$x = \cosh u, y = \sinh u$$

(Notice the close analogy with the circumference.)

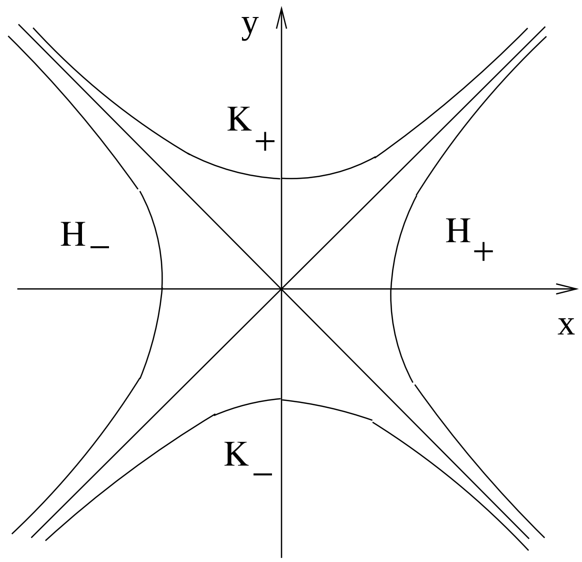
b) Parametrize $H_- = H \cap \{x < 0\}$, the left branch of the hyperbola.

c) Parametrize $K_+ = \{(x, y) : y^2 - x^2 = 1, y > 0\}$, the upper branch of the hyperbola $y^2 - x^2 = 1$.

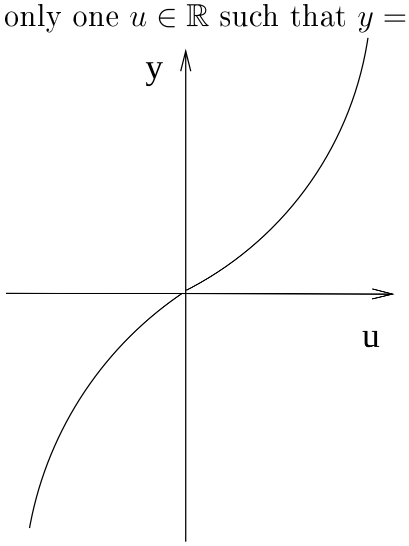
d) Parametrize $K_- = \{(x, y) : y^2 - x^2 = 1, y < 0\}$, the lower branch of the preceding hyperbola.

e) Let R be the region swept by the radius vector from the origin to the point $\gamma(u) = (\cosh u, \sinh u), u \in [0, a]$. Show that the area of R is $a/2$.

Solution:



a) Reminding the graph of $y = \sinh u$ we see that for every $y \in \mathbb{R}$ there is only one $u \in \mathbb{R}$ such that $y = \sinh u$:



The x coordinate of $P \in H_+$ is:

$x^2 - y^2 = 1 \Leftrightarrow x^2 - \sinh^2 u = 1 \Leftrightarrow x^2 = 1 + \sinh^2 u = \cosh^2 u \Leftrightarrow |x| = \cosh u$

But $x > 0$ in $H_+$and then $x = |x| = \cosh u$; we have the parametrization of the right branch of the hyperbola:

$$\gamma(u) = (\cosh u, \sinh u), u \in \mathbb{R}$$

b) If $P \in H_-$ then $x < 0$ and $x = -|x| = -\cosh u$ and a parametrization of the left branch of the hyperbola is

$$\gamma_1(u) = (-\cosh u, \sinh u), u \in \mathbb{R}$$

c) Here for every $x \in \mathbb{R}$ there is a unique $u \in \mathbb{R}$ such that $x = \sinh u$ and the y coordinate of a point of K_+ must satisfy

$$y^2 - x^2 = 1 \Leftrightarrow y^2 = 1 + \sinh^2 u = \cosh^2 u \Leftrightarrow |y| = \cosh u$$

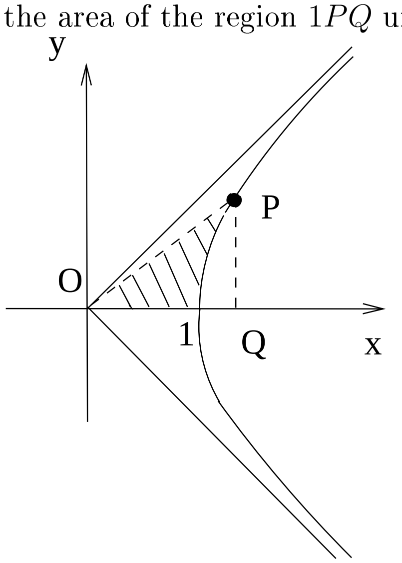
and a parametrization of the upper branch of the hyperbola is

$$\Gamma(u) = (\sinh u, \cosh u), u \in \mathbb{R}$$

d) Finally it is clear that a parametrization of K_- is

$$\Gamma_1(u) = (\sinh u, -\cosh u), u \in \mathbb{R}$$

e) It suffices to subtract from $\frac{1}{2}\cosh u \sinh u$, the area of the triangle OPQ, the area of the region $1PQ$ under the hyperbola



the area of the region $1PQ$ is

$$\begin{aligned}\int_1^{\cosh u} \sqrt{x^2-1}\,dx &= \{x = \cosh u\} = \int_0^u \sqrt{\cosh^2 u - 1}\sinh u du = \\ &= \int_0^u \sinh^2 u du = (\frac{\sinh 2u}{4} - \frac{u}{2})\,|_0^u = \frac{\sinh 2u}{4} - \frac{u}{2} \\ &=\end{aligned}$$

and the sought for area is

$$\text{A} \ = \frac{1}{2}\cosh u \sinh u - (\frac{\sinh 2u}{4} - \frac{u}{2}) = \frac{u}{2}$$

□

Problem 8: A method.

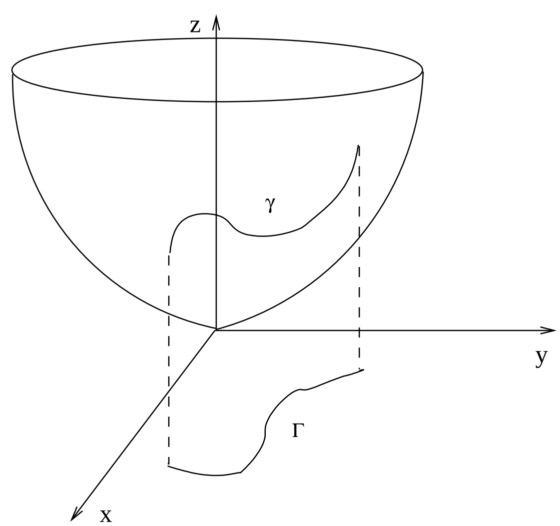
a) The orthogonal projection on the plane $z = 0$ of a parametrized curve $\gamma(t)$ contained in the paraboloid $z = x^2+y^2$ is $\Gamma(t) = (e^{-t}\cos t, e^{-t}\sin t, 0)$; find γ.

b) Point a) suggests that if we are able to parametrize the projection on the plane $z = 0$, then we can 'climb' and parametrize the curve. Parametrize the intersection of the sphere

$$S_{\sqrt{2}} = \{(x, y, z) : x^2 + y^2 + z^2 \quad \} = \quad 2$$

and the plane $P = \{(x, y, z) : x + y + z = 0\}$.

Solution:



a) Being $z = x^2 + y^2$ we must have

$$\gamma(t) = (e^{-t}\cos t, e^{-t}\sin t, e^{-2t}(\cos^2 t + \sin^2 t)) = (e^{-t}\cos t, e^{-t}\sin t, e^{-2t})$$

b) Eliminating z from the system

$$\left.\begin{array}{rcl} x^2+y^2+z^2 & = & 2 \\ x+y+z & = & 0 \end{array}\right\}$$

we have the *projecting cylinder*, that contains the intersection curve:

$$x^2+y^2+(-x-y)^2=2$$

that is

$$x^2+y^2+xy=1$$

The intersection of this cylinder with $z=0$ is the projection of the curve, a conic we want to parametrize. Completing squares

$$(x+\frac{1}{2}y)^2+\frac{3}{4}y^2=1$$

and the change of variables

$$\left\{\begin{array}{rcl} u & = & x+\frac{1}{2}y \\ v & = & y \end{array}\right\}$$

shows the ellipse

$$u^2+\frac{v^2}{(\frac{2}{\sqrt{3}})^2}=1$$

that we know how to parametrize:

$$\varphi(t)=(u(t),v(t))=(\cos t,\frac{2}{\sqrt{3}}\sin t),\ t\in[0,2\pi]$$

In the (x,y) coordinates we have

$$\psi(t)=(x(t),y(t))=(\cos t-\frac{1}{\sqrt{3}}\sin t,\frac{2}{\sqrt{3}}\sin t),\ t\in[0,2\pi]$$

and climbing to the plane we obtain finally

$$\gamma(t)=(\cos t-\frac{1}{\sqrt{3}}\sin t,\frac{2}{\sqrt{3}}\sin t,-\cos t-\frac{1}{\sqrt{3}}\sin t)$$

□

Problem 9: Minimum distance.

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parametrized curve that doesn't pass through the origin. If $\gamma(t_0)$ is the point nearest the origin (does it exist? is it unique?) and we assume $\gamma'(t_0) \neq 0$, prove that $\gamma(t_0)$ and $\gamma'(t_0)$ are perpendicular. Can we generalize this result? If the curve is on a sphere centered at the origin, is the result still valid? Is the reciproque correct in this case?

Solution:

The continuous function d on the compact set $[a, b]$

$$\begin{array}{rccl} d: & [a,b] & \to & \mathbb{R}_+ \\ & t & \mapsto & d(\gamma(t), \mathbf{0}) \end{array}$$

accesses the absolute extremums. This proves the existence of a point at a minimum distance from the origin. A circumference centered at the origin has all its points at a minimum distance (and a maximum distance as well), showing that the extremum points need not be unique.

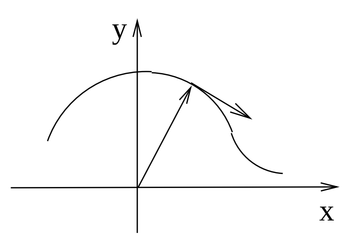
If t_0 is a value of the parameter giving the absolute minimum of d and, moreover, it is an *interior* point of $[a, b]$, we have:

$$d'(t_0) = 0$$

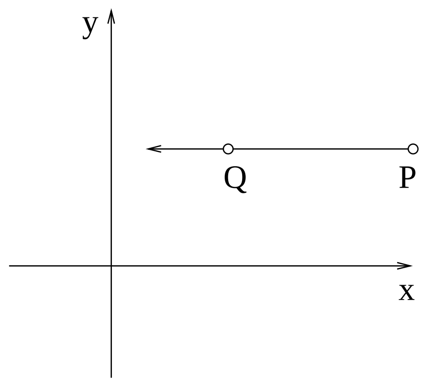
and then

$$d^2(t) = \gamma(t) \cdot \gamma(t) \Rightarrow 0 = 2d(t_0)d'(t_0) = 2\gamma(t_0) \cdot \gamma'(t_0),$$

shows the orthogonality of $\gamma(t_0)$ and $\gamma'(t_0)$ at the minimum.



If the absolute minimum is accessed at $t_0 = a$ or at $t_0 = b$ the orthogonality may fail:



The result is true for each *local* extremum of d corresponding to parameter values *interior* to $[a,b]$. For instance, for an ellipse centered at $\mathbf{0}$ the radius vector is orthogonal to the ellipse in the vertices.

If a curve C is on a sphere centered at the origin, each point of C is a local minimum (and maximum) of d and we have

$$\gamma(t)\perp\gamma'(t),\ \forall t\in[a,b]$$

Reciprocally if $\gamma(t)$ is a parametrized curve such that $\gamma(t)\neq 0,\ \gamma(t)\perp\gamma'(t),\ \forall t\in [a,b]$, we have

$$\frac{d}{dt}(d^2(\gamma(t),\mathbf{0})) = 2\gamma(t)\cdot\gamma'(t) = 0$$

showing that the distance to the origin is constant and that γ is on a sphere.

Note: There is a more geometrical way to prove the orthogonality of the radius vector and the tangent vector, using the pattern of the tangent level curve. The level curves of the distance function are circumferences centered at $\mathbf{0}$. The extremums of the distance to the origin along the curve, are found where the curve is tangent to the level curve. Then the tangent vector to the curve is as well tangent to the circumference, whence the result (see Polya).

Analitically the pattern translates to the simplest case of Lagrange's undetermined multiplier method.

□

Problem 10:

Prove that the trace of the parametrized curve, expressed in polar coordinates

$$r(\theta) = \frac{9}{5-4\cos\theta},\ \theta\in[0,2\pi]$$

is an ellipse and obtain the cartesian equation.

Solution:

The traversed path is contained in an ellipse:

$$\begin{array}{rcl} 5r - 4r\cos\theta & = & 9 \\ 5r & = & 9 + 4x \\ 25r^2 & = & 81 + 72x + 16x^2 \\ 25(x^2 + y^2) & = & 81 + 72x + 16x^2 \\ \dots & \dots & \dots \\ \frac{(x-4)^2}{5^2} & + \frac{y^2}{3^2} = & 1, \end{array}$$

the ellipse with semiaxes $5, 3$ and center at $(4, 0)$; when $\theta \in [0, 2\pi]$ we traverse the whole curve.

□

1.2 Cycloids

Cycloids are curves with remarkable properties; we shall meet some of them in the following problems.

Problem 11: Etymology.

Fix a point of light P to the tyre of a bycicle and look in the darkness the trajectory of P, assuming the wheel turns without sliding on a flat ground. The path followed by P is properly named a *cycloid*.

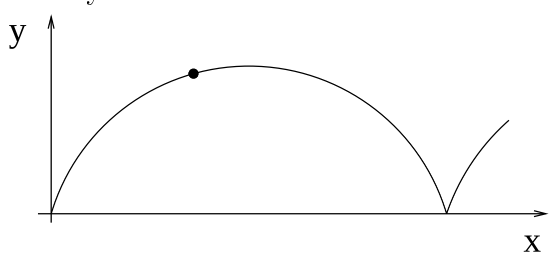
a) Make a sketch of the cycloid.

b) Parametrize it taking the ground as Ox axis.

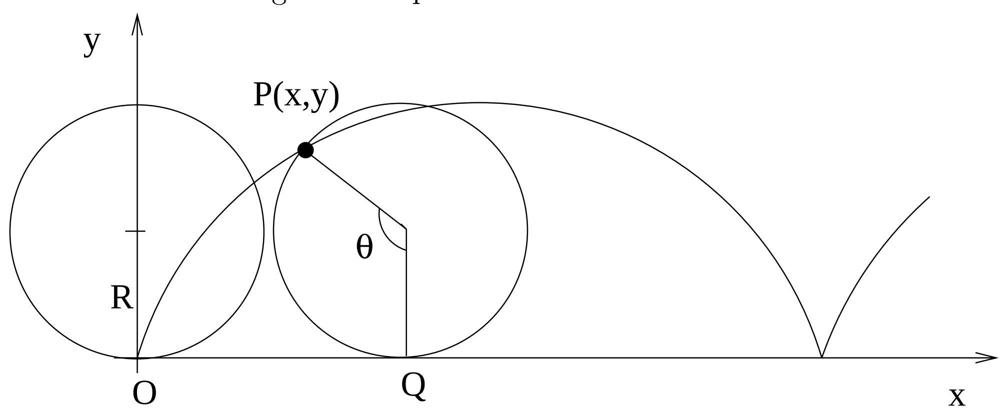
c) Is the parametrization differentiable, regular ?

Solution:

a) The cycloid looks like this



b) Take the turned angle as the parameter:



As there is no sliding $OQ = R\theta$ and the coordinates of P are:

$$\gamma(\theta) = \begin{cases} x(\theta) & = & R(\theta - \sin\theta) \\ y(\theta) & = & R(1 - \cos\theta) \end{cases}, \theta \in \mathbb{R}$$

c) Where the wheel ends a complete turn we have a sharp point; indeed the direction of the tangent vector $\gamma'(\theta) = R(1 - \cos\theta, \sin\theta)$ satisfyes

$$\lim_{\theta \to 2\pi^-} \frac{\sin\theta}{1 - \cos\theta} = \tfrac{0}{0}$$
$$\lim_{\theta \to 2\pi^+} \frac{\sin\theta}{1 - \cos\theta} = \tfrac{0}{0}$$

Using l'Hôpital's rule we have respectively

$$\lim_{\theta \to 2\pi^-} \frac{\cos\theta}{\sin\theta} = -\infty$$
$$\lim_{\theta \to 2\pi^+} \frac{\cos\theta}{\sin\theta} = +\infty$$

and the direction changes abruptly, in a discontinuous form. Nevertheless the derivative exists:

$$\gamma'(2\pi) = (0,0)$$

and, as we have seen in preceding problems, it is precisely the vanishing of the derivative what makes the differentiability possible; of course the parametrization is not regular at the values of the parameter $\theta = n2\pi$, $n \in \mathbb{Z}$.

□

Problem 12: Parametrization of the cycloid by arc-length.

a) Compute the length of the arc of a cycloid corresponding to a complete turn of the generating wheel.

b) Reparametrize by arc-length.

Solution:

a) The formula for the arc-length of a parametrized curve is $L = \int_a^b |\gamma'(t)|dt$ (see p.92). Let's apply it:

$$\gamma'(\theta) = R(1 - \cos\theta, \sin\theta)$$

$$|\gamma'(\theta)| = R\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = R\sqrt{2}\sqrt{1-\cos\theta}$$

$$L = R\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = \{1-\cos\theta = 2\sin^2\frac{\theta}{2}\} =$$

$$= R\sqrt{2}\int_0^{2\pi}\sqrt{2}|\sin\frac{\theta}{2}|d\theta = 2R\int_0^{2\pi}\sin\frac{\theta}{2}d\theta = 8R$$

b) For $\theta \in [0, 2\pi]$ the arc-length parameter is

$$s(\theta) = \int_0^\theta |\gamma'(t)|dt = 2R\int_0^\theta \sin\frac{t}{2}dt = 4R(1-\cos\frac{\theta}{2})$$

$$\theta = 2\arccos(1 - \frac{s}{4R})$$

and the reparametrization is:

$$x(s) = R(2\arccos(1 - \frac{s}{4R}) - \sin(2\arccos(1 - \frac{s}{4R}))) =$$

$$= 2R\arccos(1 - \frac{s}{4R}) - 2R\sin(\arccos(1 - \frac{s}{4R}))\cos(\arccos(1 - \frac{s}{4R}))) =$$

$$= 2R\arccos(1 - \frac{s}{4R}) - 2R(1 - \frac{s}{4R})\sqrt{1 - (1 - \frac{s}{4R})^2}$$

The other coordinate is:

$$y(s) = R(1 - \cos(2\arccos(1 - \frac{s}{4R}))) =$$

$$= R(1 - \cos^2(\arccos(1 - \frac{s}{4R})) + \sin^2(\arccos(1 - \frac{s}{4R}))) =$$

$$= R(1 - (1 - \frac{s}{4R})^2 + 1 - (1 - \frac{s}{4R})^2) = 2R(1 - (1 - \frac{s}{4R})^2) =$$

$$= s - \frac{s^2}{8R}$$

□

Problem 13: Tautochronous property of cycloids (tauto=equal, chronos=time).

Consider in a vertical plane cartesian axes with Oy oriented down and the arc of cycloid

$$\gamma(\theta) = (\theta - \sin\theta, 1 - \cos\theta), 0 \leq \theta \leq 2\pi$$

Prove that a ball left alone with zero velocity from any point of the cycloid under the action of gravity, arrives at the downmost point in a time independent of the starting place.

Hint: Use energy conservation to show that the celerity after a fall of *deepness* h is $\sqrt{2gh}$.

Solution:

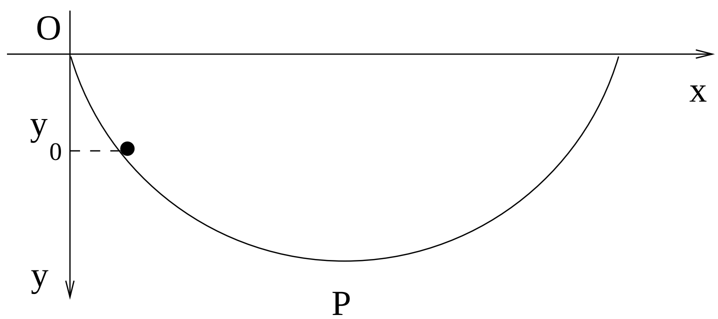
If a mass falls to ground starting from a height H the total energy is then $E_T = mgH$. When it has fallen a distance h the potential energy will be

$mg(H-h)$ and the kinetic energy will be $\frac{1}{2}mv^2$. From the principle of energy conservation we obtain

$$mgH = mg(H-h) + \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gh}$$

Notice that this result is independent of the path followed by the mass.

Now assume a ball starts with null velocity from a point of the cycloid of coordinate y_0.



The celerity at a point with coordinate y is:

$$v = \frac{ds}{dt} = \sqrt{2g}\sqrt{y-y_0} \Rightarrow \frac{dt}{ds} = \frac{1}{\sqrt{2g}\sqrt{y-y_0}}$$

We separate the variables

$$dt = \frac{1}{\sqrt{2g}\sqrt{y-y_0}} ds$$

and integrate respect to t with limits 0 and $T(y_0)$ (the time of arrival at P) and respect to s with limits s_0 (the arc-length of y_0) and 4 (the arc-length of P):

$$T(y_0) = \int_0^{T(y_0)} dt = \int_{s_0}^4 \frac{1}{\sqrt{2g}\sqrt{y-y_0}} ds$$

The expressions of y, y_0 in terms of the arc-length are:

$$y = s - \frac{s^2}{8} = -\frac{1}{8}(s^2 - 8s) = -\frac{1}{8}((s-4)^2 - 16)$$

$$y_0 = s_0 - \frac{s_0^2}{8} = -\frac{1}{8}((s_0-4)^2 - 16)$$

$$\sqrt{y - y_0} = \frac{1}{\sqrt{8}}\sqrt{(s_0-4)^2 - (s-4)^2} = \frac{|s_0-4|}{\sqrt{8}}\sqrt{1 - (\frac{s-4}{s_0-4})^2}$$

Then

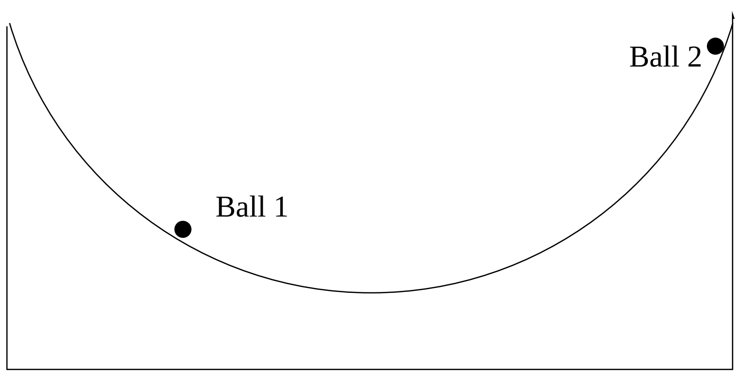
$$T(y_0) = \frac{1}{\sqrt{2g}}\frac{\sqrt{8}}{|s_0-4|}\int_{s_0}^{4}\frac{1}{\sqrt{1-(\frac{s-4}{s_0-4})^2}}ds = \{ \begin{array}{ccc} \frac{s-4}{s_0-4} & = & u \\ ds & = & (s_0-4)du \end{array} \} =$$

$$= \frac{2}{\sqrt{g}}\frac{1}{|s_0-4|}\int_{1}^{0}\frac{1}{\sqrt{1-u^2}}(s_0-4)du$$

We may safely assume that $0 \leq s_0 \leq 4$; then $\frac{s_0-4}{|s_0-4|} = -1$ and we have

$$T(y_0) = \frac{2}{\sqrt{g}}\int_{0}^{1}\frac{1}{\sqrt{1-u^2}}du = \frac{2}{\sqrt{g}}(\arcsin 1 - \arcsin 0) = \frac{2}{\sqrt{g}}\frac{\pi}{2} = \frac{\pi}{\sqrt{g}}$$

a value independent of the starting point. This is then the tautochronous property of the cycloids. Put it another way: two balls left alone from different heights will arrive at the downmost point at the same instant, they will collide at P. Should we have started with a cycloid generated by a wheel of radius R the result would be $T(y_0) = \frac{\pi}{\sqrt{g}}\sqrt{R}$.

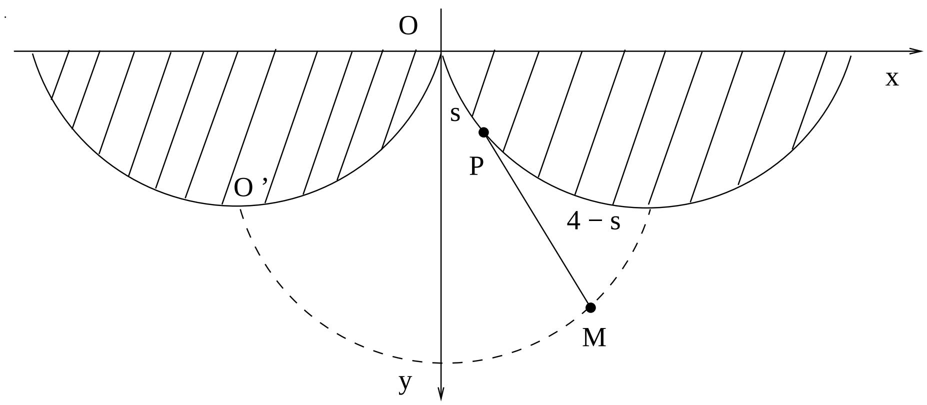


At several science museums there is a gadget that allows the visualization of this fact: two independent elevators leave two balls at the heigth we choose. Then a switch liberates both balls at the same time and we can see the collision taking place exactly at the downmost point of the cycloid.

□

Problem 14: Huygens's pendulum.

The period of a pendulum depends on the amplitude of the oscillations; the formula $T = 2\pi\sqrt{\frac{g}{l}}$ is only an approximation that comes from $\sin\theta \simeq \theta$, quite a rough one for a clock. Using two solid cycloids Huygens constructed a pendulum whose period was independent of the amplitude. In the following figure the length of the rope is half the length of an arc of cycloid



Prove that the path of the suspended mass M is a cycloid, and explain why the period is independent of the oscillations amplitude.

Solution:

Let us assume the radius of the generating wheel is 1 unit, and the length of the rope is 4 units. The parametrization of the right hand side cycloid is

$$\gamma(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$$

with tangent vector

$$\gamma'(\theta) = (1 - \cos\theta, \sin\theta)$$

$$|\gamma'(\theta)| = \sqrt{2}\sqrt{1-\cos\theta} = 2\sin\frac{\theta}{2},\ 0 \leq \theta \leq 2\pi$$

In the figure the rope touches the cycloid from O to P, and the rest of the rope is tangent to the cycloid. We choose as direction vector of the tangent line the *unit* vector

$$\mathbf{v} = \frac{1}{2\sin\frac{\theta}{2}}(1-\cos\theta, \sin\theta) = (\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$$

The tangent line through P is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix} + \lambda \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Reminding that the arc-length parameter for a cycloid is $s = 4(1 - \cos\frac{\theta}{2})$, we see that the point M traces the curve given for the value of λ

$$\lambda = 4 - s = 4 - 4(1 - \cos\frac{\theta}{2}) = 4\cos\frac{\theta}{2}$$

So M is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix} + 4\cos\frac{\theta}{2} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \theta + \sin\theta \\ 3 + \cos\theta \end{pmatrix}$$

Taking a new origin at $O' = (-\pi, 2)$, or what is the same thing, making the change of variables

$$\begin{cases} X &= x + \pi \\ Y &= y - 2 \end{cases}$$

we obtain a parametrization of the path followed by M:

$$\Gamma(\theta) = (\theta + \sin\theta + \pi, 3 + \cos\theta - 2)$$

If we change the parameter to $\varphi = \theta + \pi$ we get

$$\Gamma(\varphi) = (\varphi + \sin(\varphi - \pi), 1 + \cos(\varphi - \pi)) = (\varphi - \sin\varphi, 1 - \cos\varphi),$$

a cycloid.

The period of this pendulum is independent of the amplitude because of the tautochronous property.

□

Problem 15: Papiroflexy.

a) Show that the orthogonal projection of a helix on a plane parallel to the axis of the helix is the graph of a function sin.

b) Take a point of the helix and the tangent line there as the direction of a projection on a plane orthogonal to the axis of the helix. Show that the projection of the helix is a cycloid.

Let us 'construct' the helix by means of a film tranparency as explained in problem 1. If we put a white paper parallel to the helix axis we will see the sinus function of point a). If we put a white paper on a table and our helix with its axis perpendicular to the table, then looking with one eye and adjusting our point of view we can see clearly the cycloid of point b) with its sharp points.

Solution:

a) If we consider the parametrization of the helix

$$\gamma(t) = (\cos t, \sin t, t)$$

then the projection on the plane yz is the curve $(\sin t, t)$, the graf of the desired sinus function.

b) The tangent vector to the helix at the point of parameter t is

$$\gamma'(t) = (-\sin t, \cos t, 1)$$

Assume we fix the point $(1, 0, 0)$ whose parameter is $t = 0$. The tangent vector is

$$\gamma'(0) = (0, 1, 1)$$

and the tangent line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

From the third equation we see that the projection on the plane $z = 0$ takes place when $\lambda = -t$ and then

$$x = \cos t, y = \sin t - t$$

Changing the parameter to $\tau = -t$ gives

$$x = \cos \tau, y = \tau - \sin \tau,$$

then changing the orientation of the Ox axis produces

$$x = -\cos \tau, y = \tau - \sin \tau,$$

and, finally, choosing a new origin at $(-1,0)$ we obtain

$$X = 1 - \cos\tau, Y = \tau - \sin\tau,$$

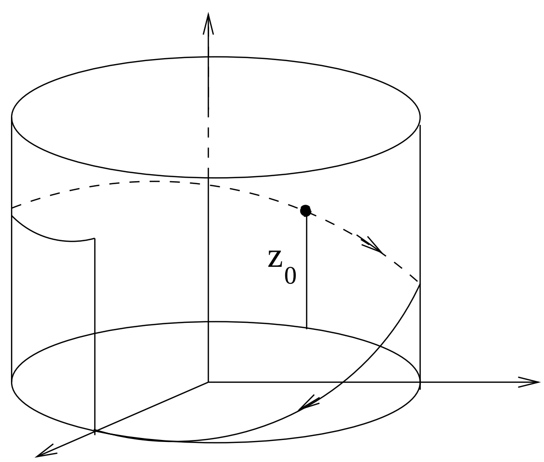
a cycloid.

□

Problem 16: Not every curve has the tautochronous property.

Consider the helix $\gamma(t) = (\cos t, \sin t, t)$ and a point mass that falls along it under the action of gravity. Assuming that the point starts from a height $z = z_0 > 0$ with vanishing initial speed, compute the time elapsed to arrive at the height $z = 0$.

Solution:



The norm of the tangent vector is

$$\gamma'(t) = (-\sin t, \cos t, 1), |\gamma'(t)| = \sqrt{2}$$

and measuring lengths from $\gamma(0) = (1,0,0)$ we have

$$s(t) = \int_0^t \sqrt{2}dt = \sqrt{2}t$$

We know from the problem in p.35, that at a height z the celerity will be

$$\frac{ds}{dt} = -\sqrt{2g(z_0 - z)}$$

Height z corresponds to the parameter $t = z$ and to the arc-length parameter $s = \sqrt{2}z$. Then

$$\frac{ds}{dt} = -\sqrt{2g(\frac{s_0 - s}{\sqrt{2}})} = -C\sqrt{s_0 - s}\,,\; C = \sqrt{\sqrt{2}g}$$

Separating variables and integrating

$$\int_0^{T(z_0)} dt = -\frac{1}{C}\int_{s_0}^{0} \frac{ds}{\sqrt{s_0 - s}}$$

Finally

$$T(z_0) = -\frac{1}{C}(-2\sqrt{s_0 - s})|_{s_0}^{0} = \frac{2}{C}\sqrt{s_0} = 2\sqrt{\frac{z_0}{g}}$$

that depends on z_0.

Actually cycloids are the only curves possessing the tautochronous property (see L.Landau and E.Lifchitz, Mécanique, Editions en langues étrangères, Moscou, p.87).

□

Problem 17: Cycloid on a circumference =: epicycloid.

A 'bycicle' wheel of radius r turns without sliding on the exterior of a circumference of radius R. The path traversed by a point P on the wheel is a curve named epicycloid.

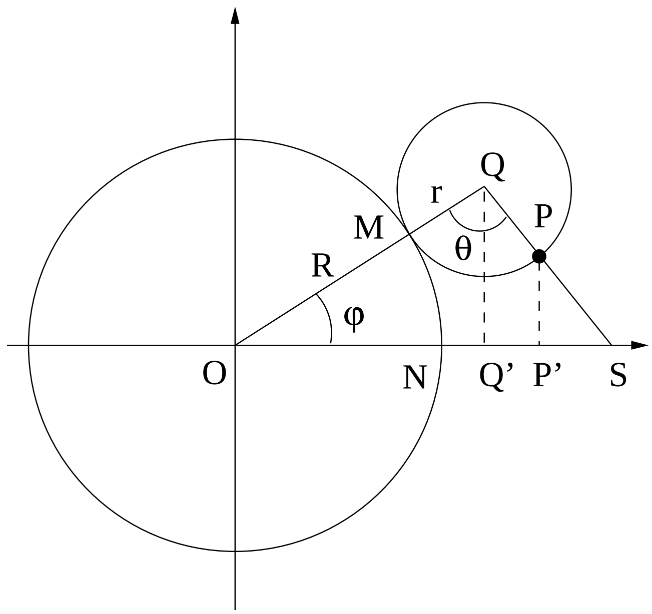
a) Parametritze the epicycloid.

b) Show that if $\frac{R}{r} = n \in \mathbb{N}$ the wheel returns to its starting position. Compute the length of the path followed by the point P until that happens.

c) What if $\frac{R}{r} \in \mathbb{Q} \setminus \mathbb{N}$?

d) Same question when $\frac{R}{r} \in \mathbb{R} \setminus \mathbb{Q}$.

Solution:



a) The arcs of circumference MN and MP have the same length because there is no sliding, so

$$R\varphi = r\theta$$

To compute P we need the angle

$$\widehat{Q'QP} = \varphi + \theta - \frac{\pi}{2}$$

The coordinates of P in terms of φ are:

$$\begin{aligned} x &= (R+r)\cos\varphi + r\sin(\varphi+\theta-\frac{\pi}{2}) = \\ &= (R+r)\cos\varphi - r\cos(\varphi+\theta) = \\ &= (R+r)\cos\varphi - r\cos(1+\frac{R}{r})\varphi \\ y &= (R+r)\sin\varphi - r(1+\frac{R}{r})\sin\varphi \end{aligned}$$

and we have the parametrization of the epicycloid

$$\gamma(\varphi) = ((R+r)\cos\varphi - r\cos(1+\frac{R}{r})\varphi, (R+r)\sin\varphi - r\sin(1+\frac{R}{r})\varphi), \varphi \in \mathbb{R}$$

b) For $\varphi = 2\pi$ we have $\theta = \frac{R}{r}\varphi = n2\pi$: the turning wheel returns to the starting position in n complete turns. A way to visualize this result is to stretch both circumferences as segments, one for the turning wheel and another for the supporting circumference. The result amounts to say that if $\frac{R}{r} = n \in \mathbb{N}$ then the length of the wheel divides the length of the circumference, an obvious thing. Following this point of view it is easy to examine what happens when $\frac{r}{R} = n \in \mathbb{N}$ and translate that, if so needed, to the world of the circumferences.
Now we compute the length of one of the n arcs of the epicycloid; write the parametrization in the form

$$\gamma(\varphi) = ((R+r)\cos\varphi - r\cos(1+n)\varphi, (R+r)\sin\varphi - r\sin(1+n)\varphi)$$

The tangent vector is

$$\begin{aligned}
x'(\varphi) &= -(R+r)\sin\varphi + r(1+n)\sin(1+n)\varphi = \\
&= -(R+r)\sin\varphi + (r+R)\sin(1+n)\varphi = \\
&= (R+r)(\sin(1+n)\varphi - \sin\varphi) \\
y'(\varphi) &= (R+r)\cos\varphi - r(1+n)\cos(1+n)\varphi = \\
&= (R+r)\cos\varphi - (r+R)\cos(1+n)\varphi = \\
&= (R+r)(\cos\varphi - \cos(1+n)\varphi)
\end{aligned}$$

and its length is

$$\begin{aligned}
|(x', y')\,| &= (R+r)\sqrt{2 - 2(\sin\varphi\sin(1+n)\varphi + \cos\varphi\cos(1+n)\varphi)} = \\
&= \sqrt{2}(R+r)\sqrt{1-\cos n\varphi} = \sqrt{2}(R+r)\sqrt{2\sin^2\frac{n}{2}\varphi} \\
&= 2(R+r)|\sin\frac{n}{2}\varphi|
\end{aligned}$$

The length of one arc is then

$$\begin{aligned}
l &= 2(R+r)\int_0^{2\pi/n} |\sin\frac{n}{2}\varphi| d\varphi = \\
&= 2(R+r)\int_0^{2\pi/n} \sin\frac{n}{2}\varphi d\varphi = \\
&= 2(R+r)\frac{2}{n}(-\cos\frac{n}{2}\varphi)\,\Big|_0^{2\pi/n} = \\
&= 4\frac{R+r}{n}(-\cos\pi + 1) = 8\frac{(R+r)}{n}
\end{aligned}$$

The whole length up to the return event is

$$L = 8(r + R)$$

c) In case $\frac{R}{r} \in \mathbb{Q} \setminus \mathbb{N}$ let $\frac{R}{r} = \frac{p}{q}$ (irreducible), and assume to fix the scene that $p > q$. From the 'segment' point of view
we may assume (using an adequate unit) that the wheel's segment measures q while the circumference's segment measures p. The first return point takes place for the smaller integers m, n such that

$$nq = mp$$

and, being $\frac{p}{q}$ irreducible, $n = p, m = q$ is the obvious solution. The wheel has made p complete turns and has completed q windings around the circumference.
From the angle point of view we can say that there is a contact of P with the supporting circumference whenever

$$\begin{array}{rcllllll} \theta & = & 0, & 2\pi, & 2 \cdot 2\pi, & 3 \cdot 2\pi, & \ldots, & m \cdot 2\pi, \quad \ldots \\ \varphi & = & 0, & 2\pi\frac{q}{p}, & 2 \cdot 2\pi\frac{q}{p}, & 3 \cdot 2\pi\frac{q}{p}, & \ldots, & m \cdot 2\pi\frac{q}{p}, \quad \ldots \end{array}$$

The epicycloid closes when two contacts have the same $\varphi \pmod{2\pi}$ which happens for the first time when $m = p$. We have $\varphi = 2\pi q$, that reveals the q times that the wheel winds around the circumference, and then $\theta = p \cdot 2\pi$ showing the p complete turns that the wheel has made. The reader can fill in what happens when $p < q$.

d) If $\frac{R}{r} \in \mathbb{R}/\mathbb{Q}$ the contact points of P with the basis circumference form a dense set. This means that there are contact points in every interval, however small, on the circumference. Notice first that they are *all* different; if we had two coinciding contact points then

$$m \cdot 2\pi\frac{r}{R} \equiv n \cdot 2\pi\frac{r}{R} \pmod{2\pi} \Leftrightarrow \exists N \in \mathbb{N} : (m - n) \cdot 2\pi\frac{r}{R} = N2\pi \Leftrightarrow$$

$$\Leftrightarrow (m - n) \cdot \frac{r}{R} = N$$

but this is impossible since r/R is irrational.
Now, fix a positive integer k and divide the circumference in k equal angular sectors of amplitude $2\pi/k$. Among the $k+1$ first contacts there

will be two in the same sector (this is known as Dirichlet's principle: if we put $k+1$ balls in k boxes, there will be two or more balls at least in one box). Let those contacts correspond to $\varphi = n \cdot 2\pi\frac{r}{R}$ and $\varphi' = m \cdot 2\pi\frac{r}{R}$; then $\alpha = (m-n)2\pi\frac{r}{R}$ is a contact in the first sector and the contacts

$$\alpha, 2\alpha, \ldots, n\alpha, \ldots$$

differ less than $2\pi/k$.
Finally given $\epsilon > 0$, choose k such that $2\pi/k < \epsilon$; then in any interval of angular amplitude less than ϵ there is a contact which was to be proved.

□

Problem 18: Cycloid under circumference:=hypocycloid.

A wheel of radius r turns without sliding on the interior of a circumference of radius R. The path followed by a point P on the wheel is a curve named hypocycloid.

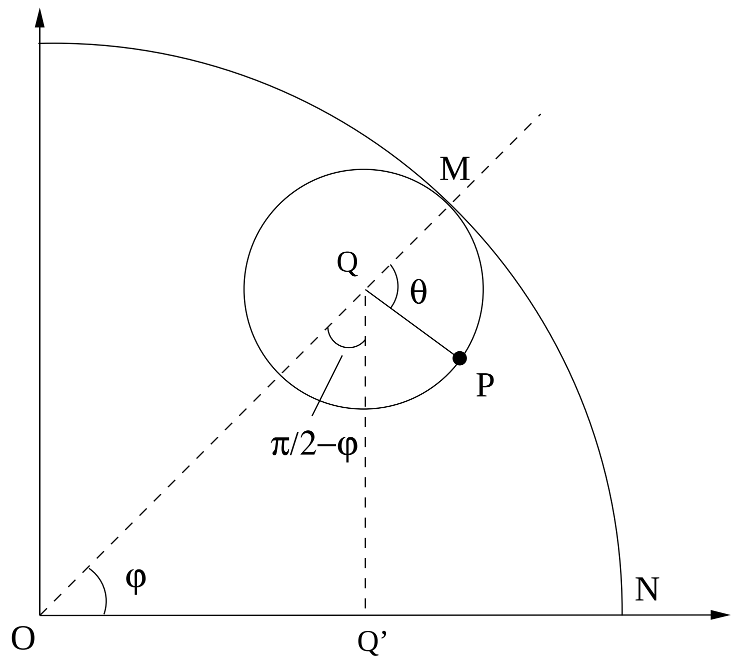
a) Parametrize the hypocycloid.

b) Show that if $\frac{R}{r} = n \in \mathbb{N}$ the wheel returns to its starting position. Compute the length of the path followed by the point P until that happens.

c) What if $\frac{R}{r} \in \mathbb{Q} \setminus \mathbb{N}$?

d) Same question when $\frac{R}{r} \in \mathbb{R} \setminus \mathbb{Q}$

Solution:



a) The arcs of circumference MN and MP have the same length because there is no sliding, so

$$R\varphi = r\theta$$

To compute P we need the angle

$$\widehat{Q'QP} = \pi - \theta - (\pi/2 - \varphi) = \frac{\pi}{2} + \varphi - \theta$$

Then

$$\begin{aligned} x &= (R-r)\cos\varphi + r\sin(\frac{\pi}{2} + \varphi - \theta) = \\ &= (R-r)\cos\varphi + r\cos(\varphi - \theta) = \\ &= (R-r)\cos\varphi + r\cos\varphi(1 - \frac{R}{r}) \\ y &= (R-r)\sin\varphi + r\sin\varphi(1 - \frac{R}{r}) \end{aligned}$$

gives a parametrization of the hypocycloid.

b) To compute the asked for length we have

$$\begin{aligned} x'(\varphi) &= -(R-r)\sin\varphi - (r-R)\sin\varphi(1-\frac{R}{r}) \\ y'(\varphi) &= (R-r)\cos\varphi + (r-R)\cos\varphi(1+\frac{R}{r}) \end{aligned}$$

$$|(x',y')\,|^2 = (R-r)^2+(r-R)^2+2(R-r)(r-R)(\cos\varphi\cos\varphi(1-\frac{R}{r})+\sin\varphi\sin\varphi(1-\frac{R}{r})) =$$

$$= 2(R-r)^2(1-\cos(\frac{R}{r}\varphi))$$

$$|(x',y')\,| = \sqrt{2}(R-r)\sqrt{1-\cos(\frac{R}{r}\varphi)} = 2(R-r)|\sin(\frac{R}{2r}\varphi)|$$

The length of one turn of the wheel is

$$l = 2(R-r)\int_0^{2\pi r/R} |\sin(\frac{R}{2r}\varphi)|d\varphi = -2(R-r)2\frac{r}{R}\cos(\frac{R}{2r}\varphi)|_0^{2\pi r/R} =$$

$$= -4(R-r)\frac{r}{R}(\cos\pi - 1) = 8\frac{r}{R}(R-r)$$

The whole length up to the return event is

$$L = n8\frac{r}{R}(R-r) = 8(R-r)$$

c) Let $\frac{R}{r} = \frac{p}{q}$ (irreducible). Assume $p > q$ and by the same reasoning of the preceding problem we see that the hypocycloid closes when the wheel has made p complete turns.

d) Exactly as in the preceding problem we may see that the contacts are dense in the circumference.

□

Before leaving the cycloids we should mention the brachystochrone property: being given two points P, Q in a vertical plane what is the curve connecting P and Q along which the time of fall of a mass under gravity's action is minimum? A straight line segment? An arc of a parabola? As the reader probably suspects it is an arc of a cycloid. Around 1620 Johan Bernouilli working on a challenge of Newton proved that fact using a heuristic argument rooted in the refraction laws (!) (see [Pol] p.177).

Cycloids & co are a particular case of a bigger family of curves named trocoids. The reader wanting to experiment vividly with those curves may do so in

http://temasmatematicos.uniandes.edu.co

a nice work by Aquiles Páramo.

Chapter 2

Vector fields

2.1 Fields

$\boxed{\mathbb{T}}$

- A *wheat field* has a wheat spike at each point (cum grano salis).

- A *scalar field* in an open set $U \subset \mathbb{R}^n$ has a scalar at each point $\mathbf{x} \in U$; so it is simply a function that takes *numerical* values, a real function:

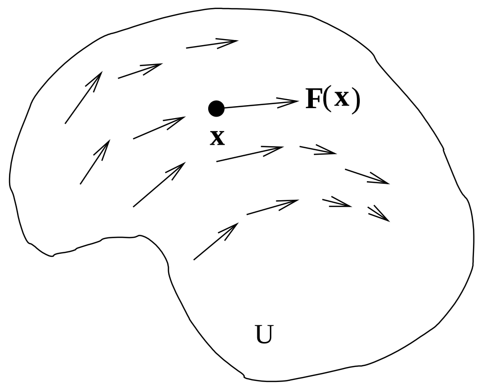
$$\boxed{\begin{array}{rccc} f: & U & \to & \mathbb{R} \\ & \mathbf{x} & \mapsto & f(\mathbf{x}) \end{array}}$$

- A *vector field* in an open set $U \subset \mathbb{R}^n$ has a vector at each point $\mathbf{x} \in U$; it is a function that takes vector values:

$$\boxed{\begin{array}{rccc} \mathbf{F}: & U & \to & \mathbb{R}^n \\ & \mathbf{x} & \mapsto & \mathbf{F}(\mathbf{x}) \end{array}}$$

  A field (scalar or vector) is of class C^k or C^∞ if the function defining the field is.

Geometrically a vector field assigns to each point $\mathbf{x} \in U$ a vector $\mathbf{F}(\mathbf{x})$ that we shall draw emanating from $\mathbf{x}$:



Problem 19: Radial fields and central fields.

a) Find the general form of a radial field (:= that has at each point the direction of the position vector) defined in $\mathbb{R}^n - \{\mathbf{0}\}$.

b) Find the general form of a central field (:= that has at each point the direction of the position vector and has constant module on each sphere centered at $\mathbf{0}$; briefly: it is radial and spherically simmetric) defined in $\mathbb{R}^n - \{\mathbf{0}\}$.

Solution:

We shall often use the familiar 'physics' notation $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. In $\mathbb{R}^n$ it is $\mathbf{r} = (x_1, \ldots, x_n)$,$r = |\mathbf{r}| = \sqrt{(x_1)^2 + \cdots + (x_n)^2}$. Then:

a) $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}$, $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ being an arbitrary function.

b) $\mathbf{F}(\mathbf{r}) = \phi(r)\mathbf{r}$, $\phi : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ being an arbitrary function.

□

Force fields versus velocity fields

$\boxed{\mathbb{T}}$ We can look at a vector field $\mathbf{F}(\mathbf{x})$ in an open set $U \subset \mathbb{R}^n$ from two points of view:

a) As a *force field* (gravitational, electric, magnetic or other), the vector $\mathbf{F}(\mathbf{x})$ represents the force exerted on a unit particle (unit mass, charge or pole intensity, etc.) at the point $\mathbf{x}$. We often integrate force fields along curves.

b) As a *velocity field,* the vector $\mathbf{F}(\mathbf{x})$ represents the velocity of the particles of a fluid when they pass through $\mathbf{x}$. We often integrate velocity fields on surfaces.

□

Problem 20: Work and flux.

a) Let $\mathbf{F}$ be a constant force field; compute the work done by the field in moving a unit particle along a segment $[\mathbf{p}, \mathbf{q}]$.

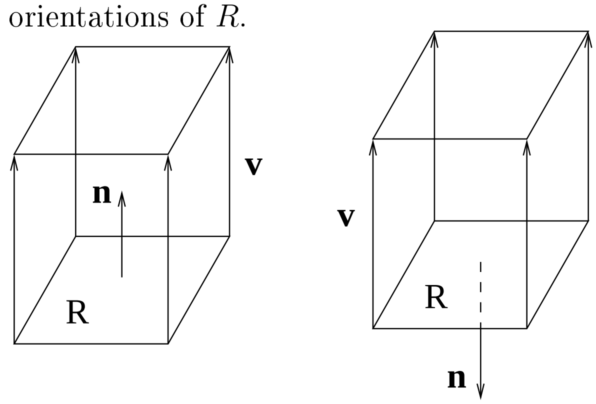
b) Let $\mathbf{v}$ be a constant velocity field; compute the volume of fluid that crosses a rectangular surface R per unit time.

Solution:

a) The force exerted on the unit particle is $\mathbf{F}$ and the work done by the field is due to the component of the field in the direction of the segment. Then

$$W = (\mathbf{F} \cdot \frac{\overrightarrow{\mathbf{pq}}}{|\overrightarrow{\mathbf{pq}}|})|\overrightarrow{\mathbf{pq}}| = \mathbf{F} \cdot \overrightarrow{\mathbf{pq}}$$

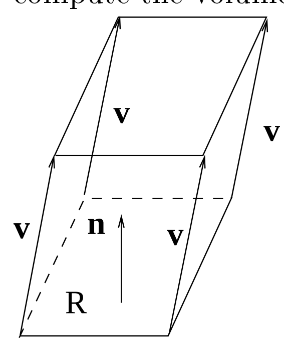
b) We must previously choose a sense of crossing the surface. Then we shall count the volume that has crossed the surface as positive if it does in the sense choosen and as negative if the volume crosses in the opposite sense. To assign a sense we choose a unit vector $\mathbf{n}$ perpendicular to the surface. In the following figure we show the two possible orientations of R.



Assume first that **v** is perpendicular to R and that it has the same sense as **n** as shown in the first figure. The particles occupying now positions on R will be a unit of time later at positions on the upper face of the parallalepiped shown. The volume that has crossed R in a unit of time is:

$$\phi = \text{Area }(R)|\mathbf{v}| = \text{Area }(R)\mathbf{n}\cdot\mathbf{v}$$

If **n** and **v** have opposite senses the formula will take that into account and we shall obtain a negative value.
If **v** is not perpendicular to R but has the same sense as **n**, we have to compute the volume of the following figure:



wich is

$$\phi = \text{Area }(R)\cdot\text{height}= \text{Area }(R)\,\mathbf{n}\cdot\mathbf{v}$$

and, as in the preceding case, the formula takes into account when **v** and **n** have opposite senses, giving a negative value.

□

Observation:

The formula in a) is the foundation of the integration of a field along a curve. The formula in b) is the foundation of the integration of a field on a surface.

2.2 Newtonian fields

$\boxed{\mathbb{T}}$ Newtonian fields are an important example of force fields. A newtonian field is a central field of the form $\mathbf{F}(\mathbf{r}) = \frac{1}{r^3}\mathbf{r}$; well known examples are the gravitational field of a point mass, the electrostatic field of a point charge and the fields from a magnetic pole. A basic reference for those fields is [Kell].

□

2.2.1 Newton's law for a particle

𝕋 A mass point M at the point $\mathbf{q}$ exerts on a mass point m at the point $\mathbf{p}$ at a distance r a gravitational attraction of strength

$$F = G\frac{Mm}{r^2},$$

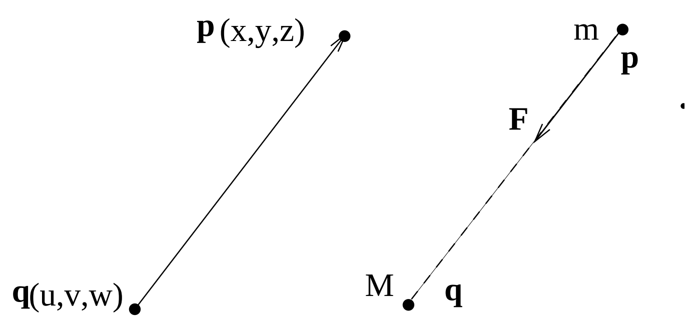
directed along the line through $\mathbf{p}$ and $\mathbf{q}$ and oriented from $\mathbf{p}$ to $\mathbf{q}$. Of course the principle of action and reaction tells us that the particle at $\mathbf{p}$ exerts an equal strength attraction on the particle at $\mathbf{q}$. If we want to mentally isolate the attraction of $\mathbf{q}$ on $\mathbf{p}$ we call $\mathbf{q}$ the *attracting particle* and $\mathbf{p}$ the *attracted particle* (and so we avoid the wandering between two forces).

Then the attracting particle of mass M at $\mathbf{q} = (u, v, w)$ exerts on the attracted particle of mass m at $\mathbf{p} = (x, y, z)$ a force

$$\mathbf{F} = -G\frac{Mm}{r^2}\frac{\mathbf{r}}{r},$$

where

$$\mathbf{r} = (x-u, y-v, z-c), r = |\mathbf{r}| = \sqrt{(x-u)^2 + (y-v)^2 + (z-c)^2}$$



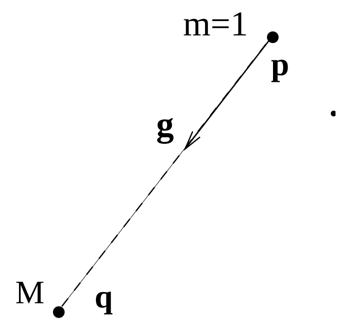
We shall write the preceding formula as

$$\mathbf{F} = -GMm\frac{\mathbf{r}}{r^3}$$

Note that the origin of $\mathbf{r}$ is at the *source of the field* (the attracting mass) and its end point lies where we want to compute the exerted force. We shall use this convention and we even give it a name: *from the source to the point.*

Definition: The *gravitational field* at the point $\mathbf{p} = (x, y, z)$ created by a point mass M at the point $\mathbf{q} = (u, v, w)$ is the force exerted on a *unit* mass at $\mathbf{p}$:

$$\mathbf{g}(x, y, z) = -GM\frac{\mathbf{r}}{r^3}$$



If instead of a unit mass at $\mathbf{p}$ we have there a mass m, the gravitational attraction is $\mathbf{F} = m\mathbf{g}$.

The constant G is the gravitational constant and is extremely small; that, among other things, makes its precise determination difficult. The value of G in the c.g.s system of units is $G = 6.664 \times 10^{-8}$. It is probably the worse known physical constant; recent results show discrepancies in the fifth decimal place.

Let us define the *attraction unit* as the gravitational force exerted by a mass point of 1g on an identical particle at 1cm; then $G = 1$ and we get rid of the constant. The equivalence is

$$1 \text{ attraction unit } = 6.664 \times 10^{-8}\text{din}$$

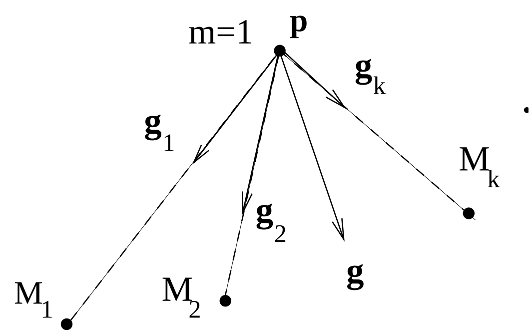
In the new unit we have

$$\boxed{\mathbf{F} = -Mm\frac{\mathbf{r}}{r^3}}$$

$$\boxed{\mathbf{g} = -M\frac{\mathbf{r}}{r^3}}$$

The gravitational field satisfies the *superposition principle*: the field created at $\mathbf{p}$ by masses $M_1, \ldots, M_k$ at the points $\mathbf{q}_1, \ldots, \mathbf{q}_k$ is the addition of the fields generated at **p** by each particle:

$$\mathbf{g}(x,y,z) = \sum_{i=1}^{k} \mathbf{g}_i(x,y,z) = \sum_{i=1}^{k} -M_i \frac{\mathbf{r}_i}{r_i^3}$$



This field is defined in the open set $U = \mathbb{R}^3 - \{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$.

□

2.2.2 Newton's law for extended bodies

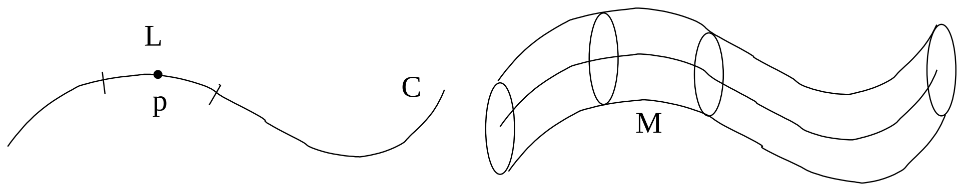
$\boxed{\mathbb{T}}$ Take two bodies, divide them in small elements (as is done in integral calculus) and assume the mass of each element concentrated at a point in the element. In that way we obtain two systems of particles.

Then the attraction of one body on the other is the limit of the attraction that its system of particles exerts on the system of particles of the other body, when the diameter of the elements tend to zero.

Remark that this law is not derivable from Newton's law for particles for the reason that, minute as can be the elements of the decomposition, they never become point masses.

Linear density

Consider a curve C and the same curve expanded to a tube of mass M. Considering the mass concentrated at C we obtain the concept of a *material wire*, call it l.



To calculate the density of the material wire at a point $\mathbf{p} \in C$, take segments of the curve containing $\mathbf{p}$; the average linear density of each segment is $\lambda = \frac{m}{L}$, m being the mass of the segment and L its length. The *linear density* at $\mathbf{p}$ is

$$\lambda(\mathbf{p}) = \lim_{l \to 0} \frac{m}{L}$$

□

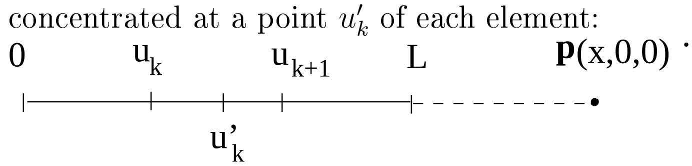
Problem 21: Gravitational field of a segment.

Consider a homogeneous mass segment (:=the mass of any piece is proportional to the length of the piece; the proportionallity constant λ is the linear density). Compute the gravitational field due to the mass at a point of the straight line containing the segment, but exterior to it.

Solution:

Take the segment on the positive Ox axis with its left end at the origin of coordinates. Let's follow the instructions given in Newton's law for extended bodies:

- Assume first that $L < x$. Divide the segment in elements $[u_k, u_{k+1}]$ of length $\Delta u_k = u_{k+1} - u_k$ and assume their masses $\Delta m_k = \lambda \Delta u_k$ concentrated at a point u'_k of each element:



- For every such mass point compute the gravitational field generated at the point $\mathbf{p} = (x, 0, 0), x > L$ exterior to the segment and add those contributions:

$$\Delta \mathbf{g} = \sum_k -\Delta m_k \frac{(x - u'_k, 0, 0)}{(x - u'_k)^3} = \sum_k -\lambda \Delta u_k \frac{(1, 0, 0)}{(x - u'_k)^2}$$

- We let the diameter of the elements tend to zero:

$$\mathbf{g}(x,0,0) = \lim_{\Delta u_k \to 0} \sum_k -\lambda \frac{(1,0,0)}{(x-u'_k)^2} \Delta u_k = \int_0^L -\lambda \frac{(1,0,0)}{(x-u)^2} du =$$

$$= -\lambda \frac{(u,0,0)}{(x-u)}|_{u=0}^{u=L} = (-\frac{\lambda L}{x(x-L)}, 0, 0) = (-\frac{M}{x(x-L)}, 0, 0)$$

- Similarly we obtain the field at $\mathbf{p} = (x,0,0), x < 0$ adding the contributions of the elements

$$\Delta \mathbf{g} = \sum_k -\Delta m_k \frac{(x-u'_k, 0, 0)}{(-(x-u'_k))^3} = \sum_k \lambda \Delta u_k \frac{(1,0,0)}{(x-u'_k)^2}$$

  and following the preceding line we arrive at

$$\mathbf{g}(x,0,0) = (\frac{M}{x(x-L)}, 0, 0)$$

Both results are summarized in the formula

$$\mathbf{g}(x,0,0) = (-\frac{M}{\mid x \mid (x-L)}, 0, 0), x < 0 \text{ or } L < x$$

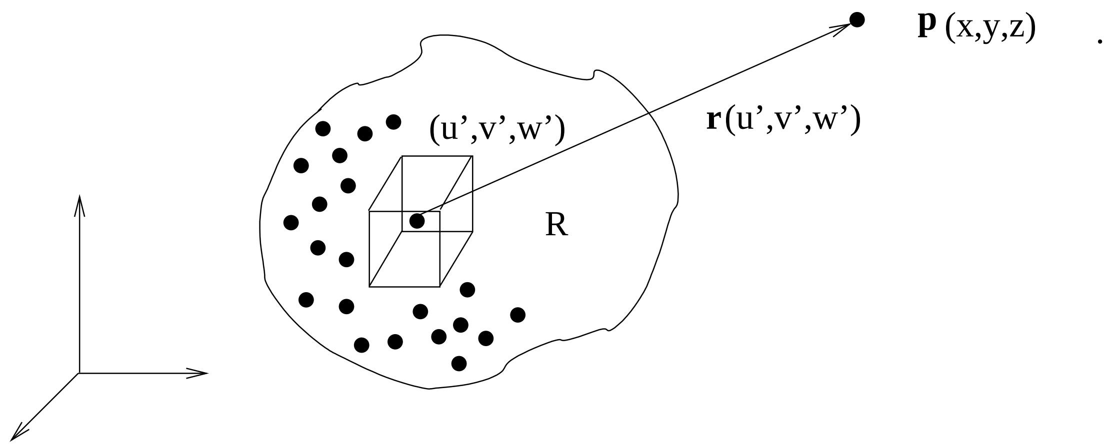
□

Problem 22: Gravitational field of a mass distribution.

Consider a continuous mass distribution in a region $R \subset \mathbb{R}^3$; that is to say that we are given the density of the distribution $\chi(x,y,z)$, a continuous function. Compute the gravitational field generated by the distribution at an exterior point $\mathbf{p} = (x,y,z)$.

Solution:

We use Newton's law for extended bodies:



- Divide R in elements ΔV; being χ continuous, the mean value theorem for integrals applies, so that the mass of an element will be $\chi(u'_k, v'_k, w'_k)\Delta V$ for a certain point (u'_k, v'_k, w'_k) in the element. Consider the mass of the element concentrated at that point.

- For every such mass point compute the gravitational field generated at the point $\mathbf{p} = (x, y, z)$ and add those contributions:
$$\Delta \mathbf{g} = -\sum_k \chi(u'_k, v'_k, w'_k)\Delta V \frac{\mathbf{r}(u'_k, v'_k, w'_k)}{|\mathbf{r}(u'_k, v'_k, w'_k)|^3}$$
where $\mathbf{r}(u'_k, v'_k, w'_k) = (x - u'_k, y - v'_k, z - w'_k)$. The Δ in front of $\mathbf{g}$ reminds that we are still in the elements level.

- We let the diameter of the elements tend to zero:
$$\mathbf{g}(x, y, z) = -\int\int\int_R \chi(u, v, w) \frac{\mathbf{r}(u, v, w)}{|\mathbf{r}(u, v, w)|^3} du dv dw$$
or in components:
$$\mathbf{g}(x, y, z) = -\int\int\int_R \chi(u, v, w) \frac{(x - u, y - v, z - w)}{(\sqrt{(x - u)^2 + (y - v)^2 + (z - w)^2})^3} du dv dw$$
Now the Δ has disappeared since we are now in the 'real', continuous world.

□

2.2.3 Electrostatic field

Problem 23: Electrostatic field.

By analogy with the gravitational field, write formulae for:

a) The electrostatic field of a point charge.

b) The electrostatic field of a system of point charges.

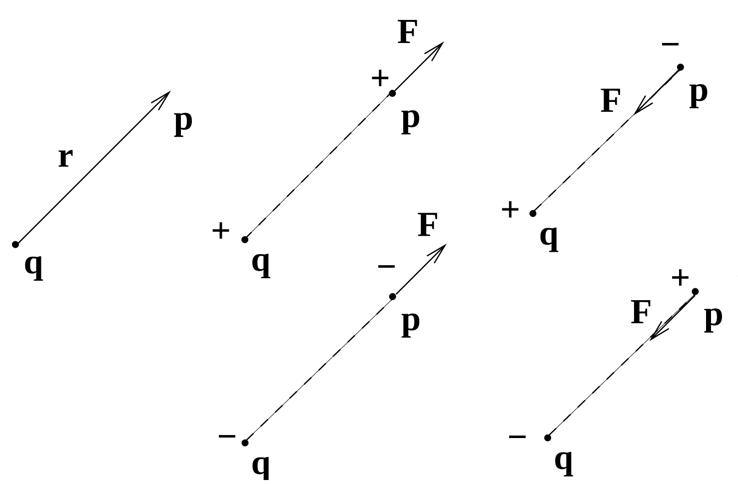
c) The electrostatic field of a charge distribution.

Solution:

Coulomb's law gives the atraction/repulsion that a charge q at $\mathbf{q}$ exerts on a charge q' at $\mathbf{p}$

$$\mathbf{F} = \epsilon q q' \frac{\mathbf{r}}{r^3}$$

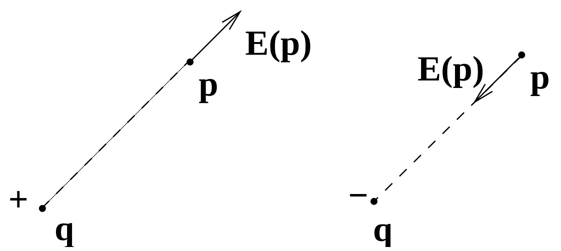
Now q, q' can be negative and we don't need the minus sign that is required in Newton's gravitational law. To convince oneself of that it suffices to examine next figure



Choosing adequate units, as we shall assume, we have $\epsilon = 1$. The *electrostatic field* produced at point $\mathbf{p}$ by a charge q at point $\mathbf{q}$ is the force exerted by this charge on the unit *positive* charge at point $\mathbf{p}$.

a) Coulomb's law gives

$$\mathbf{E}(x,y,z) = q\frac{\mathbf{r}}{r^3}$$



The force exerted on a charge q' at $\mathbf{p}$ is $\mathbf{F} = q'\mathbf{E}$.

b) If the charges $q_1, \ldots, q_k$ are at the points $\mathbf{q}_1, \ldots, \mathbf{q}_k$, the principle of superposition gives:

$$\mathbf{E}(\mathbf{p}) = \sum_i q_i \frac{\mathbf{r}_k}{{r_k}^3}$$

c) For a continuous distribution with density function χ, using a Coulomb's law for extended charges, the electric field is written thus:

$$\mathbf{E}(x,y,z) = \int\int\int_R \chi(u,v,w)\frac{(x-u, y-v, z-w)}{(\sqrt{(x-u)^2+(y-v)^2+(z-w)^2})^3}dudvdw$$

or in vector form

$$\mathbf{E}(\mathbf{p}) = \int\int\int_R \chi(u,v,w)\frac{\mathbf{r}(u,v,w)}{|\mathbf{r}(u,v,w)|^3}dudvdw = \int\int\int_R \chi\frac{\mathbf{r}}{r^3}$$

□

2.3 Velocity fields

2.3.1 Fluid

$\boxed{\mathbb{T}}$ The movement of a fluid is a 'material' model of some usual more abstract concepts such as: flux of a vector field through an oriented surface, uniparametric groups, associated fields, solution of a system of differential equations etc. Moreover the model gives an intuitive view of a certain derivative of a field.

Assume a particle P of the fluid occupies the position (x_0, y_0, z_0) at the instant t_0 (which we call *initial instant*). The positions successively occupied by P as time goes by form the *trajectory* of P, a curve that passes through (x_0, y_0, z_0) at the instant t_0:

$$\begin{cases} x(t) &= x(x_0, y_0, z_0, t) \\ y(t) &= y(x_0, y_0, z_0, t) \\ z(t) &= z(x_0, y_0, z_0, t) \end{cases}$$

Defining:

$$\phi(x_0, y_0, z_0, t) = (x(x_0, y_0, z_0, t), y(x_0, y_0, z_0, t), z(x_0, y_0, z_0, t)),$$

we synthetise all the trajectories in a unique function that we shall want to be differentiable, the *flow*:

$$\boxed{\begin{array}{cccc} \phi : & \mathbb{R}^3 \times \mathbb{R} & \rightarrow & \mathbb{R}^3 \\ & (x_0, y_0, z_0, t) & \mapsto & \phi(x_0, y_0, z_0, t) \end{array}}$$

- The flow gives the *trajectory* of a point P:

$$\begin{array}{cccc} \gamma_P : & \mathbb{R} & \rightarrow & \mathbb{R}^\mathbf{3} \\ & t & \mapsto & \gamma_P(t) = \phi(P, t) \end{array}$$

  Notice that $\gamma_P(t_0) = P$.

- The flow generates a *transformation during the time* t (starting at $t = t_0$):

$$\begin{array}{cccc} \phi_t : & \mathbb{R}^\mathbf{3} & \rightarrow & \mathbb{R}^\mathbf{3} \\ & Q & \mapsto & \phi_t(Q) = \phi(Q, t) \end{array}$$

  Notice that $\phi_{t_0}(Q) = Q$ or $\phi_{t_0} = \text{Id}_{\mathbb{R}^3}$.

Besides being differentiable we want ϕ to be such that if we compose a transformation during the time t with a transformation during the time t' we obtain a transformation during the time $t + t'$:

$$(\phi_{t'} \circ \phi_t)(Q) = \phi_{t+t'}(Q)$$

If ϕ has this property it also satisfies the commutative property $\phi_{t'} \circ \phi_t = \phi_t \circ \phi_{t'}$.

□

Problem 24: Trajectories and transformations.

a) Show that the function $\phi(x_0, y_0, z_0, t) = (x_0 e^t, y_0 e^{-t}, z_0), t_0 = 0$ is a flow.

b) Show that the trajectories are plane curves (and the movement of the fluid is then called *planar*).

c) Find the position at the instant $t = 0.1$ of the particle that was at $(2, 3, 1)$ at the instant $t = 0$.

d) What was the position at $t = 0$ of a particle that is at $(1, 2, 3)$ when $t = 5$?

e) Find the transformation during the time $t = 4$.

Solution:

a) The function ϕ satisfies

   i) $\phi(x_0, y_0, z_0, 0) = (x_0, y_0, z_0)$.

   ii) ϕ is differentiable.

   iii) The law of composition:

$$\begin{aligned} \phi_{t'}(\phi_t(x_0, y_0, z_0)) &= \phi_{t'}(x_0 e^t, y_0 e^{-t}, z_0) = \\ &= (x_0 e^t e^{t'}, y_0 e^{-t} e^{-t'}, z_0) = \phi_{t+t'}(x_0, y_0, z_0) \end{aligned}$$

b) The z-component of ϕ is constant; the movement takes place in the plane $z = z_0$.

c) $\phi(2, 3, 1, 0.1) = (2e^{0.1}, 3e^{-0.1}, 1)$.

d) Just solve the system

$$\begin{cases} 1 &= x_0 e^5 \\ 2 &= y_0 e^{-5} \\ 3 &= z_0 \end{cases}$$

that is

$$x_0 = e^{-5}, y_0 = 2e^5, z_0 = 3$$

e) $\phi_4(x, y, z) = \phi(x, y, z, 4) = (xe^4, ye^{-4}, z)$

□

2.3.2 Velocity field of a fluid

$\boxed{\mathbb{T}}$ The velocity at the instant t of the particle that for $t = t_0$ was at (x_0, y_0, z_0) is

$$\frac{d\phi}{dt} = (\frac{dx}{dt}(x_0, y_0, z_0, t), \frac{dy}{dt}(x_0, y_0, z_0, t), \frac{dz}{dt}(x_0, y_0, z_0, t))$$

Let's denote by

$$\mathbf{v}(x, y, z, t) = (X(x, y, z, t), Y(x, y, z, t), Z(x, y, z, t))$$

the velocity of the particle that is at (x, y, z) at the instant t, a vector field called the *velocity field* of the flow.

□

Problem 25: Velocity field.

Compute the velocity field of the flow

$$\phi(x_0, y_0, z_0, t) = (x_0 e^t, y_0 e^{-t}, z_0), t_0 = 0.$$

Solution:

The velocity at the instant t of the particle that for $t = t_0$ was at (x_0, y_0, z_0) is

$$\frac{d\phi}{dt} = (x_0 e^t, -y_0 e^{-t}, 0)$$

Now we find the position for $t = 0$ of the particle that is at (x, y, z) at the instant t. To this end solve the system $(x_0 e^t, y_0 e^{-t}, z_0) = (x, y, z)$ to obtain $x_0 = xe^{-t}, y_0 = ye^t, z_0 = z$. Then

$$\mathbf{v}(x, y, z) = (xe^{-t}e^t, -ye^t e^{-t}, 0) = (x, -y, 0)$$

This velocity field is independent of t and the field is called *stationary.*

□

Problem 26: Nonstationary velocity field.

Compute the velocity field of the flow

$$\phi(x_0, y_0, z_0, t) = (x_0 + t, y_0 + t^2, z_0), t_0 = 0.$$

Solution:

We proceed as in the preceding problem

$$\frac{d\phi}{dt} = (1, 2t, 0),$$

that does not depend on (x_0, y_0, z_0) and the velocity field is

$$\mathbf{v}(x, y, z) = (1, 2t, 0),$$

an example of a *nonstationary* field.

□

Problem 27:

Compute the velocity field of the fluids

a) $\phi(x_0, y_0, z_0, t) = (\frac{x_0+y_0}{2}e^t + \frac{x_0-y_0}{2}e^{-t}, \frac{x_0+y_0}{2}e^t - \frac{x_0-y_0}{2}e^{-t}, z_0), t_0 = 0.$

b) $\phi(x_0, y_0, z_0, t) = (x_0 + \sin t, y_0 + 1 - \cos t, z_0), t_0 = 0.$

Solution:

a) First

$$\frac{d\phi}{dt} = (\frac{x_0 + y_0}{2}e^t - \frac{x_0 - y_0}{2}e^{-t}, \frac{x_0 + y_0}{2}e^t + \frac{x_0 - y_0}{2}e^{-t}, 0),$$

and now we compute the position at $t = 0$ solving the system:

$$\left\{ \begin{array}{lcl} x & = & \frac{x_0+y_0}{2}e^t + \frac{x_0-y_0}{2}e^{-t} \\ y & = & \frac{x_0+y_0}{2}e^t - \frac{x_0-y_0}{2}e^{-t} \\ z & = & z_0 \end{array} \right. ;$$

adding and subtracting the first two equations:

$$\left\{ \begin{array}{lcl} x_0 + y_0 & = & (x + y)e^{-t} \\ x_0 - y_0 & = & (x - y)e^t \end{array} \right.$$

and the velocity field is:

$$\begin{aligned} \mathbf{v}(x, y, z, t) &= (\frac{x + y}{2}e^{-t}e^t - \frac{x - y}{2}e^t e^{-t}, \frac{x + y}{2}e^{-t}e^t + \frac{x - y}{2}e^t e^{-t}, 0) = \\ &= (y, x, 0) \end{aligned}$$

b) Now we have

$$\frac{d\phi}{dt} = (\cos t, \sin t, 0)$$

that is independent of the initial position. Then:

$$\mathbf{v}(x, y, z, t) = (\cos t, \sin t, 0),$$

a nonstationary field.

□

Observation:

The data of a first order system of ordinary differential equations is a vector field; the solution is a flow (at least under suitable conditions) that has the given field as its velocity field. For instance:

Problem 28: Flow generated by a field.

In the system $\frac{d\mathbf{x}}{dt} = \mathbf{x}$ the given field is $\mathbf{v}(\mathbf{x}) = \mathbf{x}$. Solve this system in $\mathbb{R}^2$ and obtain a flow that has $\mathbf{v}$ as its velocity field.

Solution:

In components the system is:

$$\begin{cases} \frac{dx}{dt} & = & x \\ \frac{dy}{dt} & = & y \end{cases}$$

that is easily solved because the equations are uncoupled. The solutions of the first equation are $x(t) = x_0 e^t$ and those of the second are $y(t) = y_0 e^t$. Each curve $\gamma(t) = (x_0 e^t, y_0 e^t)$ is a solution of the system. The flow is then

$$\phi(x_0, y_0, t) = (x_0 e^t, y_0 e^t), t_0 = 0$$

□

Problem 29:

In the system $\frac{d\mathbf{x}}{dt} = (-y, x)$ the given field is $\mathbf{v}(\mathbf{x}) = (-y, x)$. Solve this system in $\mathbb{R}^2$ and obtain a fluid that has $\mathbf{v}$ as its velocity field.

Solution:

In components the system is:

$$\begin{cases} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{cases}$$

Now, in contrast with the preceding problem, the equations are coupled. Taking into account that $x(t), y(t)$ must be differentiable functions satisfying the system, we see that $\frac{dx}{dt}, \frac{dy}{dt}$ are differentiable functions as well. Differentiating the first equation we obtain

$$\frac{d^2x}{dt^2} = -x,$$

the harmonic oscillator equation; we know that the solution is

$$x(t) = A\cos t + B\sin t$$

and we obtain $-y(t)$ differentiating $x(t)$:

$$y(t) = A\sin t - B\cos t$$

Let (x_0, y_0) be the position at $t = 0$; then $A = x_0, B = -y_0$ and the solution is

$$\mathbf{x}(t) = (x_0\cos t - y_0\sin t, x_0\sin t + y_0\cos t)$$

The flow generating the field is

$$\phi(x_0, y_0, t) = (x_0\cos t - y_0\sin t, x_0\sin t + y_0\cos t), t_0 = 0$$

□

2.4 Field derivatives

$\boxed{\mathbb{T}}$ There are several differentiation operators acting on fields. The operator ∇ (nabla)

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y})$$

is useful when writing the derivatives of fields. Let $U \subset \mathbb{R}^3$ be an open set; define:

- If $f : U \to \mathbb{R}$ is a differentiable scalar field

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}) \text{ the gradient of } f\text{, a vector field.}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \text{ the laplacian of } f\text{, a scalar field.}$$

- If $\mathbf{F} : U \to \mathbb{R}^3, \mathbf{F} = (X, Y, Z)$ is a differentiable vector field

$$\text{rot } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ X & Y & Z \end{vmatrix} \text{ the rotational of } \mathbf{F}\text{, a vector field.}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \partial_x X + \partial_y Y + \partial_z Z \text{ the divergence of } \mathbf{F}\text{, a scalar field.}$$

For fields in $\mathbb{R}^2$ there are gradient, laplacian and divergence operators, but the rotational is not defined.

□

Problem 30: Gradient.

Remind the notation $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$; compute the gradient of the following scalar fields:

a) $f(x, y, z) = r$

b) $f(x, y, z) = \log r$ (logarithmic potential)

c) $f(x, y, z) = \frac{1}{r}$ (newtonian potential)

d) $f(x, y, z) = \frac{1}{r^n}, n = 2, 3, \ldots$

e) $f(x_1, \ldots, x_n) = \frac{1}{r^{n-2}}$ (generalized newtonian potential in $\mathbb{R}^n$)

Solution:

It is useful to have in mind the derivatives

$$\partial_x r = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}\ ,\ \partial_y r = \frac{y}{r}\ ,\ \partial_z r = \frac{z}{r}.$$

a)
$$\nabla r = \frac{1}{r}(x,y,z) = \frac{\mathbf{r}}{r} =: \mathbf{e}_r, \text{ a central field.}$$

b)
$$\nabla \log r = \frac{1}{r^2}(x,y,z) = \frac{\mathbf{r}}{r^2} = \frac{1}{r}\frac{\mathbf{r}}{r} = \frac{1}{r}\mathbf{e}_r, \text{ a central field.}$$

c)
$$\nabla(\frac{1}{r}) = -\frac{1}{r^3}(x,y,z) = -\frac{\mathbf{r}}{r^3} = -\frac{1}{r^2}\mathbf{e}_r,$$

precisely the gravitational field at point $\mathbf{r}$ of a unit mass at the origin.

d)
$$\nabla(\frac{1}{r^n}) = -n\frac{1}{r^{n+2}}(x,y,z) = -n\frac{1}{r^{n+1}}\mathbf{e}_r, \text{ a central field.}$$

e)
$$\nabla(\frac{1}{r^{n-2}}) = (2-n)\frac{1}{r^n}(x_1,\ldots,x_n) = \frac{2-n}{r^{n-1}}\mathbf{e}_r$$

Notice that from a) we may deduce that $\nabla\phi(r) = \phi'(r)\mathbf{e}_{\mathbf{r}}$.

□

Problem 31: Rotational and divergence.

Compute the rotational and the divergence of the following fields:

a) $\mathbf{F}(x,y,z) = (x,y,0)$

b) $\mathbf{F}(x,y,z) = (-y,x,0)$

c) $\mathbf{F}(x,y,z) = (x,y,z)$

d) $\mathbf{F}(x,y,z) = (y,z,x)$

e) $\mathbf{F}(x,y,z) = (z,x,y)$

f) $\mathbf{v}(x,y,z) = (a,b,c)\times(x,y,z) = \mathbf{w}\times\mathbf{r}$

Solution:

a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & 0 \end{vmatrix} = (0,0,0)$$
$$\nabla \cdot \mathbf{F} = \partial_x x + \partial_y y + \partial_z 0 = 2$$

b)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0,0,2)$$
$$\nabla \cdot \mathbf{F} = \partial_x(-y) + \partial_y x + \partial_z 0 = 0$$

c)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = (0,0,0)$$
$$\nabla \cdot \mathbf{F} = 3$$

d)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = (-1,-1,-1)$$
$$\nabla \cdot \mathbf{F} = 0$$

e)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} = (1,1,1)$$
$$\nabla \cdot \mathbf{F} = 0$$

f)

$$\mathbf{v} = (bz - cy, cx - az, ay - bx)$$
$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a,b,c)$$
$$\nabla \cdot \mathbf{v} = 0$$

□

Problem 32: Leibniz's rule $(uv)' = u'v + uv'$.

Let f, g be numerical funcions defined in $\mathbb{R}^3$ and let $\mathbf{F}, \mathbf{G}$ be vector fields in $\mathbb{R}^3$; observe the following schema

$$\begin{array}{lcll} \nabla f & \to & \nabla(fg), & \nabla(\mathbf{F}\cdot\mathbf{G}) \\ \nabla\cdot\mathbf{F} & \to & \nabla\cdot(f\mathbf{F}), & \nabla\cdot(\mathbf{F}\times\mathbf{G}) \\ \nabla\times\mathbf{F} & \to & \nabla\times(f\mathbf{F}), & \nabla\times(\mathbf{F}\times\mathbf{G}) \end{array}$$

and then show:

a) $\nabla(fg) = (\nabla f)g + f(\nabla g)$

b) $\nabla\cdot(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}$

c) $\nabla\times(f\mathbf{F}) = (\nabla f)\times\mathbf{F} + f(\nabla\times\mathbf{F})$

d) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})$

e) $\nabla\times(\mathbf{F}\times\mathbf{G}) = (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$

f) $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$

Solution:

Let $\mathbf{F} = (X, Y, Z)$ and $\mathbf{G} = (P, Q, R)$.

a)

$$\nabla(fg) = (\partial_x(fg), \partial_y(fg), \partial_z(fg)) =$$
$$= (g\partial_x f + f\partial_x g, g\partial_y f + f\partial_y g, g\partial_y f + f\partial_y g) = g\nabla f + f\nabla g$$

We may say that Leibniz's rule is satisfied.

b)

$$\begin{array}{rcl} \nabla\cdot(f\mathbf{F}) & = & \partial_x(fX) + \partial_y(fY) + \partial_z(fZ) = \\ & = & X\partial_x f + Y\partial_y f + Z\partial_z f + f(\partial_x X + \partial_y Y + \partial_z Z) = \\ & = & \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F} \end{array}$$

and we have Leibniz's rule satisfied again. Notice that once we know that this is so, then the formula is easy to remember.

c) Both terms of the formula are additive in $\mathbf{F}$. Then, as we can write any field in the form $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, it suffices to prove the equality for the fields $X\mathbf{i}, Y\mathbf{j}, Z\mathbf{k}$; for instance if $\mathbf{F} = X\mathbf{i}$ we have

$$\nabla \times (fX\mathbf{i}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ fX & 0 & 0 \end{pmatrix} = (0, \partial_z(fX), -\partial_y(fX))$$

Observe that in those simple cases we can calculate thus:

$$\nabla \times (fX\mathbf{i}) = (\partial_x \mathbf{i} + \partial_y \mathbf{j} + \partial_z \mathbf{k}) \times (fX\mathbf{i}) = -\partial_y(fX)\mathbf{k} + \partial_z(fX)\mathbf{j}$$

We do so in the next derivation; the second term is:

$$\begin{aligned} & (\nabla f) \times (X\mathbf{i}) + f(\nabla \times (X\mathbf{i})) = \\ = \ & (\partial_x f\mathbf{i} + \partial_y f\mathbf{j} + \partial_z f\mathbf{k}) \times (X\mathbf{i}) + f((\partial_x \mathbf{i} + \partial_y \mathbf{j} + \partial_z \mathbf{k}) \times (X\mathbf{i})) = \\ = \ & -X(\partial_y f)\mathbf{k} + X(\partial_z f)\mathbf{j} - f(\partial_y X)\mathbf{k} + f(\partial_z X)\mathbf{j} = \\ = \ & \partial_z(fX)\mathbf{j} - \partial_y(fX)\mathbf{k} \end{aligned}$$

We proceed in the same way with $Y\mathbf{j}, Z\mathbf{k}$ and so we have proved the formula. Again Leibniz's rule is satisfied.

d) As in c) both terms in the formula are additive in $\mathbf{F}$ and $\mathbf{G}$. Now we must consider all possible couples of fields choosen among $X\mathbf{i}, Y\mathbf{j}, Z\mathbf{k}$ and $P\mathbf{i}, Q\mathbf{j}, R\mathbf{k}$.
If there is a repeated basic vector, such as $X\mathbf{i}$ and $P\mathbf{i}$, the left hand term vanishes and for the right hand term we have as well

$$(\nabla \times \mathbf{F}) \cdot \mathbf{G} = (\nabla \times X\mathbf{i}) \cdot P\mathbf{i} = ((\partial_z X)\mathbf{j} - (\partial_y X)\mathbf{k}) \cdot P\mathbf{i} = 0$$

Choose now a couple such as $X\mathbf{i}$ and $Q\mathbf{j}$; the left hand term is:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \nabla \cdot (X\mathbf{i} \times Q\mathbf{j}) = \nabla \cdot (XQ\mathbf{k}) = \partial_z(XQ)$$

and the right hand term is:

$$\begin{aligned} (\nabla \times X\mathbf{i}) \cdot Q\mathbf{j} - X\mathbf{i} \cdot (\nabla \times Q\mathbf{j}) \ &= \ ((\partial_z X)\mathbf{j} - (\partial_y X)\mathbf{k}) \cdot Q\mathbf{j} - \\ & \quad -((\partial_z Q)\mathbf{i} - (\partial_x Q)\mathbf{k}) \cdot X\mathbf{i} = \\ &= \ Q\partial_z X + X\partial_z Q = \partial_z(XQ) \end{aligned}$$

and we have Leibniz's rule for those two fields. Obviously the same is true for all other couples and Leibniz rule works for any two fields. In applying it we have to be carefull; if we start the differentiation with $(\nabla \cdot \mathbf{F}) \times \mathbf{G}$, we see that it has no sense (because $\nabla \cdot \mathbf{F}$ is a scalar) and again the the formula is then 'evident'.

e) Once more both terms are additive in $\mathbf{F}$ and in $\mathbf{G}$ and we can proceed as in d). If there is a repeated basic vector, such as $X\mathbf{i}$ and $P\mathbf{i}$ we have:

$$\nabla \times (X\mathbf{i} \times P\mathbf{i}) = 0$$

and the right term is

$$(\nabla \cdot P\mathbf{i})X\mathbf{i} - (\nabla \cdot X\mathbf{i})P\mathbf{i} + (P\mathbf{i} \cdot \nabla)X\mathbf{i} - (X\mathbf{i} \cdot \nabla)P\mathbf{i} =$$
$$= (X\partial_x P)\mathbf{i} - (P\partial_x X)\mathbf{i} + (P\partial_x X)\mathbf{i} - (X\partial_x P)\mathbf{i} = 0$$

For a couple such as $X\mathbf{i}$ and $Q\mathbf{j}$ we have:

$$\nabla \times (X\mathbf{i} \times Q\mathbf{j}) = \nabla \times (XQ\mathbf{k}) = \partial_y(XQ)\mathbf{i} - \partial(XQ)\mathbf{j}$$

and

$$(\nabla \cdot Q\mathbf{j})X\mathbf{i} - (\nabla \cdot X\mathbf{i})Q\mathbf{j} + (Q\mathbf{j} \cdot \nabla)X\mathbf{i} - (X\mathbf{i} \cdot \nabla)Q\mathbf{j} =$$
$$= X(\partial_y Q)\mathbf{i} - Q(\partial_x X)\mathbf{j} + Q(\partial_y X)\mathbf{i} - X(\partial_x Q)\mathbf{j} =$$
$$= \partial_y(XQ)\mathbf{i} - \partial_x(XQ)\mathbf{j}$$

We proceed in the same way with the other couples and so we have proved the formula. In this case Leibniz's rule is *not* satisfied.

f) Following the same line of the preceding case, for $\mathbf{F} = X\mathbf{i}$ and $\mathbf{G} = P\mathbf{i}$ we have:

$$\nabla(X\mathbf{i} \cdot P\mathbf{i}) = \nabla(XP) = \partial_x(XP)\mathbf{i} + \partial_y(XP)\mathbf{j} + \partial_z(XP)\mathbf{k}$$

and the right hand term is:

$$(X\mathbf{i} \cdot \nabla)P\mathbf{i} + (P\mathbf{i} \cdot \nabla)X\mathbf{i} + X\mathbf{i} \times (\nabla \times P\mathbf{i}) + P\mathbf{i} \times (\nabla \times X\mathbf{i}) =$$
$$= (X\partial_x P)\mathbf{i} + (P\partial_x X)\mathbf{i} + X\mathbf{i} \times (\partial_z P)\mathbf{j} - (\partial_y P)\mathbf{k} + P\mathbf{i} \times (\partial_z X)\mathbf{j} - (\partial_y X)\mathbf{k} =$$
$$= (X\partial_x P)\mathbf{i} + (P\partial_x X)\mathbf{i} + (X\partial_y P)\mathbf{j} + (X\partial_z P)\mathbf{k} + (\partial_y X)\mathbf{j} + (P\partial_z X)\mathbf{k} =$$
$$= \partial_x(XP)\mathbf{i} + \partial_y(XP)\mathbf{j} + \partial_z(XP)\mathbf{k}$$

For couples such as $\mathbf{F} = X\mathbf{i}$ i $\mathbf{G} = P\mathbf{j}$ we have:

$$\nabla(X\mathbf{i} \cdot P\mathbf{j}) = \nabla(0) = 0$$

and the right hand term is:

$$(X\mathbf{i}\cdot\nabla)P\mathbf{j}+(P\mathbf{j}\cdot\nabla)X\mathbf{i}+X\mathbf{i}\times(\nabla\times P\mathbf{j})+P\mathbf{j}\times(\nabla\times X\mathbf{i})=$$
$$=(X\partial_x P)\mathbf{j}+(P\partial_y X)\mathbf{i}+X\mathbf{i}\times(\partial_x P\mathbf{k}-\partial_z P\mathbf{i})+P\mathbf{j}\times(-\partial_y X\mathbf{k}+\partial_z X\mathbf{j})=$$
$$=(X\partial_x P)\mathbf{j}+(P\partial_y X)\mathbf{i}-(X\partial_x P)\mathbf{j}-(P\partial_y X)\mathbf{i}=0$$

So Leibniz's rule is *not* true in this case.

Observation:

The language of differential forms allows shorter and clearer proofs of many of the preceding facts. You may want to look at *Algebraic and differential forms* by the same author.

□

Problem 33: Rotational and divergence.

Compute the rotational and the divergence of the following fields:

a) $\mathbf{F}(\mathbf{r})=\mathbf{r}$

b) $\mathbf{F}(\mathbf{r})=\frac{1}{r}\mathbf{r}$

c) $\mathbf{F}(\mathbf{r})=-\frac{1}{r^2}\mathbf{r}$

d) $\mathbf{F}(\mathbf{r})=-\frac{1}{r^3}\mathbf{r}$

e) $\mathbf{F}(\mathbf{r})=(x^2+y^2+z^2)(3\mathbf{i}+4\mathbf{j}+5\mathbf{k})$

f) $\mathbf{F}(\mathbf{r})=\frac{1}{x^2+y^2+z^2}(yz,zx,xy)$

Solution:

The first four fields have the form $\mathbf{F}(\mathbf{r})=f(r)\mathbf{r}$ and from the preceding problem, points b) and c), we have:

$$\nabla\times(f(r)\mathbf{r})=\nabla f\times\mathbf{r}+f(r)\nabla\times\mathbf{r}=f'(r)\frac{\mathbf{r}}{r}\times\mathbf{r}+0=0$$

$$\nabla\cdot(f(r)\mathbf{r})=(\nabla f)\cdot\mathbf{r}+f\nabla\cdot\mathbf{r}=f'(r)\frac{\mathbf{r}}{r}\cdot\mathbf{r}+3f(r)=rf'(r)+3f(r)$$

We obtain

a)

$$\nabla \times (1\mathbf{r}) = \mathbf{0}$$
$$\nabla \cdot (1\mathbf{r}) = 3$$

b)

$$\nabla \times (-\frac{1}{r}\mathbf{r}) = \mathbf{0}$$
$$\nabla \cdot (-\frac{1}{r}\mathbf{r}) = r\frac{1}{r^2} - 3\frac{1}{r} = -\frac{2}{r}$$

c)

$$\nabla \times (-\frac{\mathbf{r}}{r^2}) = \mathbf{0}$$
$$\nabla \cdot (-\frac{\mathbf{r}}{r^2}) = -(-r\frac{2}{r^3} + 3\frac{1}{r^2}) = -\frac{1}{r^2}$$

d)

$$\nabla \times (-\frac{\mathbf{r}}{r^3}) = \mathbf{0}$$
$$\nabla \cdot (-\frac{\mathbf{r}}{r^3}) = -(-r\frac{3}{r^4} + 3\frac{1}{r^3}) = 0$$

e)

$$\mathbf{F}(\mathbf{r}) = r^2(3,4,5)$$

We can apply Leibniz's rule:

$$\begin{aligned}
\nabla \times (r^2(3,4,5)) &= (\nabla r^2) \times (3,4,5) + r^2\nabla \times (3,4,5) = \\
&= 2r\frac{\mathbf{r}}{r} \times (3,4,5) = \mathbf{r} \times (6,8,10) = \\
&= (10y - 8z, 6z - 10x, 8x - 6y)
\end{aligned}$$

The divergence can be computed directly:

$$\nabla \cdot \mathbf{F} = \partial_x(3r^2) + \partial_y(4r^2) + \partial_z(5r^2) = 6r\frac{x}{r} + 8r\frac{y}{r} + 10r\frac{z}{r} = 6x + 8y + 10z$$

or we may use Leibniz's rule once more:

$$\nabla \cdot \mathbf{F} = 2r\frac{\mathbf{r}}{r} \cdot (3,4,5) + r^2\nabla \cdot (3,4,5) == 6x + 8y + 10z$$

f)

$$\mathbf{F}(\mathbf{r}) = \frac{1}{r^2}(yz, zx, xy)$$

$$\begin{aligned} \nabla \times (\frac{1}{r^2}(yz, zx, xy)) &= \nabla(\frac{1}{r^2}) \times (yz, zx, xy) + \frac{1}{r^2}\nabla \times (yz, zx, xy) = \\ &= -\frac{2}{r^3}\frac{\mathbf{r}}{r} \times (yz, zx, xy) + \frac{1}{r^2}\nabla \times (yz, zx, xy) \end{aligned}$$

We evaluate both terms:

$$\begin{aligned} \nabla \times (yz, zx, xy) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & zx & xy \end{vmatrix} = \mathbf{0} \\ -2\frac{\mathbf{r}}{r^4} \times (yz, zx, xy) &= -\frac{2}{r^4}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \\ &= -\frac{2}{r^4}(x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)) \end{aligned}$$

and then

$$\nabla \times \mathbf{F}(\mathbf{r}) = -\frac{2}{r^4}(x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2))$$

The divergence is:

$$\begin{aligned} \nabla \cdot (\frac{1}{r^2}(yz, zx, xy)) &= \nabla(\frac{1}{r^2}) \cdot (yz, zx, xy) + \frac{1}{r^2}\nabla \cdot (yz, zx, xy) = \\ &= -\frac{2}{r^3}\frac{\mathbf{r}}{r} \cdot (yz, zx, xy) = -\frac{2}{r^4}3xyz \end{aligned}$$

□

Problem 34: Around the vector product.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be vectors in $\mathbb{R}^3$ and $\mathbf{E}$ a vector field in $\mathbb{R}^3$; prove the following formulae:

a) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \det \begin{pmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{pmatrix}$

b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

c) $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$

Solution:

a) The left hand term

$$\varphi(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$$

is a linear function in each entry. And so is the right hand term

$$\psi(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \det\begin{pmatrix} \mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D} \\ \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D} \end{pmatrix}$$

Since both terms are linear in each entry to prove the desired equality it suffices to verify it on a basis, the canonical one $\mathbf{i},\mathbf{j},\mathbf{k}$ say. We can save work if we observe that both functions are alternate in $\mathbf{A},\mathbf{B}$ and alternate in $\mathbf{C},\mathbf{D}$ for then we know that $(\mathbf{i},\mathbf{j},\mathbf{j},\mathbf{k}) = -(\mathbf{j},\mathbf{i},\mathbf{j},\mathbf{k}) = -(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{j})$ etc. In brief, we only have to check the following combinations:

$$\begin{array}{ccc} (\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{j}) & (\mathbf{i},\mathbf{k},\mathbf{i},\mathbf{j}) & (\mathbf{j},\mathbf{k},\mathbf{i},\mathbf{j}) \\ (\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{k}) & (\mathbf{i},\mathbf{k},\mathbf{i},\mathbf{k}) & (\mathbf{j},\mathbf{k},\mathbf{i},\mathbf{k}) \\ (\mathbf{i},\mathbf{j},\mathbf{j},\mathbf{k}) & (\mathbf{i},\mathbf{k},\mathbf{j},\mathbf{k}) & (\mathbf{j},\mathbf{k},\mathbf{j},\mathbf{k}) \end{array}$$

Let us do it for the first:

$$\varphi(\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{j}) = (\mathbf{i}\times\mathbf{j})\cdot(\mathbf{i}\times\mathbf{j}) = \mathbf{k}\cdot\mathbf{k} = 1$$

and

$$\psi(\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{j}) = \det\begin{pmatrix} \mathbf{i}\cdot\mathbf{i} & \mathbf{i}\cdot\mathbf{j} \\ \mathbf{j}\cdot\mathbf{i} & \mathbf{j}\cdot\mathbf{j} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

The other cases are similar.

b) If $\mathbf{B}\times\mathbf{C} = 0$ then $\mathbf{B} = \lambda\mathbf{C}$ and both terms vanish. If $\mathbf{B}\times\mathbf{C} \neq 0$ then $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is orthogonal to $\mathbf{B}\times\mathbf{C}$ wich tells us that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \in \langle\mathbf{B},\mathbf{C}\rangle$:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = a\,\mathbf{B} + b\,\mathbf{C},$$

To find a, b it seems reasonable to make the scalar product with $\mathbf{B}$ and $\mathbf{C}$:

$$\left.\begin{array}{rcl} a\,\mathbf{B}\cdot\mathbf{B} + b\,\mathbf{C}\cdot\mathbf{B} & = & (\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))\cdot\mathbf{B} \\ a\,\mathbf{B}\cdot\mathbf{C} + b\,\mathbf{C}\cdot\mathbf{C} & = & (\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))\cdot\mathbf{C} \end{array}\right\}$$

Using the cyclic permutability of a triple product

$$(\mathbf{P}\times\mathbf{Q})\cdot\mathbf{R}=(\mathbf{Q}\times\mathbf{R})\cdot\mathbf{P}=(\mathbf{R}\times\mathbf{P})\cdot\mathbf{Q},$$

we can change the independent terms of the system to

$$(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))\cdot\mathbf{B}=(\mathbf{B}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C})=\det\begin{pmatrix}\mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C}\\ \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C}\end{pmatrix}$$

$$(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))\cdot\mathbf{C}=(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C})=\det\begin{pmatrix}\mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C}\\ \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C}\end{pmatrix}.$$

The determinant of the system is

$$\Delta=\det\begin{pmatrix}\mathbf{B}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{B}\\ \mathbf{B}\cdot\mathbf{C} & \mathbf{C}\cdot\mathbf{C}\end{pmatrix}$$

and applying Cramer's rule we obtain:

$$a=\frac{1}{\Delta}\det\begin{pmatrix}\det\begin{pmatrix}\mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C}\\ \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C}\end{pmatrix} & \mathbf{C}\cdot\mathbf{B}\\ \det\begin{pmatrix}\mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C}\\ \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C}\end{pmatrix} & \mathbf{C}\cdot\mathbf{C}\end{pmatrix}=$$

$$=\frac{1}{\Delta}\det\begin{pmatrix}(\mathbf{B}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{C}) & (\mathbf{B}\cdot\mathbf{C})\\ (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{C}) & (\mathbf{C}\cdot\mathbf{C})\end{pmatrix}=$$

$$=\frac{1}{\Delta}\det\begin{pmatrix}(\mathbf{B}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) & (\mathbf{B}\cdot\mathbf{C})\\ (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{C}) & (\mathbf{C}\cdot\mathbf{C})\end{pmatrix}=\frac{1}{\Delta}(\mathbf{A}\cdot\mathbf{C})\Delta=\mathbf{A}\cdot\mathbf{C}$$

In a similar way one obtains $b=-\mathbf{A}\cdot\mathbf{B}$ and b) is proved. There are much shorter proofs, but this one only needs endurance.

c) The laplacian of $\mathbf{F}=(X,Y,Z)$ is the *field*

$$\nabla^2\mathbf{F}=(\nabla^2X,\nabla^2Y,\nabla^2Z)$$

Both terms in the formula

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$

are linear in $\mathbf{E}$ and it suffices to check the equality on terms of the form $f(x,y,z)\mathbf{i}, g(x,y,z)\mathbf{j}, h(x,y,z)\mathbf{k}$. For instance if $\mathbf{E} = g\mathbf{j}$ the left hand member is

$$\nabla \times (\nabla \times g\mathbf{j}) = \nabla \times (-\partial_z g, 0, \partial_x g) = (\partial^2{}_{yx} g, -\partial^2{}_{xx} g - \partial^2{}_{zz} g, \partial^2{}_{yz} g)$$

and the right hand member is

$$\begin{aligned} \nabla(\nabla \cdot g\mathbf{j}) - \nabla^2 g\mathbf{j} &= \nabla(\partial_y g) - (\partial^2{}_{xx} g + \partial^2{}_{yy} g + \partial^2{}_{zz} g)\mathbf{j} = \\ &= (\partial^2{}_{xy} g, \partial^2{}_{yy} g, \partial^2{}_{zy} g) - (0, \partial^2{}_{xx} g + \partial^2{}_{yy} g + \partial^2{}_{zz} g, 0) = \\ &= (\partial^2{}_{xy} g, -\partial^2{}_{xx} g - \partial^2{}_{zz} g, \partial^2{}_{zy} g) \end{aligned}$$

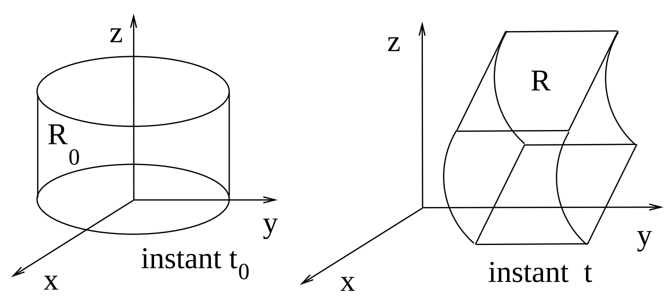
and they coincide whenever there is equality of mixed second order derivatives.

□

2.5 Fluid expansion; divergence

$\boxed{\mathbb{T}}$ Next we give a 'physical' interpretation of the divergence using a fluid as a model.

In the going by of time any region of the fluid changes its form. Lets think about a fluid region R_0 at the instant $t = t_0$ being colored. At a later instant t the coloring will show R, a different region; what is the time rate of change of the region's volume in terms of the velocity field?



□

Problem 35: Movement of regions in a fluid.

Consider the flow $\phi(x_0, y_0, z_0, t) = (x_0e^t, y_0e^{-t}, z_0), t_0 = 0$, and for $t = 0$ a region R_0. Find the tranformed region R at time t, the volume of this region and the time rate of change of the region's volume, in particular at $t = 0$, when:

a) $R_0 = \{(x_0, y_0, z_0) : x_0^2 + y_0^2 = a^2, 0 \leq z_0 \leq 1\}$, a straight circular cylinder of radius a and height 1.

b) $R_0 = \{(x_0, y_0, z_0) : 0 \leq x_0 \leq a, 0 \leq y_0 \leq a, 0 \leq z_0 \leq a\}$, a cube of side a with a vertex at the origin.

c) $R_0 = \{(x_0, y_0, z_0) :\mid x_0 \mid\leq 1, \mid y_0 \mid\leq 1, \mid z_0 \mid\leq 1\}$, a cube of side 2 centered at the origin.

d) $R_0 = \{(x_0, y_0, z_0) : x_0^2 + y_0^2 + z_0^2 = a^2\}$, a sphere of radius a centered at the origin.

Solution:

The point that was at (x_0, y_0, z_0) at the instant $t = 0$, is at $x = x_0e^t, y = y_0e^{-t}, z = z_0$ at the instant t. To find the equation of the transformed region we make the substitutions $x_0 = xe^{-t}, y_0 = ye^t, z_0 = z$:

a)

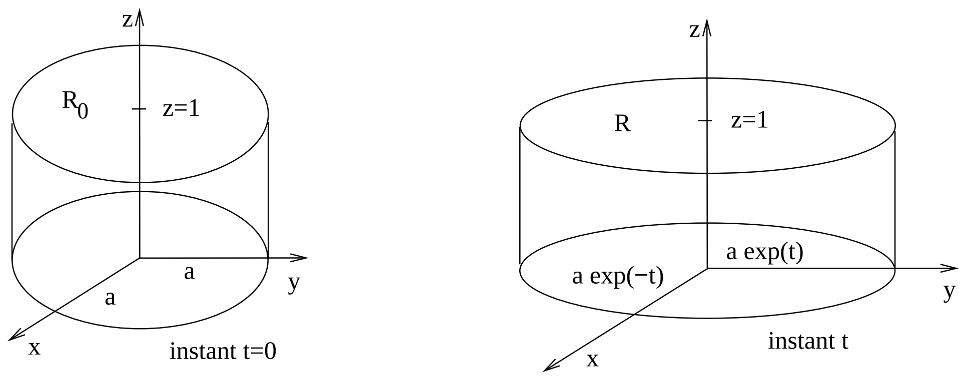
$$(xe^{-t})^2 + (ye^t)^2 = a^2, 0 \leq z \leq 1$$

$$x^2e^{-2t} + y^2e^{2t} = a^2, 0 \leq z \leq 1$$

that we can rewrite as

$$\frac{x^2}{(ae^t)^2} + \frac{y^2}{(ae^{-t})^2} = 1, 0 \leq z \leq 1,$$

showing an elliptic cylinder with volume $V(t) = \pi ae^tae^{-t} = \pi a^2$; then $\frac{dV}{dt} = \frac{d}{dt}(\pi a^2) = 0$ and in particular $\frac{d}{dt}|_{t=0}(\pi a^2) = 0$.



b) We proceed as in a) :

$$0 \le xe^{-t} \le a, 0 \le ye^{t} \le a, 0 \le z \le a$$

$$0 \le x \le ae^{t}, 0 \le y \le ae^{-t}, 0 \le z \le a$$

a parallelepiped with volume $V(t) = ae^t ae^{-t} a = a^3$; now $\frac{dV}{dt} = \frac{d}{dt}(a^3) = 0$ and $\frac{d}{dt}|_{t=0}(a^3) = 0$.

c) Analogously

$$| xe^{-t} | \le 1, | ye^{t} | \le 1, | z | \le 1$$

$$| x | \le e^{t}, | y | \le e^{-t}, | z | \le 1$$

a parallelepiped with volume $V(t) = e^t e^{-t} = 1$ and $\frac{dV}{dt} = \frac{d}{dt}(1) = 0$, $\frac{d}{dt}|_{t=0}(1) = 0$.

d) In this case

$$(xe^{-t})^2 + (ye^{t})^2 + z^2 = a^2$$

$$\frac{x^2}{(ae^t)^2} + \frac{y^2}{(ae^{-t})^2} + \frac{z^2}{a^2} = 1$$

an ellipsoid of volume $V(t) = \frac{4}{3}\pi ae^t ae^{-t} a = \frac{4}{3}\pi a^3$ and

$$\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi a^3) = 0$$

$$\frac{d}{dt}|_{t=0}(\frac{4}{3}\pi a^3) = 0$$

Why is it that all derivatives vanish?

□

Problem 36:

Do the same as in the preceding problem for the flow

$$\phi(x_0, y_0, z_0, t) = (x_0 + t, y_0 e^t, z_0), t_0 = 0.$$

Solution:

a)
$$(x-t)^2 + (ye^{-t})^2 = a^2, 0 \le z \le 1$$
$$\frac{(x-t)^2}{a^2} + \frac{y^2}{(ae^t)^2} = 1$$

an elliptic cylinder centered at $(t, 0, 0)$ with volume $V(t) = \pi a a e^t = \pi a^2 e^t$ and derivative $\frac{d}{dt}(\pi a^2 e^t) = \pi a^2 e^t$ and

$$\frac{d}{dt}|_{t=0}(\pi a^2 e^t) = \pi a^2.$$

b)
$$0 \le x - t \le a, 0 \le ye^{-t} \le a, 0 \le z \le a$$
$$t \le x \le a + t, 0 \le y \le ae^t, 0 \le z \le a$$

a parallelepiped with a vertex at $(t, 0, 0)$, volume $V(t) = aae^t a = a^3 e^t$ and derivative $\frac{d}{dt}(a^3 e^t) = a^3 e^t$ which at the instant $t = 0$ is

$$\frac{dV}{dt}|_{t=0} = a^3$$

c)
$$| x - t | \le 1, | y | \le e^t, | z| \le 1$$

a parallelepiped with volume $V(t) = 8e^t$ and $\frac{d}{dt}(8e^t) = 8e^t$ and

$$\frac{d}{dt}|_{t=0}(8e^t) = 8.$$

d)
$$(x-t)^2 + (ye^{-t})^2 + z^2 = a^2$$
$$\frac{(x-t)^2}{a^2} + \frac{y^2}{(ae^t)^2} + \frac{z^2}{a^2} = 1$$

an ellipsoid with volume $V(t) = \frac{4}{3}\pi a a e^t a = \frac{4}{3}\pi a^3 e^t$ with $\frac{d}{dt}(\frac{4}{3}\pi a^3 e^t) = \frac{4}{3}\pi a^3 e^t$ and then

$$\frac{d}{dt}|_{t=0}(\frac{4}{3}\pi a^3 e^t) = \frac{4}{3}\pi a^3$$

□

$\boxed{\mathbb{T}}$ In points a), b) an d) of the preceding problem the time rate of change of volume at the instant $t = 0$ depends on the bigness of the figure we color (it depends on a). To get rid of this fact we center our attention on

$$\frac{1}{V}\frac{dV}{dt}|_{t=0}$$

Then we have, respectively:

$$\frac{1}{\pi a^2}\frac{dV}{dt}|_{t=0} = \frac{\pi a^2}{\pi a^2} = 1$$

$$\frac{1}{a^3}\frac{dV}{dt}|_{t=0} = \frac{a^3}{a^3} = 1$$

$$\frac{1}{8}\frac{dV}{dt}|_{t=0} = \frac{8}{8} = 1$$

$$\frac{1}{\frac{4}{3}\pi a^3}\frac{dV}{dt}|_{t=0} = \frac{\frac{4}{3}\pi a^3}{\frac{4}{3}\pi a^3} = 1$$

Relation of $\frac{1}{V}\frac{dV}{dt}|_{t=0}$ with the divergence of the field $\mathbf{v}(\mathbf{x})$

Consider the flow

$$\phi(x_0, y_0, z_0, t) = (x(x_0, y_0, z_0, t), y(x_0, y_0, z_0, t), z(x_0, y_0, z_0, t)), t_0$$

and a region R_0 at the initial instant t_0. Let

$$\phi_t(x_0, y_0, z_0) = (x(x_0, y_0, z_0, t), y(x_0, y_0, z_0, t), z(x_0, y_0, z_0, t))$$

be the transformation during time t, $J_t(\mathbf{x}_0) = \det(\frac{\partial \phi_t}{\partial x_0}, \frac{\partial \phi_t}{\partial y_0}, \frac{\partial \phi_t}{\partial z_0})|_{(\mathbf{x}_0, t)}$ the jacobian determinant of the transformation, and $R(t) = \phi_t(R_0)$ the tranformed region. Then the change of variables theorem for integrals gives:

$$V(t) = \text{Vol }(R(t)) = \int\int\int_R dxdydz = \int\int\int_{R_0} |J_t(\mathbf{x}_0)| dx_0 dy_0 dz_0$$

Assuming $J_t(\mathbf{x}_0) > 0$ in R we have

$$\frac{dV}{dt} = \int\int\int_{R_0} \frac{d(J_t(\mathbf{x}_0))}{dt} dx_0 dy_0 dz_0$$

- Let's compute the integrand; we have:

$$\frac{d(J_t(\mathbf{x}_0))}{dt} =$$

$$= \det(\frac{\partial^2 \phi_t}{\partial t \partial x_0}, \frac{\partial \phi_t}{\partial y_0}, \frac{\partial \phi_t}{\partial z_0})|_{(\mathbf{x}_0,t)} + \det(\frac{\partial \phi_t}{\partial x_0}, \frac{\partial^2 \phi_t}{\partial t \partial y_0}, \frac{\partial \phi_t}{\partial z_0})|_{(\mathbf{x}_0,t)} + \det(\frac{\partial \phi_t}{\partial x_0}, \frac{\partial \phi_t}{\partial y_0}, \frac{\partial^2 \phi_t}{\partial t \partial z_0})|_{(\mathbf{x}_0,t)}$$

As we want to compute $\frac{dV}{dt}|_{(\mathbf{x}_0,t_0)}$ let us work for $t = t_0$; for instance

$$\begin{aligned} \frac{\partial \phi_t}{\partial x_0}|_{(\mathbf{x}_0,t_0)} &= \lim_{h\to 0} \frac{\phi(x_0+h, y_0, z_0, t_0) - \phi(x_0, y_0, z_0, t_0)}{h} = \\ &= \lim_{h\to 0} \frac{(x_0+h, y_0, z_0, t_0) - (x_0, y_0, z_0, t_0)}{h} = (1,0,0) \end{aligned}$$

and analogously:

$$\frac{\partial \phi_t}{\partial y_0}|_{t=t_0} = (0,1,0), \frac{\partial \phi_t}{\partial z_0}|_{t=t_0} = (0,0,1)$$

- For the second derivatives notice that $\frac{\partial \phi_t}{\partial t}|_{(\mathbf{x}_0,t)}$ is the velocity at the instant t of the particle that was at $\mathbf{x}_0$ at the instant t_0; then

$$\frac{\partial \phi_t}{\partial t}|_{(\mathbf{x}_0,t_0)} = (X(\mathbf{x}_0,t_0), Y(\mathbf{x}_0,t_0), Z(\mathbf{x}_0,t_0)),$$

the velocity field at $\mathbf{x}_0$.

- Thus

$$\frac{d(J_t(\mathbf{x}_0))}{dt}|_{t=t_0} = \det\begin{pmatrix} \frac{\partial X}{\partial x_0} & 0 & 0 \\ \frac{\partial Y}{\partial x_0} & 1 & 0 \\ \frac{\partial Z}{\partial x_0} & 0 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & \frac{\partial X}{\partial y_0} & 0 \\ 0 & \frac{\partial Y}{\partial y_0} & 0 \\ 0 & \frac{\partial Z}{\partial y_0} & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 & \frac{\partial X}{\partial z_0} \\ 0 & 1 & \frac{\partial Y}{\partial z_0} \\ 0 & 0 & \frac{\partial Z}{\partial z_0} \end{pmatrix} =$$

$$= (\frac{\partial X}{\partial x_0} + \frac{\partial Y}{\partial y_0} + \frac{\partial Z}{\partial z_0})|_{(\mathbf{x}_0,t_0)}$$

and

$$\frac{dV}{dt}|_{t=t_0} = \int\int\int_{R_0} (\frac{\partial X}{\partial x_0} + \frac{\partial Y}{\partial y_0} + \frac{\partial Z}{\partial z_0})|_{(\mathbf{x}_0,t_0)} dx_0 dy_0 dz_0$$

Now the mean value theorem for integrals asserts there is a point $P \in R_0$ such that

$$\frac{dV}{dt} = (\frac{\partial X}{\partial x_0} + \frac{\partial Y}{\partial y_0} + \frac{\partial Z}{\partial z_0})|_P V$$

Finally letting $V \to 0$, at each point $\mathbf{x}_0 \in R_0$ we have

$$\lim_{V \to 0} \frac{1}{V}\frac{dV}{dt} = \frac{\partial X}{\partial x_0} + \frac{\partial Y}{\partial y_0} + \frac{\partial Z}{\partial z_0} = \nabla \cdot \mathbf{v}$$

and we see that the divergence of the velocity field of the fluid is the instantaneous time rate of change of volume per unit volume.

□

Problem 37:

Check this last result in the two preceding problems.

Solution:

a) The velocity field of the flow $\phi(x_0, y_0, z_0, t) = (x_0 e^t, y_0 e^{-t}, z_0), t_0 = 0$ is $\mathbf{v}(\mathbf{x}) = (x, -y, 0)$ by problem **25**, and its divergence is $\nabla \cdot \mathbf{v} = 1 - 1 = 0$ that agrees whith problem **35**.

b) For $\phi(x_0, y_0, z_0, t) = (x_0 + t, y_0 e^t, z_0), t_0 = 0$ we have

$$\frac{d\phi}{dt} = (1, y_0 e^t, 0)$$

and the velocity field is $\mathbf{v}(\mathbf{x}) = (1, y, 0)$; then $\nabla \cdot \mathbf{v} = 1$ in agreement with problem **36**.

□

Chapter 3

Integration of fields over curves

3.1 Integration of scalar fields

$\boxed{\mathbb{T}}$ Let $\gamma : [a, b] \to U$ be a $\mathcal{C}^1$ parametrized curve in the open set $U \subset \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a continuous function. The integral of f along γ is

$$\int_\gamma f dl = \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

If C is a curve in the open set $U \subset \mathbb{R}^n$ the integral of f along C is

$$\boxed{\int_C f dl = \int_a^b f(\gamma(t))|\gamma'(t)|dt}$$

$\gamma : [a, b] \longrightarrow U$ being a parametrization of C of class $\mathcal{C}^1$; this definition makes sense because it is independent of which one among the equivalent parametrizations of C we choose (see problem p.90).

We can as well integrate on a piecewise $\mathcal{C}^1$ curve: we simply integrate on each of the subintervals where the curve is $\mathcal{C}^1$ and add the results.

Taking $f \equiv 1$ we obtain the *length* of the curve; to have the true geometric length we use regular parametrizations.

$$\boxed{L = \int_a^b |\gamma'(t)|dt}$$

Note that as $|\gamma'(t)|$ is the celerity, $|\gamma'(t)|dt$ is the length of that part of the curve traversed during the small time interval dt. We get the total length adding (i.e.: integrating).

Problem 38: Scalar field.

Compute $\int_C f dl$ in each of the following cases (γ is a parametrization of C):

a) $f(x, y, z) = x + y + z, \gamma(t) = (\cos t, \sin t, t), 0 \leq t \leq 2\pi$.

b) $f(x, y, z) = z, \gamma(t) = (t \cos t, t \sin t, t), 0 \leq t \leq T$.

c) $f(x, y) = 2x - y, \gamma(t) = (t^4, t^4), -1 \leq t \leq 1$.

d) $f(x, y) = x^2 + y^2, C$ the square with vertexs $(\pm 1, 0), (0, \pm 1)$ traversed in the positive sense.

e) $f(x, y, z) = x + y, C$ the part of the circumference

$$\begin{cases} x^2 + y^2 + z^2 &= R^2 \\ y &= x \end{cases}$$

in the first octant.

Solution:

a) $\gamma'(t) = (-\sin t, \cos t, 1)$,

$$|\gamma'(t)| = \sqrt{2}$$

$$\int_C (x + y + z) dl = \int_0^{2\pi} (\cos t + \sin t + t)\sqrt{2} dt = 2\pi^2\sqrt{2}$$

b) $\gamma'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$,

$$|\gamma'(t)| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} = \sqrt{2 + t^2}$$

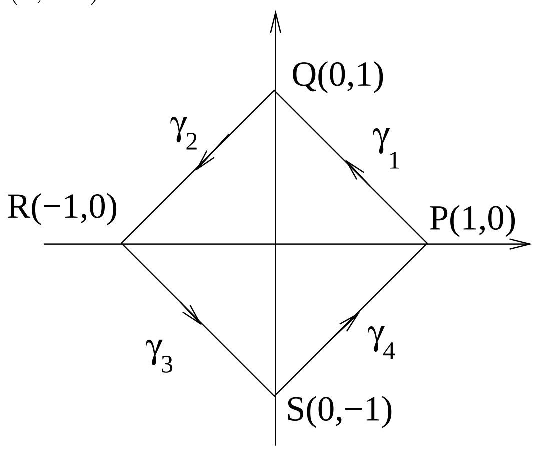
$$\int_C z dl = \int_0^T t\sqrt{2 + t^2} dt = \frac{1}{2} \int_0^T 2t\sqrt{2 + t^2} dt = \frac{1}{2}\frac{2}{3}(2+t^2)^{3/2}|_0^T = \frac{1}{3}[(2+T^2)^{3/2} - 2^{3/2}]$$

c) $\gamma'(t) = (4t^3, 4t^3)$,

$$|\gamma'(t)| = 4\sqrt{2t^6} = 4\sqrt{2}|t|^3$$

$$\int_C (2x - y) dl = \int_{-1}^1 (2t^4 - t^4) 4\sqrt{2}|t|^3 = 4\sqrt{2}(2 \int_0^1 t^7 dt) = \sqrt{2}$$

d) We traverse the square in the order $P = (1,0), Q = (0,1), R = (-1,0), S = (0,-1)$.



Observe that if γ_1 parametrizes the segment $[P,Q]$ then $\gamma_3 = -\gamma_1$ parametrizes $[R,S]$; analogously, if γ_2 parametrizes the segment $[Q,R]$ then $\gamma_4 = -\gamma_2$ parametrizes $[S,P]$. We have:

$$\gamma_1(t) = (1-t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-t \\ t \end{pmatrix}, 0 \le t \le 1$$

$$\gamma_2(t) = (1-t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ 1-t \end{pmatrix}, 0 \le t \le 1$$

and

$$\gamma_1'(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad , \quad \gamma_2'(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
$$|\gamma_1'(t)| = \sqrt{2} \quad , \quad |\gamma_2'(t)| = \sqrt{2}$$

Now we can compute the integrals:

$$\int_{\gamma_1} (x^2+y^2)dl = \int_0^1 [(1-t)^2 + t^2]\sqrt{2}dt = \sqrt{2}\int_0^1 (2t^2 - 2t + 1)dt = \frac{2\sqrt{2}}{3}$$

and, taking into account that the squares in the integrand eliminate minus signs, we also have $\int_{\gamma_3}(x^2+y^2)dl = \frac{2\sqrt{2}}{3}$. On the other side

$$\int_{\gamma_2} (x^2+y^2)dl = \int_0^1 ((-t)^2+(1-t)^2)\sqrt{2}dt = \sqrt{2}\int_0^1 (2t^2-2t+1)dt = \frac{2\sqrt{2}}{3}$$

and we also have $\int_{\gamma_4}(x^2+y^2)dl = \frac{2\sqrt{2}}{3}$. Finally

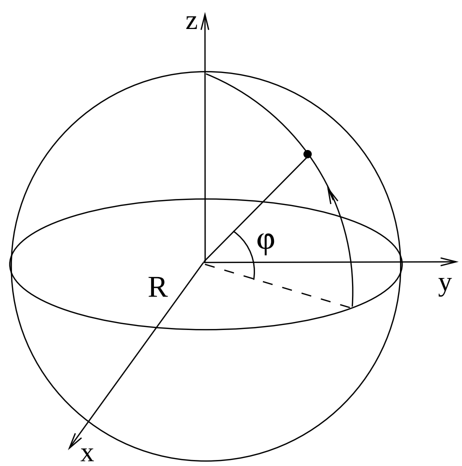
$$\int_C (x^2+y^2)dl = \frac{8\sqrt{2}}{3}$$

e) Take the parametrization

$$\gamma(\varphi) = (\frac{R}{\sqrt{2}}\cos\varphi, \frac{R}{\sqrt{2}}\cos\varphi, R\sin\varphi), 0 \le \varphi \le \frac{\pi}{2}$$

$$\gamma'(\varphi) = (-\frac{R}{\sqrt{2}}\sin\varphi, -\frac{R}{\sqrt{2}}\sin\varphi, R\cos\varphi)$$

$$\mid \gamma'(\varphi) \mid = \sqrt{\frac{R^2}{2}\sin^2\varphi + \frac{R^2}{2}\sin^2\varphi + R^2\cos^2\varphi} = R$$



We obtain

$$\int_\gamma (x+y)dl = \int_0^{\pi/2} \frac{2R}{\sqrt{2}}\cos\varphi R d\varphi = \sqrt{2}R^2\int_0^{\pi/2}\cos\varphi d\varphi = \sqrt{2}R^2$$

□

Problem 39: Independence of the parametrization.

Prove that

$$\int_C f dl$$

does not depend on the parametrization of the curve C.

Solution:

Let $\gamma(t), t \in [a,b]$ be a parametrization of C, $t = h(\tau)$ a change of variable

$$\begin{array}{rccc} h: & [a,b] & \to & [c,d] \\ & t & \mapsto & \tau = h(t) \end{array}$$

and $\Gamma(\tau)$ the corresponding reparametrization: $\gamma(t) = \Gamma(h(t))$. Then

$$\gamma'(t) = \Gamma'(h(t))h'(t)$$

To use the change of variables theorem for integrals in one variable notice that as h is a diffeomorphism it has a nonvanishing derivative. Then if $h' > 0$, h is increasing, $h(a) = c, h(b) = d$ and if $h' < 0$, h is decreasing and $h(a) = d, h(b) = c$. Now apply the theorem:

$$I = \int_c^d f(\Gamma(\tau))|\Gamma'(\tau)|d\tau = \left\{ \begin{array}{ccc} \tau & = & h(t) \\ d\tau & = & h'(t)dt \end{array} \right\} =$$

$$= \left\{ \begin{array}{ll} \int_a^b f(\Gamma(h(t)))|\Gamma'(h(t))|h'(t)dt & \text{if} \quad h' > 0 \\ \int_b^a f(\Gamma(h(t)))|\Gamma'(h(t))|h'(t)dt & \text{if} \quad h' < 0 \end{array} \right\}$$

In the first case $|h'(t)| = h'(t)$ and

$$I = \int_a^b f(\Gamma(h(t)))|\Gamma'(h(t))h'(t)|dt = \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

In the second case $|h'(t)| = -h'(t)$ and

$$I = \int_b^a f(\Gamma(h(t)))|\Gamma'(h(t))|h'(t)dt = -\int_a^b f(\Gamma(h(t)))|\Gamma'(h(t))|h'(t)dt =$$

$$= \int_a^b f(\Gamma(h(t)))|\Gamma'(h(t))|(-h'(t))dt = \int_a^b f(\Gamma(h(t)))|\Gamma'(h(t))h'(t)|dt =$$

$$= \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

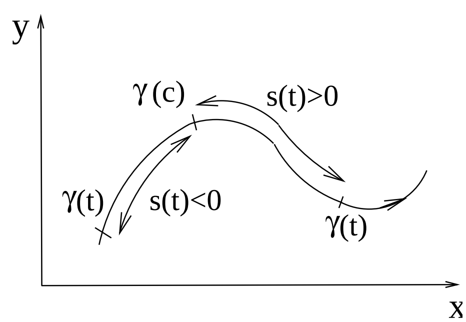
□

3.1.1 Arc-length

$\boxed{\mathbb{T}}$ Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular parametrized curve; the function

$$s(t) = \int_c^t |\gamma'(t)|dt,\ c, t \in [a,b]$$

measures the length of the arc of the curve from the point $\gamma(c)$ to the point $\gamma(t)$ and is called the *arc-length parameter* of the curve:



Notice that the arc-length parameter has a positive value for $t > c$ and a negative value for $t < c$. If we choose $c = b$ then all the values of the arc-length parameter will be negative, while they will all be positive if $c = a$. Think about the curve as a deformed interval of $\mathbb{R}$ with its own origin (the point $\gamma(c)$), its positive points and its negative points, as shown in the preceding figure.

If we assume γ to be regular then $s'(t) = |\gamma'(t)| > 0$, the arc-length parameter is a strictly increasing function and so it is inversible. Substituting $t = t(s)$ into γ we obtain the *arc-length reparametrization*.

Problem 40: Reparametrization by arc-length.

Compute the length and reparametrize by arc-length:

a) A circumference of radius R.

b) The arc of the helix $\gamma(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$.

c) A complete turn of the helix $\gamma(t) = (a\cos t, a\sin t, bt)$ for $0 \leq t \leq 2\pi$.

d) The arc of a spiral $\gamma(t) = (e^t\cos t, e^t\sin t)$, $0 \leq t \leq T$.

Solution:

a) Take the parametrization

$$\gamma(t) = (R\cos t, R\sin t),\ t \in [0, 2\pi]$$

$$\gamma'(t) = (-R\sin t, R\cos t),\ |\gamma'(t)| = R.$$

Then

$$L = \int_0^{2\pi} R dt = 2\pi R$$

$$s(t) = \int_0^t R dt = Rt \Rightarrow t = \frac{s}{R}$$

Substituting $t = t(s)$ into the parametrization Γ we obtain the reparametrization by arc-length :

$$\Gamma(s) = (R\cos\frac{s}{R}, R\sin\frac{s}{R}),\ s \in [0, 2\pi R]$$

b)

$$\gamma'(t) = (-\sin t, \cos t, 1),\ |\gamma'(t)| = \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2} dt = t\sqrt{2} \Rightarrow t = \frac{s}{\sqrt{2}}$$

The reparametrization by arc-length is:

$$\Gamma(s) = (\cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}}),\ s \in [0, 2\pi\sqrt{2}]$$

c)

$$\gamma'(t) = (-a\sin t, a\cos t, b),\ |\gamma'(t)| = \sqrt{a^2+b^2}$$

$$L = \int_0^{2\pi} \sqrt{a^2+b^2} dt = 2\pi\sqrt{a^2+b^2}$$

$$s(t) = \int_0^t \sqrt{a^2+b^2} dt = \sqrt{a^2+b^2} t \Rightarrow t = \frac{s}{\sqrt{a^2+b^2}}$$

The reparametrization by arc-length is:

$$\Gamma(s) = (a\cos(\frac{s}{\sqrt{a^2+b^2}}), a\sin(\frac{s}{\sqrt{a^2+b^2}}), b\frac{s}{\sqrt{a^2+b^2}}),\ s \in [0, 2\pi\sqrt{a^2+b^2}]$$

Notice that if $\sqrt{a^2+b^2} = 1$ the helix is already parametrized by arc-length.

d)

$$\gamma'(t) = e^t(\cos t - \sin t, \cos t + \sin t),\ |\gamma'(t)| = e^t\sqrt{2}$$

$$L = \int_0^T \sqrt{2}e^t dt = \sqrt{2}(e^T - 1)$$

$$s(t) = \int_0^t \sqrt{2}e^t dt = \sqrt{2}(e^t - 1) \Rightarrow t = \log(\frac{s}{\sqrt{2}} + 1)$$

The reparametrization by arc-length is:

$$\Gamma(s) = (\frac{s}{\sqrt{2}} + 1)(\cos(\log(\frac{s}{\sqrt{2}} + 1)), \sin(\log(\frac{s}{\sqrt{2}} + 1)))$$

□

Problem 41: Parameter arc.

Discuss the following assertion: a regular parametrized curve $\gamma : [a,b] \to \mathbb{R}^n$ is parametrized by arc-length iff $|\gamma'(t)| = 1$, that is to say iff the celerity along the curve is unity.

Solution:

a) For $\gamma(t)$ to be parametrized by arc-length it should be a reparametrization by arc-length of a certain parametrization $\alpha(\tau), \tau \in [c,d]$. If that is the case we have

$$t(\tau) = \int_\xi^\tau |\alpha'(\tau)| d\tau, \xi \in [c,d]$$

$$\frac{dt}{d\tau} = |\alpha'(\tau)| > 0 \Rightarrow \frac{d\tau}{dt} = \frac{1}{|\alpha'(\tau)|}$$

and

$$\gamma(t) = \alpha(\tau(t)) \Rightarrow \gamma'(t) = \alpha'(\tau(t))\tau'(t)$$

and we obtain

$$|\gamma'(t)| = |\alpha'(\tau(t))| \frac{1}{|\alpha'(\tau(t))|} = 1$$

From a physical point of view we have a 'clock' s (the arc-length) that runs so as to have a displacement along the curve equal in measure to the 'time' elapsed. Now it's clear-cut that the celerity will be 1.

b) Reciprocally, assume that $|\gamma'(t)| = 1$; then the arc-length parameter from $c \in [a, b]$ is

$$s(t) = \int_c^t |\gamma'(t)| dt = t - c \Rightarrow t = s + c$$

and the new parametrization $\Gamma(s) = \gamma(s + c)$ just shows that we take the origin of the new measure of lengths at the point of the curve corresponding to $t = c$. We may reasonably say that the curve is parametrized by arc-length. In the particular case that $0 \in [a, b]$ we can take the parameter arc $s(t) = \int_0^t |\gamma'(t)| dt$ and then $s = t$, the curve is parametrized by arc-length.

□

Problem 42: Length of a graph.

If $f : [a, b] \to \mathbb{R}$ is a $\mathcal{C}^1$ function, the length of its graph is that of the parametrized curve $\gamma(t) = (t, f(t))$, $t \in [a, b]$.

a) Show that the length of f's graph is

$$\boxed{L = \int_a^b \sqrt{1 + (f'(t))^2} dt}$$

b) Compute the length of the graph of $f(t) = \cosh t$ over the interval $[0, 1]$.

Solution:

a)

$$\gamma'(t) = (1, f'(t)), |\gamma'(t)| = \sqrt{1 + (f'(t))^2}$$

$$L = \int_a^b \sqrt{1 + (f'(t))^2} dt$$

b)

$$\begin{aligned} L &= \int_0^1 \sqrt{1+\sinh^2 t}dt = \int_0^1 \sqrt{\cosh^2 t}dt = \int_0^1 \cosh t\, dt = \\ &= \sinh t|_0^1 = \sinh 1 = \frac{1}{2}(e - \frac{1}{e}) \end{aligned}$$

□

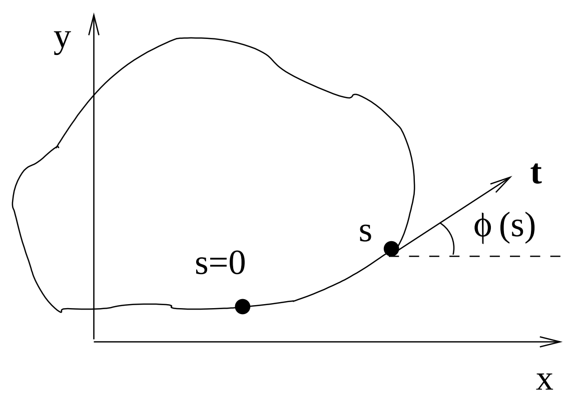
Problem 43:

Let C be a plane closed curve that has a regular parametrization and let ϕ be the angle the tangent vector $\mathbf{t}$ makes with the Ox axis. Prove that

$$\int_C \cos\phi dl = \int_C \sin\phi dl = 0$$

Solution:

Let's make a figure



Let $\gamma(s)$ be a parametrization of C by arc-length, defined in $[0, L]$ (L the length of C); in this way the tangent vector $\mathbf{t} = \gamma'(s)$ will be unitary and $\cos\phi$ and $\sin\phi$ will be easily computable. We have:

$$\begin{aligned} \cos\phi(s) &= \mathbf{t}(s)\cdot\mathbf{i} = (x'(s), y'(s))\cdot(1,0) = x'(s) \\ \int_C \cos\phi(s)ds &= \int_0^L x'(s)ds = x(L) - x(0) = 0\,(C\,\text{is closed}) \end{aligned}$$

Analogously

$$\begin{aligned} \sin\phi(s) &= \mathbf{t}(s)\cdot\mathbf{j} = (x'(s), y'(s))\cdot(0,1) = y'(s) \\ \int_C \sin\phi(s)ds &= \int_0^L y'(s)ds = y(L) - y(0) = 0\,(C\text{ is closed}) \end{aligned}$$

□

Problem 44: A curve in polar coordinates.

a) Show that the integral of a continuous function $f(x, y)$ along the curve given in polar coordinates by

$$r = r(\theta), \theta_1 \le \theta \le \theta_2$$

is

$$\int_{\theta_1}^{\theta_2} f(r(\theta)\cos\theta, r(\theta)\sin\theta)\sqrt{r(\theta)^2 + (\frac{dr}{d\theta})^2}d\theta$$

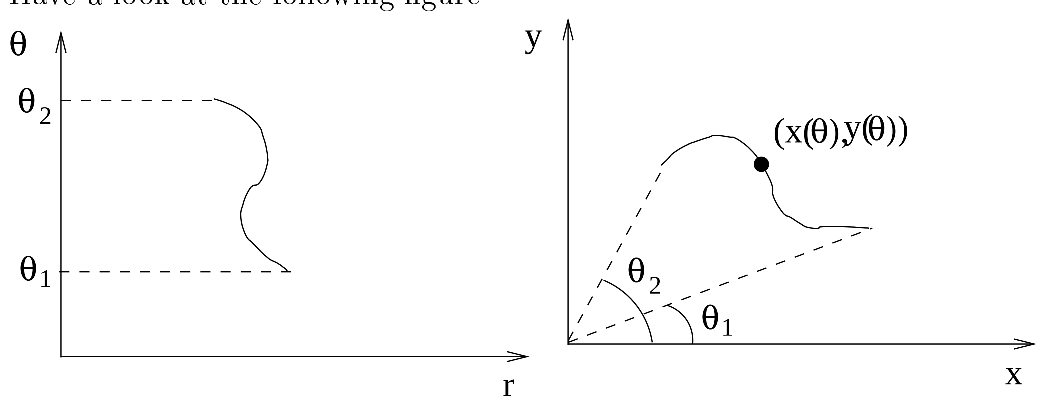
b) Compute the length of a cardioid $r = 1 + \cos\theta$, $0 \le \theta \le 2\pi$.

c) Compute $\int_C \arctan(\frac{y}{x})\,dl$ along the curve $r = 2\theta$, $0 \le \theta \le 2$.

d) Find a formula for the curve $r = r(t), \theta = \theta(t)$.

Solution:

a) Have a look at the following figure



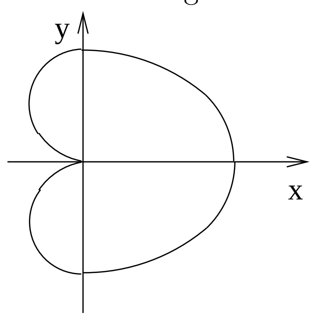
In cartesian coordinates we have

$$\begin{aligned}\gamma(\theta) &= (r(\theta)\cos\theta, r(\theta)\sin\theta)\\ \gamma'(\theta) &= (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta)\\ |\gamma'(\theta)| &= \sqrt{r^2 + r'^2}\end{aligned}$$

and

$$\int_C f dl = \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$$

b) We make a figure and use the preceding formula



$$\begin{aligned}L &= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta d\theta} = \int_0^{2\pi} \sqrt{2(1+\cos\theta)}d\theta =\\ &= 2\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}d\theta = 2\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2(\frac{\theta}{2})}d\theta =\\ &= 4\int_0^{\pi} \cos(\frac{\theta}{2})d\theta = 8\sin(\frac{\theta}{2})|_0^{\pi} = 8\end{aligned}$$

c) Still the same formula

$$\begin{aligned}\int_C \arctan(\frac{y}{x})\, dl &= \int_0^2 \arctan(\tan\theta)\sqrt{4\theta^2+4}\, d\theta =\\ &= \int_0^2 2\theta\sqrt{c1+\theta^2}2d\theta = \frac{2}{3}(1+\theta^2)^{3/2}|_0^2 =\\ &= \frac{2}{3}(5^{3/2} - 1)\end{aligned}$$

d) In cartesian coordinates we have the curve

$$\begin{aligned}\gamma(t) &= (r(t)\cos\theta(t), r(t)\sin\theta(t)) \\ \gamma'(t) &= (r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t), r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t)) \\ |\gamma'(t)| &= \sqrt{r'^2 + r^2\theta'^2}\end{aligned}$$

and the formula is

$$\int_C f dl = \int_{t_1}^{t_2} f(r\cos\theta, r\sin\theta)\sqrt{r'^2 + r^2\theta'^2}dt$$

□

3.1.2 Averages

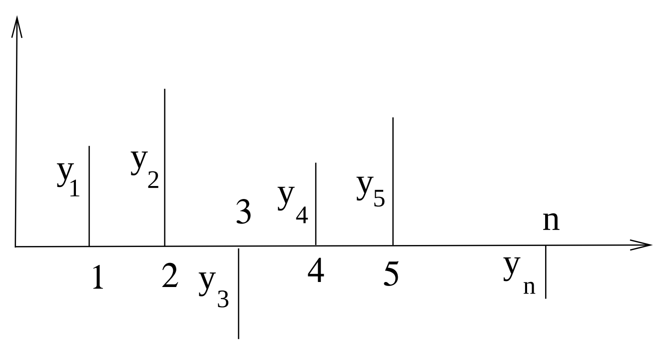
[T] We start with the elementary idea that the average of two numbers a, b is $\frac{a+b}{2}$ and develop several associated ideas.

Average of n numbers

The average of the numbers $y_1, \ldots, y_n$ is

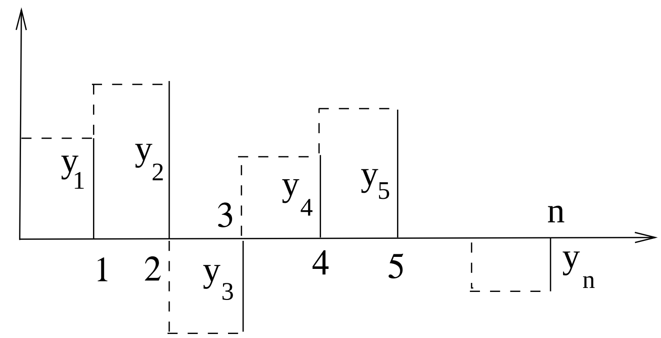
$$\langle y \rangle = \frac{y_1 + \cdots + y_n}{n}$$

We can think those numbers as a finite quantity of magnitudes discretely distributed:



If we form rectangles of basis 1 and heights y_j, the total algebraic area (:= areas *below* the axis counted as negative) is

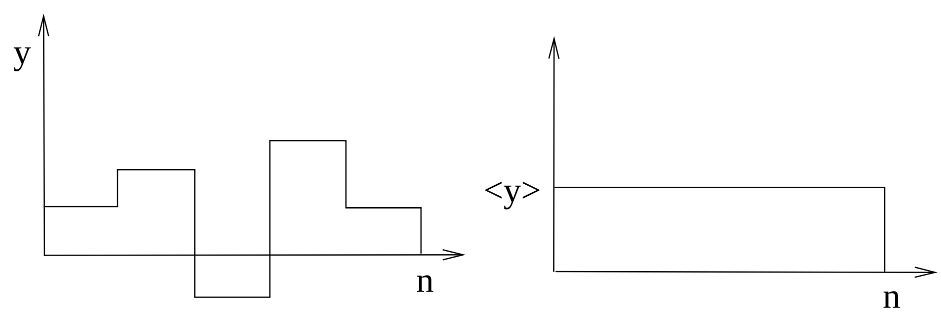
$$A = 1 \cdot y_1 + \cdots + 1 \cdot y_n$$



From the definition of the average we obtain:

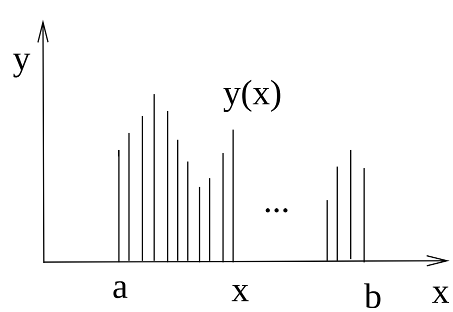
$$A = y_1 + \cdots + y_n = n\langle y\rangle$$

that tells us that $\langle y\rangle$ is the height of a rectangle with basis n that has the same algebraic area:



Average of a function

Let us draw a magnitude continuously distributed on the segment $[a, b]$:

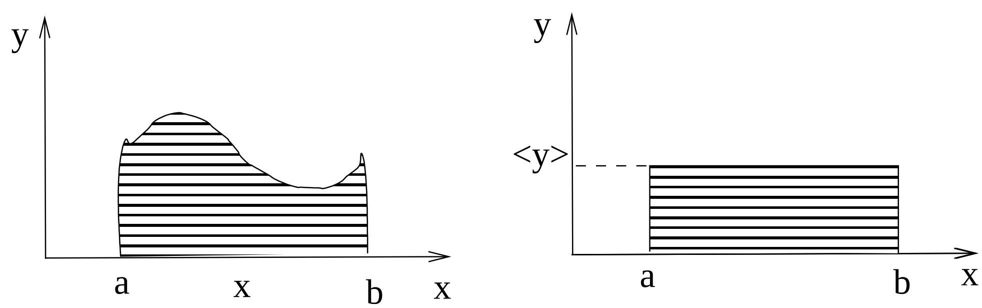


We have a bar at *each* point of the segment; that is we have a function $y(x)$ defined in $[a,b]$. By analogy with the discrete case we define the average of the function $y(x)$ on $[a,b]$ by

$$\langle y\rangle = \frac{'\sum_i' y(i)}{\sharp(i)} = \frac{\int_a^b y(x)dx}{b-a}$$

where loosely speaking $\sharp(i)$ is the 'number of indexes'; the length of the interval is a measure of this number. $'\sum'$ is the corresponding analog to the sum of the discrete case; the integral of $y(x)$ does that. The analogy takes roots if we notice that in the discrete case we can write $y_1,\ldots,y_n$ in the form $y(1),\ldots,y(n)$, a function defined in $\{1,2,\ldots,n\}$.

Geometrically $\langle y\rangle$ is the height of a rectangle with basis $[a,b]$ that has the same algebraic area as that under the graph of the function:



The first mean value theorem for integrals tells us that if y is continuous there is $c\in[a,b]$ such that $\langle y\rangle = y(c)$: the average value is accessed.

Average on a curve

The average of a function f defined on a curve is a concept close to the preceding one. It is clear that we should define

$$\langle f\rangle = \frac{1}{L}\int_C f dl$$

□

Problem 45: Average on a curve.

Let $\gamma(t) = (\sin t, \cos t, t)$, $t \in [0, 2\pi]$ be a parametrization of an helix; compute the average of f when

a) $f(x, y, z) = x + y + z$

b) $f(x, y, z) = \cos z$

Solution:

a)

$$\begin{aligned}\int_C (x+y+z)dl &= \int_0^{2\pi} (\sin t + \cos t + t)\sqrt{\cos^2 t + \sin^2 t + 1}dt = \\ &= \sqrt{2}\frac{t^2}{2}|_0^{2\pi} = 2\pi^2\sqrt{2}\end{aligned}$$

$$\begin{aligned}L &= \int_C 1dl = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2} \\ \langle f \rangle &= \frac{2\pi^2\sqrt{2}}{2\pi\sqrt{2}} = \pi\end{aligned}$$

b)

$$\int_C \cos z dl = \int_0^{2\pi} \cos t\,\sqrt{2}dt = 0 \Rightarrow \langle f \rangle = 0$$

□

3.1.3 Averages with weights

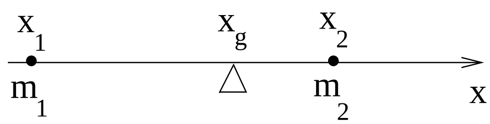
$\boxed{\mathbb{T}}$ Discrete case

To assign weights m_1, m_2 to the numbers x_1, x_2 is something like counting the number x_1 "m_1 times" and the number x_2 "m_2 times" (we have written " " because in general m_1, m_2 need not be integer numbers). From this point of view the average should be defined as

$$\langle x \rangle_m = \frac{x_1 + \overset{m_1)}{\cdots} + x_1 + x_2 + \overset{m_2)}{\cdots} + x_2}{1 + \overset{m_1)}{\cdots} + 1 + 1 + \overset{m_2)}{\cdots} + 1} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Geometrically, we put weights (or masses) m_1, m_2 at the points of the line with coordinates x_1, x_2.

The point that equilibrates the fulcrum of the figure is called the system's center of mass.



The center of mass coordinate x_g must satisfy the fulcrum law:

$$(x_g - x_1)m_1 = (x_2 - x_g)m_2$$

or

$$x_g = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

We see that the coordinate of the center of mass coincides with the average value with weights, a somewhat remarcable result.

□

Problem 46: Center of mass.

Define the center of mass of

a) k points in a line.

b) k points in space.

Solution:

a) Consider masses $m_1, \ldots, m_k$ at the points with coordinates $x_1, \ldots, x_k$. Then

$$\langle x \rangle_m = \frac{m_1x_1 + \cdots + m_kx_k}{m_1 + \cdots + m_k}$$

b) Consider masses $m_1, \ldots, m_k$ at the points with position vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$. Now

$$\langle \mathbf{x} \rangle_m = \frac{m_1\mathbf{x}_1 + \cdots + m_k\mathbf{x}_k}{m_1 + \cdots + m_k}$$

□

Continuous case

$\boxed{\mathbb{T}}$ If a material wire has linear density λ (see p.57), its total mass is

$$M = \int_C \lambda dl$$

Notice that any kind of density (lineal, surface, volume, probability, etc.) ends up being integrated, like in the preceding formula. Speaking about densities notice also that we can think about the velocity as being the density of space with respect to time.

Consider a material segment $[a, b]$ (we could equally well say 'consider a mass distribution on the segment $[a, b]$'). By analogy with the discrete case we define the center of mass of the segment through

$$\langle x \rangle_\lambda = \frac{'\sum_i' x(i)\sharp(i)}{'\sum_i' \sharp(i)} = \frac{\int_a^b x\lambda(x)dx}{\int_a^b \lambda(x)dx}$$

$\lambda(x)$ being the linear density of the segment and the denominator its total mass.

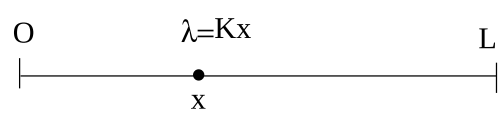
$\square$

Problem 47: Center of mass of a segment.

Compute the center of mass coordinate of a material segment whose linear density λ is proportional to the distance from one end.

Solution:

Take the origin of coordinates at the end where the density vanishes:



Then $\lambda(x) = Kx$ and

$$M = \int_0^L Kxdx = K\frac{L^2}{2}$$

$$\int_0^L xKxdx = K\frac{L^3}{3}$$

$$\langle x\rangle_\lambda = \frac{K\frac{L^3}{3}}{K\frac{L^2}{2}} = \frac{2}{3}L$$

□

$\boxed{\mathbb{T}}$ If we have a material wire C in $\mathbb{R}^3$ with linear density λ, the center of mass coordinates are

$$\langle x\rangle_\lambda = \frac{\int_C x\lambda dl}{\int_C \lambda dl},\ \langle y\rangle_\lambda = \frac{\int_C y\lambda dl}{\int_C \lambda dl},\ \langle z\rangle_\lambda = \frac{\int_C z\lambda dl}{\int_C \lambda dl},$$

or, expressed vectorially:

$$\langle \mathbf{x}\rangle_\lambda = \frac{\int_C \mathbf{x}\lambda dl}{\int_C \lambda dl}$$

□

Problem 48: A special density.

Compute de center of mass of a material wire extended along the circumference of radius R and center at $(0,0)$ if the linear density is $\lambda(x,y) = |x| + |y|$.

Solution:

Parametrize the circumference

$$\gamma(t) = (R\cos t, R\sin t),\ t \in [0, 2\pi]$$

Then

$$M = \int_C \lambda dl = \int_0^{2\pi} (|R\cos t| + |R\sin t|)Rdt = R^2\int_0^{2\pi}(|\cos t| + |\sin t|)dt = 8R^2$$

and the center of mass coordinates are

$$\langle x\rangle_\lambda = \frac{1}{M}\int_C x\lambda dl = \frac{1}{M}\int_0^{2\pi} R\cos t(|R\cos t| + |R\sin t|)Rdt = 0$$

and, analogously $\langle y\rangle_\lambda = 0$.

□

Problem 49: Center of mass of a material wire.

Let C be the parametrized curve

$$\gamma(t) = (0, R\sin t, R\cos t),\ t \in [0,\pi],\ R > 0$$

a) Compute the average of each coordinate.

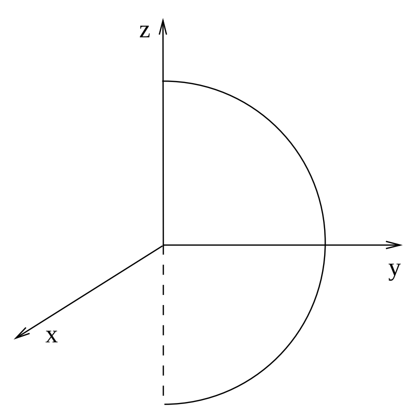
b) If the curve is a material wire with constant linear density $\lambda = 2$, compute the total mass and the coordinates of the center of mass.

c) Same question if the density is the restriction to C of the function $\lambda(x,y,z) = x + y + z$.

d) Same question if the density at each point is the double of the arc-length measured from the point $(0,0,R)$.

Solution:

The curve is a semicircumference in the plane yz:



a) From the picture we see that $\langle x\rangle = \langle z\rangle = 0$. Let's see this analytically:

$$\gamma'(t) = (0, R\cos t, -R\sin t),\ |\gamma'(t)| = \sqrt{R^2\cos^2 t + R^2\sin^2 t} = R$$

$$\begin{aligned}
\int_C x dl &= 0 \\
\int_C y dl &= \int_0^\pi R\sin t\, R dt = 2R^2 \\
\int_C z dl &= \int_0^\pi R\cos t\, R dt = 0 \\
L &= \pi R
\end{aligned}$$

and we obtain

$$\langle x\rangle = 0, \langle y\rangle = \frac{2R^2}{\pi R} = \frac{2R}{\pi}, \langle z\rangle = 0$$

b)

$$\begin{aligned}
\int_C x 2 dl &= 0 \\
\int_C y 2 dl &= 4R^2 \\
\int_C z 2 dl &= 0 \\
M &= \int_C 2 dl = 2\pi R
\end{aligned}$$

and

$$\langle x\rangle_\lambda = \langle z\rangle_\lambda = 0,\ \langle y\rangle_\lambda = \frac{4R^2}{2\pi R} = \frac{2R}{\pi}$$

c)

$$\begin{aligned}
\int_C x(x+y+z) dl &= 0 \\
\int_C y(x+y+z) dl &= \int_0^\pi R\sin t(R\sin t + R\cos t) R dt = \frac{\pi R^3}{2} \\
\int_C z(x+y+z) dl &= \int_0^\pi R\cos t(R\sin t + R\cos t) R dt = \frac{\pi R^3}{2} \\
M &= \int_C (x+y+z) dl = \int_0^\pi (R\sin t + R\cos t) R dt = 2R^2
\end{aligned}$$

and

$$\langle x\rangle_\lambda = 0, \langle y\rangle_\lambda = \langle z\rangle_\lambda = \frac{\frac{\pi R^3}{2}}{2R^2} = \frac{\pi R}{4}$$

d) The arc-length parametrization is

$$\Gamma(s) = (0, R\sin(\frac{s}{R}), R\cos(\frac{s}{R})),\ s \in [0, R\pi]$$

Then

$$\begin{aligned}
\int_C x\lambda dl &= 0 \\
\int_C y\lambda dl &= \int_0^{R\pi} R\sin(\frac{s}{R})2sds = 2\pi R^3 \\
\int_C z\lambda dl &= \int_0^{R\pi} R\cos(\frac{s}{R})2sds = -4R^3 \\
M &= \int_0^{R\pi} 2sds = \pi^2 R^2
\end{aligned}$$

and

$$\langle x\rangle_\lambda = 0,\ \langle y\rangle_\lambda = \frac{2\pi R^3}{\pi^2 R^2} = \frac{2R}{\pi},\ \langle z\rangle_\lambda = \frac{-4R^3}{\pi^2 R^2} = -\frac{4R}{\pi^2}$$

□

Problem 50: A mass.

Find the mass of a material wire c extended along the intersection of the sphere $x^2 + y^2 + z^2 = 2$ and the plane $x + y + z = 0$ if the linear density is the restriction to the wire of the function $\lambda(x, y, z) = x^2$.

Solution:

We give two solutions:

a) Project the curve on the plane $z = 0$, parametrize the projection and 'climb' to the curve; see p.28. There we met a parametrization of c.

$$\begin{aligned}
\gamma(t) &= (\cos t - \frac{1}{\sqrt{3}}\sin t, \frac{2}{\sqrt{3}}\sin t, -\cos t - \frac{1}{\sqrt{3}}\sin t), t \in [0, 2\pi] \\
\gamma'(t) &= (-\sin t - \frac{1}{\sqrt{3}}\cos t, \frac{2}{\sqrt{3}}\cos t, \sin t - \frac{1}{\sqrt{3}}\cos t)
\end{aligned}$$

and then

$$|\ \gamma'(t)\ |= \sqrt{2}$$

The mass is

$$\begin{aligned} M &= \int_0^{2\pi} (\cos t - \frac{1}{\sqrt{3}}\sin t)^2\sqrt{2}dt = \\ &= \int_0^{2\pi} (\cos^2 t + \frac{1}{3}\sin^2 t - \frac{2}{\sqrt{3}}\cos t \sin t)\sqrt{2}dt = \\ &= \sqrt{2}(\pi + \frac{1}{3}\pi) = \frac{4\sqrt{2}}{3}\pi \end{aligned}$$

b) The intersection curve is a meridian of the sphere. If we take new coordinates $(\bar{x}, \bar{y}, \bar{z})$ such that the plane $x + y + z = 0$ becomes the plane $\bar{z} = 0$ then we will see the intersection curve as a circumference of radius 2 centered at the origin and this we know ho to parametrize. We must choose $\bar{\mathbf{e}}_3 = \frac{1}{\sqrt{3}}(1, 1, 1)$ and we can find $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2$ because they are perpendicular to $\bar{\mathbf{e}}_3$ and unit vectors. The matrix of the change of basis appears to be:

$$C = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

The parametrization of c in the bar coordinates is

$$\begin{aligned} \Gamma(t) &= (\sqrt{2}\cos t, \sqrt{2}\sin t, 0),\ t \in [0, 2\pi] \\ \Gamma'(t) &= (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0) \\ |\ \Gamma'(t)\ | &= \sqrt{2} \end{aligned}$$

The linear density must satisfy $\bar{\lambda}(\bar{x}, \bar{y}, \bar{z}) = x^2$; the change of coordinates gives

$$x = -\frac{1}{\sqrt{2}}\bar{x} - \frac{1}{\sqrt{6}}\bar{y} + \frac{1}{\sqrt{3}}\bar{z}$$

and

$$x^2 = \frac{1}{2}\bar{x}^2 + \frac{1}{6}\bar{y}^2 + \frac{1}{3}\bar{z}^2 + \frac{1}{\sqrt{3}}\bar{x}\,\bar{y} - \frac{2}{\sqrt{6}}\bar{x}\,\bar{z} - \frac{2}{\sqrt{18}}\bar{y}\,\bar{z}.$$

Then

$$\begin{aligned} M &= \int_c (\frac{1}{2}\bar{x}^2 + \frac{1}{6}\bar{y}^2 + \frac{1}{3}\frac{1}{3}\bar{z}^2 + \frac{1}{\sqrt{3}}\bar{x}\,\bar{y} - \frac{2}{\sqrt{6}}\bar{x}\,\bar{z} - \frac{2}{\sqrt{18}}\bar{y}\,\bar{z})dl = \\ &= \int_0^{2\pi} (\frac{1}{2}2\cos^2 t + \frac{1}{6}2\sin^2 t + \frac{1}{\sqrt{3}}2\sin t\cos t)\sqrt{2}dt = \frac{4\sqrt{2}}{3}\pi \end{aligned}$$

□

3.1.4 Gravitational field of wires

Gravitational field

$\boxed{\mathbb{T}}$ If we have a material wire C with linear density λ we can use Newton's law for extended bodies (see p.57) to compute the gravitational field at a point; we arrive at

$$\boxed{\mathbf{g}(x, y, z) = -\int_C \lambda \frac{\mathbf{r}}{r^3} dl}$$

where in the integral (x, y, z) is where we want to know the field, (u, v, w) is a variable point in the material wire, $\mathbf{r} = (x-u, y-v, z-w)$ and $r = |\mathbf{r}|$. We are still applying the convention 'from the source to the point' (see p.55).

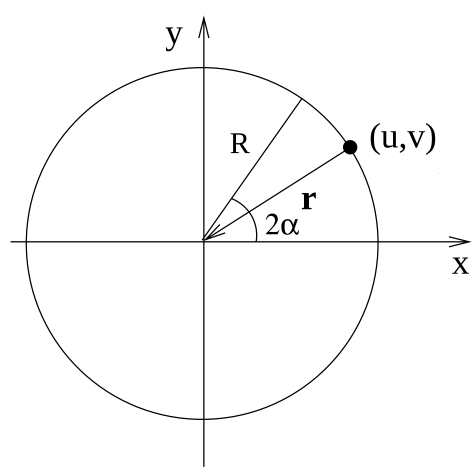
□

Problem 51:

Let C be a material wire with the form of an arc of a circumference of radius R. Assuming the linear density λ to be constant, compute the gravitational field at the center.

Solution:

This is clearly a bidimensional problem; take the circumference in the xy plane with its center at the origin and an arc of angle 2α.



Parametrize that arc by

$$\gamma(\theta) = (R\cos\theta, R\sin\theta), \theta \in [0, 2\alpha]$$

The field at the center is

$$\mathbf{g}(0,0) = -\int_C \lambda \frac{(-u,-v)}{R^3} dl$$

and its components are:

$$X = \int_C \frac{\lambda}{R^3} u dl = \frac{\lambda}{R^3} \int_0^{2\alpha} R\cos\theta \, R d\theta = \frac{\lambda}{R} \sin 2\alpha$$

$$Y = \int_C \frac{\lambda}{R^3} v dl = \frac{\lambda}{R^3} \int_0^{2\alpha} R\sin\theta \, R d\theta = \frac{\lambda}{R}(1 - \cos 2\alpha)$$

The module of the field has a value

$$|\mathbf{g}|^2 = \frac{2\lambda^2}{R^2}(1 - \cos 2\alpha) \Rightarrow |\mathbf{g}| = \frac{\sqrt{2}\lambda}{R}\sqrt{2\sin^2\alpha} = \frac{2\lambda}{R}|\sin\alpha|$$

The angle of $\mathbf{g}$ with the Ox axis satisfies

$$\tan\beta = \frac{Y}{X} = \frac{1 - \cos 2\alpha}{\sin 2\alpha} = \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha \Rightarrow \beta = \alpha$$

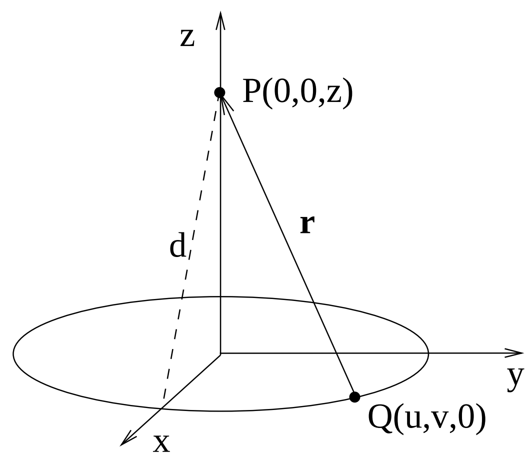
where we can see that β is half the total amplitude of the arc as was expectable from the symmetry.

□

Problem 52: Gravitational field of a circumference at points of its axis.

Let C be a material wire that has the form of a circumference and constant linear density λ. Compute the gravitational field at the points of its axis.

Solution:



A parametrization of the circumference is

$$\begin{aligned} \gamma(t) &= (R\cos t, R\sin t, 0), t \in [0, 2\pi] \\ dl &= Rdt \end{aligned}$$

The gravitational field at a point in the axis is

$$\mathbf{g}(0,0,z) = -\int_C \lambda \frac{(-u,-v,z)}{r^3} dl$$

If $\mathbf{g} = (X, Y, Z)$ it is clear by symmetry that $X = Y = 0$; the only nonvanishing coordinate is

$$Z(0,0,z) = -\lambda \int_C \frac{z}{r^3} dl = -\lambda \int_0^{2\pi} \frac{z}{(\sqrt{u^2+v^2+z^2})^3} Rdt = -\lambda \frac{2\pi R z}{d^3} = -\frac{M}{d^3} z$$

where $d = \sqrt{R^2+z^2}$ and $M = \lambda 2\pi R$ is the wire's mass.

As a check note that for $z = 0$ we have $d^3 = R^3$and $Z(0,0,0) = 0$.

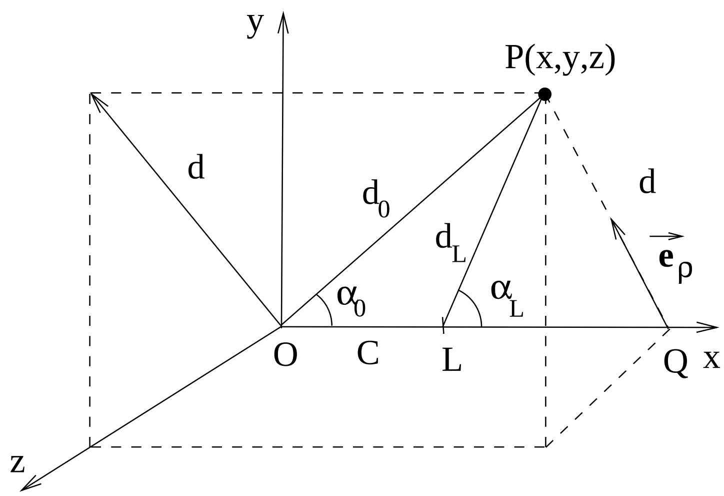
□

Problem 53: Gravitational field of a segment.

Compute the gravitational attraction of a material segment C with constant linear density, at points of space exterior to the segment.

Solution:

Put the segment C on the Ox axis, $C = [0, L]$:



and parametrize it by $\gamma(u) = (u, 0, 0), u \in [0, L], dl = du$.

The gravitational field created by the segment at points out of the segment is

$$\mathbf{g}(x, y, z) = -\int_C \lambda \frac{\mathbf{r}}{r^3} dl = -\int_0^L \lambda \frac{(x-u, y, z)}{r^3} du$$

To compute the components of $\mathbf{g} = (X, Y, Z)$ put $d = \sqrt{y^2 + z^2}$, $d_0 = |(x, y, z)| = \sqrt{x^2 + d^2}$, $d_L = \sqrt{(x-L)^2 + d^2}$; we can see those magnitudes in the figure. We have:

a)

$$\begin{aligned} X &= -\int_0^L \lambda \frac{x-u}{((x-u)^2 + d^2)^{3/2}} du = -\lambda((x-u)^2 + d^2)^{-1/2}|_0^L = \\ &= -\lambda\left(\frac{1}{d_L} - \frac{1}{d_0}\right) \end{aligned}$$

As a check we see that this component of the field vanishes on the bisector plane of the segment $x = \frac{L}{2}$, as expected by symmetry.

b)

$$\begin{aligned} Y &= -\int_0^L \lambda \frac{y}{((x-u)^2+d^2)^{3/2}} du = \\ &= -\lambda y \int_0^L \frac{1}{((x-u)^2+d^2)^{3/2}} du \end{aligned}$$

We must end up with an integral of the form

$$\begin{aligned} \int \frac{dz}{(z^2+1)^{3/2}} &= \left\{ \begin{array}{rcl} z &=& \tan t \\ dz &=& \frac{1}{\cos^2 t} dt \end{array} \right\} = \int \frac{1}{\cos^2 t} \cdot \frac{1}{(\tan^2 t+1)^{3/2}} dt = \\ &= \int \cos t dt = \sin t = \frac{z}{\sqrt{z^2+1}} \end{aligned}$$

We now use this result to find Y:

$$\begin{aligned} \int_0^L \frac{1}{((x-u)^2+d^2)^{3/2}} du &= \frac{1}{d^3} \int_0^L \frac{1}{((\frac{x-u}{d})^2+1)^{3/2}} du = \left\{ \begin{array}{rcl} \frac{x-u}{d} &=& v \\ du &=& -d\, dv \end{array} \right\} = \\ &= -\frac{1}{d^2} \int_{\frac{x}{d}}^{\frac{x-L}{d}} \frac{1}{(v^2+1)^{3/2}} dv = -\frac{1}{d^2} \frac{v}{\sqrt{v^2+1}} \Big|_{\frac{x}{d}}^{\frac{x-L}{d}} = \\ &= -\frac{1}{d^2} \left(\frac{x-L}{d_L} - \frac{x}{d_0} \right) \end{aligned}$$

and finally we obtain the Y component of the field:

$$Y = \lambda \frac{y}{d^2} \left(\frac{x-L}{d_L} - \frac{x}{d_0} \right)$$

c) Analogously

$$Z = \lambda \frac{z}{d^2} \left(\frac{x-L}{d_L} - \frac{x}{d_0} \right)$$

- We have solved the problem and we now give more geometrical content to the solution. The triangle PQL is rectangle at Q and

$$\frac{LQ}{d_L} = \frac{x-L}{d_L} = \cos \alpha_L, \alpha_L = \angle(\text{axis segment}, \text{line } d_L)$$

The triangle POQ is rectangle at Q and

$$\frac{OQ}{d_L} = \frac{x}{d_0} = \cos \alpha_0, \alpha_0 = \angle(\text{axis segment}, \text{line } d_0)$$

The *radial part* of the field is $\mathbf{g}_\rho = (0, Y, Z)$ because in terms of a unit radial vector $\mathbf{e}_\rho = \frac{(0.y.z)}{d}$ we have

$$\begin{aligned}
\mathbf{g}_\rho &= \lambda(\frac{x-L}{d_L} - \frac{x}{d_0})\frac{1}{d}\mathbf{e}_\rho = \\
&= \lambda(\cos\alpha_L - \cos\alpha_0)\frac{1}{d}\mathbf{e}_\rho
\end{aligned}$$

The module of this field is constant on a circumference with center at $(x, 0, 0)$ passing through P. It is a radial field, type '$\frac{1}{r}$' . If we put $\mathbf{g}_s = -\lambda(\frac{1}{d_L} - \frac{1}{d_0})(1, 0, 0)$ the whole field can be written as

$$\mathbf{g} = \mathbf{g}_s + \mathbf{g}_\rho$$

- At points of the segment's axis we have $d_L = \mid x - L \mid, d_0 = \mid x \mid$. There is no radial component and we obtain

$$\mathbf{g}(x, 0, 0) = (-\lambda(\frac{1}{\mid x - L \mid} - \frac{1}{\mid x \mid}), 0, 0)$$

Now if $x > L$

$$\begin{aligned}
X(x, 0, 0) &= -\lambda(\frac{1}{x-L} - \frac{1}{x}) = -\lambda\frac{x-(x-L)}{x(x-L)} = \\
&= -\lambda\frac{x-(x-L)}{x(x-L)} = -\frac{\lambda L}{x(x-L)} = \\
&= -\frac{M}{x(x-L)}
\end{aligned}$$

- If $x < 0$

$$X(x, 0, 0) = -\lambda(\frac{1}{L-x} + \frac{1}{x}) = \frac{M}{x(x-L)}$$

that coincides with the result in problem on p.58.

□

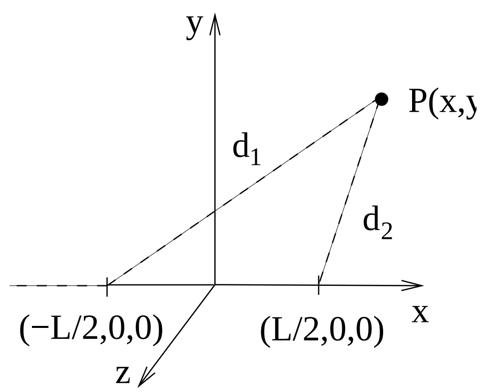
Logarithmic field

Problem 54: Field of an infinite wire.

At each point in free space (:= points free of masses) find the limit position of the gravitational field generated by a material segment of constant linear density λ, when we make the length of the segment go to infinity. This is the field of an infinite wire. Notice a complement to the present problem in p.120.

Solution:

Let a segment extend between the points $(-L/2, 0, 0)$ and $(L/2, 0, 0)$ and let $d = \sqrt{y^2 + z^2}$, $d_1 = \sqrt{(x + \frac{L}{2})^2 + d^2}$, $d_2 = \sqrt{(x - \frac{L}{2})^2 + d^2}$.



From the preceding problem we know that the field $\mathbf{g} = (X, Y, Z)$ generated by the segment is

$$\begin{aligned} X &= -\lambda(\frac{1}{d_2} - \frac{1}{d_1}) \\ Y &= \lambda\frac{y}{d^2}(\frac{x - \frac{L}{2}}{d_2} - \frac{x + \frac{L}{2}}{d_1}) \\ Z &= \lambda\frac{z}{d^2}(\frac{x - \frac{L}{2}}{d_2} - \frac{x + \frac{L}{2}}{d_1}) \end{aligned}$$

Let the field of the infinite wire be $\mathbf{g}_E = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. To compute it we let L tend to infinity; as $\lim_{L\to\pm\infty} d_1 = \lim_{L\to\pm\infty} d_2 = \infty$ we have:

a)

$$\mathcal{X} = \lim_{L\to\infty} X = \lim_{L\to\infty} -\lambda(\frac{1}{d_2} - \frac{1}{d_1}) = 0$$

and the field is perpendicular to the wire, a reasonable result by symmetry.

b)

$$\begin{aligned}\mathcal{Y} &= \lim_{L\to\infty} Y = \lim_{L\to\infty} \lambda\frac{y}{d^2}(\frac{x-\frac{L}{2}}{d_2} - \frac{x+\frac{L}{2}}{d_1}) = \\ &= \lambda\frac{y}{d^2}\lim_{L\to\infty}(-\frac{L}{2d_2} - \frac{L}{2d_1})\end{aligned}$$

Lets compute first

$$\lim_{L\to\infty}\frac{L}{d_2} = \lim_{L\to\infty}\frac{L}{\sqrt{(x-\frac{L}{2})^2+d^2}} = \lim_{L\to\infty}\frac{1}{\sqrt{(\frac{x}{L}-\frac{1}{2})^2+\frac{d^2}{L^2}}} = 2$$

$$\lim_{L\to\infty}\frac{L}{d_1} = \lim_{L\to\infty}\frac{L}{\sqrt{(x+L)^2+d^2}} = \lim_{L\to\infty}\frac{1}{\sqrt{(\frac{x}{L}+\frac{1}{2})^2+\frac{d^2}{L^2}}} = 2$$

Then

$$\mathcal{Y} = -2\lambda\frac{y}{d^2}$$

c) Analogously

$$\mathcal{Z} = -2\lambda\frac{z}{d^2}$$

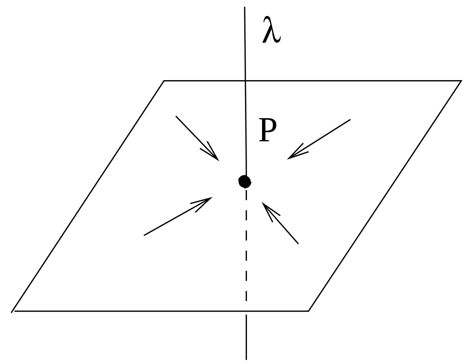
Summing up

$$\mathbf{g}_E(x,y,z) = -2\lambda(0, \frac{y}{d^2}, \frac{z}{d^2})$$

Let us focus the attention on the plane perpendicular to the wire through the origin; put $\mathbf{r} = (y, z)$ and then

$$\mathbf{g}_E(0,y,z) = \mathbf{L}(\mathbf{r}) = -2\lambda\frac{\mathbf{r}}{r^2} = -2\lambda\frac{1}{r}\frac{\mathbf{r}}{r}$$

a central field analogous to the newtonian field but now of the form $\frac{1}{r}$. The point mass 2λ generating this field is named a *logarithmic particle* (because its potential is logarithmic).



□

Problem 55: Logarithmic field.

Find the field generated at $\mathbf{p}$ by a logarithmic particle of mass m at $\mathbf{q}$. Show that this field has zero divergence in $\mathbb{R}^2 - \{\mathbf{q}\}$.

Solution:

Let $\mathbf{q} = (a, b), \mathbf{p} = (x, y), \mathbf{r} = (x - a, y - b), r = |\mathbf{r}|, \mathbf{L} = (X, Y)$. Then

$$\begin{aligned} X(x, y) &= -m\frac{x - a}{(x - a)^2 + (y - b)^2} \\ Y(x, y) &= -m\frac{y - b}{(x - a)^2 + (y - b)^2} \end{aligned}$$

The divergence is

$$\begin{aligned} \partial_x X &= -m\frac{(x - a)^2 + (y - b)^2 - 2(x - a)^2}{((x - a)^2 + (y - b)^2)^2} = -m\frac{(y - b)^2 - (x - a)^2}{((x - a)^2 + (y - b)^2)^2} \\ \partial_y Y &= -m\frac{(x - a)^2 + (y - b)^2 - 2(y - b)^2}{((x - a)^2 + (y - b)^2)^2} = -m\frac{(x - a)^2 - (y - b)^2}{((x - a)^2 + (y - b)^2)^2} \end{aligned}$$

so

$$\text{div } \mathbf{L} = 0$$

□

Problem 56: Field of a logarithmic circumference.

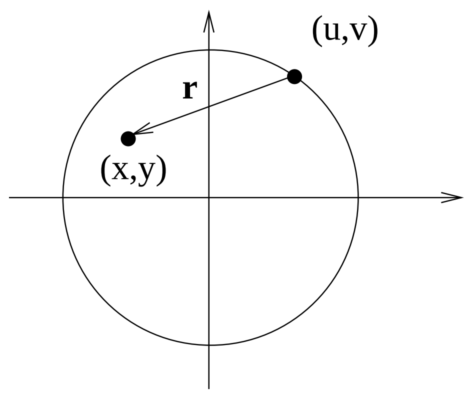
Let C be a material circumference with constant linear mass density λ constituted of logarithmic particles. Compute the field at points exterior to the wire.

Solution:

First at all we parametrize C

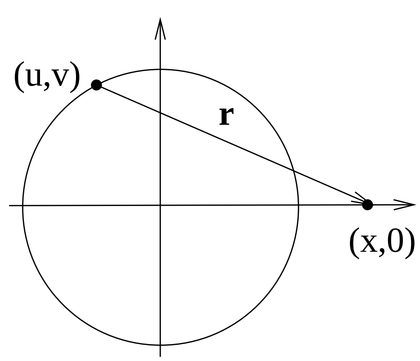
$$\gamma(\theta) = (R\cos\theta, R\sin\theta), \theta \in [0, 2\pi]$$

.



The field is $\mathbf{L}(x,y) = -\int_C \lambda \frac{(x-u,y-v)}{r^2} dl$; by symmetry we need to compute only

$$\mathbf{L}(x,0) = -\int_C \lambda \frac{(x-u,v)}{r^2} dl,\ x > 0:$$



On the circumference we have

$$\begin{aligned} r^2 &= (x-u)^2 + v^2 = \\ &= (x - R\cos\theta)^2 + R^2\sin^2\theta = \\ &= x^2 + R^2 - 2xR\cos\theta \end{aligned}$$

and

$$\begin{aligned} X(x,0) &= -\int_C \lambda \frac{x-u}{r^2} dl = \\ &= -\lambda \int_0^{2\pi} \frac{x - R\cos\theta}{x^2 + R^2 - 2xR\cos\theta} R d\theta = \begin{cases} 0 & \text{if } 0 < x < R \\ -\frac{M}{x} & \text{if } R < x \end{cases} \end{aligned}$$

where $M = \lambda 2\pi R$. This integral is best evaluated using complex variable techniques (do the substitution $z = e^{i\theta}$ and integrate along the unit circle using the residue theorem).

On another hand:

$$\begin{aligned} Y(x,0) &= -\int_C \lambda \frac{-v}{r^2} dl = -\int_0^{2\pi} \lambda \frac{-R\sin\theta}{(x - R\cos\theta)^2 + R^2\sin^2\theta} R d\theta \\ &= 0 \text{ (we integrate over a period an odd function respect to } \pi) \end{aligned}$$

We obtain

$$\mathbf{L}(x,y) = \mathbf{L}(\mathbf{r}) = \begin{cases} 0 & \text{if } \ 0 \le r < R \\ -\frac{M}{r}\frac{\mathbf{r}}{r} & \text{if } \ R < r \end{cases}$$

□

Problem 57: Infinite wire in $\mathbb{R}^4$.

As the field of an infinite wire in $\mathbb{R}^3$ generates in $\mathbb{R}^2$ a logarithmic field we might suspect that the gravitational field of an infinite wire in $\mathbb{R}^4$ generates the ordinary gravitational field in $\mathbb{R}^3$.

Assume a newtonian gravitational attraction in $\mathbb{R}^4 = \{(x,y,z,u) : x,y,z,u \in \mathbb{R}\}$ to be of the form '$\frac{1}{r^3}$' and take an infinite material wire of linear density λ along the axis $(0,0,0,u)$. Show that the field in $\mathbb{R}^3$ (the hiperplane $u = 0$) has the form '$\frac{1}{r^2}$'. Compare with p.116.

Solution:

We denote by G' the gravitational constant in $\mathbb{R}^4$and assume the attraction between two point masses m, m' to be

$$\mathbf{F} = -G' \frac{mm'}{r^3} \frac{\mathbf{r}}{r}$$

Then the field at a point $(x, y, z, 0)$ of the hiperplane $u = 0$ is:

$$\mathbf{g}(x, y, z, 0) = -G' \int_{-\infty}^{\infty} \frac{\mathbf{r}}{r^4} \lambda dl, \; \mathbf{r} = (x, y, z, -u), r = |\mathbf{x}| = \sqrt{d^2 + u^2}, d^2 = x^2 + y^2 + z^2$$

Let us first compute the integral

$$\int_0^a \frac{1}{r^4} du = \int_0^a \frac{1}{(d^2 + u^2)^2} du$$

Using Hermitte's method write

$$\frac{1}{(d^2 + u^2)^2} = \frac{d}{du} (\frac{Au + B}{d^2 + u^2}) + \frac{Cu + D}{d^2 + u^2}$$

and after finding the indeterminate coefficients A, B, C, D we obtain

$$\frac{1}{(d^2 + u^2)^2} = \frac{d}{du} (\frac{(1/2d^2)u}{d^2 + u^2}) + \frac{1/2d^2}{d^2 + u^2}$$

Then

$$\begin{aligned} \int_0^a \frac{1}{r^4} du &= (\frac{1}{2d^2} \frac{u}{d^2 + u^2} + \frac{1}{2d^3} \arctan \frac{u}{d})|_0^a = \\ &= \frac{1}{2d^2} \frac{a}{d^2 + a^2} + \frac{1}{2d^3} \arctan \frac{a}{d} \end{aligned}$$

$$\lim_{a \to +\infty} \int_0^a \frac{1}{r^4} du = \frac{1}{2d^3} \frac{\pi}{2}$$

and due to the evenness of the integrand

$$\lim_{a \to +\infty} \int_{-a}^0 \frac{1}{r^4} du = \frac{1}{2d^3} \frac{\pi}{2}$$

We have obtained

$$\int_{-\infty}^{+\infty} \frac{1}{r^4} du = \frac{\pi}{2d^3}$$

Then, as the symmetry already shows, the fourth component of the field vanishes and we have

$$\begin{aligned}\mathbf{g}(x,y,z,0) &= -G'\int_{-\infty}^{\infty}\frac{\mathbf{r}}{r^4}\lambda dl = \\ &= -G'\frac{\lambda\pi}{2d^3}(x,y,z,0)\end{aligned}$$

If $\mathbf{r}=(x,y,z)$ we can write

$$\mathbf{g}(x,y,z) = -G'\frac{\lambda\pi}{2}\frac{\mathbf{r}}{\mathbf{r}^3}$$

and we see that the field induced in $\mathbb{R}^3$ has the form of the familiar newtonian gravitational field.

□

3.2 Integration of vector fields

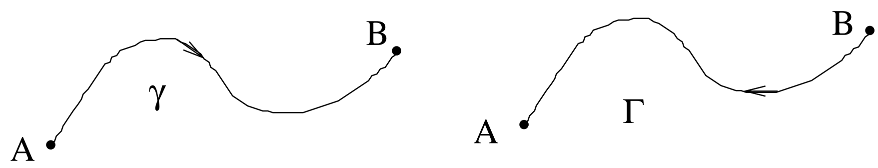
$\boxed{\mathbb{T}}$ Let $\gamma:[a,b]\to U$ be a $\mathcal{C}^1$ parametrized curve in the open set $U\subset\mathbb{R}^n$ and $\mathbf{F}$ a continuous vector field in U. The *line integral* of $\mathbf{F}$ along γ (or circulation of $\mathbf{F}$ a long γ, or work done by $\mathbf{F}$) is

$$\boxed{\int_\gamma \mathbf{F}\cdot d\mathbf{l} = \int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)dt}$$

The boldface in $d\mathbf{l}$ reminds us that it is a *vector* element of line. For a rigorous study of that concept see [Jän], pp.169,173.

Two parametrizations γ and Γ are *positively* equivalent if the change of variable $h:[a,b]\to[c,d]$ doing the reparametrization satisfies $h'(t)>0$. An oriented curve C is the collection of all positively equivalent parametrized curves.

Let $\gamma:[a,b]\to U$ be a parametrization of the oriented curve C; then $A=\gamma(a)$ is the origin and $B=\gamma(b)$ the end point of the oriented curve. Now let $k:[a,b]\to[c,d]$ be a change of variable with $k'(t)<0$, $t\in[a,b]$; then the equivalence class of the reparametrized curve Γ is that of the opposite curve C^-. It has the origin at $\Gamma(c)=\gamma(k^{-1}(c))=\gamma(b)=B$, the end point of C, and the end point at $\Gamma(d)=\gamma(k^{-1}(d))=\gamma(a)=A$:



Let C be an oriented curve in the open set $U \subset \mathbb{R}^n$; the integral of $\mathbf{F}$ along C is

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{l} = \int_\gamma \mathbf{F} \cdot d\mathbf{l}}$$

$\gamma : [a, b] \to U$ being a parametrization of C. This definition is independent of the parametrization choosen (positively equivalent); see p.128.

□

3.2.1 Medley

Problem 58:

Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute the line integral of $\mathbf{F}$ along the following oriented curves:

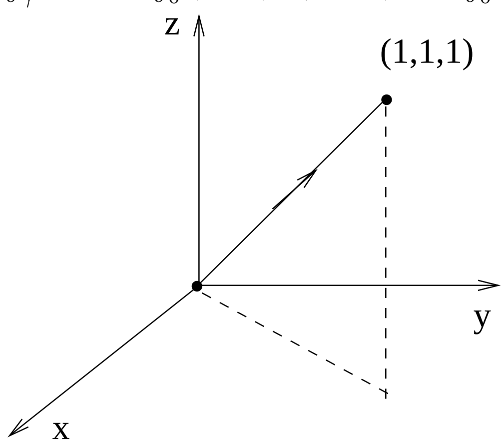
a) $\gamma(t) = (t, t, t), 0 \le t \le 1$.

b) $\gamma(t) = (\cos t, \sin t, 0), 0 \le t \le 2\pi$.

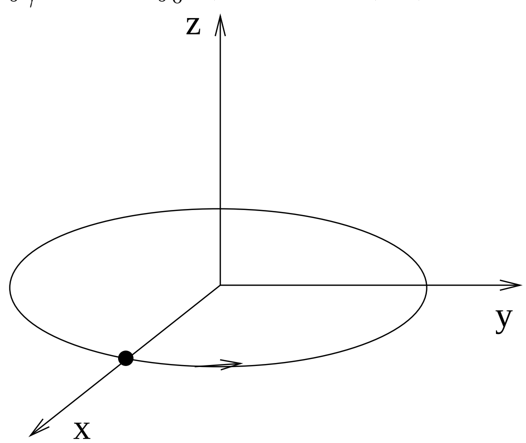
c) The arc of parabola $y = x^2, z = 0$, from $x = -1$ up to $x = 2$.

Solution:

a) $\int_\gamma \mathbf{F} \cdot d\mathbf{l} = \int_0^1 (t,t,t) \cdot (1,1,1) dt = \int_0^1 3t dt = \frac{3}{2}$.

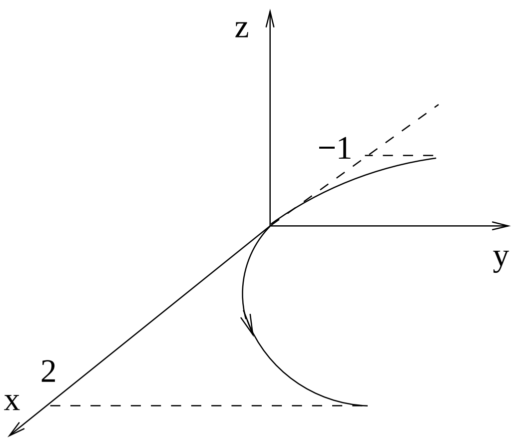


b) $\int_\gamma \mathbf{F} \cdot d\mathbf{l} = \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) dt = \int_0^{2\pi} 0 dt = 0$.



c) Parametrize the arc of parabola by $\gamma(t) = (t, t^2, 0), -1 \leq t \leq 2$ and then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{l} = \int_{-1}^{2} (t, t^2, 0) \cdot (1, 2t, 0) dt = \int_{-1}^{2} (t + 2t^3) dt = 9$$



□

Problem 59:

Let $R > 0$ and C the semicircumference

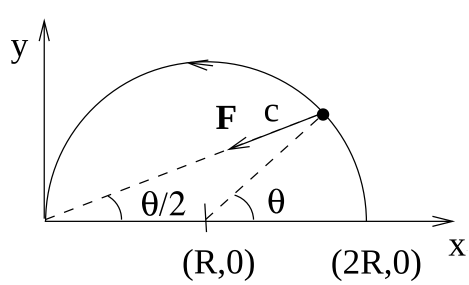
$$(x - R)^2 + y^2 = R^2,\ y \geq 0$$

traversed from $(2R, 0)$ up to $(0, 0)$. Let $\mathbf{F}$ be a vector field with direction and sense the same as those for going from (x, y) to $(0, 0)$, and constant module c. Compute

$$\int_C \mathbf{F} \cdot d\mathbf{l}$$

Solution:

We make a figure



θ is the polar angle of points of C as seen from the center of the circumference. The field is

$$\mathbf{F}(x, y) = -c \quad (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$$

and a parametrization of C is

$$\begin{aligned} \gamma(\theta) &= (R + R\cos\theta, R\sin\theta),\ \theta \in [0, \pi] \\ \gamma'(\theta) &= (-R\sin\theta, R\cos\theta) \end{aligned}$$

The circulation is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{l} &= -c\int_0^\pi (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}) \cdot (-R\sin\theta, R\cos\theta)d\theta = \\ &= -cR\int_0^\pi (-\cos\frac{\theta}{2}\sin\theta + \sin\frac{\theta}{2}\cos\theta)d\theta = \\ &= cR\int_0^\pi \sin\frac{\theta}{2}d\theta = 2c \end{aligned}$$

□

Problem 60: Agnesi's curve.

A straight line r passing through the origin cuts at the point A the circumference S

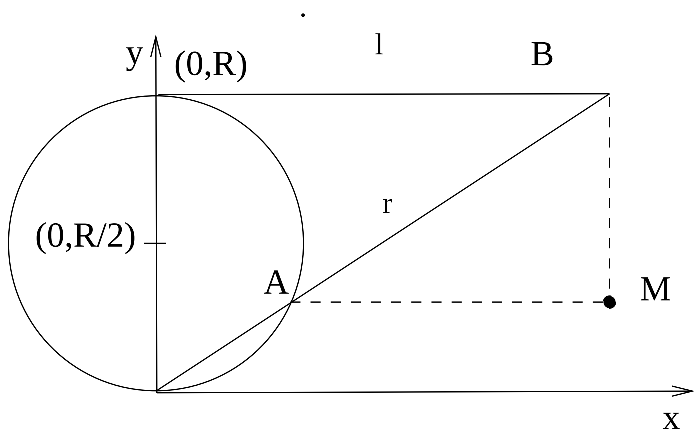
$$x^2 + (y - R/2)^2 = R^2/4$$

and cuts l, the tangent line to S at $(0, R)$, at the point B. The lines through A and B and respectively parallel to Ox and Oy cut in M. The path followed by M as r varies is Agnesi's curve.

a) Parametrize Agnesi's curve.

b) Find the line integral of $\mathbf{F}(x, y) = (y, x)$ along Agnesi's curve between $(-R, R/2)$ and $(R, R/2)$.

Solution:



a) Let $y = mx$ be the equation of r. Geometrically we see that $m = 0$ must be excluded for both lines are paral.lel. From the figure as well, it is easy to see that

$$\lim_{m\to\pm\infty} M(m) = (0, R)$$

Now:
y coordinate of A

$$\left.\begin{array}{rcl} x^2 + (y - R/2)^2 & = & R^2/4 \\ y & = & mx \end{array}\right\} \Rightarrow y = \frac{m^2}{1+m^2}R$$

x coordinate of B

$$x = \frac{1}{m}R$$

Parametrization of the curve

$$\gamma(m) = (\frac{1}{m}R, \frac{m^2}{1+m^2}R), m \in (-\infty, +\infty) \setminus \{0\}.$$

If we prefer to use an angular parameter $m = \tan\theta$ we have

$$\Gamma(\theta) = (R\cot\theta, R\sin^2\theta), \theta \in [-\pi/2, \pi/2] \setminus \{0\}.$$

Notice those formulae give $\Gamma(\pm\pi/2) = (0, R)$ that equals the limit point.

b) The point $(-R, R/2)$ corresponds to a value of the parameter $\theta_1 = 3\pi/4$ and the point $(R, R/2)$ corresponds to $\theta_2 = \pi/4$. On another hand

$$\Gamma'(\theta) = (-R\frac{1}{\sin^2\theta}, 2R\sin\theta\cos\theta)$$

We have:

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_{3\pi/4}^{\pi/4} (R\sin^2\theta, R\cot\theta)\cdot(-R\frac{1}{\sin^2\theta}, 2R\sin\theta\cos\theta)d\theta =$$

$$= R^2\int_{3\pi/4}^{\pi/4}(2\cos^2\theta - 1)d\theta = R^2\int_{3\pi/4}^{\pi/4}\cos 2\theta d\theta = R^2$$

□

Problem 61:

Let C be an oriented curve and $\mathbf{F}$ a vector field. Show that the definition

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_\gamma \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,dt$$

does not depend on the positively equivalent parametrization γ used.

Solution:

Let $\gamma(t), t\in[a,b]$ be a parametrization of C, $\tau = h(t)$ a change of variable such that $h'(t) > 0$

$$\begin{array}{rccc} h: & [a,b] & \longrightarrow & [c,]d \\ & t & \longmapsto & \tau = h(t) \end{array},$$

and $\Gamma(\tau)$ the corresponding reparametrization. Then $\gamma(t) = \Gamma(h(t)), \gamma'(t) = \Gamma'(h(t))h'(t)$ and by the change of variable theorem for integrals we have

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{l} = \int_c^d \mathbf{F}(\Gamma(\tau))\cdot\Gamma'(\tau)d\tau = \left\{\begin{array}{ccc} \tau & = & h(t) \\ d\tau & = & h'(t)dt \end{array}\right\} =$$

$$= \int_a^b \mathbf{F}(\Gamma(h(t)))\cdot\Gamma'(h(t))h'(t)dt =$$

$$= \int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)dt = \int_\gamma \mathbf{F}\cdot d\mathbf{l}$$

If $h'(t) < 0$ we have

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{l} = \int_c^d \mathbf{F}(\Gamma(\tau)) \cdot \Gamma'(\tau) d\tau = \left\{ \begin{array}{ccc} \tau & = & h(t) \\ d\tau & = & h'(t)dt \end{array} \right\} =$$

$$= \int_b^a \mathbf{F}(\Gamma(h(t))) \cdot \Gamma'(h(t))h'(t)dt =$$

$$= -\int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)dt = -\int_\gamma \mathbf{F} \cdot d\mathbf{l}$$

showing that

$$\int_{C^-} \mathbf{F} \cdot d\mathbf{l} = -\int_C \mathbf{F} \cdot d\mathbf{l}$$

□

Notation

$\boxed{\mathbb{T}}$ Writing $\gamma(t) = (x_1(t), \dots, x_n(t))$ and $\gamma'(t) = (\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt})$ one has

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{l} &= \int_a^b (F_1, \dots, F_n) \cdot (\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt})dt = \\ &= \int_a^b F_1 dx_1 + \dots + F_n dx_n = \\ &= \int_C F_1 dx_1 + \dots + F_n dx_n \end{aligned}$$

□

Problem 62:

Evaluate the following line integrals:

a) $\int_\gamma xdy - ydx$, $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

b) $\int_C yzdx + zxdy + xydz$, C the triangle $P = (1,0,0), Q = (0,1,0), R = (0,0,1)$.

c) $\int_C xdz$, C the arc of the curve resulting from the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = y$ satisfying $x \geq 0, y \geq 0, z \geq 0$ (Viviani's curve).

Solution:

a)

$$\int_\gamma xdy - ydx = \int_0^{2\pi} (\cos t \cos t - \sin t(-\sin t))dt = 2\pi$$

b) We are not given an orientation on the curve; we use the one given by PQR. Let us integrate along each side; on the segment PQ this is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1-t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-t \\ t \\ 0 \end{pmatrix}, \ t \in [0,1]$$

$$\int_C yzdx + zxdy + xydz = \int_0^1 0\,dt = 0$$

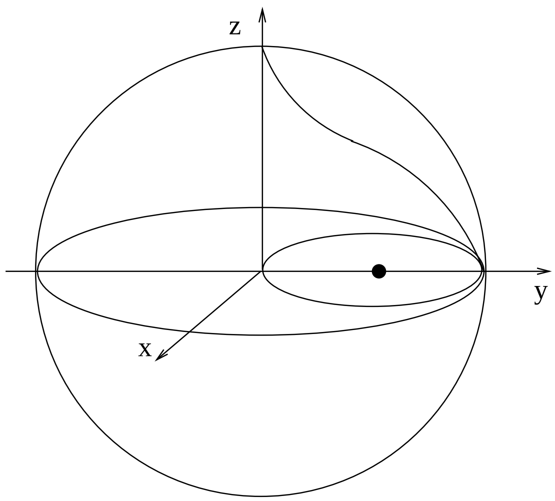
along QR we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1-t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1-t \\ 0 \end{pmatrix}, \ t \in [0,1]$$

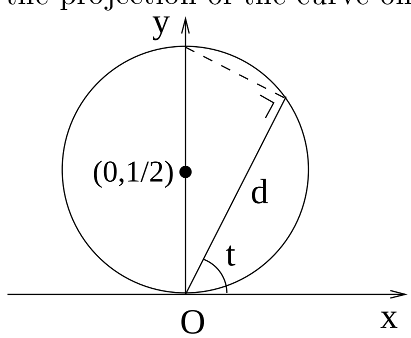
$$\int_C yzdx + zxdy + xydz = \int_0^1 0\,dt = 0$$

etc. and the integral vanishes.
Alternatively we may observe that the segments lie in the coordinate planes. Thus PQ is in the $z = 0$ plane and the integral amounts to $\int_C xydz$ that vanishes because $dz = 0$.

c) Taking into account that the cylinder is $x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$ and that the curve lies in the first octant we obtain the following figure:



i) This problem is related to the one in p.28. So we first parametrize the projection of the curve on $z = 0$ using polar coordinates.



In the figure we see that $d = \sin t$ and a parametrization is

$$\gamma(t) = (\sin t \cos t, \sin^2 t),\ t \in [0, \frac{\pi}{2}],$$

and 'climbing' to the sphere we obtain a parametrization of Viviani's curve

$$\begin{aligned} \Gamma(t) &= (\sin t \cos t, \sin^2 t, \sqrt{1 - (\sin^2 t \cos^2 t + \sin^4 t}) = \\ &= (\sin t \cos t, \sin^2 t, \cos t) \end{aligned}$$

Now we can integrate

$$\begin{aligned} \int_C x dz &= \int_0^{\pi/2} \sin t \cos t (-\sin t) dt = \int_0^{\pi/2} -\sin^2 t \cos t \, dt = \\ &= -\frac{\sin^3 t}{3}\Big|_0^{\pi/2} = -\frac{1}{3} \end{aligned}$$

ii) Alternatively we may identify the points in the first octant of the sphere through its spherical coordinates

$$\begin{array}{rcl} x & = & \sin\varphi\cos\theta \\ y & = & \sin\varphi\sin\theta \\ z & = & \cos\varphi \end{array}$$

with $\varphi \in [0, \frac{\pi}{2}]$, $\theta \in [0, \frac{\pi}{2}]$, take into account that the points of C satisfy $x^2 - y + y^2 = 0$, and obtain the relation

$$\sin^2\varphi\cos^2\theta - \sin\varphi\sin\theta + \sin^2\varphi\sin^2\theta = 0$$

or

$$\sin\varphi(\sin\varphi - \sin\theta) = 0 \Rightarrow \left\{ \begin{array}{c} \sin\varphi = 0 \Rightarrow \varphi = 0 \\ \sin\varphi = \sin\theta \Rightarrow \varphi = \theta \end{array} \right.$$

The case $\varphi = 0$ corresponds to the north pole; when we integrate one point doesn't matter. The case $\varphi = \theta$ leads to the same parametrization:

$$\Gamma(\theta) = (\sin\theta\cos\theta, \sin^2\theta, \cos\theta),\ \theta \in [0, \frac{\pi}{2}].$$

□

Problem 63:

Let C be the intersection of the cylinder $x^2+y^2 = 1$ with the plane $z = ax+by$. Find a, b with $a^2 + b^2 = 1$ such that

$$I = \int_C y dx + (z - x)dy - y dz = 0$$

Solution:

Still using the method in p.28 we parametrize C by:

$$\gamma(\theta) = (\cos\theta, \sin\theta, a\cos\theta + b\sin\theta),\ \theta \in [0, 2\pi]$$

$$\gamma'(\theta) = (-\sin\theta, \cos\theta, -a\sin\theta + b\cos\theta)$$

Computing I :

$$I = \int_0^{2\pi} (\sin\theta, (a-1)\cos\theta + b\sin\theta, -\sin\theta)\cdot(-\sin\theta, \cos\theta, -a\sin\theta + b\cos\theta)d\theta =$$

$$= \int_0^{2\pi} (a-1)d\theta = 2\pi(a-1)$$

and we see that it suffices to take $a = 1, b = 0$.

□

Problem 64:

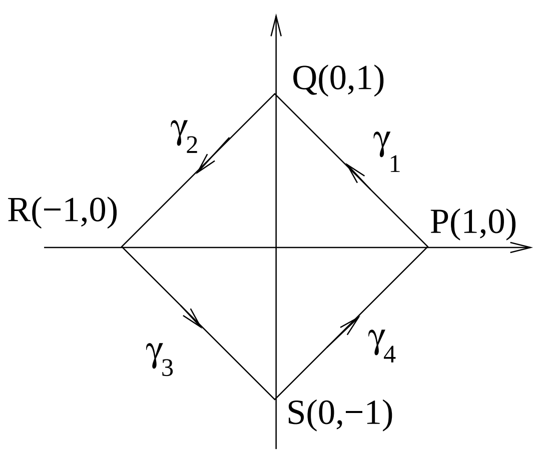
Compute the integral

$$I = \int_C \frac{dx + dy}{|x| + |y|}$$

C being the square with vertexs $(1,0), (0,1), (-1,0), (0,-1)$.

Solution:

Take the segments C_1, C_2, C_3, C_4 with the orientation of the figure:



Remind (see p.88) that if γ_1 parametrizes C_1, then $-\gamma_1$ parametrizes C_3; and the same happens with C_2 and C_4. Then

Parametrization of C_1

$$\gamma_1(t) = \begin{pmatrix} x \\ y \end{pmatrix} = (1-t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-t \\ t \end{pmatrix}, \gamma_1' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Parametrization of C_2

$$\gamma_2(t) = \begin{pmatrix} x \\ y \end{pmatrix} = (1-t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ 1-t \end{pmatrix}, \gamma_2' = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Now compute the integrals

$$\begin{aligned} \int_{C_1} \frac{dx+dy}{|x|+|y|} &= \int_0^1 \frac{-dt+dt}{|1-t|+|t|} = 0 \\ \int_{C_3} \frac{dx+dy}{|x|+|y|} &= \int_0^1 \frac{dt-dt}{|t-1|+|-t|} = 0 \end{aligned}$$

$$\begin{aligned} \int_{C_2} \frac{dx+dy}{|x|+|y|} &= \int_0^1 \frac{-dt-dt}{|-t|+|1-t|} = -2\int_0^1 \frac{dt}{t+(1-t)} = -2 \\ \int_{C_4} \frac{dx+dy}{|x|+|y|} &= \int_0^1 \frac{dt+dt}{|t|+|t-1|} = 2\int_0^1 \frac{dt}{t+(1-t)} = 2 \end{aligned}$$

Adding the partial results:

$$I = 0$$

□

Problem 65: Dependence on path.

Evaluate the work done by the field $\mathbf{F}(x,y,z) = (y,0,0)$ in moving a unit mass point from $(0,0,0)$ up to $(1,1,0)$ along:

a) The polygonal line with vertices at $(0,0,0), (1,0,0), (1,1,0)$.

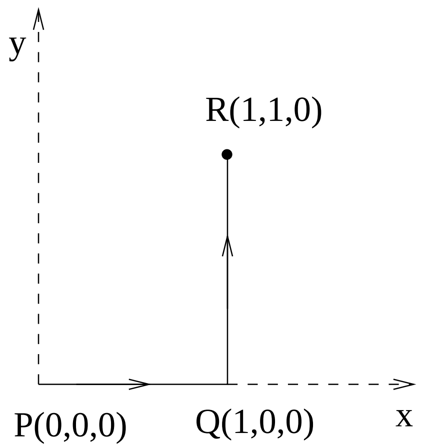
b) The polygonal line with vertices at $(0,0,0), (0,1,0), (1,1,0)$.

c) The parabola $y = x^2, z = 0$ from $(0,0,0)$ up to $(1,1,0)$.

d) Show there are paths joining $(0,0,0)$ to $(1,1,0)$ along which the work done by $\mathbf{F}$ is as big as we please.

Solution:

a) Parametrize the path piecewise:

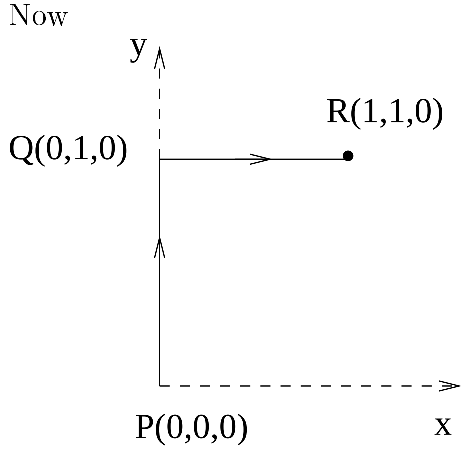


$$\begin{aligned} [P,Q]: \ \gamma_1(t) &= (t,0,0),\ t \in [0,1] \\ [Q,R]: \ \gamma_2(t) &= (1,t,0),\ t \in [0,1] \end{aligned}$$

and compute the work integrating:

$$W = \int_0^1 (0,0,0) \cdot (1,0,0) dt + \int_0^1 (t,0,0) \cdot (0,1,0) dt = 0$$

b) Now

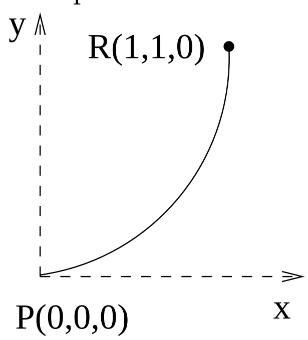


$$\begin{aligned} [P,Q]: \ \gamma_1(t) &= (0,t,0),\ t \in [0,1] \\ [Q,R]: \ \gamma_2(t) &= (t,1,0),\ t \in [0,1] \end{aligned}$$

and integrate the field:

$$W = \int_0^1 (t,0,0)\cdot(0,1,0)dt + \int_0^1 (1,0,0)\cdot(1,0,0)dt = 1$$

c) The path is



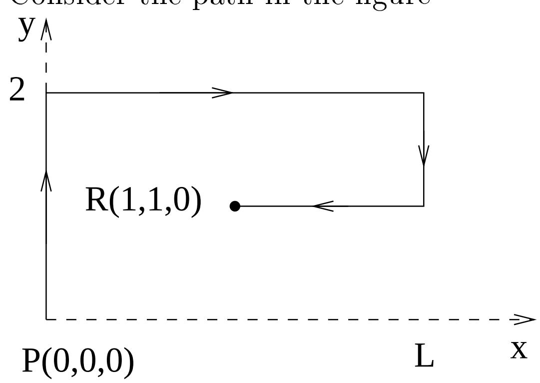
which we parametrize by

$$\gamma(t) = (t, t^2, 0),\ t \in [0,1]$$

and the work is

$$W = \int_0^1 (t^2,0,0)\cdot(1,2t,0)dt = 1/3$$

d) Consider the path in the figure



In the vertical segments $dx = 0$ and the field doesn't do work along them. Along the horizontal segments the work is

$$W = 2L - 1\frac{L}{2} = \frac{3}{2}L$$

a value we can make as big as we please taking L big.

□

3.2.2 Scalar potential

$\boxed{\mathbb{T}}$ We say that a vector field in an open set $\mathbf{F} = (F_1, \ldots F_n) \in \mathcal{C}^1(U)$, $U \subset \mathbb{R}^n$ has potential $V \in \mathcal{C}^2(U)$ if

$$\boxed{\mathbf{F} = \nabla V}$$

or, in components,

$$F_i = \frac{\partial V}{\partial x_i}, i = 1, \ldots, n.$$

V is a bit like a primitive of $\mathbf{F}$; a single function alone has the information about the n component functions of the field. Fields that have a potential function are named *gradient* fields.

If $\mathbf{F} \in \mathcal{C}^1(U)$ has a potential V, being of class $\mathcal{C}^2$ we can invert the order of derivation:

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial F_j}{\partial x_i}$$

We shall name that necessary condition for the existence of a potential the *mixed derivatives condition*; when $U \subset \mathbb{R}^3$ we may summarize it as rot $\mathbf{F} = 0$.

$U \subset \mathbb{R}^3$ is called simply connected if any closed curve in U can be 'filled' with a surface contained in U. Then the necessary condition is a sufficient one as well:

$$\mathbf{F} \text{ is a gradient} \Leftrightarrow \text{rot } \mathbf{F} = \mathbf{0}$$

The previous equivalence remains true for $U \subset \mathbb{R}^2$. In this case we call U simply connected if it has only one 'piece' and, moreover, it has no holes. Then the mixed derivatives condition is a sufficient one for the existence of a potential.

Let $U \subset \mathbb{R}^3$ be open and $\mathbf{F} \in \mathcal{C}^1(U)$; the following conditions are equivalent

a) $\mathbf{F}$ has a potential V.

b) $\int_C \mathbf{F} \cdot d\mathbf{l} = 0$, for every closed curve $C \subset U$.

c) $\int_C \mathbf{F} \cdot d\mathbf{l}$ depends only on the endpoints of C.

Fields satisfying the third condition are called *conservative fields*; the equivalence above says that gradient fields and conservative fields are the same.

□

Problem 66: Finding the field from a potential.

Find the fields that have the following potentials:

a) $V(x,y) = \log\frac{1}{r}, \mathbf{r} = (x,y), r = \mid \mathbf{r} \mid$ logarithmic potencial.

b) $V(x,y,z) = \frac{1}{r}, \mathbf{r} = (x,y,z), r = \mid \mathbf{r} \mid$ newtonian potencial.

c) $V(x_1,\ldots,x_n) = \frac{1}{n-2}r^{-(n-2)}, n \geq 3, \mathbf{r} = (x_1,\ldots,x_n), r = \mid \mathbf{r} \mid$ generalized newtonian potential.

Solution:

a) $\mathbf{F}(\mathbf{r}) = \nabla(\log\frac{1}{r}) = r\nabla(\frac{1}{r}) = r(-\frac{1}{r^2})\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{r^2}.$

b) $\mathbf{F}(\mathbf{r}) = \nabla(\frac{1}{r}) = -\frac{1}{r^2}\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{r^3}$

c) $\mathbf{F}(\mathbf{r}) = \nabla(\frac{1}{n-2}r^{-(n-2)}) = -\frac{1}{n-2}(n-2)r^{-(n-1)}\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{r^n}$

□

$\boxed{\mathbb{T}}$ The line integral of a gradient field $\mathbf{F}$ with potential V is particularly simple. Observe first that if γ is a parametrized curve

$$\frac{d}{dt}V(\gamma(t)) = \frac{\partial V}{\partial x_1}\gamma_1' + \cdots + \frac{\partial V}{\partial x_n}\gamma_n' = \nabla V \cdot \gamma'$$

and then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= \int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)dt = \\ &= \int_a^b \nabla V(\gamma(t))\cdot\gamma'(t)dt = \\ &= \int_a^b \frac{d}{dt}V(\gamma(t))dt = V(\gamma(b)) - V(\gamma(a))\end{aligned}$$

which reinforces the idea of the potential being a primitive of the field.

□

Problem 67:

Let $V(x,y,z) = \frac{x}{r}$; compute

a) $\int_C \nabla V \cdot d\mathbf{l}$, C being the arc of the circumference with center at $(0,0,0)$ joining $(1,0,0)$ to $(-1,0,0)$.

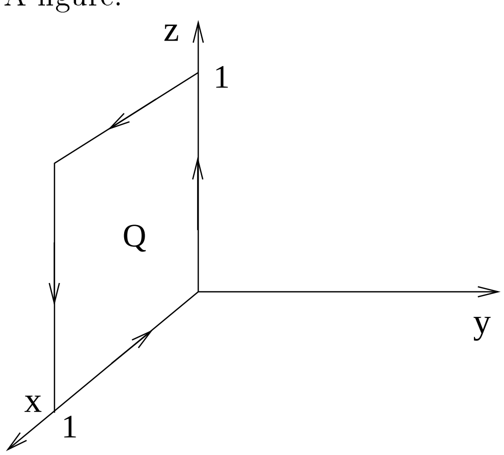
b) $\int_Q r^3 \nabla V \cdot d\mathbf{l}$, Q being the boundary of the square $Q = \{(x,y,z) : 0 \leq x \leq 1, y = 0, 0 \leq z \leq 1\}$.

Solution:

a) We need only the end points:

$$\int_C \nabla V \cdot d\mathbf{l} = V(-1,0,0) - V(1,0,0) = -1 - 1 = -2$$

b) A figure:



Now we must do some computations

$$\begin{aligned}
\nabla(\frac{x}{r}) &= (\frac{r^2 - x^2}{r^3}, -\frac{xy}{r^3}, -\frac{xz}{r^3}) \\
&= \frac{1}{r^3}(y^2 + z^2, -xy, -xz) \\
\mathbf{G} = r^3 \nabla(\frac{x}{r}) &= (y^2 + z^2, -xy, -xz)
\end{aligned}$$

The field we want to integrate is $\mathbf{G}(x,0,z) = (z^2, 0, -xz)$ because $Q \subset \{y = 0\}$. We parametrize the boundary of Q and integrate:

$$\gamma_1(t) = (0,0,t), t \in [0,1] \qquad \int_0^1 (t^2, 0, 0) \cdot (0,0,1)dt = 0$$

$$\gamma_2(t) = (t,0,1), t \in [0,1] \qquad \int_0^1 (1, 0, -t) \cdot (1,0,0)dt = 1$$

$$\gamma_3(t) = (1,0,1-t), t \in [0,1] \qquad \int_0^1 ((1-t)^2, 0, t-1) \cdot (0,0,-1)dt = 1/2$$

$$\gamma_4(t) = (1-t,0,0), t \in [0,1] \qquad \int_0^1 (0,0,0) \cdot (-1,0,0)dt = 0$$

Finally

$$\int_Q r^3 \nabla V \cdot d\mathbf{l} = 3/2$$

□

Problem 68: Integral of conservative fields.

a) Compute the line integral $\int_C ydx + xdy$ along the segment joining $(0,0)$ to $(2,3)$.

b) Compute the line integral $\int_C xdx + ydy + zdz$ along the segment joining $(0,0,0)$ to $(1,2,3)$.

Solution:

We need not parametrize C because both fields have a potential that we can find by inspection:

a) $V(x,y) = xy$ is a potential function of $\mathbf{F}(x,y) = (y,x)$. Then

$$\int_C ydx + xdy = V(2,3) - V(0,0) = 6$$

b) $V(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$ is a potential function of $\mathbf{F}(x,y,z) = (x,y,z)$. So

$$\int_C xdx + ydy + zdz = V(1,2,3) - V(0,0,0) = 7$$

□

Problem 69:

Show that the field defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$

$$\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

satisfies the mixed derivatives condition but does not have a potential.

Solution:

It satisfies the condition:

$$\left.\begin{array}{rcl} \frac{\partial F_1}{\partial y} & = & \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial F_2}{\partial x} & = & \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{array}\right\} \Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

If $\mathbf{F}$ had a potential its integral along any closed curve should vanish. But consider the curve:

$$\gamma(t) = (\cos t, \sin t), t \in [0, 2\pi],$$

and the line integral

$$\int_\gamma \mathbf{F} \cdot d\mathbf{l} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^{2\pi} dt = 2\pi \neq 0$$

We see that $\mathbf{F}$ cannot have a potential; note that $\mathbf{F}$ is defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ which is not simply connected for it has a hole at the origin.

□

To obtain a potential of a gradient field it suffices to compute its line integral along a curve that connects a choosen fixed point with the point where we want the potential. This is possible when U is a connected set; if it is not we apply the procedure to each connected component.

Problem 70: First method to find a potential.

a) Show that the following fields have a potential and find it.

  i) $\mathbf{F}(x,y) = (y^2 + 2xy, 2xy + x^2)$

  ii) $\mathbf{F}(x,y,z) = (y+z, z+x, x+y)$

b) If the field $\mathbf{F} = (P, Q)$ defined in the whole of $\mathbb{R}^2$ has a potential, find a potential that takes the value 0 at $\mathbf{p} = (a, b)$.

Solution:

a) The fields are defined in the whole of $\mathbb{R}^2$ or of $\mathbb{R}^3$. As they are simply connected sets, it suffices to show that they satisfy the mixed derivatives condition.

i) We have:

$$\partial_y(y^2 + 2xy) = 2y + 2x = \partial_x(2xy + x^2)$$

and so $\mathbf{F}$ has a potential. Using the origin as the fixed point we may integrate along a segment:

$$\begin{aligned}
\gamma(t) &= (tx, ty), 0 \leq t \leq 1 \\
\gamma'(t) &= (x, y) \\
V(x, y) &= \int_0^1 ((ty)^2 + 2(tx)(ty), 2(tx)(ty) + (tx)^2) \cdot (x, y) dt = \\
&= (y^2 x + 2x^2 y + 2xy^2 + x^2 y) \int_0^1 t^2 dt = xy^2 + x^2 y
\end{aligned}$$

ii) Now we have:

$$\begin{aligned}
\partial_y(y + z) = \ & 1 \ = \partial_x(z + x) \\
\partial_z(y + z) = \ & 1 \ = \partial_x(x + y) \\
\partial_z(z + x) = \ & 1 \ = \partial_y(x + y)
\end{aligned}$$

To find a potential we proceed as in i)

$$\begin{aligned}
\gamma(t) &= (tx, ty, tz), 0 \leq t \leq 1 \\
\gamma'(t) &= (x, y, z)
\end{aligned}$$

and

$$\begin{aligned}
V(x, y, z) &= \int_0^1 (ty + tz, tz + tx, tx + ty) \cdot (x, y, z) dt = \\
&= 2(xy + yz + zx) \int_0^1 t dt = \\
&= xy + yz + zx
\end{aligned}$$

b) To obtain the value 0 at $\mathbf{p} = (a, b)$, we choose $\mathbf{p}$ as the fixed point:

$$\begin{aligned} \gamma(t) &= (1-t)\mathbf{p} + t\mathbf{x}, 0 \leq t \leq 1 \\ \gamma'(t) &= \mathbf{x} - \mathbf{p} \end{aligned}$$

and obtain

$$\begin{aligned} V(x,y) &= \int_0^1 (P((1-t)\mathbf{p} + t\mathbf{x}), Q((1-t)\mathbf{p} + t\mathbf{x})) \cdot (\mathbf{x} - \mathbf{p}) dt = \\ &= (x-a) \int_0^1 (P((1-t)\mathbf{p} + t\mathbf{x}) dt + (y-b) \int_0^1 (Q((1-t)\mathbf{p} + t\mathbf{x}) dt \end{aligned}$$

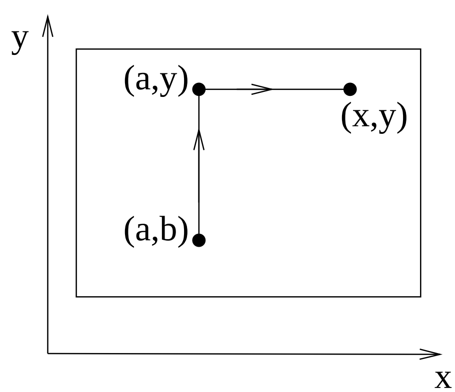
In particular if $\mathbf{p} = (0, 0)$ we obtain the formula

$$V(x,y) = x \int_0^1 P(t\mathbf{x}) dt + y \int_0^1 Q(t\mathbf{x}) dt$$

We can use this method in the case of n variables.

□

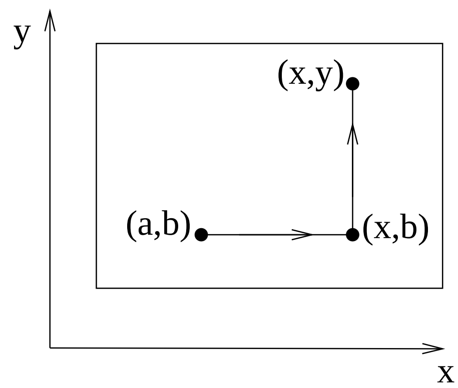
$\boxed{\mathbb{T}}$ If we want the potential of a conservative field in a rectangle, the following paths are useful:



and we have a potencial

$$V(x,y) = \int_a^x P(x,y) dx + \int_b^y Q(a,y) dy,$$

The path



produces the potential

$$V(x,y) = \int_a^x P(x,b)dx + \int_b^y Q(x,y)dy$$

Both integrations coincide because the integration depends only on the end points.

$\square$

Problem 71: Second method to find a potential.

a) Let $\mathbf{F} = (P,Q) \in \mathcal{C}^1(U), U \subset \mathbb{R}^2$ being an open set, a field that satisfies the mixed derivatives condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Let $\int P(x,y)dx$ be *any* primitive of P respect to x defined in U. Show that in any neighborhood of each point there is a function $C(y)$ such that

$$\int P(x,y)dx + C(y)$$

is a potencial for $\mathbf{F}$.

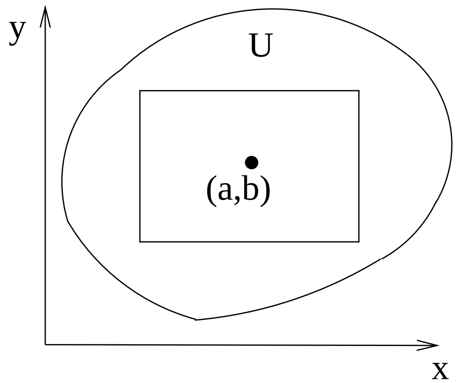
b) Decide whether the following fields have a potential, and in the affirmative case compute it:

i) $\mathbf{F}(x,y) = (y^2 + 2xy, x^2 + 2xy)$

ii) $\mathbf{F}(x,y,z) = (y+z, z+x, x+y)$.

Solution:

a) Let $(a, b) \in U$ and choose an open rectangle K centered at (a, b) and contained in U; we find a potential V in K.



If $\int P(x, y)dx$ is an x-primitive of P, any other primitive will differ from it in a constant that depending on y. The function $\int_a^x P(u, y)du + \int_b^y Q(a, v)dv$ is an x-primitive of P as well as a potential for the field; so

$$\int P(x, y)dx - (\int_a^x P(u, y)du + \int_b^y Q(a, v)dv) = -C(y)$$

and then

$$\int P(x, y)dx + C(y) = \int_a^x P(u, y)du + \int_b^y Q(a, v)dv$$

is a potential.

b) Both fields satisfy the mixed derivatives condition and as we have seen in a) they have locally a potential.

i)

$$\begin{aligned} \partial_x V &= y^2 + 2xy \Rightarrow \\ V(x, y) &= \int (y^2 + 2xy)dx + C(y) = y^2 x + x^2 y + C(y) \end{aligned}$$

To compute $C(y)$, we impose the other condition $V(x, y)$ must satisfy:

$$\begin{aligned} \partial_y V &= x^2 + 2xy \Rightarrow \\ 2yx + x^2 + C'(y) &= x^2 + 2xy \Rightarrow C'(y) = 0 \Rightarrow C(y) = \text{const} \end{aligned}$$

We have obtained the potencial

$$V(x,y) = y^2x + x^2y + \text{const}$$

ii) $\partial_x V = y+z \Rightarrow V(x,y,z) = \int (y+z)dx + C(y,z) = (y+z)x + C(y,z)$
To compute $C(y,z)$ we impose two conditions $V(x,y,z)$ must satisfy:

$$\begin{aligned} \partial_y V &= z+x \Rightarrow x + \partial_y C(y,z) = z+x \Rightarrow \partial_y C(y,z) = z \\ \partial_z V &= x+y \Rightarrow x + \partial_z C(y,z) = x+y \Rightarrow \partial_z C(y,z) = y \end{aligned}$$

And now we have a similar problem to that in a):

$$\begin{aligned} C(y,z) &= \int z dy + D(z) = zy + D(z) \\ y + D'(z) &= y \Rightarrow D'(z) = 0 \Rightarrow D(z) = \text{const} \end{aligned}$$

The potential is:

$$V(x,y,z) = (y+z)x + zy + \text{const}$$

□

Problem 72: Integral of conservative fields.

See whether the following integrals are independent of C:

a) $\int_C (\sin y\, e^{x \sin y}, x \cos y\, e^{x \sin y}) \cdot d\mathbf{l}$, between $(0,0)$ and $(1,\pi)$.

b) $\int_C (xy^2z^2, x^2yz^2, x^2y^2z) \cdot d\mathbf{l}$, between $(1,1,1)$ and $(1,2,3)$.

and in the affirmative case compute them.

Solution:

Both fields are conservative:

a) $V(x,y) = e^{x \sin y}$ is a potential and we have

$$\begin{aligned} \int_C (\sin y\, e^{x \sin y}, x \cos y\, e^{x \sin y}) \cdot d\mathbf{l} &= V(1,\pi) - V(0,0) = \\ &= 1 - 1 = 0 \end{aligned}$$

b) $V(x,y,z) = \frac{x^2y^2z^2}{2}$ is a potential and we have

$$\begin{aligned}\int_C (xy^2z^2, x^2yz^2, x^2y^2z)\cdot d\mathbf{l} &= V(1,2,3) - V(1,1,1) = \\ &= 18 - \frac{1}{2} = \frac{35}{2}\end{aligned}$$

□

Problem 73:

Compute the line integral

$$I = \int_C (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz,$$

C being the arc of the circular helix parametrized by $\gamma(t) = (a\cos t, a\sin t, \frac{h}{2\pi}t)$, from the point $A = (a,0,0)$ up to the point $B = (a,0,h)$.

Solution:

Let us see whether $\mathbf{F}(x,y,z) = (x^2 - yz, y^2 - xz, z^2 - xy)$ has a potential:

$$\text{rot } \mathbf{F} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 - yz & y^2 - xz & z^2 - xy \end{pmatrix} = (0,0,0)$$

and being a differentiable field defined in all of $\mathbb{R}^3$ it has a potential. We compute it integrating along the segment $\gamma(t) = (tx, ty, tz), t \in [0,1]$:

$$\begin{aligned}V(x,y,z) &= \int_0^1 t^2(x^2 - yz, y^2 - xz, z^2 - xy)\cdot(x,y,z)dt = \\ &= \frac{1}{3}(x^3 + y^3 + z^3 - 3xyz)\end{aligned}$$

Then

$$I = V(a,0,h) - V(a,0,0) = \frac{1}{3}(a^3 + h^3 - a^3) = \frac{1}{3}h^3$$

□

Problem 74:

a) See whether the following integrals are independent of C:

   i)
$$I = \int_C (2xyz + \sin x)dx + xz^2dy + x^2ydz$$

   ii)
$$J = \int_C ydx + xdy + xyzdz$$

b) Same question if $C \subset \{(x, y, z) : z = 0\}$.

Solution:

a)

   i) rot $(2xyz + \sin x, xz^2, x^2y) = \mathbf{0}$, $V(x, y, z) = x^2yz - \cos x$ is a potential and there is independence of C

   ii) rot $(y, x, xyz) = (xz, -yz, 0)$ doesn't vanish everywhere and there is dependence on C.

b)

   i) The field has a potential and that fact doesn't depend on where C is.

   ii) Now we are in the plane $z = 0$ and $\frac{\partial}{\partial x}(y) = 0 = \frac{\partial}{\partial y}(x)$; a potential exists, namely $V(x, y) = xy$ and there is independence of the integral on C.

□

Problem 75:

Consider the plane $z = 0$ in $\mathbb{R}^3$ and another plane π passing through the Ox axis. Being given $\mathbf{X} = (x, y, z) \in \mathbb{R}^3$,

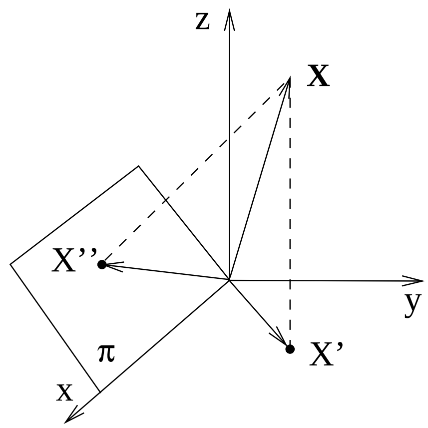
a) Compute $\mathbf{X}'$, $\mathbf{X}$", the orthogonal projections of $\mathbf{X}$ on $z = 0$ and π respectively.

b) Show that the field

$$\mathbf{F} = \mathbf{X}' + \mathbf{X}"$$

has a potential and find it.

Solution:



a) Clearly $\mathbf{X}'(x, y, z) = (x, y, 0)$; to find $\mathbf{X}"$ let the equation of π be

$$ay + bz = 0, a^2 + b^2 = 1$$

and select an ON basis of π, for instance

$$(1, 0, 0), (0, b, -a)$$

Then the projection is

$$\begin{aligned} \mathbf{X}"(x, y, z) &= ((x, y, z) \cdot (1, 0, 0))(1, 0, 0) + ((x, y, z) \cdot (0, b, -a))(0, b, -a) = \\ &= (x, b(by - az), -a(by - az)) \end{aligned}$$

The field is

$$\mathbf{F}(x, y, z) = (2x, y + b(by - az), -a(by - az))$$

b) Let us see first whether it can have a potential:

$$\text{rot } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2x & y + b(by - az) & -a(by - az) \end{pmatrix} = (-ab+ab, 0, 0) = \mathbf{0}$$

and, moreover, the field is everywhere defined and so has a potential $V(x, y, z)$. We compute it

$$\partial_x V = 2x \Rightarrow V = x^2 + \varphi(y, z)$$

Now we impose the other two conditions to V and obtain

$$\begin{aligned} \partial_y V &= y + b(by - az) : \ \partial_y \varphi = y + b(by - az) \\ \partial_z V &= z - a(by - az) : \ \partial_z \varphi = -a(by - az) \end{aligned}$$

Integrating the first equation

$$\varphi(y, z) = (1 + b^2)\frac{y^2}{2} - abyz + \psi(z)$$

and substituting into the second

$$\begin{aligned} -aby + \psi'(z) &= -a(by - az) \\ \psi'(z) &= a^2 z \\ \psi(z) &= a^2 \frac{z^2}{2} \end{aligned}$$

Finally

$$V(x, y, z) = x^2 + (1 + b^2)\frac{y^2}{2} - abyz + a^2\frac{z^2}{2}$$

□

Problem 76:

a) Let $\mathbf{F}$ be a nowhere vanishing conservative field in $\mathbb{R}^3$. The *field* lines are those curves tangent to the direction of the field at each point. Show they can't be closed curves.

b) Let $f \in \mathcal{C}^1(\mathbb{R}^3)$ and C be a curve orthogonal to the level surfaces of f (at critical points of f there is no orthogonal direction, so we assume that C passes through no such points). Show that C cannot be closed.

Solution:

a) Let C be a closed field line; being $\mid \mathbf{F} \mid \neq 0$ we have

$$0 < \int_C \mid \mathbf{F} \mid dl$$

and we can arrange the orientation of C to have

$$0 < \int_C \mid \mathbf{F} \mid dl = \int_C \mathbf{F} \cdot \mathbf{t} dl = \int_C \mathbf{F} \cdot d\mathbf{l}$$

But $\mathbf{F}$ being assumed conservative we have $\int_C \mathbf{F} \cdot d\mathbf{l} = 0$, a contradiction.

b) The curve has the direction of $\mathbf{F} = \nabla f$ at each one of its points and $\mathbf{F}$ doesn't vanish. Then we can apply a).

□

Problem 77:

Let $\mathbf{F}$ be a vector field with fixed direction but whose module and sense at each point depend on the distance to a fixed reference plane orthogonal to the direction of $\mathbf{F}$, and the dependence is $\mathcal{C}^1$.

a) Show that $\mathbf{F}$ has a potential.

b) Is the result true if the dependence on the distance is only $\mathcal{C}^0$?

Solution:

Choose the Oz axis in the direction of the field and choose the origin in the reference plane. Then

$$\mathbf{F}(x, y, z) = f(z)(0, 0, 1)$$

a) If the dependence is $\mathcal{C}^1$, so is the field; moreover it has rot $\mathbf{F} = 0$ in $\mathbb{R}^3$. It follows that $\mathbf{F}$ is conservative.

b) A potential is:

$$V(x, y, z) = \int_0^z f(t) dt$$

□

Problem 78: Potential of a central field.

For a central field $\mathbf{F}$ find a potential. Apply the result to find potentials for the following fields

a) The gravitational field of a point mass m at the origin.

b) The electrostatic field of a point charge q at the origin.

c) A central field with center at the origin that satisfies Hooke's law.

Solution:

Consider a central field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}$, f being a continuous function and assume it defined only in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. Let us try to guess a potential; it has to do with $f(r)$. What about f as a potential?

$$\nabla f(r) \quad = \quad f'(r)\frac{\mathbf{r}}{r}$$

It does not work first at all because of the derivative of f that isn't even assumed to exist. What if we take a primitive of f? Let $\varphi(r) = \int f(r)dr$ (recall that f is continuous); in this case

$$\nabla\varphi(r) = f(r)\frac{\mathbf{r}}{r}$$

and we are nearer. As $\frac{\mathbf{r}}{r}$ is the gradient of r we guess that $V(r) = \int rf(r)dr$ will be a potential. Let us try:

$$\nabla V(r) = rf(r)\frac{\mathbf{r}}{r} = f(r)\mathbf{r} = \mathbf{F}(\mathbf{r})$$

a) The field is $\mathbf{g}(\mathbf{r}) = -m\frac{\mathbf{r}}{r^3}$, so $f(r) = -m\frac{1}{r^3}$ and we have $V(r) = -m\int r\frac{1}{r^3}dr = m\frac{1}{r}$.

b) Now $\mathbf{E}(\mathbf{r}) = q\frac{\mathbf{r}}{r^3}$ and by analogy with a) $V(r) = -q\frac{1}{r}$. In electricity texts one sees $V(r) = q\frac{1}{r}$ but $\mathbf{E} = -\nabla V$.

c) The field is $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}$, $f(r) = -k$ and $V(r) = -\frac{1}{2}kr^2$.

□

Problem 79:

Let $\mathbf{F}$ have potential V; prove the following are equivalent:

a) $\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r}$ ($\mathbf{F}$ is radial).

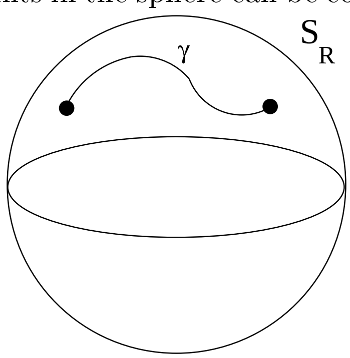
b) $\mathbf{F}(\mathbf{r}) = h(r)\mathbf{r}$ ($\mathbf{F}$ is central).

c) $V(\mathbf{r}) = f(r)$ (V is spherically symmetric).

Solution:

b)⇒a): Evident.

a)⇒c): It suffices to show that V is constant on each sphere S_R. Two points in the sphere can be connected by a curve in the sphere $\gamma : [a, b] \to S_R$:



and we prove V is constant on γ. Differentiate V on γ;

$$\begin{aligned}\frac{d}{dt}(V(\gamma(t))) &= \nabla V(\gamma(t)) \cdot \gamma'(t) = \\ &= g(\gamma(t))\gamma(t) \cdot \gamma'(t) = 0\end{aligned}$$

because $\gamma(t)$ and $\gamma'(t)$ are a radius vector and a tangent vector to the sphere respectively; they are orthogonal.

c)⇒b): From $V(\mathbf{r}) = f(r)$ compute $\mathbf{F}$.

$$\frac{\partial V}{\partial x_i} = f'(r)\frac{x_i}{r} \Rightarrow \mathbf{F}(\mathbf{r}) = \frac{f'(r)}{r}\mathbf{r}$$

□

Problem 80: Energy conservation.

Let $\mathbf{F} = -\nabla V$ be a vector field and consider a mass point m in that field submitted to Newton's law. Let $\mathbf{x}(t)$ be the movement of the mass point.

a) Show that the energy $E = \frac{1}{2}m \mid \dot{\mathbf{x}}(t) \mid^2 + V(\mathbf{x}(t))$ is constant.

b) Show that if the particle moves on an equipotential surface then the celerity is constant.

c) If $\mathbf{y}(t)$ is a trajectory of $\mathbf{F}$ looked upon as a velocity field show that $V(\mathbf{y}(t))$ is nonincreasing.

Solution:

a) Write $E = \frac{1}{2}m\dot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t) + V(\mathbf{x}(t))$ and differentiate

$$\frac{dE}{dt} = \frac{1}{2}m2\dot{\mathbf{x}}(t) \cdot \ddot{\mathbf{x}}(t) + \nabla V(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$$

Newton's law is $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}(t)) = -\nabla V(\mathbf{x}(t))$ and substituting

$$\frac{dE}{dt} = \dot{\mathbf{x}}(t) \cdot (-\nabla V(\mathbf{x}(t)) + \nabla V(\mathbf{x}(t))) = 0$$

and E is constant along the movement.

b) If the mass point moves on an equipotential surface, being the energy constant, the kinetic term is constant and so is the celerity.

c) The law of the trajectory is now $\dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t))$ and

$$\begin{aligned}\frac{d}{dt}V(\mathbf{y}(t)) &= \nabla V(\mathbf{y}(t)) \cdot \dot{\mathbf{y}}(t) = \nabla V(\mathbf{y}(t)) \cdot \mathbf{F}(\mathbf{y}(t)) = \\ &= -\mathbf{F}(\mathbf{y}(t)) \cdot \mathbf{F}(\mathbf{y}(t)) = - \mid \mathbf{F}(\mathbf{y}(t)) \mid^2 \leq 0\end{aligned}$$

□

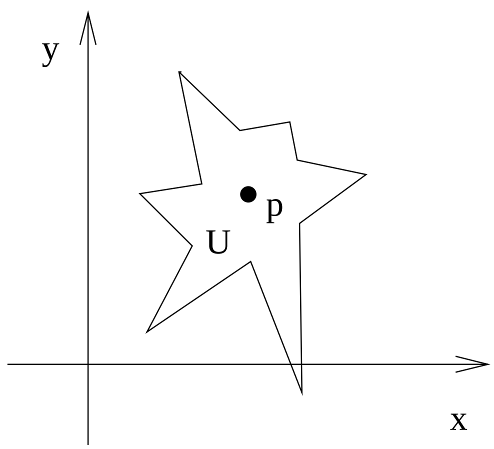
3.2.3 Vector potential

$\boxed{\mathbb{T}}$ A vector field $\mathbf{F} \in \mathcal{C}^1(U)$ in the open set $U \subset \mathbb{R}^3$ has a vector potential $\mathbf{A} \in \mathcal{C}^2(U)$ if

$$\boxed{\mathbf{F} = \text{rot } \mathbf{A}}$$

Not every vector field has a vector potential because taking the divergence of both terms in the preceding expression gives div $\mathbf{F}$ = div (rot $\mathbf{A}$) = 0, a necessary condition for a vector potential to exist. A field such that div$\mathbf{F} \neq 0$ can't have a vector potential.

An open set $U \subset \mathbb{R}^n$ is star shaped if there is a point $\mathbf{p} \in U$ that can 'see' all the points in U, that is for every $\mathbf{x} \in U$ one has $[\mathbf{p}, \mathbf{x}] \subset U$.



For such sets the necessary condition is sufficient as well, that is:

$$\mathbf{F} \text{ has vector potential} \Leftrightarrow \text{div } \mathbf{F} = \mathbf{0}$$

Let $U \subset \mathbb{R}^3$ be an open set and $\mathbf{F} \in \mathcal{C}^1(U)$; the following conditions are equivalent:

a) $\mathbf{F}$ has a vector potential $\mathbf{A}$: rot $\mathbf{A} = \mathbf{F}$.

b) $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0$, for every closed surface in U.

c) $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ depends only on ∂S.

□

First method for vector potential calculus

Problem 81:

Consider the closed ball $\bar{U} = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < R\}$ and $\mathbf{F} \in \mathcal{C}^1(\bar{U})$ such that div $\mathbf{F} = 0$. We want to prove that

$$\boxed{\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})dt}$$

is a vector potential for $\mathbf{F}$. Show:

a) If $\mathbf{M} \in \mathcal{C}^1(U)$ and $\mathbf{N}(\mathbf{x}) = \int_0^1 \mathbf{M}(t\mathbf{x})dt$ then rot $\mathbf{N} = \int_0^1 (\text{rot } \mathbf{M})(t\mathbf{x})dt$.

b) $\nabla \times (\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})) = 2t\mathbf{F}(t\mathbf{x}) + t^2 \frac{d}{dt}\mathbf{F}(t\mathbf{x})$.

c) rot $\mathbf{A} = \mathbf{F}$.

d) $\mathbf{A}, \mathbf{A}_1$ are vector potentials (of the same field) $\Leftrightarrow \mathbf{A}_1 = \mathbf{A} + \nabla f$ locally (f an arbitrary function).

e) Find a vector potential of $\mathbf{F} = (0, 0, 1)$.

f) Find a vector potential of $\mathbf{F} = (0, 0, 1)$ satisfying div$\mathbf{A} = 2x + y - 1$.

g) Find a vector potential of $\mathbf{F}(x, y, z) = (2x, -y, -z)$.

Solution:

a) Let us check the first component; using Leibniz's rule for the differentiation of integrals depending on parameters we have

$$\begin{aligned} (\text{rot}\mathbf{N})_1 &= \partial_y N_3 - \partial_z N_2 \\ \partial_y N_3 &= \int_0^1 \partial_y M_3(t\mathbf{x})dt \\ \partial_z N_2 &= \int_0^1 \partial_z M_2(t\mathbf{x})dt \end{aligned}$$

and

$$\begin{aligned} (\text{rot } \mathbf{N})_1 &= \int_0^1 (\partial_y M_3 - \partial_z M_2)(t\mathbf{x})dt = \\ &= \int_0^1 (\text{rot } \mathbf{M})_1(t\mathbf{x})dt \end{aligned}$$

We can proceed in the same way with the other components.

b) We use the formula (see p.72)

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

thus

$$\nabla \times (\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})) = (\nabla \cdot (t\mathbf{x}))\mathbf{F}(t\mathbf{x}) - (\nabla \cdot \mathbf{F}(t\mathbf{x}))t\mathbf{x} + (t\mathbf{x} \cdot \nabla)\mathbf{F}(t\mathbf{x}) - (\mathbf{F}(t\mathbf{x}) \cdot \nabla)t\mathbf{x}$$

Compute the different terms

$$\begin{aligned} \nabla \cdot (t\mathbf{x}) &= 3t \\ \nabla \cdot \mathbf{F}(t\mathbf{x}) &= t\text{div }\mathbf{F} = 0 \\ t\mathbf{x} \cdot \nabla &= tx\partial_x + ty\partial_y + tz\partial_z \\ \mathbf{F}(t\mathbf{x}) \cdot \nabla &= F_1(t\mathbf{x})\partial_x + F_2(t\mathbf{x})\partial_y + F_3(t\mathbf{x})\partial_z \end{aligned}$$

and obtain

$$\begin{aligned} \nabla \times (\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})) &= 3t\mathbf{F}(t\mathbf{x}) + (tx\partial_x + ty\partial_y + tz\partial_z)\mathbf{F}(t\mathbf{x}) \\ &\quad -(F_1(t\mathbf{x})\partial_x + F_2(t\mathbf{x})\partial_y + F_3(t\mathbf{x})\partial_z)t\mathbf{x} \end{aligned}$$

The first operator acting on the first component of $\mathbf{F}(t\mathbf{x})$:

$$(tx\partial_x + ty\partial_y + tz\partial_z)F_1(t\mathbf{x}) = t^2(x(\partial_x F_1)_{t\mathbf{x}} + y(\partial_y F_1)_{t\mathbf{x}} + z(\partial_z F_1)_{t\mathbf{x}})$$

that we can write as

$$t^2 \frac{d}{dt} F_1(t\mathbf{x}) = t^2((\partial_x F_1)_{t\mathbf{x}} x + (\partial_y F_1)_{t\mathbf{x}} y + (\partial_z F_1)_{t\mathbf{x}} z),$$

and analogous results for the other components F_2, F_3. The other operator is

$$(F_1(t\mathbf{x})\partial_x + F_2(t\mathbf{x})\partial_y + F_3(t\mathbf{x})\partial_z)t\mathbf{x} = t(F_1(t\mathbf{x}), F_2(t\mathbf{x}), F_3(t\mathbf{x})) = t\mathbf{F}(t\mathbf{x})$$

Finally

$$\begin{aligned} \nabla \times (\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})) &= 2t\mathbf{F}(t\mathbf{x}) + t^2 \frac{d}{dt}\mathbf{F}(t\mathbf{x}) = \\ &= \frac{d}{dt}(t^2 \mathbf{F}(t\mathbf{x})) \end{aligned}$$

c)

$$\begin{aligned}(\text{rot } \mathbf{A})(\mathbf{x}) &= \int_0^1 \nabla \times (\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x}))dt = \\ &= \int_0^1 \frac{d}{dt}(t^2\mathbf{F}(t\mathbf{x}))dt = (t^2\mathbf{F}(t\mathbf{x}))|_0^1 = \mathbf{F}(\mathbf{x})\end{aligned}$$

d) If $\mathbf{A}_1 = \mathbf{A} + \nabla f$ where f is an arbitrary function, then $\mathbf{A}_1$ is a vector potential:

$$\text{rot } \mathbf{A}_1 = \text{rot } \mathbf{A} + \text{rot } (\nabla f) = \text{rot } \mathbf{A}$$

Reciprocally if $\mathbf{A}, \mathbf{A}_1$ are two vector potentials

$$\text{rot}(\mathbf{A}_1 - \mathbf{A}) = \text{rot } \mathbf{A}_1 - \text{rot } \mathbf{A} = \mathbf{F} - \mathbf{F} = \mathbf{0}$$

and $\mathbf{A}_1 - \mathbf{A} = \nabla f$ for some f.

e) div $\mathbf{F} = 0$ and we use the formula just proved:

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \int_0^1 (0,0,1) \times (tx, ty, tz)dt = \\ &= \int_0^1 (-ty, tx, 0)dt = \frac{1}{2}(-y, x, 0)\end{aligned}$$

f) We look for an f such that $\mathbf{A}_1 = \frac{1}{2}(-y, x, 0) + \nabla f$ satisfies

$$\text{div } \mathbf{A}_1 = \text{div}(\frac{1}{2}(-y, x, 0) + \nabla f) = \nabla^2 f = 2x + y - 1$$

that is called a Poisson equation; the variables are separated and we easily find:

$$f(x, y, x) = \frac{x^3}{3} + \frac{y^3}{6} - \frac{z^2}{2}$$

Finally

$$\mathbf{A}_1 = \frac{1}{2}(-y, x, 0) + (x^2, \frac{y^2}{2}, -z)$$

g) Now $\mathbf{F}(x, y, z) = (2x, -y, -z)$

$$\mathbf{F}(t\mathbf{x}) \times (t\mathbf{x}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2tx & -ty & -tz \\ tx & ty & tz \end{pmatrix} = (0, -3t^2xz, 3t^2xy)$$

$$\mathbf{A}(\mathbf{x}) = \int_0^1 (0, -3t^2xz, 3t^2xy)dt = (0, -xz, xy)$$

$\square$

Second method for vector potential calculus

$\boxed{\mathbb{T}}$ Let $U \subset \mathbb{R}^3$ be a star shaped open set and let $\mathbf{F} = (P, Q, R) \in \mathcal{C}^1(U)$ be such that div $\mathbf{F} = 0$; then we know that $\mathbf{F}$ has a vector potential $\mathbf{A} = (X, Y, Z)$. Write the components of the equality rot $\mathbf{A} = \mathbf{F}$ and obtain the partial differential equations system

$$\left.\begin{array}{rcl} \partial_y Z - \partial_z Y &=& P \\ \partial_z X - \partial_x Z &=& Q \\ \partial_x Y - \partial_y X &=& R \end{array}\right\}$$

We know that an arbitrary gradient can be added to $\mathbf{A}$; using that fact we can assume that $X = 0$; then

$$\left.\begin{array}{rcl} \partial_x Z &=& -Q \\ \partial_x Y &=& R \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} Z &=& \int_{x_0}^x -Q(t,y,z)dt + \varphi(y,z) \\ Y &=& \int_{x_0}^x R(t,y,z)dt + \psi(y,z) \end{array}\right\}$$

To find φ, ψ we impose Y, Z to satisfy the first equation $\partial_y Z - \partial_z Y = P$; we have

$$\partial_y Z - \partial_z Y = \int_{x_0}^x -\frac{\partial Q(t,y,z)}{\partial y}dt + \frac{\partial \varphi(y,z)}{\partial y} - (\int_{x_0}^x \frac{\partial R(t,y,z)}{\partial z}dt + \frac{\partial \psi(y,z)}{\partial z}) =$$

$$= \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial z} - (\int_{x_0}^x (\frac{\partial Q(t,y,z)}{\partial y} + \frac{\partial R(t,y,z)}{\partial z})dt) =$$

$$= \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial z} + \int_{x_0}^x \frac{\partial P(t,y,z)}{\partial x}dt = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial z} + P(x,y,z) - P(x_0,y,z)$$

In the middle line we have used that div $\mathbf{F} = 0$. Then the first equation is satisfied if

$$\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial z} = P(x_0, y, z)$$

Choosing $\psi = 0$ we obtain the solution

$$\varphi(y,z) = \int_{y_0}^y P(x_0, t, z)dt$$

and a vector potential is

$$\mathbf{A} = (0, \int_{x_0}^{x} R(t,y,z)dt, \int_{x_0}^{x} -Q(t,y,z)dt + \int_{y_0}^{y} P(x_0,t,z)dt)$$

□

Problem 82: Vector potentials.

Find a vector potential for each of the following fields:

a) $\mathbf{F}(x,y,z) = (y,z,x)$

b) $\mathbf{F}(x,y,z) = (0,0,1)$

c) $\mathbf{F}(x,y,z) = (x^2,0,-y^2)$

d) $\mathbf{F}(x,y,z) = (2x,-y,-z)$

Solution:

Let $\mathbf{A} = (X,Y,Z)$ and choose $X = 0$; then rot $\mathbf{A} = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, -\frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x})$

a)

$$\begin{array}{lll} \frac{\partial Y}{\partial x} = x & \Rightarrow & Y = \int x dx + \varphi(y,z) = \frac{x^2}{2} + \varphi(y,z) \\ \frac{\partial Z}{\partial x} = -z & \Rightarrow & Z = -\int z dx = -xz \end{array}$$

We impose the first condition

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = y : -\frac{\partial \varphi}{\partial z} = y \Rightarrow \varphi = -yz$$

and obtain the vector potential

$$\mathbf{A} = (0, \frac{x^2}{2} - yz, -xz)$$

b)

$$\begin{array}{lll} \frac{\partial Y}{\partial x} = 1 & \Rightarrow & Y = x + \varphi(y,z) \\ \frac{\partial Z}{\partial x} = 0 & \Rightarrow & Z = 0 \end{array}$$

The first equation is

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = 0 : -\frac{\partial \varphi}{\partial z} = 0 \Rightarrow \varphi = 0$$

and

$$\mathbf{A} = (0, x, 0)$$

c) div $\mathbf{F} \neq 0$ and $\mathbf{F}$ cannot have a vector potential.

d)

$$\begin{array}{lll} \frac{\partial Y}{\partial x} = -z & \Rightarrow & Y = -\int z dx + \varphi(y,z) = -xz + \varphi(y,z) \\ \frac{\partial Z}{\partial x} = y & \Rightarrow & Z = -\int y dx = -xy \end{array}$$

and

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = 2x : x + x - \frac{\partial \varphi}{\partial z} = 2x \Rightarrow \varphi = 0$$

Finally

$$\mathbf{A} = (0, -xz, xy)$$

□

3.2.4 Newtonian and logarithmic potentials

$\boxed{\mathbb{T}}$ We know that

$$U(\mathbf{r}) = m\frac{1}{r}$$

is the potential of the gravitational field created by a mass point m at the origin. As the field has the superposition property, the potential has it as well. We use the same principle for continuous distributions (see [Kell]).

□

Problem 83: Potential of wires.

Find the potential of

a) A material wire with constant linear density λ.

b) A homogeneous material segment with constant linear density λ with origin and end points at $(0, 0, a), (0, 0, b)$.

c) A material circumference with constant linear density. Find the potential at a point of the axis.

Solution:

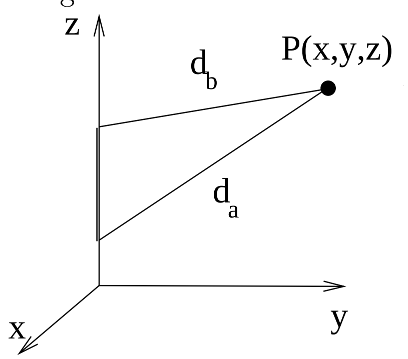
a) Using superposition in the continuous case we have

$$U(x,y,z) = \int_C \lambda \frac{1}{r} dl, r = \mid (x-u, y-v, z-w) \mid$$

and if $\gamma(t) = (u(t), v(t), w(t)), t \in [a,b]$ is a parametrization of C then

$$U(x,y,z) = \int_a^b \lambda \frac{1}{r(\gamma(t))} \gamma'(t) dt$$

b) A figure:



Parametrize the segment putting $\gamma(t) = (0,0,t), t \in [a,b], dl = dt$ and let $d^2 = x^2 + y^2$ where d is the distance from (x,y,z) to the axis. Then

$$\begin{aligned} U(x,y,z) &= \lambda \int_a^b \frac{1}{\sqrt{d^2 + (z-t)^2}} dt = \left\{ \begin{array}{rcl} t-z &=& u \\ dt &=& du \end{array} \right\} = \\ &= \lambda \int_{a-z}^{b-z} \frac{1}{\sqrt{d^2 + u^2}} du \end{aligned}$$

We compute a primitive function of the integrand

$$\begin{aligned} \int \frac{1}{\sqrt{d^2+u^2}} du &= \left\{ \begin{array}{rcl} u &=& d \sinh v \\ du &=& d \cosh v dv \end{array} \right\} = \int \frac{d \cosh v}{\sqrt{d^2(1+\sinh^2 v)}} dv = \\ &= \int \frac{\cosh v}{\cosh v} dv = v = \sinh^{-1} \frac{u}{d} \end{aligned}$$

that is

$$\frac{e^v - e^{-v}}{2} = \frac{u}{d}$$

and isolating v

$$v = \log \frac{u + \sqrt{d^2 + u^2}}{d} = \log(u + \sqrt{d^2 + u^2}) - \log d$$

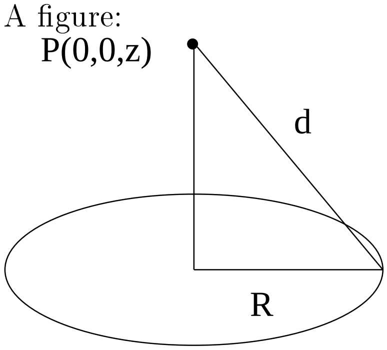
The potential is

$$U(x) = \lambda \int_{a-z}^{b-z} \frac{1}{\sqrt{d^2 + u^2}} du = \lambda[\log(u + \sqrt{d^2 + u^2}) - \log d]_{a-z}^{b-z}$$

$$\begin{aligned} U(x, y, z) &= \lambda \log \frac{b - z + \sqrt{d^2 + (z-b)^2}}{a - z + \sqrt{d^2 + (z-a)^2}} = \\ &= \lambda \log \frac{b - z + d_b}{a - z + d_a} \end{aligned}$$

d_a and d_b being the distances form (x, y, z) to the end points of the segment.

c) A figure:



A parametriztion of the wire is $\gamma(t) = (R\cos t, R\sin t, 0)$,$t \in [0, 2\pi]$ and then

$$U(0, 0, z) = \int_C \lambda \frac{1}{r} dl = \lambda \int_0^{2\pi} \frac{1}{d} R dt = 2\pi R \lambda \frac{1}{d} = M \frac{1}{d}$$

the same potential as that of a mass M at the origin.

□

Problem 84:

Check that $U(x, y) = \log \frac{1}{r}$ is a potential for the field $\mathbf{L}(x, y) = -\frac{\mathbf{r}}{r^2}$ generated by a logarithmic particle of unit mass at the origin.

Solution:

$$\frac{\partial U}{\partial x} = r\frac{\partial}{\partial x}(\frac{1}{r}) = r(-\frac{1}{r^2})\frac{x}{r} = -\frac{x}{r^2}$$

$$\frac{\partial U}{\partial y} = r\frac{\partial}{\partial y}(\frac{1}{r}) = r(-\frac{1}{r^2})\frac{y}{r} = -\frac{y}{r^2}$$

□

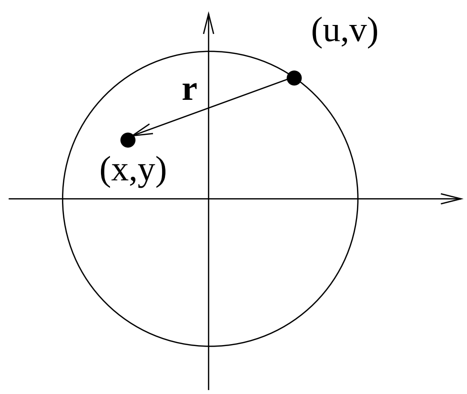
Problem 85: Logarithmic circumference.

Let C be a circumference made of logarithmic particles; find the potential at points in the plane of the circumference. The following integration formula may be useful:

$$\int_0^{2\pi} \log(1 - e\cos\theta)d\theta = 2\pi \log\frac{1+\sqrt{1-e^2}}{2}, 0 \leq e < 1$$

Solution:

Let $\mathbf{r} = (x-u, y-v), r = \mid \mathbf{r} \mid$



Because of the symmetry it suffices to compute $U(x, 0), x > 0$. Parametrizing the circumference by

$$\gamma(\theta) = (R\cos\theta, R\sin\theta), \theta \in [0, 2\pi], dl = Rd\theta$$

we have

$$\begin{aligned} U(x,0)=\int_C \lambda \log\frac{1}{r}dl &= \lambda\int_0^{2\pi}\log(\frac{1}{\sqrt{(x-R\cos\theta)^2+R^2\sin^2\theta}})Rd\theta = \\ &= -\frac{1}{2}\lambda R\int_0^{2\pi}\log(x^2+R^2-2xR\cos\theta)d\theta = \\ &= -\frac{1}{2}\lambda R\int_0^{2\pi}\log[(x^2+R^2)(1-\frac{2xR\cos\theta}{x^2+R^2})]d\theta = \\ &= -\frac{1}{2}\lambda R(\int_0^{2\pi}\log(x^2+R^2)d\theta+\int_0^{2\pi}\log(1-\frac{2xR\cos\theta}{x^2+R^2})d\theta \end{aligned}$$

If $e=\frac{2xR}{x^2+R^2}$, then $0\le e<1$ and we are allowed to use the formula:

$$U(x,0)=-\frac{1}{2}\lambda R(2\pi\log(x^2+R^2)-2\pi\log\frac{1+\sqrt{1-e^2}}{2})$$

A short computation gives

$$\frac{1+\sqrt{1-e^2}}{2}=\frac{1}{2}(1+\frac{\mid x^2-R^2\mid}{x^2+R^2})$$

and then

$$\begin{aligned} U(x,0) &= -\frac{M}{2}(\log(x^2+R^2)+\log\frac{1}{2}(1+\frac{\mid x^2-R^2\mid}{x^2+R^2})) = \\ &= -\frac{M}{2}\log[(x^2+R^2)\frac{1}{2}(1+\frac{\mid x^2-R^2\mid}{x^2+R^2})] = \\ &= -\frac{M}{2}\log\frac{x^2+R^2+\mid x^2-R^2\mid}{2} \end{aligned}$$

Two cases are in view:

a) Interior points $0\le x<R$

$$U(x,0)=-\frac{M}{2}\log R^2=M\log\frac{1}{R}$$

The potential is constant in the interior; the field vanishes there. This situation is analogous to that of a material sphere in $\mathbb{R}^3$ which has vanishing newtonian field in the interior.

b) Exterior points $R < x$

$$U(x,0) = -\frac{M}{2}\log x^2 = M\log\frac{1}{x}$$

or, for any exterior point

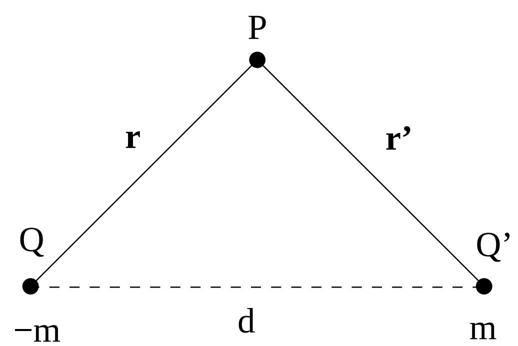
$$U(\mathbf{r}) = M\log\frac{1}{r}, \mathbf{r} = (x,y), r = \mid \mathbf{r} \mid$$

and we see that it is the same potential as that of a logarithmic particle of mass M at the center.

□

Dipoles

$\boxed{\mathbb{T}}$ We want to describe the potential generated by two equal point masses of contrary sign (!) $-m, m$ when they approach along a line. In the limit we have a dipole and the line is the dipole axis.



Now the potential at P generated by both masses is

$$U(P) = -\frac{m}{r} + \frac{m}{r'}$$

and if we let $d \to 0$ we shall have $U(P) \to 0$. To avoid that we define $\mu = md$, the *dipole moment*, and maintain it constant in passing to the limit. Writing the potential of the two masses in terms of the dipole moment we have

$$U(P) = \frac{\mu}{d}(\frac{1}{r'} - \frac{1}{r})$$

Put $-m$ at point $Q = (-\frac{d}{2}, 0, 0)$ and m at point $Q' = (\frac{d}{2}, 0, 0)$; then

$$\begin{aligned} r(x,y,z) &= \sqrt{(x + \frac{d}{2})^2 + y^2 + z^2} \\ r'(x,y,z) &= \sqrt{(x - \frac{d}{2})^2 + y^2 + z^2} \end{aligned}$$

and noticing that $r'(x,y,z) = r(x-d,y,z)$, we obtain the potential of the dipole at point $P = (x,y,z)$:

$$U(x,y,z) = \lim_{d \to 0} \mu \frac{\frac{1}{r(x-d,y,z)} - \frac{1}{r(x,y,z)}}{d} = \frac{\partial}{\partial x}(\frac{1}{r})$$

We have an x-derivative because to simplify we have choosen Q and Q' along the x-axis. The potential of a dipole with moment μ and axis $\mathbf{u}$ (a unitari vector from the negative mass point to the positive one) is

$$\boxed{U = \mu \frac{\partial}{\partial \mathbf{u}}(\frac{1}{r})}$$

which we shall write as well as

$$U = \mu \nabla (\frac{1}{r}) \cdot \mathbf{u}$$

□

Problem 86: Field of a dipole.

a) Compute the field of a dipole with moment μ and axis $\mathbf{u} = (1,0,0)$.

b) Do the same for $\mathbf{u} = \frac{1}{\sqrt{3}}(1,1,1)$.

Solution:

a)

$$U = \mu \frac{\partial}{\partial \mathbf{u}}(\frac{1}{r}) = \mu \frac{\partial}{\partial x}(\frac{1}{r}) = -\mu \frac{x}{r^3}$$

and the components of the dipole field are

$$\begin{aligned}
X &= \frac{\partial U}{\partial x} = -\mu\frac{r^3 - 3r^2\frac{x}{r}x}{r^6} = -\mu\frac{r^2-3x^2}{r^5} \\
Y &= \frac{\partial U}{\partial y} = -\mu\frac{-3r^2\frac{y}{r}x}{r^6} = -\mu\frac{3xy}{r^5} \\
Z &= \frac{\partial U}{\partial z} = -\mu\frac{-3r^2\frac{z}{r}x}{r^6} = -\mu\frac{3xz}{r^5}
\end{aligned}$$

b)

$$U = \mu\frac{\partial}{\partial \mathbf{u}}(\frac{1}{r}) = -\mu\frac{\mathbf{r}}{r^3}\cdot\frac{1}{\sqrt{3}}(1,1,1) = -\mu\frac{1}{r^3}\frac{1}{\sqrt{3}}(x+y+z)$$

and the components of the field are

$$\begin{aligned}
X &= \frac{\mu}{\sqrt{3}}(\frac{3}{r^4}\frac{x}{r}(x+y+z) - \frac{1}{r^3}) \\
Y &= \frac{\mu}{\sqrt{3}}(\frac{3}{r^4}\frac{y}{r}(x+y+z) - \frac{1}{r^3}) \\
Z &= \frac{\mu}{\sqrt{3}}(\frac{3}{r^4}\frac{z}{r}(x+y+z) - \frac{1}{r^3})
\end{aligned}$$

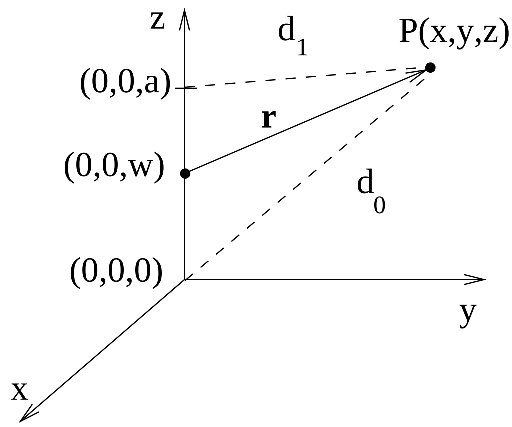
□

Problem 87: A wire of dipoles.

Let C be a segment of length a constituted of dipoles with constant linear density of moments μ with all the axes in the segment's direction.

Solution:

Take the segment along the Oz axis; with the notation as in the figure we have $\mathbf{r} = (x, y, z - w)$



The potential of the dipole segment is

$$\begin{aligned} U(x,y,z) &= \int_C \mu \frac{\partial}{\partial \mathbf{u}} (\frac{1}{r}) dl = \int_C \mu \frac{\partial}{\partial w} (\frac{1}{r}) dl = \\ &= \int_0^a \mu \frac{\partial}{\partial w} (\frac{1}{\sqrt{x^2+y^2+(z-w)^2}}) dw = \\ &= \mu \frac{1}{\sqrt{x^2+y^2+(z-w)^2}} \Big|_{w=0}^{w=a} = \frac{\mu}{d_1} - \frac{\mu}{d_0} \end{aligned}$$

Bar magnets have two poles and the interaction between poles follows Newton-Coulomb law with the intensity of the poles substituting the charge. Both poles of a magnet have the same intensity and opposite signs. The segment going from the negative pole to the positive one is called the axis of the magnet.

Cutting a magnet in two pieces we have two poles in each piece, with moreless the same intensity. We infer that a magnet can be seen as formed by microscopic magnets with aligned axes. At the endpoints the forces among poles are not compensated and we have the two poles of the magnett.

If we represent the microscopic magnets through dipoles, the result of the preceding problem shows that our point of view is consistent.

□

Problem 88: Logarithmic dipole.

Define the logarithmic dipole for the potential theory in the plane. Compute the field of this dipole.

Solution:

By analogy with dipoles in space we define the potential of a logarithmic dipole as

$$U(\mathbf{r}) = \mu \frac{\partial}{\partial \mathbf{u}} (\log \frac{1}{r})$$

Assume $\mathbf{u} = (a, b), a^2 + b^2 = 1$. Then

$$\begin{aligned} U = \mu \mathbf{u} \cdot \nabla (\frac{1}{r}) &= \mu\, (a, b) \cdot (\frac{\partial}{\partial x} (\log \frac{1}{r}), \frac{\partial}{\partial y} (\log \frac{1}{r})) \\ &= \mu(a, b) \cdot (-\frac{x}{r^2}, -\frac{y}{r^2}) = -\frac{\mu}{r^2}(ax + by) \end{aligned}$$

and the field is

$$\frac{\partial U}{\partial x} = -\mu(-\frac{2}{r^3}\frac{x}{r}(ax + by) + \frac{a}{r^2}) = -\frac{\mu}{r^4}(a(y^2 - x^2) - 2bxy)$$

$$\frac{\partial U}{\partial y} = -\frac{\mu}{r^4}(-2axy + b(x^2 - y^2))$$

□

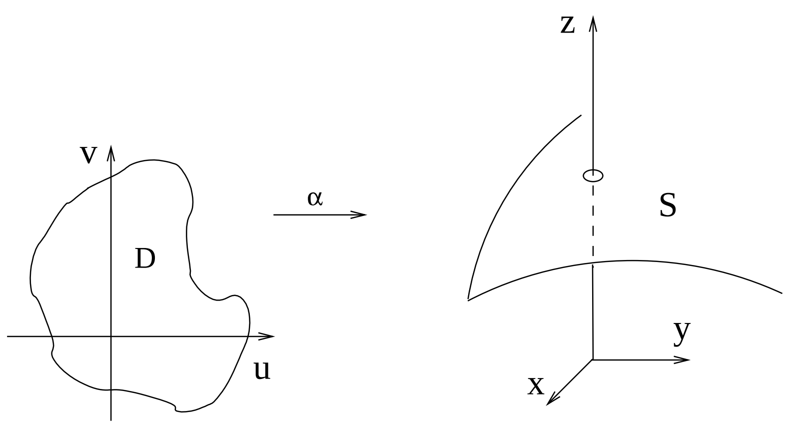
Chapter 4

Surfaces

4.1 Surfaces

$\boxed{\mathbb{T}}$ A *parametrized surface* of $\mathbb{R}^3$ is a diferenciable function defined in a region $D \subset \mathbb{R}^2$ with values in $\mathbb{R}^3$:

$$\boxed{\begin{array}{lccll} \alpha: & D & \subset \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ & (u,v) & & \mapsto & \alpha(u,v) = (x(u,v), y(u,v), z(u,v)) \end{array}}$$



- The parametrization is *regular* at those points of D where

$$\text{rank} \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix} = 2$$

This condition says that the vectors $\partial_u\alpha, \partial_v\alpha$ are linearly independent. If $\mathbf{p} \in \alpha(D)$ corresponds to a regular point, the plane

$$T_{\mathbf{p}} = \mathbf{p} + \langle \partial_u\alpha, \partial_v\alpha \rangle$$

is called the *tangent plane* at $\mathbf{p}$, and the vector $\mathbf{N} = \partial_u\alpha \times \partial_v\alpha$ is the *normal vector associated* to the parametrization. We call the parametrization regular if it is everywhere regular

- The parametrization is *simple* if it is injective.

We will use mainly regular and simple parametrizations.

□

Problem 91: Spherical parametrization of a sphere.

Parametrize the unit sphere

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

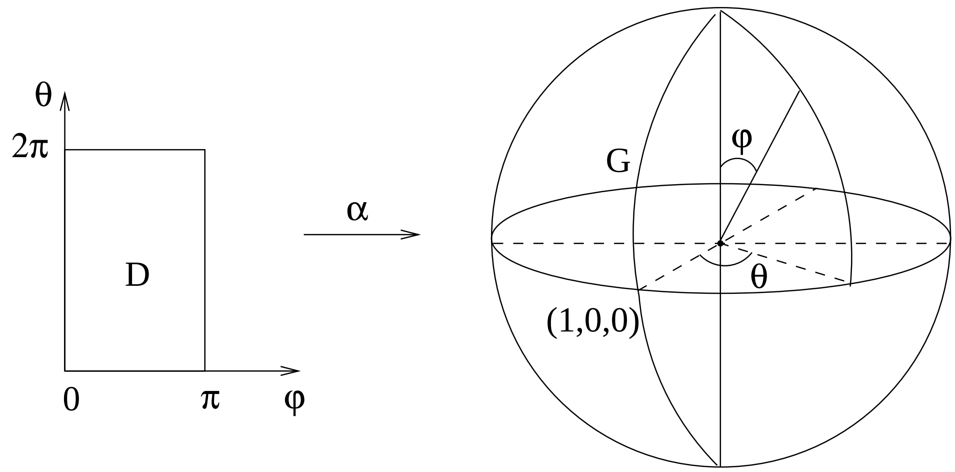
using spherical coordinates and find out if it is differentiable, regular and simple.

Solution:

Using the spherical coordinates φ= colatitude, θ= longitude (see the figure below) a parametrization of the sphere is:

$$\begin{aligned} \alpha(\varphi, \theta) &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \\ D &= \{(\varphi, \theta); 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\} \end{aligned}$$

which is differentiable at every point of D (in fact it is differentiable in $\mathbb{R}^2$)



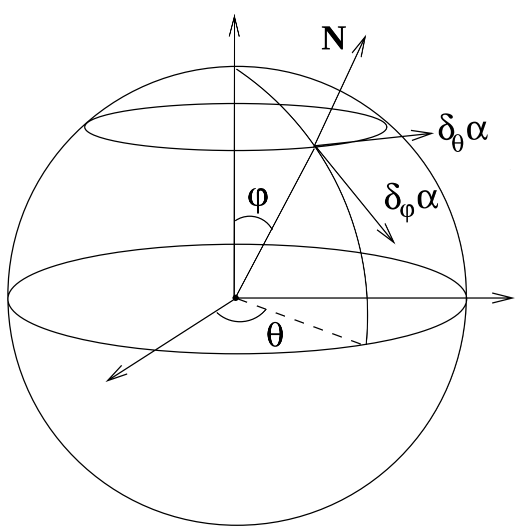
- Geometrically we see that α is injective in $\overset{\circ}{D}= (0,\pi)\times(0,2\pi)$, the interior of D, but the image $\alpha(\overset{\circ}{D})$ excludes the 'Greenwich semimeridian'

$$G = \{(\sin\varphi, 0, \cos\varphi) : \varphi \in [0,\pi]\}$$

  α is a bijection of $\overset{\circ}{D}= (0,\pi)\times(0,2\pi)$ onto $S^2 - G$.

- The tangent vector to the meridians is $\partial_\varphi\alpha$ and the tangent vector to the parallels is $\partial_\theta\alpha$:

$$\begin{aligned}\partial_\varphi\alpha &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)\\ \partial_\theta\alpha &= (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)\end{aligned}$$



The normal vector associated to the parametrization and its norm are:

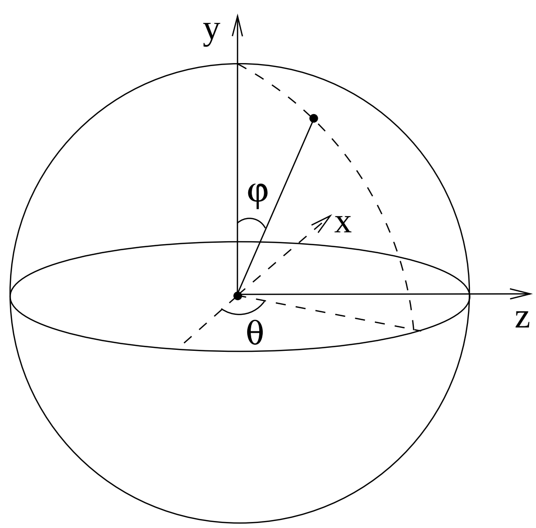
$$\begin{aligned}\mathbf{N} &= \partial_\varphi\alpha \times \partial_\theta\alpha = (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)\\ |\mathbf{N}| &= (\sin^4\varphi + \sin^2\varphi\cos^2\varphi)^{1/2} = |\sin\varphi| = \sin\varphi\end{aligned}$$

We see that whenever $0 < \varphi < \pi$ we have $|\mathbf{N}| \neq 0$ and then $\mathbf{N} \neq \mathbf{0}$: the vectors $\partial_\varphi\alpha$ and $\partial_\theta\alpha$ are linearly independent and the parametrization is *regular* at interior points of D. Note that $\mathbf{N}$ points to the exterior of the sphere as can be seen writing (remind that $\sin\varphi > 0$)

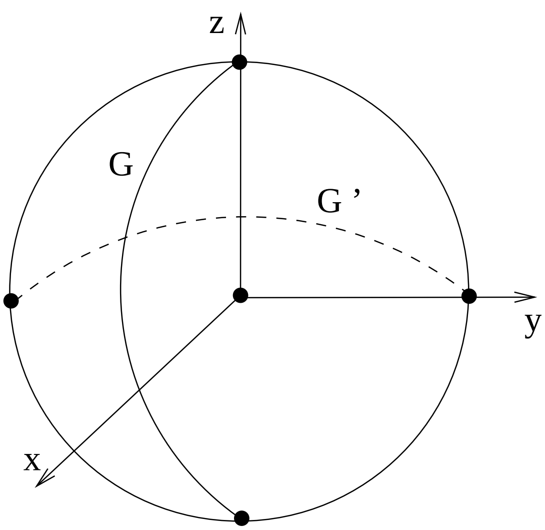
$$\mathbf{N} = \sin\varphi(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = (\sin\varphi)\alpha(\varphi, \theta)$$

- $\alpha : \overset{\circ}{D} \to S^2$ is a *map* (or *local chart* see [Jän]) of S^2 it being understood that a map need not cover the whole sphere, as happens with ordinary geographical maps. A collection of maps that cover the whole sphere is called an *atlas.*
  Now if we want an atlas of the sphere we need more maps apart from α. For instance if we define the colatitude φ with respect to the Oy axis and take the negative Ox axis to measure longitudes, we have the parametrization

$$\beta(\varphi, \theta) = (-\sin\varphi\sin\theta, \cos\varphi, \sin\varphi\cos\theta), (\varphi, \theta) \in D = [0, \pi] \times [0, 2\pi]$$



defined in $\overset{\circ}{D} = (0, \pi) \times (0, 2\pi)$. This new parametrization doesn't cover the semimeridian $G' = \{\theta = 0\}$. Then α, β are an atlas of S^2:



□

Problem 92: Cartesian parametrization of the sphere.

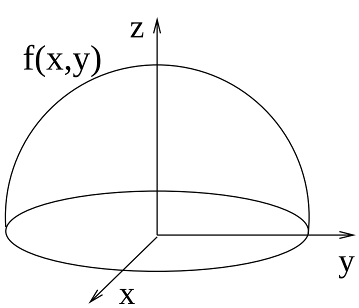
Parametrize the sphere S^2 as the graph of a certain function and find out wether it is regular and simple.

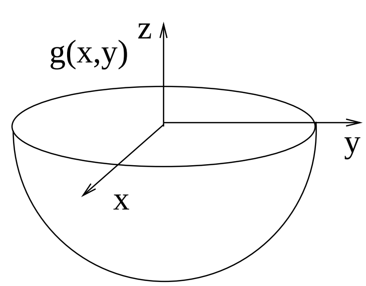
Solution:

From the sphere's equation $x^2 + y^2 + z^2 = 1$ we can isolate several functions:

a) In $D_1 = \{(x, y) : x^2 + y^2 < 1\}$

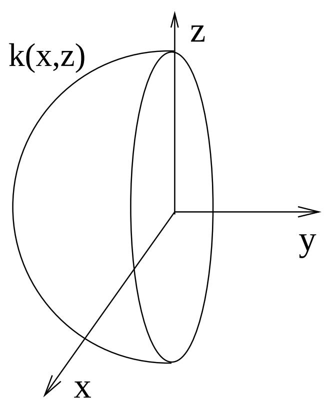
$$\begin{aligned} f(x,y) &= \sqrt{1-(x^2+y^2)} \\ g(x,y) &= -\sqrt{1-(x^2+y^2)} \end{aligned}$$

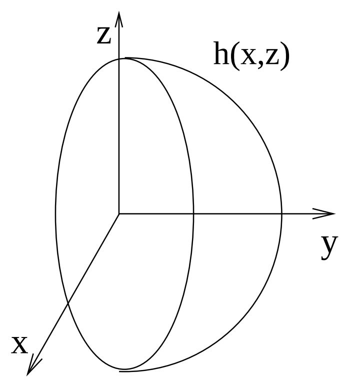




b) In $D_2 = \{(x,z) : x^2 + z^2 < 1\}$

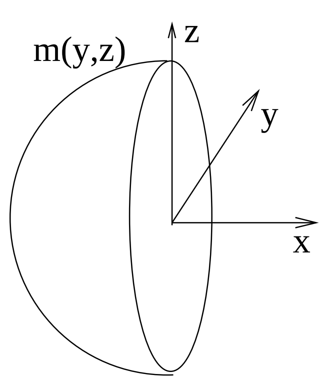
$$\begin{aligned} h(x,z) &= \sqrt{1-(x^2+z^2)} \\ k(x,z) &= -\sqrt{1-(x^2+z^2)} \end{aligned}$$

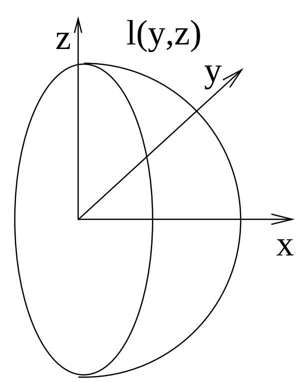




c) In $D_3 = \{(y,z) : y^2 + z^2 < 1\}$

$$\begin{aligned} l(y,z) &= \sqrt{1-(y^2+z^2)} \\ m(y,z) &= -\sqrt{1-(y^2+z^2)} \end{aligned}$$





Let us use as a model

$$\alpha(x,y) = (x,y,f(x,y)) = (x,y,\sqrt{1-(x^2+y^2)}), (x,y) \in D_1$$

- α is a parametrization of the upper semisphere (excluding the equator), differentiable in $\mathring{D}_1$, and α is clearly injective; it is a bijection of $\mathring{D}_1$ onto $S^2 \setminus S^2_-$.

- The tangent vectors to the coordinate curves are

$$\begin{aligned}\partial_x \alpha &= (1, 0, \frac{x}{\sqrt{1-x^2-y^2}}) \\ \partial_y \alpha &= (0, 1, \frac{y}{\sqrt{1-x^2-y^2}})\end{aligned}$$

  and the associated normal vector and its norm are

$$\begin{aligned}\mathbf{N} &= (\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1) \\ |\mathbf{N}| &= \frac{1}{\sqrt{1-x^2-y^2}}\end{aligned}$$

  We see that the normal vector $\mathbf{N}$ points to the exterior of the sphere; again $|\mathbf{N}| \neq 0$ and the parametrization is *regular.*

We can proceed in the same way with the other maps; altogether they are an atlas of S^2.

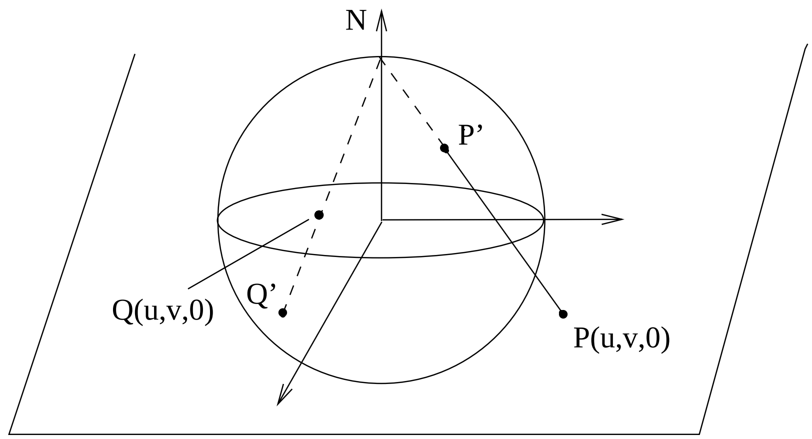
$\square$

Problem 93: Stereographic projection on the sphere.

The *stereographic projection* from the north pole $N = (0, 0, 1)$ of the plane $\pi = \{z = 0\}$ on the unit sphere S^2 sends each point P of π to the intersection point P' of the straight line through P and N with the sphere. Compute the equations of the projection so obtaining a parametrization of S^2 (whose domain is $D = \mathbb{R}^2$).

Solution:

First make a figure:



We find the projection in two different ways:

a) The line uniting $P=(u,v,0)$ and N is

$$X(t)=N+t(P-N)=(tu,tv,1-t),t\in\mathbb{R}$$

We look for the point $X(t)\in S^2$

$$t^2(u^2+v^2)+(1-t)^2=1\Rightarrow t=0,t=\frac{2}{1+u^2+v^2}$$

the P' coordinates are:

$$\alpha(u,v)=(\frac{2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{u^2+v^2-1}{1+u^2+v^2}),(u,v)\in\mathbb{R}^2.$$

It is geometrically clear that α is injective and that it's trace is S^2-N. The tangent vectors to the coordinate curves are

$$\begin{aligned}\partial_u &= (\frac{2(1-u^2+v^2)}{(1+u^2+v^2)^2},\frac{e-4uv}{(1+u^2+v^2)^2},\frac{4u}{(1+u^2+v^2)^2})\\ \partial_v &= (\frac{-4uv}{(1+u^2+v^2)^2},\frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2},\frac{4v}{(1+u^2+v^2)^2})\end{aligned}$$

and the associated normal vector is

$$\mathbf{N}=-4(\frac{2u}{(1+u^2+v^2)^3},\frac{2v}{(1+u^2+v^2)^3},\frac{u^2+v^2-1}{(1+u^2+v^2)^3})$$

But $\mathbf{N}=\mathbf{0}$ can only happen if $u=v=u^2+v^2-1=0$ which is impossible, and we conclude that the parametrization is everywhere regular.

b) For another derivation we remind the equations of the projection, from the north pole, of the Ox axis on S^1:

$$\gamma(t) = (\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}), t \in \mathbb{R}$$

Take polar coordinates in the plane:

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta \end{aligned}$$

From the formula of γ we see that the point $(\rho, 0)$ is projected on

$$(\frac{2\rho}{1+\rho^2}, 0, \frac{\rho^2-1}{1+\rho^2}) \in S^2$$

Then $P = (\rho\cos\theta, \rho\sin\theta)$ is projected to

$$\begin{aligned} (\frac{2\rho}{1+\rho^2}\cos\theta, \frac{2\rho}{1+\rho^2}\sin\theta, \frac{\rho^2-1}{1+\rho^2}) &= \\ = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}) &\in S^2 \end{aligned}$$

□

Problem 94: Parametrization of an ellipsoid.

By analogy with the angular coordinate in the parametrization of the ellipse and with the spherical coordinates, parametrize the ellipsoid:

$$E = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, a > b > c > 0\}$$

Find out if the parametrization is regular, injective.

Solution:

The analogy suggests the parametrization

$$\alpha(\varphi, \theta) = (a\sin\varphi\cos\theta, b\sin\varphi\sin\theta, c\cos\varphi),\ (\varphi, \theta) \in D = [0, \pi] \times [0, 2\pi],$$

a $\mathcal{C}^\infty$ function such that $\alpha(\varphi,\theta)\in E$ because

$$\frac{(a\sin\varphi\cos\theta)^2}{a^2}+\frac{(b\sin\varphi\sin\theta)^2}{b^2}+\frac{(c\cos\varphi)^2}{c^2}=$$
$$=(\sin\varphi\cos\theta)^2+(\sin\varphi\sin\theta)^2+(\cos\varphi)^2=$$
$$=\sin^2\varphi+\cos^2\varphi=1$$

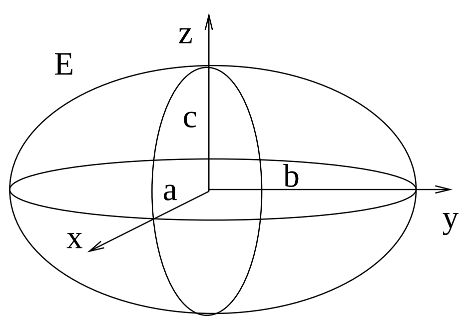
a) α is injective in $\overset{\circ}{D}=(0,\pi)\times(0,2\pi)$ because if $(x,y,z)\in\alpha(D)$ there is a unique $\varphi\in(0,\pi)$ such that $z=c\cos\varphi$. Then the equations

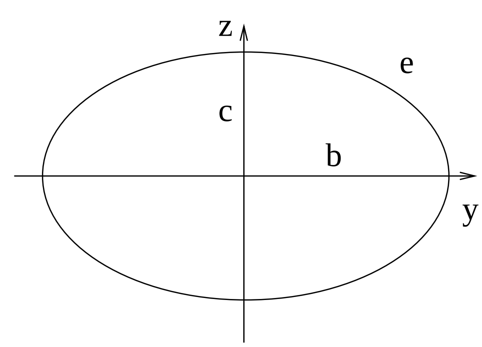
$$\left.\begin{array}{rcl} x &=& a\sin\varphi\cos\theta \\ y &=& b\sin\varphi\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} \cos\theta &=& \frac{x}{a\sin\varphi} \\ \sin\theta &=& \frac{y}{b\sin\varphi} \end{array}\right\}$$

produce a unique $\theta\in(0,2\pi)$ satisfying them.

b) To see wether it is exhaustive we take a look at the geometrical meaning of the coordinates φ,θ. Cutting E with the plane $x=0$ we obtain an ellipse e:

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$





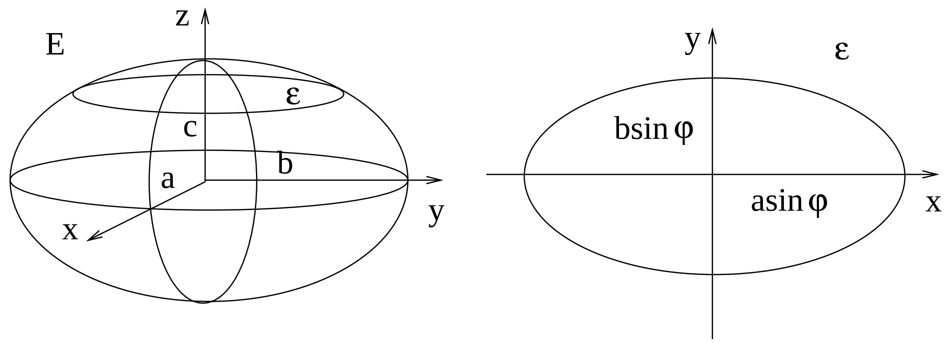
whose right part $(y>0)$ we parametrize by

$$\gamma(\varphi)=(b\sin\varphi,c\cos\varphi),\ \varphi\in[0,\pi]$$

φ being the colatitude in the circumscribed circumference (see Problem 5). Cutting E with the plane $z=c\ \cos\varphi$ we obtain the ellipse ϵ:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\cos^2\varphi=1$$

$$\frac{x^2}{(a\sin\varphi)^2} + \frac{y^2}{(b\sin\varphi)^2} = 1$$



We parametrize ϵ taking $\theta \in [0, 2\pi]$ as a parameter:

$$\Gamma(\theta) = (a\sin\varphi\cos\theta, b\sin\varphi\sin\theta)$$

And taking into account that ϵ is in the plane $z = c\,\cos\varphi$ we obtain the parametrization α. If

$$N = (0, 0, c), S = (0, 0, -c), G = \{(a\sin\varphi, 0, c\cos\varphi) : 0 < \varphi < \pi\}$$

we see that α is a bijection from $\overset{\circ}{D}$ to $E - \{N, S, G\}$.

The matrix of the tangent vectors is

$$\begin{pmatrix} a\cos\varphi\cos\theta & -a\sin\varphi\sin\theta \\ b\cos\varphi\sin\theta & b\sin\varphi\cos\theta \\ -c\sin\varphi & 0 \end{pmatrix}$$

Whose minors are $\Delta_1 = \frac{ab}{2}\sin 2\varphi, \Delta_2 = bc\sin^2\varphi\cos\theta, \Delta_3 = ac\sin^2\varphi\sin\theta$. Δ_1 vanishes only for $\varphi = \frac{\pi}{2}$; but for that value $\Delta_2 = bc\cos\theta, \Delta_3 = ac\sin\theta$ don't vanish simultaneously. The parametrization is everywhere regular.

□

Problem 95: Cones.

A cone S with directrix a plane curve C and vertex at $\mathbf{p} = (a, b, c) \in \mathbb{R}^3$ consists of the half lines emanating from $\mathbf{p}$ and passing through the points of C.

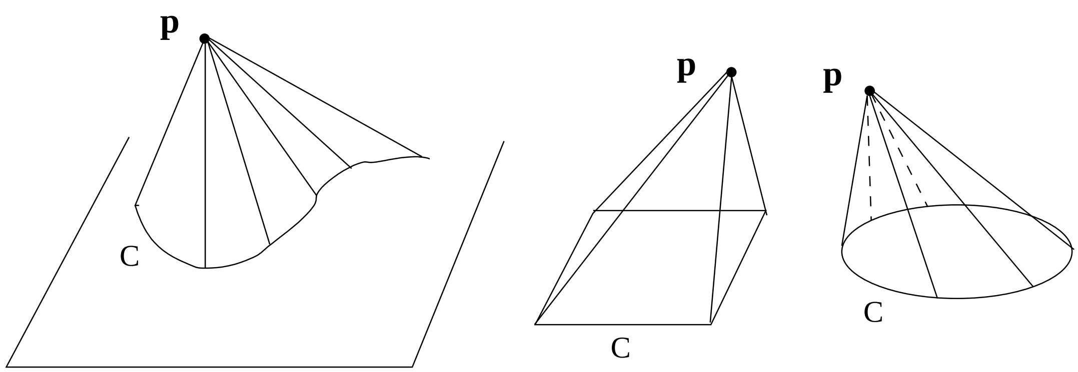
a) Parametrize S.

b) Parametrize the cone with vertex $\mathbf{p}$ and directrix the parabola $y^2 = 2px, z = 0$.

c) Parametrize the cone with vertex $\mathbf{p}$ and directrix the ellipse $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$.

Solution:

Here are several cones:



a) Let C be given by

$$\gamma(t) = (x(t), y(t), z(t)), t \in [a, b]$$

The points of the cone are $X = (1 - s)\mathbf{p} + s\gamma(t)$, and in components

$$\begin{aligned} \alpha(s, t) &= ((1 - s)a + sx(t), (1 - s)b + sy(t), (1 - s)c + sz(t)) \\ (s, t) &\in \mathbb{R}_+ \times [a, b] \end{aligned}$$

b) Parametrize the parabola, $\gamma(y) = (\frac{y^2}{2p}, y, 0)$, $y \in \mathbb{R}$ and obtain a parametrization of the cone:

$$\begin{aligned} \alpha(s, y) &= ((1 - s)a + s\frac{y^2}{2p}, (1 - s)b + sy, (1 - s)c) \\ (s, y) &\in \mathbb{R}_+ \times \mathbb{R} \end{aligned}$$

c) Parametrize the ellipse, $\gamma(\theta) = (m\cos\theta, n\sin\theta, 0)$, $\theta \in [0, 2\pi]$ and obtain the parametrization of S:

$$\begin{aligned} \alpha(s,\theta) &= ((1-s)a + sm\cos\theta, (1-s)b + sn\sin\theta, (1-s)c) \\ (s,\theta) &\in \mathbb{R}_+ \times [0, 2\pi] \end{aligned}$$

□

Problem 96: Cylinders.

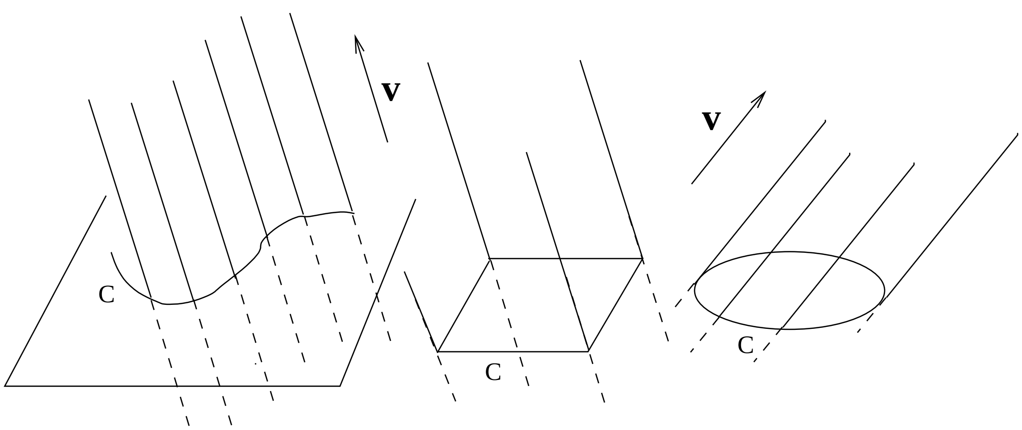
A cylinder S with directrix a plane curve C and generatrixs parallel to a given vector $\mathbf{v}$ consists of the lines that pass through the points of C and are parallel to $\mathbf{v}$.

a) Parametrize S.

b) Parametrize the cylinder with directrix $\gamma(t) = (t, t^2, 0)$ and generatrixs parallel to $\mathbf{v} = (1, 2, 3)$.

Solution:

Here are several cylinders:



a) Let C be parametrized by $\gamma(t) = (x(t), y(t), 0)$; then the points in the cylinder are

$$X = \gamma(t) + s\mathbf{v} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + s \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

or, in components:

$$\alpha(s,t) = (x(t) + sv_1, y(t) + sv_2, z(t) + sv_3)$$

b) Using a) we have

$$\alpha(s,t) = (t + s, t^2 + 2s, 3s)$$

□

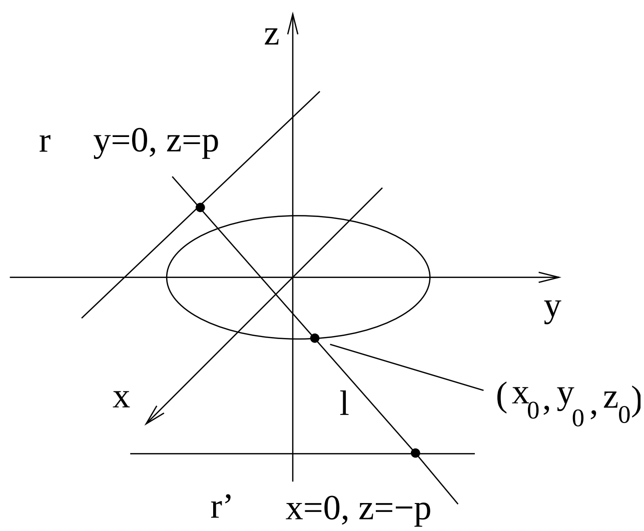
Problem 97: Parametrization of a ruled surface.

Let $p > 0$ and consider the straight lines

$$r: \quad y = 0, \quad z = p$$
$$r': \quad x = 0, \quad z = -p$$

Parametrize the surface S consisting of the straight lines that pass through a point of $S^1 = \{(x, y, 0) : x^2 + y^2 = 1\}$ and cut r and r'.

Solution:



Let $(a, b, 1)$ be the director vector of l, one of the lines in the surface, and $(x_0, y_0, 0)$ a point in S^1. The parametric equations of the line are:

$$\left.\begin{array}{rcl} x &=& x_0 + \lambda a \\ y &=& y_0 + \lambda b \\ z &=& \lambda \end{array}\right\} l$$

The line l cuts the line $y = 0, z = p$ and we have

$$\left.\begin{array}{rcl}\lambda & = & p \\ \lambda b & = & -y_0\end{array}\right\} \Rightarrow b = -\frac{1}{p}y_0$$

The line l cuts the line $x = 0, z = -p$ and we have

$$\left.\begin{array}{rcl}\lambda & = & -p \\ \lambda a & = & -x_0\end{array}\right\} \Rightarrow a = \frac{1}{p}x_0$$

The points in the surface are of the form

$$(x, y, z) = (x_0, y_0, 0) + t(\frac{1}{p}x_0, -\frac{1}{p}y_0, 1), t \in \mathbb{R}$$

As $(x_0, y_0, 0)$ is a point in S^1 we can write it in the form $(\cos\theta, \sin\theta, 0)$ and we obtain the parametrization of the surface

$$\alpha(\theta, t) = ((1 + \frac{t}{p})\cos\theta, (1 - \frac{t}{p})\sin\theta, t), t \in \mathbb{R}, \theta \in [0, 2\pi]$$

□

Problem 98:

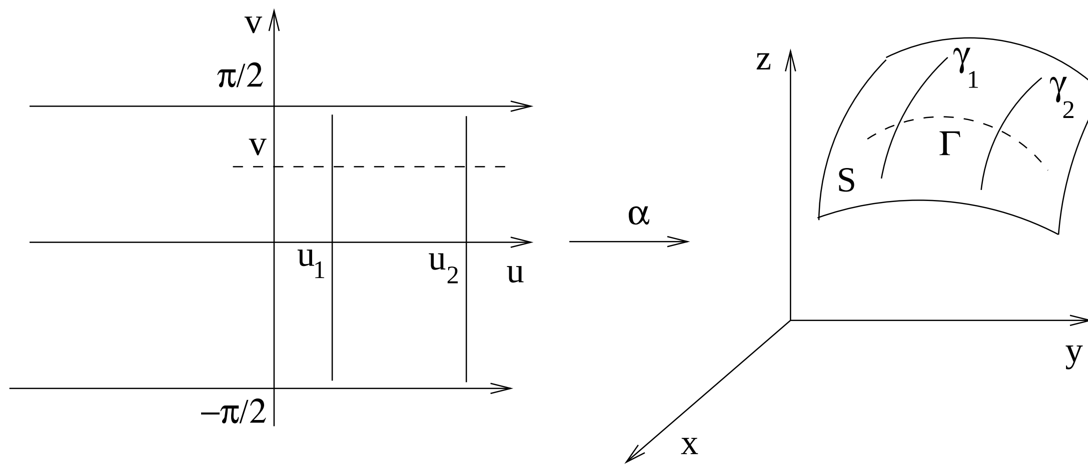
Let S be the parametrized surface

$$\alpha(u, v) = (u\cos v, u\sin v, u + \ln\cos v), u \in \mathbb{R}, v \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Fix two values u_1, u_2 and consider the curves $\gamma_i(t) = \alpha(u_i, t), t \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1, 2$. Show that the length of the arc of $\Gamma(\tau) = \alpha(\tau, v), \tau \in \mathbb{R}, v$ fixed, limited by the intersections of Γ with γ_1 and γ_2 is independent of v.

Solution:

A figure:



Let τ_1 and τ_2 the values of the parameter corresponding to the intersection of Γ with γ_1 and γ_2 respectively. To compute the length of the above mentioned arc we have

$$\begin{aligned} \Gamma(\tau) &= (\tau \cos v, \tau \sin v, \tau + \ln \cos v) \\ \Gamma'(\tau) &= (\cos v, \sin v, 1) \\ \mid \Gamma'(\tau) \mid &= \sqrt{2} \end{aligned}$$

and

$$L = \int_{\tau_1}^{\tau_2} \sqrt{2} d\tau = \sqrt{2}(\tau_2 - \tau_1)$$

We want to see that $\tau_2 - \tau_1$ doen't depend on v. The points of intersection are given respectively by

$$\begin{aligned} u_1 \cos t_1 &= \tau_1 \cos v \\ u_1 \sin t_1 &= \tau_1 \sin v \\ u_1 + \ln \cos t_1 &= \tau_1 + \ln \cos v \end{aligned}$$

and

$$\begin{aligned} u_2 \cos t_2 &= \tau_2 \cos v \\ u_2 \sin t_2 &= \tau_2 \sin v \\ u_2 + \ln \cos t_2 &= \tau_2 + \ln \cos v \end{aligned}$$

From the first and second equations we obtain $\tau_1^2 = u_1^2$ and similarly from the fourth and the fifth we have $\tau_2^2 = u_2^2$. Then $\tau_1 = \pm u_1$ and $\tau_2 = \pm u_2$.

On another side the third and sixth equations give

$$\begin{aligned} \tau_2 - \tau_1 &= u_2 - u_1 + \ln\cos t_2 - \ln\cos t_1 = \\ &= u_2 - u_1 + \ln\frac{\cos t_2}{\cos t_1} \end{aligned}$$

Now from the first and fourth equations we find

$$0 < \frac{\cos t_2}{\cos t_1} = \frac{\tau_2}{\tau_1}\cdot\frac{u_1}{u_2} = \frac{\pm u_2}{\pm u_1}\cdot\frac{u_1}{u_2} = \pm 1 = 1,$$

because the quotient is positive. Then

$$\begin{aligned} \tau_2 - \tau_1 &= u_2 - u_1 + \ln 1 = \\ &= u_2 - u_1 \end{aligned}$$

that is what we wanted to see. The argument fails if u_1 or u_2 vanish (they can't vanish both because we have *two* curves); we leave to the care of the reader to fill in this black hole if he wants to do so.

$\square$

$\boxed{\mathbb{T}}$ Two parametrizations

$$\begin{array}{cccc} \alpha: & D & \subset \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (u,v) & & \mapsto & \alpha(u,v) = (x(u,v), y(u,v), z(u,v)) \end{array}$$

and

$$\begin{array}{cccc} \beta: & D' & \subset \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (s,t) & & \mapsto & \beta(s,t) = (X(s,t), Y(s,t), Z(s,t)) \end{array}$$

are equivalent if there is a differentiable bijection h with differentiable inverse h^{-1} (a diffeomorphism, a change of variables)

$$\begin{array}{ccc} \overset{\circ}{D'} & \overset{h}{\to} & \overset{\circ}{D} \\ \beta \searrow & & \swarrow \alpha \\ & \mathbb{R}^3 & \end{array}$$

such that $\beta = \alpha \circ h$. A surface S consists of all the equivalent parametrized surfaces. The common trace of all those equivalent parametrizations is called the geometrical surface S (we shall call it S).

$\square$

Problem 99: Equivalent parametrizations.

Show that the two following parametrizations are equivalent:

$$\begin{array}{rccc} \alpha: & [0,2\pi]\times[0,1] & \to & \mathbb{R}^3 \\ & (u,v) & \mapsto & (\cos u, \sin u, v) \end{array}$$

$$\begin{array}{rccc} \beta: & [0,\pi]\times[0,1] & \to & \mathbb{R}^3 \\ & (u',v') & \mapsto & (\cos 2u', \sin 2u', v') \end{array}$$

Solution:

Clearly

$$\begin{array}{rccc} h: & (0,2\pi)\times(0,1) & \to & (0,\pi)\times(0,1) \\ & (u,v) & \mapsto & (u/2,v) \end{array}$$

is a diffeomorphism showing the equivalence:

$$\beta(h(u,v)) = \beta(u/2,v) = (\cos 2\frac{u}{2}, \sin 2\frac{u}{2}, v) = \alpha(u,v)$$

□

Problem 100: Lower semisphere.

The parametrization $\gamma(u) = (\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1})$, $u \in [-1,1]$ of the lower unit semi-circumference S^1_-, gives the parametrization of the lower unit semisphere S^2_-

$$\alpha(r,\theta) = (\frac{2r}{r^2+1}\cos\theta, \frac{2r}{r^2+1}\sin\theta, \frac{r^2-1}{r^2+1}),\ (r,\theta) \in [0,1]\times[0,2\pi].$$

We also have the spherical parametrization of the same surface

$$\beta(\varphi,\lambda) = (\sin\varphi\cos\lambda, \sin\varphi\sin\lambda, \cos\varphi),\ (\varphi,\lambda) \in [\pi/2,\pi]\times[0,2\pi].$$

Show they are equivalent.

Solution:

We want to find a diffeomorphism $h = (h^1, h^2)$

$$\begin{array}{rccccl} h: & (0,1) & \times & (0,2\pi) & \rightarrow & (\pi/2,\pi)\times(0,2\pi) \\ & (r & , & \theta) & \mapsto & h(r,\theta) = (\varphi,\lambda) \end{array}$$

such that $\alpha = \beta \circ h$. It is geometrically clear that $\theta = \lambda$ so $h^2(r,\theta) = \theta$ and we have to find out h^1, which amounts to express φ in terms of (r,θ).

We must have

$$\cos\varphi = \frac{r^2-1}{r^2+1}$$

$$\varphi = \arccos(\frac{r^2-1}{r^2+1})$$

Then

$$h(r,\theta) = (\arccos(\frac{r^2-1}{r^2+1}), \theta)$$

and we should check that it is a diffeomorphism. Cearly h is a bijection (the variables are uncoupled and each component of h is a bijection). To see h is a diffeomorphism it suffices to show that the jacobian determinant $\det h' \neq 0$, because then the inverse function theorem applies. We have

$$\det(h'(r,\theta)) = \det\begin{pmatrix} \frac{2}{r^2+1} & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{r^2+1} > 0$$

So both parametrizations are equivalent.

□

4.2 Surfaces of revolution

Problem 101: Surfaces of revolution.

Let C be a plane, simple, regular curve in the half plane $y = 0, x > 0$. Parametrize the surface of revolution S obtained revolving C around the Oz axis and decide if it is regular and simple. As an application parametrize:

a) A right circular cylinder.

b) A right circular cone with radius R and height h.

c) A right circular cone with an angle 2α at the vertex.

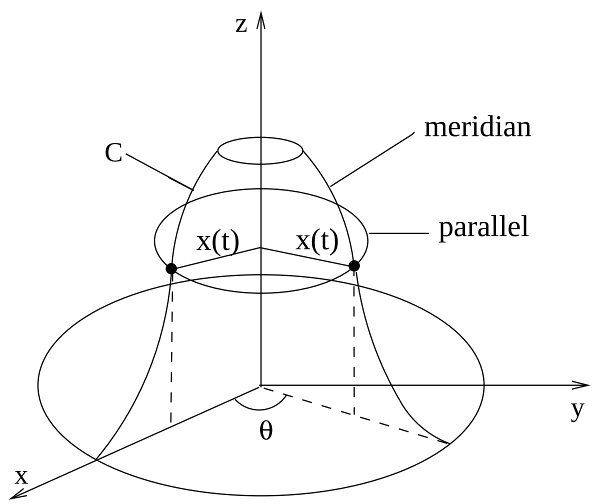
d) A torus (doughnut) is the surface of revolution obtained revolving around the Oz axis the circumference with center at $(a, 0, 0), a > 0$ and radius b, $0 < b < a$ in the xz plane. Parametrize it.

e) A circular paraboloid.

f) An ellipsoid of revolution.

g) The surface obtained revolving the graph of $z = f(y)$, $f : [a, b] \to \mathbb{R}$, $a > 0$ around the Oz axis.

Solution:



Let

$$\gamma(t) = (x(t), z(t)), t \in [a, b], x(t) > 0$$

be a parametrization of C (in the plane $y = 0$); then

$$\alpha(t, \theta) = (x(t)\cos\theta, x(t)\sin\theta, z(t)), (t, \theta) \in D = [a, b] \times [0, 2\pi]$$

is a parametrization of S, differentiable as many times as γ is. The parametrization is injective in $\mathring{D}$ and $\alpha(\mathring{D})$ is S except for the meridian corresponding to $\theta = 0$. Notice that θ still controls the 'longitude' and that t controls the 'latitude'; the curves $\theta = \text{const}$ are the *meridians* of S and the curves $t = \text{const}$ are the *parallels* of S.

The tangent vectors to the coordinate curves are

$$\begin{aligned} \partial_t \alpha &= (x'(t)\cos\theta, x'(t)\sin\theta, z'(t)) \\ \partial_\theta \alpha &= (-x(t)\sin\theta, x(t)\cos\theta, 0), \end{aligned}$$

and the associated normal vector and its norm are

$$\begin{aligned} \mathbf{N} &= \partial_t \alpha \times \partial_\theta \alpha = (-xz'\cos\theta, -xz'\sin\theta, xx') \\ |\mathbf{N}| &= (x^2 z'^2 + x^2 x'^2)^{1/2} = x\sqrt{x'^2 + z'^2} \end{aligned}$$

We see that the parametrization is regular for

$$|\mathbf{N}| = x\sqrt{x'^2 + z'^2} > 0$$

due to the regularity of the revolving curve.

a) Consider the cylinder generated by revolving the straight line $\{y = 0, x = R, R > 0\}$ around the Oz axis. The straight line in the xz plane is

$$x(t) = R, z(t) = t$$

and we obtain the cylinder's parametrization

$$\alpha(t,\theta) = (R\cos\theta, R\sin\theta, t),\ (t,\theta) \in \mathbb{R} \times [0, 2\pi]$$

b) A circular right cone with basis the circumference $x^2 + y^2 = R^2, z = 0$ and height h is obtained revolving the segment $z = -\frac{h}{R}x + h, 0 \leq x \leq R$ around the Oz axis. The segment is

$$x(t) = t,\ z(t) = -\frac{h}{R}t + h, t \in [0, R]$$

and we obtain the parametrization of the cone

$$\alpha(t,\theta) = (t\cos\theta, t\sin\theta, -\frac{h}{R}t + h),\ (t,\theta) \in [0, R] \times [0, 2\pi]$$

with vertex at $(0, 0, h)$.

c) Now $\tan\alpha = R/h$ and the parametrization is

$$\gamma(t) = (t, (\cot\alpha)t + h)$$

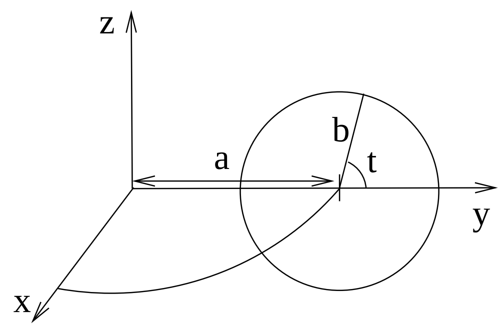
$$\alpha(t,\theta) = (t\cos\theta, t\sin\theta, -(\cot\alpha)t + h),\ (t,\theta) \in [0, h\tan\alpha] \times [0, 2\pi]$$

d) A torus is obtained revolving the circumference $(x-a)^2+z^2=b^2, y=0$ (with $0<b<a$) around the Oz axis. Parametrizing this circumference

$$\gamma(t)=(a+b\cos t,0,b\sin t), t\in(0,2\pi)$$

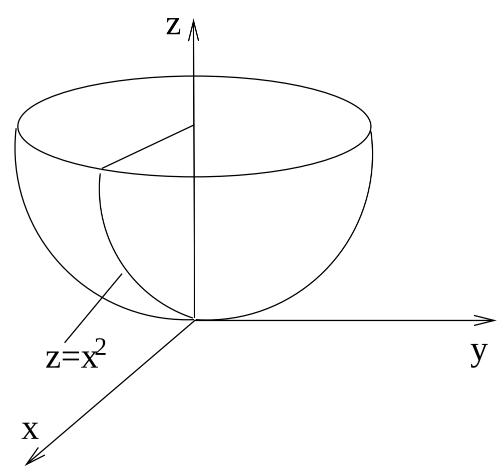
we obtain the torus parametrization

$$\alpha(t,\theta)=((a+b\cos t)\cos\theta,(a+b\cos t)\sin\theta,b\sin t),\ (t,\theta)\in[0,2\pi]\times[0,2\pi]$$



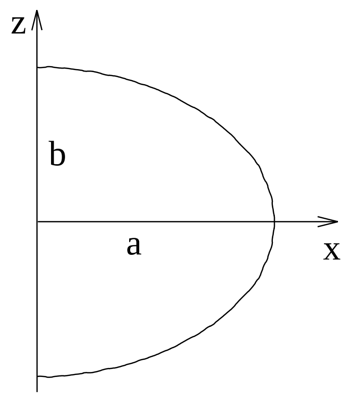
e) The circular paraboloid is obtained revolving the parabola $z=x^2, y=0$ around the Oz axis. Then S admits the parametrization

$$\alpha(t,\theta)=(t\cos\theta,t\sin\theta,t^2),\ (t,\theta)\in\mathbb{R}_+\times[0,2\pi]$$



f) Let's parametrize the generating semiellipse: $x(t)=a\cos t, z(t)=b\sin t,\ t\in[-\pi/2,\pi/2]$; then the ellipsoid of revolution is

$$\alpha(t,\theta)=(a\cos t\cos\theta,a\cos t\sin\theta,b\sin t),\ (t,\theta)\in[-\pi/2,\pi/2]\times[0,2\pi]$$



g) Let's parametrize the graph of f: $\gamma(t) = (t, f(t))$, $t \in [a, b]$. Then

$$\alpha(t, \theta) = (t \cos\theta, t \sin\theta, f(t)),\ (t, \theta) \in [a, b] \times [0, 2\pi]$$

□

Problem 102: Hyperboloid.

Using the parametrization $(\cosh u, \sinh u), u \in \mathbb{R}$ of the hiperbole $x^2 - y^2 = 1$ (see problem 7, curves) to parametrize the hyperboloid

$$H = \{(x, y, z) : x^2 + y^2 - z^2 = 1\}$$

Solution:

The intersection of H with the plane $y = 0$ is the hyperbola C with equation $x^2 - z^2 = 1$ parametrized by $x = \cosh u, z = \sinh u$. The intersection of H with the plane $z = c$ is the circumference $x^2 + y^2 = 1 + c^2$; this shows that H is a surface of revolution generated by revolving C around the Oz axis. The parametrization of H is then

$$\alpha(u, \theta) = (\cosh u \cos\theta, \cosh u \sin\theta, \sinh u), (u, \theta) \in \mathbb{R} \times [0, 2\pi]$$

α is everywhere regular because the generating curve C satisfies:

$$\begin{aligned} \gamma(u) &= (\cosh u, \sinh u) \\ \gamma'(u) &= (\sinh u, \cosh u) \\ \mid \gamma'(u) \mid^2 &= \sinh^2 u + \cosh^2 u = 2\cosh^2 u - 1 > 0 \end{aligned}$$

□

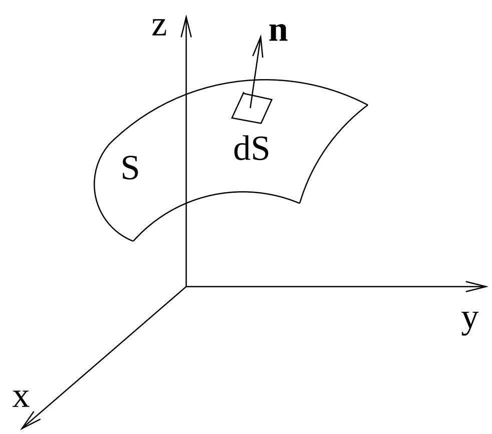
Chapter 5

Integration of fields over surfaces

5.1 Area of a surface

$\boxed{\mathbb{T}}$ Let $\alpha : D \to \mathbb{R}^3$ be a differentiable, regular and simple parametrization of a surface S and let $\mathbf{N} = \partial_u\alpha \times \partial_v\alpha$ be the normal vector associated to α; define

$$\boxed{\text{Area}(S) = \int\int_D |\mathbf{N}| dudv}$$



One can see that this definition is independent of the parametrization (see problem p.**??**). Then defining the *scalar element of area* $dS = \mid \mathbf{N} \mid dudv$ (see interesting comments in [Jän] p.173,185), we may write

$$\text{Area}(S) = \int\int_S 1 dS$$

□

Problem 103:

Using spherical coordinates compute:

a) The area of a sphere of radius R.

b) The area of the region $\mathcal{R}$ on the unit sphere S^2 limited by two meridians $\theta = \theta_1$ and $\theta = \theta_2$ where $\theta_2 - \theta_1 = \pi/6$, and the parallels corresponding to $z = 0$, $z = 1/2$.

Solution:

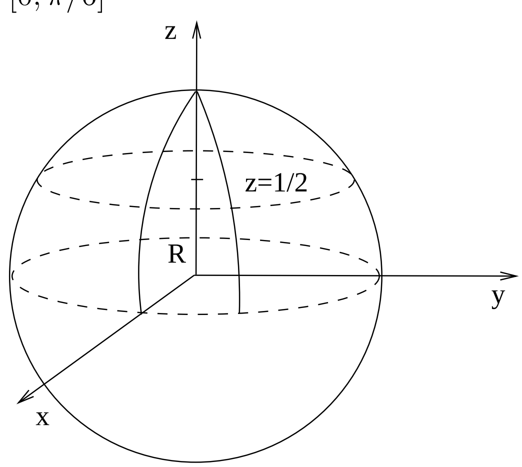
a) In the parametrization of the sphere by geographical coordinates the element of area is

$$dS = R^2 \sin\varphi d\varphi d\theta.$$

Then

$$\begin{aligned} \text{A} &= \int_0^{\pi}\int_0^{2\pi} R^2 \sin\varphi d\varphi d\theta = \\ &= 2\pi R^2 \int_0^{\pi} \sin\varphi d\varphi = 4\pi R^2 \end{aligned}$$

b) Take as region $\mathcal{R}$ the one limited by the parameters values $\varphi \in [\pi/3, \pi/2], \theta \in [0, \pi/6]$



Then

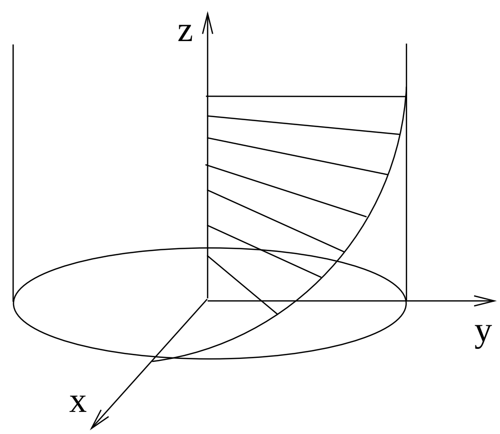
$$\begin{aligned} A &= \int_{\pi/3}^{\pi/2}\int_0^{\pi/6} \sin\varphi d\varphi d\theta = \\ &= \frac{\pi}{6}(-\cos\varphi)|_{\pi/3}^{\pi/2} = \frac{\pi}{12} \end{aligned}$$

□

Problem 104:

Joining each point of the helicoid $\gamma(t) = (\cos t, \sin t, t), t \in \mathbb{R}$ with the point $(0, 0, t)$ in the Oz axis we obtain a surface called the helicoidal ramp; compute the area of a complete turn.

Solution:



A parametrization of one turn is

$$\alpha(s, t) = (s\cos t, s\sin t, t), (s, t) \in D = [0, 1] \times [0, 2\pi]$$

The associated normal vector and its norm are:

$$\begin{aligned} \alpha_s &= (\cos t, \sin t, 0) \\ \alpha_t &= (-s\sin t, s\cos t, 1) \\ \mathbf{N} = \alpha_s \times \alpha_t &= (\sin t, -\cos t, s) \\ |\mathbf{N}| &= \sqrt{1+s^2} \end{aligned}$$

We compute the area:

$$
\begin{aligned}
A &= \int\int_D \mid \mathbf{N} \mid dsdt = \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1+s^2}dsdt = \left\{ \begin{array}{ccc} s &=& \sinh u \\ ds &=& \cosh u du \end{array} \right\} = \\
&= 2\pi \int_0^{\text{arcsinh } 1} \cosh^2 u du = 2\pi \int_0^{\text{arcsinh } 1} \frac{e^{2u}+e^{-2u}+2}{4} du = \\
&= \frac{\pi}{2}(\frac{e^{2u}}{2} - \frac{e^{-2u}}{2} + 2u) \mid_{u=0}^{u=\text{arcsinh } 1} = \frac{\pi}{2}(\sinh 2u + 2u) \mid_{u=0}^{u=\text{arcsinh } 1}
\end{aligned}
$$

and reminding that $\sinh 2u = 2\sinh u \cosh u = 2 \sinh u \sqrt{1+\sinh^2 u}$ and that arcsinh $1 = \log(1+\sqrt{2})$ we obtain

$$
\begin{aligned}
A &= \frac{\pi}{2}(2 \cdot 1 \cdot \sqrt{1+1} + 2\text{arcsinh } 1) = \\
&= \pi(\sqrt{2} + \log(1+\sqrt{2}))
\end{aligned}
$$

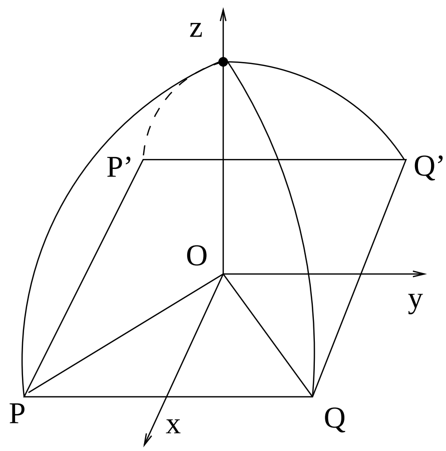
□

Problem 105: Two cylinders.

Let C_1 and C_2 be two right circular cylinders of radius R and axis Ox and Oy respectively. Compute the area of the region S cut by the solid cylinder C_1 in the surface of C_2.

Solution:

Let C_1 be the solid cylinder; a figure of the upper half of S is:



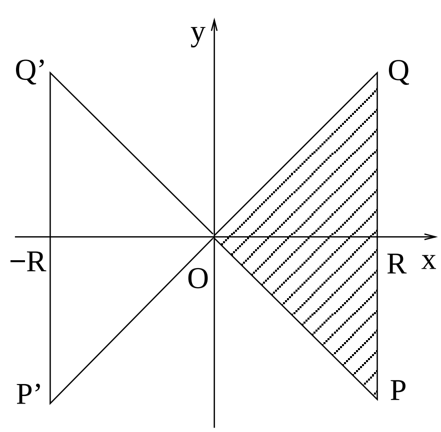
The equations of the cylinders are

$$\begin{aligned} C_1: \quad y^2 + z^2 \leq & \quad R^2 \\ C_2: \quad x^2 + z^2 = & \quad R^2 \end{aligned}$$

and the projections of the points of S on $z = 0$ satisfy

$$y^2 - x^2 \leq 0 \Leftrightarrow |y| \leq |x|$$

and thus the projected points are the triangles POQ and $P'OQ'$



Parametrize the part of the region cut in C_2 having $z > 0$:

$$\begin{aligned}
\alpha(x,y) &= (x, y, \sqrt{R^2 - x^2}) \\
\partial_x \alpha &= (1, 0, \frac{-x}{\sqrt{R^2 - x^2}}) \\
\partial_y \alpha &= (0, 1, 0) \\
\mathbf{N} = \partial_\theta \alpha \times \partial_z \alpha &= (\frac{x}{\sqrt{R^2 - x^2}}, 0, 1), \ |\mathbf{N}| = \frac{R}{\sqrt{R^2 - x^2}}
\end{aligned}$$

and consider the points of S that lie on $D = POQ$ (this is a quarter of the total surface); that area will be

$$\begin{aligned}
\text{Area} &= \int\int_D \frac{R}{\sqrt{R^2 - x^2}} dx dy = R \int_0^R dx \int_{-x}^{x} \frac{1}{\sqrt{R^2 - x^2}} dy = \\
&= R \int_0^R \frac{2x}{\sqrt{R^2 - x^2}} dx = 2R(-\sqrt{R^2 - x^2})|_0^R = 2R^2
\end{aligned}$$

Then the total area is $\text{Area}(S) = 4 \cdot 2R^2 = 8R^2$.

□

$\boxed{\mathbb{T}}$ The graph of the differentiable function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a surface S admitting the differentiable, regular and simple parametrization

$$\alpha(x,y) = (x, y, f(x,y)), (x,y) \in U.$$

Then

$$\begin{aligned}
\mathbf{N} &= (1, 0, f_x) \times (0, 1, f_y) = (-f_x, -f_y, 1) \\
|\mathbf{N}| &= \sqrt{1 + (f_x)^2 + (f_y)^2}
\end{aligned}$$

and the area of the graph is

$$\boxed{\text{Area}(S) = \int\int_D \sqrt{1 + (f_x)^2 + (f_y)^2} dx dy}$$

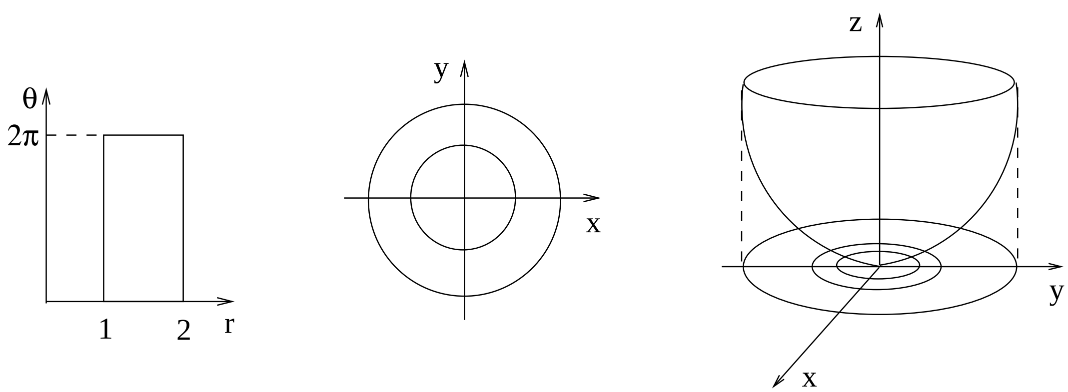
Compare with the formula that gives the length of the graph of a differentiable function $f : [a,b] \to \mathbb{R}$: $\text{Length}(C) = \int_a^b \sqrt{1 + (f')^2} dx$.

□

Problem 106:

Compute the area of that part S of the paraboloid $z = x^2 + y^2$ that lies on the annulus $D = \{(x, y) : 1 \leq \sqrt{x^2 + y^2} \leq 2\}$.

Solution:



S is the graph of the function $f(x, y) = x^2 + y^2$ and using the above formula we havande

$$\begin{aligned}
\text{Area}(S) &= \int\int_D \sqrt{1 + 4(x^2 + y^2)}dxdy = \{\text{polar coords}\} = \\
&= \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2}rdr = 2\pi\frac{1}{8}\int_1^2 8r\sqrt{1 + 4r^2}dr = \\
&= \frac{\pi}{4}\frac{2}{3}(1 + 4r^2)^{3/2}|_1^2 = \frac{\pi}{6}(17^{3/2} - 5^{3/2})
\end{aligned}$$

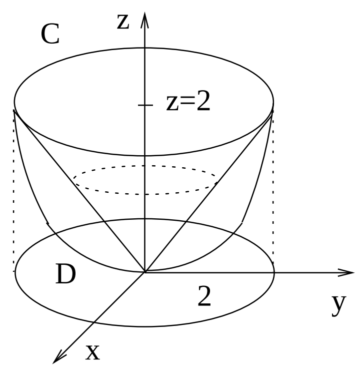
□

Problem 107:

Compute the area of the bounded region S of the paraboloid $2z = x^2 + y^2$ that lies outside the cone $z \geq \sqrt{x^2 + y^2}$.

Solution:

A figure:



The paraboloid cuts the cone along

$$z = \frac{1}{2}(x^2 + y^2) = \sqrt{x^2 + y^2} \Rightarrow \sqrt{x^2 + y^2} = 2, z = 2,$$

the circumference C in the figure. The paraboloid is the graph of the function $f(x, y) = \frac{1}{2}(x^2 + y^2)$ over the disc D of radius 2. Being given that $\partial_x f = x, \partial_y f = y$, using the formula results in

$$\begin{aligned} \text{Area } (S) &= \int\int_D \sqrt{1 + x^2 + y^2} dxdy = \{\text{polar coords}\} = \\ &= \int_0^{2\pi} \int_0^2 r\sqrt{1 + r^2} dr d\theta = 2\pi \int_0^2 r\sqrt{1 + r^2} dr = \\ &= 2\pi \frac{1}{3}(1 + r^2)^{3/2} \mid_0^2 = 2\pi \frac{1}{3}(5^{3/2} - 1) \end{aligned}$$

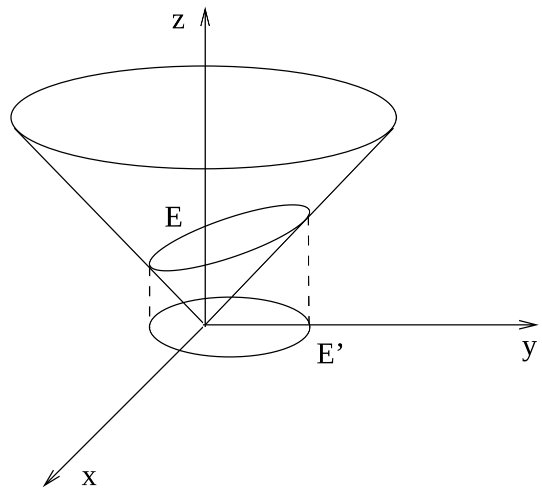
□

Problem 108:

Compute the area of S, the region of the cone $z^2 = x^2 + y^2$, $z \geq 0$ limited by the planes $z = 0, x + 2z = 3$.

Solution:

Lets do a figure



Eliminating z we get the projecting cylinder of the intersection E of the cone with the plane $x + 2z = 3$:

$$\begin{aligned} x^2 + y^2 &= (\frac{3-x}{2})^2 \\ 3x^2 + 4y^2 + 6x - 9 &= 0 \end{aligned}$$

and completing squares

$$\begin{aligned} 3(x^2 + 2x) + 4y^2 - 9 &= 0 \\ 3((x+1)^2 - 1) + 4y^2 - 9 &= 0 \\ 3(x+1)^2 + 4y^2 - 12 &= 0 \end{aligned}$$

If we make the natural change of variables $\{X = x + 1, Y = y\}$ we obtain the equation of E':

$$\begin{aligned} 3X^2 + 4Y^2 - 12 &= 0 \\ \frac{X^2}{2^2} + \frac{Y^2}{(\sqrt{3})^2} &= 1 \end{aligned}$$

To compute the area we need the equation of the cone in the new coordinates $X, Y, Z = z$:

$$Z^2 = (X-1)^2 + Y^2, Z > 0$$

Viewing it as the graph of $f(X,Y)=\sqrt{(X-1)^2+Y^2}$, we can use the formula for the area of a graph. Call $\mathcal{R}(E')$ the region enclosed by E'; then:

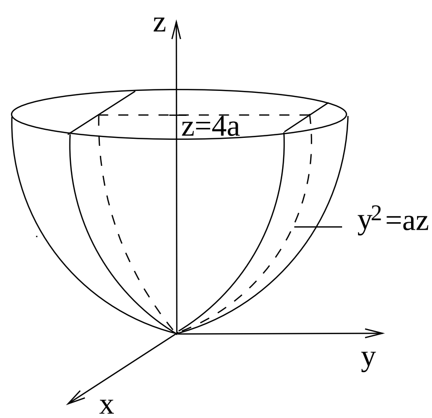
$$\begin{aligned}\text{Area}(S) &= \int\int_{\mathcal{R}(E')}\sqrt{1+\frac{(X-1)^2}{(X-1)^2+Y^2}+\frac{Y^2}{(X-1)^2+Y^2}}dXdY = \\ &= \int\int_{\mathcal{R}(E')}\sqrt{2}dXdY = \sqrt{2}\text{Area }(\mathcal{R}(E')) = \sqrt{2}\pi\cdot 2\cdot\sqrt{3} = 2\pi\sqrt{6}\end{aligned}$$

□

Problem 109:

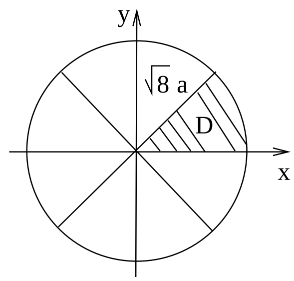
Compute the area of that part S of the paraboloid $x^2+y^2=2az$, $(a>0)$ limited by the plane $z=4a$ and the cylinder $y^2=az$.

Solution:



The paraboloid and the plane intersect along the circumference $x^2+y^2=8a^2, z=2a$, of radius $R=\sqrt{8}a$.
The projecting cylinder of the intersection of the paraboloid and the cylinder is

$$\left.\begin{aligned} z &= \tfrac{1}{a}y^2 \\ z &= \tfrac{1}{2a}(x^2+y^2)\end{aligned}\right\} \Rightarrow x^2-y^2=0 \equiv \mid x \mid = \mid y \mid$$



The paraboloid is the graph of $f(x,y)=\frac{1}{2a}(x^2+y^2)$ and taking symmetries into account we have:

$$\begin{aligned}\text{Area}(S) &= 4\int\int_D \sqrt{1+\frac{x^2+y^2}{a^2}}dxdy = \\ &= \{\text{polar coords}\} = \frac{4}{a}\int_0^{\sqrt{8}a}\int_0^{\pi/4}\sqrt{a^2+r^2}rdrd\theta = \\ &= \frac{4}{a}\frac{\pi}{4}\frac{1}{3}(a^2+r^2)^{3/2}\,|_{r=0}^{r=\sqrt{8}a} = \frac{26}{3}\pi a^2\end{aligned}$$

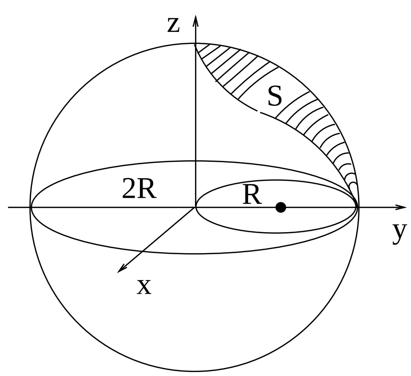
□

Problem 110: Viviani's vault.

Let S be the intersection of a solid cylinder of radius R and a semisphere of radius $2R$ with the center on the surface of the cylinder. S is called Viviani's vault; compute its area.

Solution:

We take the coordinate axes as in the following figure and compute the area of the intersection of the solid cylinder with the upper semisphere:



Let us give two points of view:

- The equation of the sphere is $x^2+y^2+z^2=4R^2$ and S is on the graph of the function

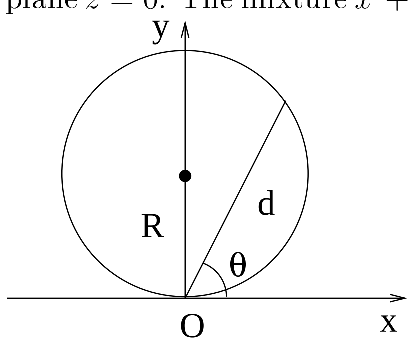
$$f(x,y)=\sqrt{4R^2-(x^2+y^2)}$$

Now we use the formula for the area of a graph:

$$\begin{aligned}\partial_x f &= \frac{-x}{\sqrt{4R^2-(x^2+y^2)}}\\ \partial_y f &= \frac{-y}{\sqrt{4R^2-(x^2+y^2)}}\\ 1+(f'_x)^2+(f'_y)^2 &= 1+\frac{x^2+y^2}{4R^2-(x^2+y^2)}=\\ &= \frac{4R^2}{4R^2-(x^2+y^2)}\end{aligned}$$

$$\text{Area}(S) = \int\int_D \frac{2R}{\sqrt{4R^2-(x^2+y^2)}}dxdy,$$

D being the disc $D((0,R,0);R)$ that is the basis of the cylinder in the plane $z=0$. The mixture x^2+y^2 suggests a change to polar coordinates:



Not to say we take the pole at $\mathbf{0}$; points in the disc $x^2+(y-R)^2\le R^2$ correspond to polar angles $\theta\in[0,\pi]$ and for each θ we have $r\in$

$[0, 2R\sin\theta]$. Then

$$\begin{aligned}
\text{Area}(S) &= \int_0^\pi d\theta \int_0^{2R\sin\theta} \frac{2R}{\sqrt{4R^2-r^2}} r dr d\theta = \\
&= -2R\int_0^\pi \sqrt{4R^2-r^2}|_0^{2R\sin\theta} d\theta = \\
&= -2R\int_0^\pi (\sqrt{4R^2-4R^2\sin^2\theta}-\sqrt{4R^2})d\theta = \\
&= -4R^2\int_0^\pi (|\cos\theta|-1)d\theta = \\
&= -4R^2(2-\pi) = 4R^2(\pi-2)
\end{aligned}$$

- In spherical coordinates the upper semisphere has the parametrization

  $\alpha(\varphi,\theta) = (2R\sin\varphi\cos\theta, 2R\sin\varphi\sin\theta, 2R\cos\varphi), (\varphi,\theta)\in(0,\pi/2)\times(0,2\pi)$

  Let P' be the projection on $z=0$ of a point $P\in S$; we have:

  $P\in S\subset S^2_{2R}\Leftrightarrow P'\in D\Leftrightarrow(2R\sin\varphi\cos\theta)^2+(2R\sin\varphi\sin\theta-R)^2\le R^2$

  As $\sin\varphi>0$ we are able to arrive at

  $$P\in S\Leftrightarrow\sin\varphi\le\sin\theta\Leftrightarrow\varphi<\theta<\pi-\varphi$$

  The element of area in spherical coordinates is $dS = 4R^2\sin\varphi d\varphi d\theta$ and the area is

  $$\begin{aligned}
  \text{Area}(S) &= \int_0^{\pi/2} d\varphi\int_\varphi^{\pi-\varphi} 4R^2\sin\varphi d\theta = 4R^2\int_0^{\pi/2}(\pi-2\varphi)\sin\varphi d\varphi = \\
  &= 4R^2\pi - 8R^2\int_0^{\pi/2}\varphi\sin\varphi d\varphi = 4R^2(\pi-2)
  \end{aligned}$$

It is noticeable that the complement of the area of Viviani's vault to the area of the quadrant where it lies is

$$\pi(2R)^2 - 4R^2(\pi-2) = 8R^2,$$

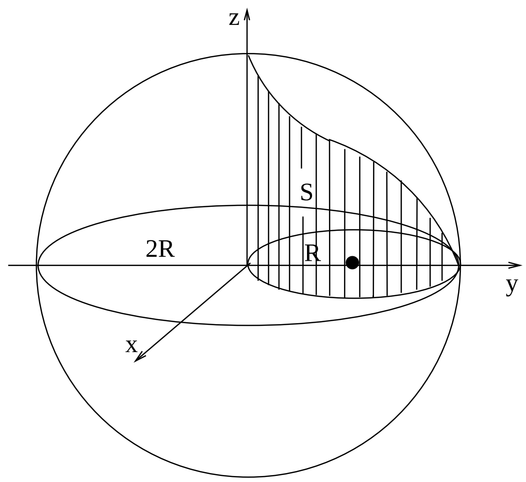
a rational function of R.

□

Problem 111: Viviani's cylinder.

Let S be the intersection of the solid upper semisphere and the surface of the cylinder in the preceding problem. Compute the area of S.

Solution:



A parametrization of the cylinder by means of the polar angle $\theta \in [0, \pi]$ (see the preceding problem) is:

$$\begin{aligned} \alpha(\theta, z) &= (2R\sin\theta\cos\theta, 2R\sin\theta\sin\theta, z) = (R\sin 2\theta, 2R\sin^2\theta, z) \\ \partial_\theta\alpha &= (2R\cos 2\theta, 2R\sin 2\theta, 0) \\ \partial_z\alpha &= (0, 0, 1) \\ \partial_\theta\alpha \times \partial_z\alpha &= (2R\sin 2\theta, -2R\cos 2\theta, 0),\ |\partial_\theta\alpha \times \partial_z\alpha| = 2R \end{aligned}$$

To compute the area of S we must find the limits of z; at the point $(2R\sin\theta\cos\theta, 2R\sin\theta\sin\theta, 0)$ in the basis of the cylinder, we can 'climb' up to the sphere, that is up to

$$\sqrt{4R^2 - 4R^2\sin^2\theta\cos^2\theta - 4R^2\sin^2\theta\sin^2\theta} = 2R \mid \cos\theta \mid$$

Finally

$$\text{Area}(S) = \int_0^\pi d\theta \int_0^{2R|\cos\theta|} 2R dz = 4R^2 \int_0^\pi |\cos\theta| d\theta = 8R^2$$

□

Problem 112: A formula.

Let S be a surface and $\alpha : D \to \mathbb{R}^3$ a regular, simple parametrization. Define

$$E = \langle \partial\alpha, \partial_u\alpha\rangle, F = \langle \partial_u\alpha, \partial_v\alpha\rangle, G = \langle \partial_v\alpha, \partial_v\alpha\rangle$$

and show that

$$\text{Area}(S) = \int\int_D \sqrt{EG - F^2}dudv$$

Solution:

It suffices to show that $|\mathbf{N}| = \sqrt{EG - F^2}$. We know (see problem on p.77) that if $\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^3$ then

$$(\mathbf{e} \times \mathbf{f}) \cdot (\mathbf{g} \times \mathbf{h}) = \det \begin{pmatrix} \mathbf{e} \cdot \mathbf{g} & \mathbf{e} \cdot \mathbf{h} \\ \mathbf{f} \cdot \mathbf{g} & \mathbf{f} \cdot \mathbf{h} \end{pmatrix}$$

Now

$$|\mathbf{N}|^2 = |\partial_u\alpha \times \partial_v\alpha|^2 = (\partial_u\alpha \times \partial_v\alpha) \cdot (\partial_u\alpha \times \partial_v\alpha) =$$

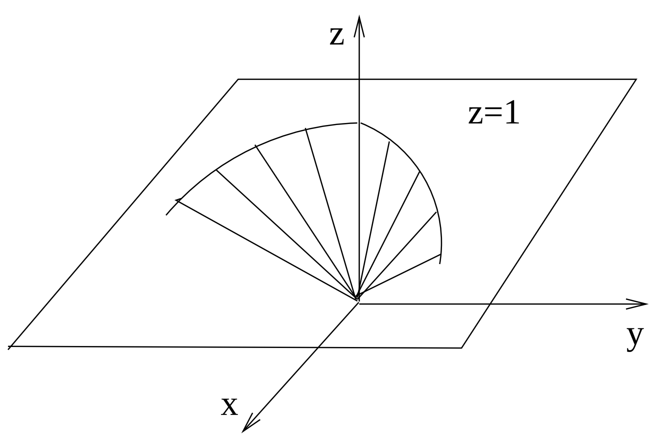
$$= \det \begin{pmatrix} \partial_u\alpha \cdot \partial_u\alpha & \partial_u\alpha \cdot \partial_v\alpha \\ \partial_u\alpha \cdot \partial_v\alpha & \partial_v\alpha \cdot \partial_v\alpha \end{pmatrix} = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2$$

□

Problem 113:

Compute the area of S, the conic surface with vertex at the origin and basis the part of the parabola $x^2 = 2y, z = 1$ that lies in the region $0 < x < 1$.

Solution:



The arc of the parabola in the region is $\gamma(t) = (\frac{t^2}{2}, t, 1), t \in [-\sqrt{2}, \sqrt{2}]$ and the cone is

$$\begin{aligned}\beta(t,s) &= (s\frac{t^2}{2}, st, s), (t,s) \in [-\sqrt{2}, \sqrt{2}] \times [0,1] \\ \partial_t\beta &= (st, s, 0) \\ \partial_s\beta &= (\frac{t^2}{2}, t, 1)\end{aligned}$$

Using the formula of the preceding problem

$$\begin{aligned}E &= \langle \partial_t\beta, \partial_t\beta\rangle = s^2(t^2+1) \\ F &= \langle \partial_t\beta, \partial_s\beta\rangle = st(\frac{t^2}{2}+1) \\ G &= \langle \partial_s\beta, \partial_s\beta\rangle = (\frac{t^2}{2}+1)^2 \\ EG - F^2 &= s^2(t^2+1)(\frac{t^2}{2}+1)^2 - s^2t^2(\frac{t^2}{2}+1)^2 = \\ &= s^2(\frac{t^2}{2}+1)^2\end{aligned}$$

and the area is

$$\begin{aligned}\text{Area } (S) &= \int_0^1 \int_{-\sqrt{2}}^{\sqrt{2}} s(\frac{t^2}{2}+1)dtds = \\ &= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (\frac{t^2}{2}+1)dt = \frac{1}{2}(\frac{t^3}{6}+t)\ |_{-\sqrt{2}}^{\sqrt{2}} = \frac{4\sqrt{2}}{3}\end{aligned}$$

□

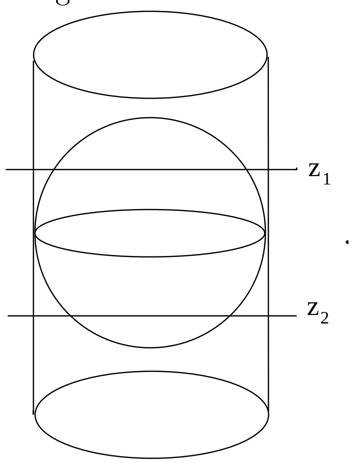
Problem 114: Sphere and cylinder: a surprising result.

a) A sphere S is inscribed in a right circular cylinder C; cut both of them with two parallel planes perpendicular to the cylinder's axis. This produces a region A on the sphere and a region B on the cylinder; show they have the same area.

b) Now let π be the orthogonal projection of S onto C from the axis. Show that π preserves areas.

Solution:

a) A figure:



We may assume that the sphere and the cylinder have radius 1 and that $z_1 > z_2$; then on the cylinder

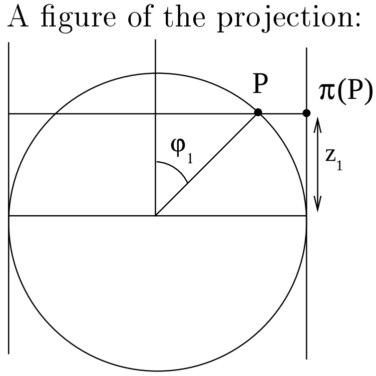
$$\text{Area}(B) = 2\pi(z_1 - z_2)$$

On the sphere, if $z_1 = \cos\varphi_1, z_2 = \cos\varphi_2$, the area of the spherical region is

$$\begin{aligned}\text{Area}(A) &= \int_0^{2\pi}\int_{\varphi_1}^{\varphi_2} \sin\varphi d\varphi d\theta = 2\pi(\cos\varphi_1 - \cos\varphi_2) = \\ &= 2\pi(z_1 - z_2)\end{aligned}$$

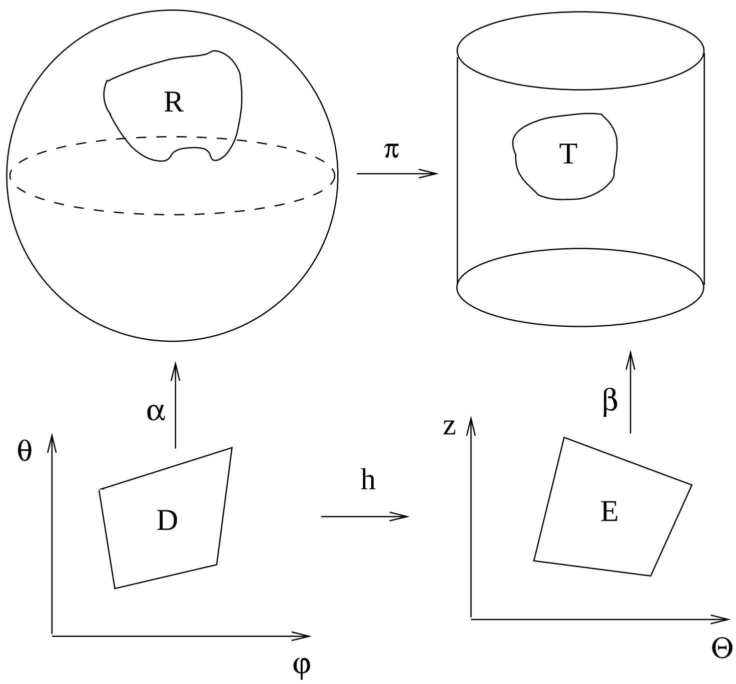
Notice: if we cut a bell 1cm wide all around the equator and a similar bell around the pole they have the same area!

b) A figure of the projection:



We parametrize S and C:

$$\begin{aligned}\alpha(\varphi,\theta) &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),\ (\varphi,\theta)\in[0,\pi]\times[0,2\pi]\\ \beta(\Theta,z) &= (\cos\Theta,\sin\Theta,z)\ (\Theta,z)\in[0,2\pi]\times[-1,1]\end{aligned}$$



In the figure $T=\pi(R)$, $R=\alpha(D)$, $T=\beta(E)$. π is a bijection

$$\pi : S-\{N,S,G\}\to C-\{L\},$$

L being the generatrix of the cylinder that corresponds to G (the meridian $\theta=0$). In coordinates the projection is

$$h(\varphi,\theta)=(\theta,\cos\varphi),$$

a differentiable function whose jacobian is

$$J=\begin{pmatrix}\partial_\varphi\Theta & \partial_\theta\Theta\\ \partial_\varphi z & \partial_\theta z\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\sin\varphi & 0\end{pmatrix}$$

Using the change of variables theorem for double integrals we have

$$\begin{aligned}\text{Area}(T) &= \int\int_E 1d\Theta dz=\int\int_D|\sin\varphi|d\varphi d\theta=\\ &= \int\int_D \sin\varphi d\varphi d\theta=\text{Area}(R)\end{aligned}$$

□

Problem 115: Surfaces of revolution.

Let $\alpha(t,\theta)=(x(t)\cos\theta, x(t)\sin\theta, z(t)), (t,\theta)\in[a,b]\times[0,2\pi]$ be a parametrization of S, the surface of revolution obtained revolving a regular, simple curve parametrized by $\gamma(t)=(x(t),z(t)), t\in[a,b],\ x(t)\geq 0$.

a) Find a formula to compute the area of S.

b) Find the area of a right circular cylinder with radius R and height h.

c) Find the area of a right circular cone with radius R and height h.

d) Find the area of a torus (make first a conjecture).

Solution:

a) The tangent vectors and the normal vector associated are

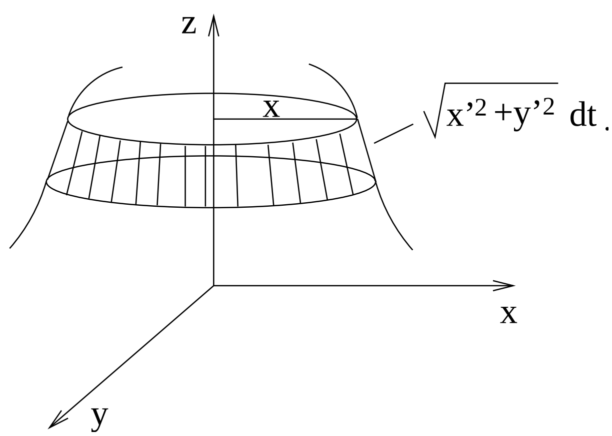
$$\begin{aligned}\partial_t\alpha &= (x'(t)\cos\theta, x'(t)\sin\theta, z'(t))\\ \partial_\theta\alpha &= (-x(t)\sin\theta, x(t)\cos\theta, 0)\\ \mathbf{N}=\partial_t\alpha\times\partial_\theta\alpha &= (-xz'\cos\theta, -xz'\sin\theta, xx')\\ dS=|\mathbf{N}|dtd\theta &= x\sqrt{x'^2+z'^2}dtd\theta\end{aligned}$$

And the area is

$$\text{Area }(S)=\int_0^{2\pi}\int_a^b x\sqrt{x'^2+z'^2}dtd\theta=2\pi\int_a^b x\sqrt{x'^2+z'^2}dt$$

Notice that $\sqrt{x'^2+z'^2}dt$ is the small arc of the curve traversed during the time dt. Then $(2\pi x)\sqrt{x'^2+z'^2}dt$ is the area of a small cylinder with radius x and height $\sqrt{x'^2+z'^2}dt$. The formula expresses the area as a sum of those areas.

$$\boxed{\text{Area}(S)=2\pi\int_a^b x\sqrt{x'^2+z'^2}dt}$$



b) A generator curve is the segment

$$\gamma(t) = (R, 0, t), t \in [0, h]$$

Using a)

$$\text{Area}(S) = 2\pi \int_0^h R dt = 2\pi Rh$$

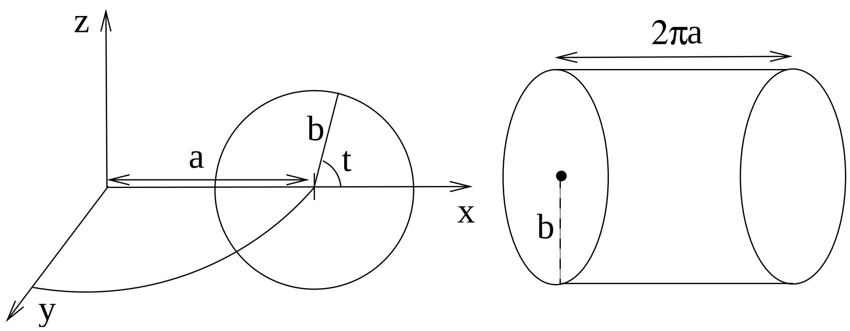
c) A generator curve is the segment

$$\gamma(t) = (t, 0, -\frac{h}{R}t + h), t \in [0, R]$$

and the formula gives:

$$\text{Area}(S) = 2\pi \int_0^R t\sqrt{1 + \frac{h^2}{R^2}} dt = 2\pi \frac{R^2}{2}\sqrt{1 + \frac{h^2}{R^2}} = \pi R\sqrt{R^2 + h^2}$$

d) If we cut and straighten the torus we obtain a cylinder of radius b and height $2\pi a$ whose area is $2\pi a \times 2\pi b$; some fibers have stretched while some others have shortened and the result is only a reasonable conjecture.



Assume the torus is generated revolving around the Oz axis the circumference $(x-a)^2+z^2=a^2, y=0$ that has a parametrization

$$\gamma(t)=(a+b\cos t,0,b\sin t), t\in(0,2\pi), 0<b<a$$

We have $x'=-b\sin t, z'=b\cos t$ and using the formula we obtain

$$\text{Area}(S)=2\pi\int_0^{2\pi}(a+b\cos t)bdt=4\pi^2ab$$

$\square$

Problem 116:

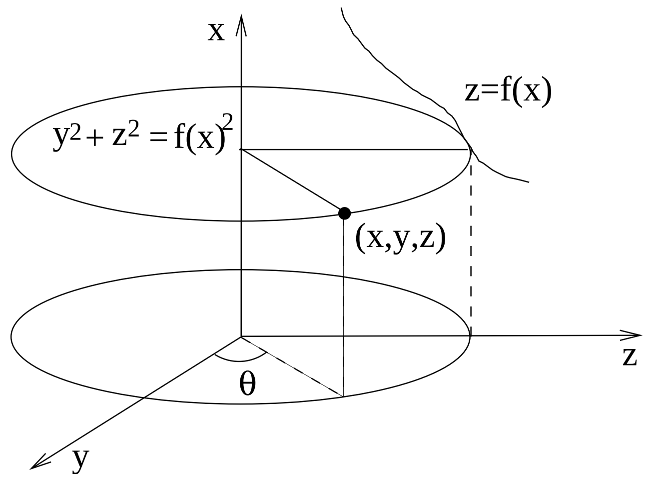
Let $f:[a,b]\to\mathbb{R}, f(x)>0$ be a differentiable function and S the surface

$$S=\{(x,y,z): y^2+z^2=(f(x))^2\}.$$

Parametrize S and find a formula for the area of S.

Solution:

At a height x we have in S the circumference $y^2+z^2=(f(x))^2$ with radius $f(x)$, and we see that S is obtained revolving the graph of $z=f(x)$ (a curve in the plane xz) around the Ox axis:



Let θ be as in the figure; we have the parametrization of S

$$\alpha(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta), x \in [a,b], \theta \in [0,2\pi]$$

and

$$\begin{aligned}
\frac{\partial\alpha}{\partial x} &= (1, f'(x)\cos\theta, f'(x)\sin\theta) \\
\frac{\partial\alpha}{\partial\theta} &= (0, -f(x)\sin\theta, f(x)\cos\theta) \\
\frac{\partial\alpha}{\partial x} \times \frac{\partial\alpha}{\partial\theta} &= ((f'(x)f(x), -f(x)\cos\theta, -f(x)\sin\theta) \\
\mid \frac{\partial\alpha}{\partial x} \times \frac{\partial\alpha}{\partial\theta} \mid &= (f'^2 f^2 + f^2)^{1/2} = f\sqrt{1+f'^2}
\end{aligned}$$

and the formula for the area is

$$\text{Area }(S) = \int_a^b \int_0^{2\pi} f\sqrt{1+f'^2}dxd\theta$$

□

5.2 Integration of scalar fields

$\boxed{\mathbb{T}}$ Let $\alpha : D \to \mathbb{R}^3$ be a differentiable, regular and simple parametrization of a surface S, $\mathbf{N} = \partial_u\alpha \times \partial_v\alpha$ the associated normal vector and f a continuous function on S; define

$$\boxed{\int\int_S f dS = \int\int_D f(\alpha(u,v))|\mathbf{N}|dudv}$$

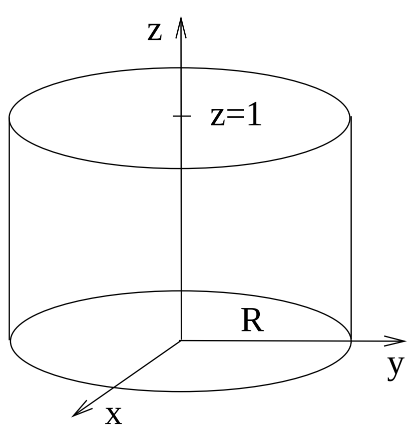
The definition is all right because the integral in the right member takes the same value for all equivalent parametrizations (see p.**??**).

□

Problem 117:

Let $f(x,y,z) = x + y + z$ and S the region of a cylinder $\{x^2 + y^2 = R^2, 0 < z < 1\}$. Compute $\int\int_S f dS$.

Solution:



Parametrize the region in the cylinder

$$\alpha(\theta, z) = (R\cos\theta, R\sin\theta, z), (\theta, z) \in (0, 2\pi) \times (0, 1)$$

The tangent vectors and the associated normal are

$$\begin{aligned} \partial_\theta \alpha &= (-R\sin\theta, R\cos\theta, 0) \\ \partial_z \alpha &= (0, 0, 1) \end{aligned}$$

$$\begin{aligned} \mathbf{N} &= (R\cos\theta, R\sin\theta, 0) \\ |\mathbf{N}| &= R \\ dS &= Rd\theta dz \end{aligned}$$

Now the integral is

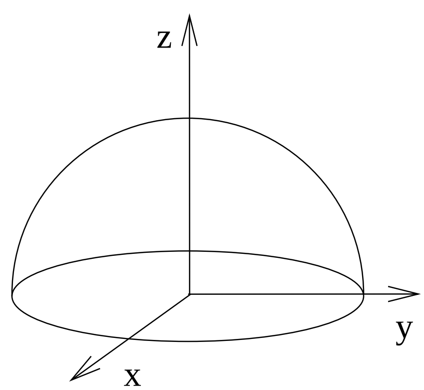
$$\begin{aligned}\int\int_S (x+y+z)dS &= \int\int_{(0,2\pi)\times(0,1)} (R\cos\theta + R\sin\theta + z)Rd\theta dz = \\ &= R^2\int_0^{2\pi}(\cos\theta+\sin\theta)d\theta + R\frac{2\pi}{2} = 4R^2 + R\pi\end{aligned}$$

□

Problem 118:

Compute $I = \int\int_S (x^2+y^2-3z^2)dS$, S being the unit upper semisphere.

Solution:



On the sphere $x^2+y^2+z^2=1$ we have $x^2+y^2-3z^2 = 1-4z^2$ and this is the function we want to integrate. In spherical coordinates we have:

$$\begin{aligned}I &= \int\int_S (1-4z^2)dS = \int_0^{2\pi} d\theta \int_0^{\pi/2}(1-4\cos^2\varphi)\sin\varphi d\varphi = \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/2}(\sin\varphi - 4\cos^2\varphi\sin\varphi)d\varphi = \\ &= 2\pi(-\cos\varphi + 4\frac{\cos^3\varphi}{3})|_0^{\pi/2} = 2\pi(1-\frac{4}{3}) = -\frac{2\pi}{3}\end{aligned}$$

□

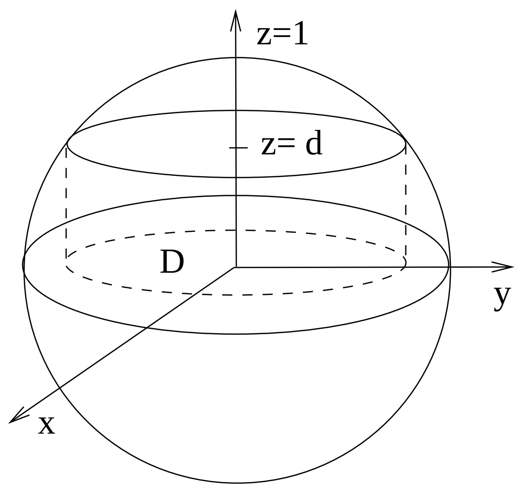
Problem 119:

Let S be the surface

$$\left.\begin{array}{rcl} x^2+y^2+z^2 & = & 4 \\ z & \geq & 1 \end{array}\right\}$$

Parametrize S as the graph of a function and compute $\int\int_S (x^2+y^2)z\,dS$.

Solution:



Noticing that when $z=1$ we have $x^2+y^2=3$ we parametrize S as the graph of the function

$$f(x,y)=\sqrt{4-(x^2+y^2)}\,,\ (x,y)\in D=D(\mathbf{0};\sqrt{3})$$

The element of area is

$$dS=\sqrt{1+\frac{x^2+y^2}{4-(x^2+y^2)}}\,dxdy=\frac{2}{\sqrt{4-(x^2+y^2)}}\,dxdy$$

and the integral is

$$\begin{aligned}\int\int_S (x^2+y^2)z\,dS &= \int\int_D (x^2+y^2)\sqrt{4-(x^2+y^2)}\frac{2}{\sqrt{4-(x^2+y^2)}}\,dxdy= \\ &= \int\int_D 2(x^2+y^2)\,dxdy=\{\text{polar coords}\}= \\ &= \int_0^{\sqrt{3}}\int_0^{2\pi} 2r^2\,r\,drd\theta=9\pi\end{aligned}$$

□

Problem 120: First Pappus-Guldin theorem.

Let S be a surface of revolution generated revolving around the Oz axis a regular, simple curve C of length L, parametrized by $\gamma(t) = (x(t), z(t)), t \in [a, b]$, $x(t) \geq 0$. Show that Area$(S) = 2\pi\langle x\rangle L$, $\langle x\rangle$ being the average value of x along C. Using this result compute the area of:

a) A sphere.

b) A cone of revolution.

c) A torus.

Solution:

A formula that gives the area of a surface of revolution is:

$$\text{Area}(S) = 2\pi \int_a^b x\sqrt{x'^2 + z'^2}dt = 2\pi \int_C xdl,$$

and as $\langle x\rangle = \frac{\int_C xdl}{L}$ the result follows.

a) Parametrize the generatrix circumference

$$\gamma(t) = (R\cos t, R\sin t), t \in [-\pi/2, \pi/2], |\gamma'(t)| = R$$

and obtain the average

$$\langle x\rangle = \frac{\int_C xdl}{\pi R} = \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} R\cos t\, Rdt = \frac{2R}{\pi}$$

The Pappus-Guldin theorem gives the area:

$$\text{Area}(S) = 2\pi\frac{2R}{\pi}\pi R = 4\pi R^2$$

b) Consider the cone generated by the segment $z = \frac{h}{R}x$, $x \in [0, R]$. The average is

$$\langle x\rangle = \frac{\int_C xdl}{L} = \frac{\int_0^R x\sqrt{1 + \frac{h^2}{R^2}}dx}{\sqrt{h^2 + R^2}} = \frac{1}{2}\frac{R^2\sqrt{1 + \frac{h^2}{R^2}}}{\sqrt{h^2 + R^2}} = \frac{1}{2}R$$

And the area

$$\text{Area}(S) = 2\pi\frac{1}{2}R\sqrt{h^2 + R^2} = \pi R\sqrt{h^2 + R^2}$$

c) It is geometrically clear that

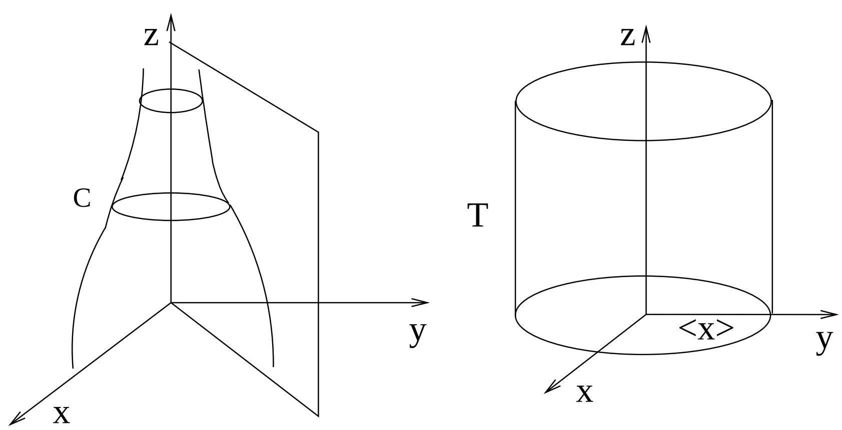
$$\langle x \rangle = a$$

and the theorem gives

$$\text{Area}(S) = 2\pi a 2\pi b = 4\pi^2 ab$$

Comment

We straighten C along a segment of length L and revolve it around the Oz axis thus generating a cylinder T that we arrange to be of radius $\langle x \rangle$. Then the first Pappus-Guldin's theorem asserts that the area of S, the surface generated by C is the same as that of T:



There is a second Pappus-Guldin theorem; let C be a simple closed curve in the half plane $y = 0, x > 0$ and R the region enclosed. Then revolving R around Oz we obtain a solid body whose *volume* is $2\pi\langle x \rangle$Area (R). If we cut and straighten the body we obtain a straight cylinder with basis R and height $2\pi\langle x \rangle$ (see p.205).

□

Problem 121: Associated normal vector.

Let

$$\alpha(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in D$$

and

$$\beta(s,t) = (X(s,t), Y(s,t), Z(s,t)), (s,t) \in D'$$

be two parametrizations of the same surface equivalent through a diffeomorphism h:

$$\begin{array}{ccc} \overset{\circ}{D} & \overset{h}{\rightarrow} & \overset{\circ}{D}{}' \\ \alpha \searrow & & \swarrow \beta \\ & \mathbb{R}^3 & \end{array}$$

Show:

a) The normal vectors $\mathbf{N}, \mathbf{M}$ respectively associated to α and β satisfy

$$\mathbf{N} = \partial_u \alpha \times \partial_v \alpha = (\det h')\partial_s \beta \times \partial_t \beta = (\det h')\mathbf{M}$$

b) Whenever α is regular so is β.

Solution:

a) We know that $\alpha = \beta \circ h$ and the chain rule gives the following matricial relation

$$\alpha'_{(u,v)} = \beta'_{h(u,v)} \cdot h'_{(u,v)}$$

that is

$$\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix} = \begin{pmatrix} \partial_s X & \partial_t X \\ \partial_s Y & \partial_t Y \\ \partial_s Z & \partial_t Z \end{pmatrix} \begin{pmatrix} \partial_u h^1 & \partial_v h^1 \\ \partial_u h^2 & \partial_v h^2 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} \mathbf{i} & \partial_u x & \partial_v x \\ \mathbf{j} & \partial_u y & \partial_v y \\ \mathbf{k} & \partial_u z & \partial_v z \end{pmatrix} = \begin{pmatrix} \mathbf{i} & \partial_s X & \partial_t X \\ \mathbf{j} & \partial_s Y & \partial_t Y \\ \mathbf{k} & \partial_s Z & \partial_t Z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial_u h^1 & \partial_v h^1 \\ 0 & \partial_u h^2 & \partial_v h^2 \end{pmatrix}$$

Taking determinants we obtain

$$\mathbf{N} = \mathbf{M}(\det h')$$

From this last equality we see that $\mathbf{N}$ and $\mathbf{M}$ have the same sense iff $\det h' > 0$ and contrary sense if $\det h' < 0$.

b) A point $(u,v) \in \overset{\circ}{D}$ is regular iff $\mathbf{N} \neq 0$ and, taking into account that $\det h' \neq 0$, that is equivalent to $\mathbf{M} \neq 0$. Both parametrizations are regular or non regular at the same points.

$\square$

Problem 122:

Prove that $\int\int_D f(\alpha(u,v))|\mathbf{N}|dudv$ takes the same value for equivalent parametrizations.

Solution:

Let $\alpha, \beta, \alpha = \beta \circ h$ as in the preceding problem; the change of variables theorem for integrals gives:

$$I = \int\int_{D'} f(\beta(s,t))|\mathbf{M}|_{(s,t)}dsdt = \int\int_D f(\beta(h(u,v)))|\mathbf{M}|_{h(u,v)} \mid \det h' \mid_{(u,v)} dudv$$

but on one hand $f(\beta(h(u,v))) = f(\alpha(u,v))$ and on the other $|\mathbf{M}|_{h(u,v)} \mid \det h' \mid_{(u,v)} = \mid \mathbf{N} \mid_{(u,v)}$. Then

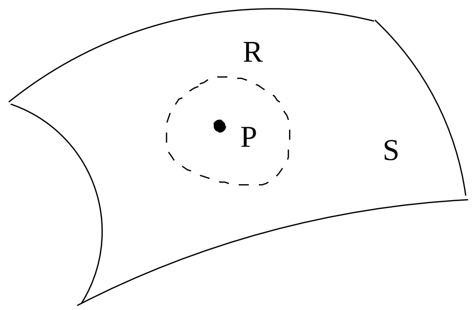
$$I = \int\int_D f(\alpha(u,v))|\mathbf{N}|dudv$$

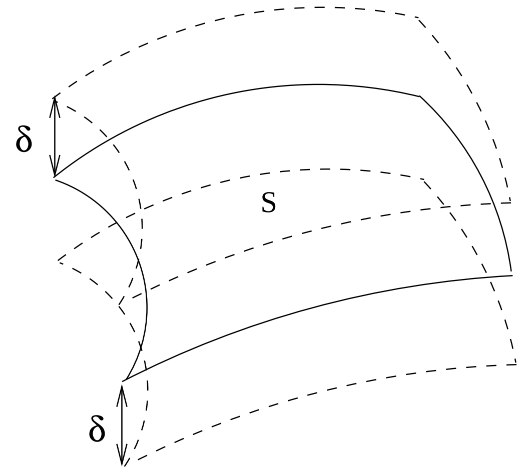
as desired.

$\square$

5.3 Material laminae

$\boxed{\mathbb{T}}$ Let S be a surface and K the body obtained translating S a distance δ on each side. Assume that K has a mass and for each region $R \subset S$ let the corresponding mass of K be concentrated in R. We have then a *material lamina*.





The surface mass density at a point P of the lamina is

$$\sigma = \lim_{\text{Area}(R)\to 0} \frac{m(R)}{\text{Area}(R)}$$

The densities are there to be integrated; for instance the total mass of the lamina is

$$M = \int\int_S \sigma dS$$

$\square$

Problem 123: Center of mass and gravitational field.

Let S be a material lamina with surface density σ. Write the formulae giving

a) The center of mass.

b) The gravitational field created by the lamina.

Solution:

a) By analogy with the unidimensional case (material wire, see p.57) we put

$$\langle x\rangle_\sigma = \frac{\int\int_S x\sigma dS}{M}, \langle y\rangle = \frac{\int\int_S y\sigma dS}{M}, \langle z\rangle_\sigma = \frac{\int\int_S z\sigma dS}{M}$$

b)

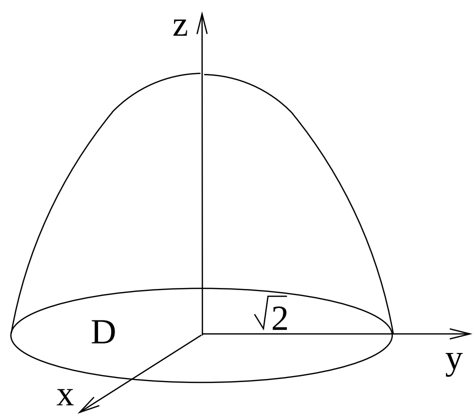
$$\mathbf{g}(x,y,z) = -\int\int_S \sigma \frac{\mathbf{r}}{r^3} dS,\ \mathbf{r} = (x-u, y-v, z-w)$$

□

Problem 124:

Find the center of mass of the region S of the paraboloid $z = 2 - (x^2 + y^2)$ limited by the plane $z = 0$.

Solution:



Parametrize S:

$$\begin{aligned} \alpha(x,y) &= (x, y, 2-(x^2+y^2)), (x,y) \in D(\mathbf{0};2) = D \\ \mathbf{N} &= (2x, 2y, 1) \\ |\,\mathbf{N}\,| &= \sqrt{1+4(x^2+y^2)} \end{aligned}$$

Consider S as a material lamina with surface density $\sigma = 1$:

$$\begin{aligned} M &= \int\int_S dS = \int\int_D \sqrt{1+4(x^2+y^2)} = \{\text{polar coords}\} = \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} r\sqrt{1+4r^2} dr d\theta = 2\pi \int_0^{\sqrt{2}} r\sqrt{1+4r^2} dr = \\ &= 2\pi \frac{1}{12}(1+4r^2)^{3/2} \,|_0^{\sqrt{2}} = \frac{26\pi}{6} \end{aligned}$$

$$\begin{aligned}\int\int_S xdS &= \int\int_D x\sqrt{1+4(x^2+y^2)}dxdy = \{\text{polar coords}\} = \\ &= \int_0^{\sqrt{2}}\int_0^{2\pi} r(r\cos\theta)\sqrt{1+4r^2}drd\theta = \\ &= (\int_0^{2\pi}\cos\theta d\theta)(\int_0^{2\pi} r^2\sqrt{1+4r^2}dr) = 0\end{aligned}$$

Then $\langle x\rangle = 0$ and analogously $\langle y\rangle = 0$, as was previsible because of the symmetry. Now

$$\begin{aligned}\int\int_S zdS &= \int\int_D (2-(x^2+y^2))\sqrt{1+4(x^2+y^2)}dxdy = \{\text{polar coords}\} = \\ &= \int_0^{\sqrt{2}}\int_0^{2\pi}(2-r^2)r\sqrt{1+4r^2}drd\theta\end{aligned}$$

and it suffices to compute $\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}dr$; by the preceding results and integrating by parts we have:

$$\int r^3\sqrt{1+4r^2}dr = r^2\frac{1}{12}(1+4r^2)^{3/2} - \int 2r\frac{1}{12}(1+4r^2)^{3/2}dr$$

The last integral is

$$\begin{aligned}\frac{2}{12}\int r(1+4r^2)^{3/2}dr &= \frac{2}{12}\frac{1}{20}(1+4r^2)^{5/2} \\ &= \frac{1}{120}(1+4r^2)^{5/2}\end{aligned}$$

Then

$$\int_0^{\sqrt{2}} r^3\sqrt{1+4r^2}dr = \frac{27}{6} - \frac{242}{120}$$

and summing up partial results we have

$$\begin{aligned}\int\int_S zdS &= 2\int_0^{\sqrt{2}}\int_0^{2\pi} r\sqrt{1+4r^2}drd\theta - \int_0^{\sqrt{2}}\int_0^{2\pi} r^3\sqrt{1+4r^2}drd\theta = \\ &= \frac{52\pi}{6} - \frac{54\pi}{6} + \frac{242\pi}{60}\end{aligned}$$

and finally (wow!)

$$\langle z\rangle = \frac{223\pi}{60}/\frac{26\pi}{6} = 223/260 \simeq 0.85$$

□

Problem 125: Field of a homogeneous disc.

Let D be a material disc of radius R with constant surface density σ.

a) Compute the gravitational field at a point P of the axis.

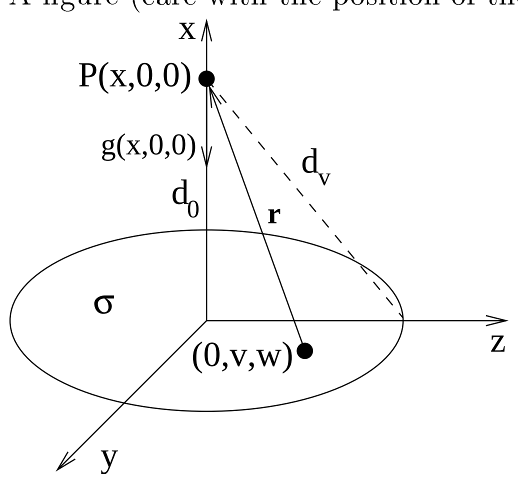
b) Compare with the field generated by a mass point at the origin with the same mass as that of the disc, when P is at an axis point infinitely far away. First do a conjecture.

c) Verify that when P approaches D, the field does *not* become infinite (that does not happen when the body is a particle or a wire).

d) Show that the field's component in the direction of the axis experiments an increment $4\pi\sigma$ when P crosses the disc from the upper space to the lower space.

Solution:

a) A figure (care with the position of the axes):



Parametrize the disc:

$$\begin{aligned} \alpha(\rho,\theta) &= (0,\rho\cos\theta,\rho\sin\theta), \rho \in (0,R), \theta \in [0,2\pi] \\ \mathbf{N} &= (0,0,\rho), \mid \mathbf{N} \mid = \rho \end{aligned}$$

From symmetry we see that the field has the form $\mathbf{g} = (X, 0, 0)$ where

$$\begin{aligned}
X(x,0,0) &= -\int\int_D \sigma \frac{x}{r^3} dS = -\int\int_D \sigma \frac{x}{(x^2+v^2+w^2)^{3/2}} dS = \\
&= -\sigma x \int_0^R \int_0^{2\pi} \frac{\rho}{(x^2+\rho^2)^{3/2}} d\rho d\theta = -2\pi\sigma x \int_0^R \frac{\rho}{(x^2+\rho^2)^{3/2}} d\rho = \\
&= -2\pi\sigma x(-(x^2+\rho^2)^{-1/2})|_{\rho=0}^{\rho=R} = 2\pi\sigma x(\frac{1}{\sqrt{R^2+x^2}} - \frac{1}{|x|})
\end{aligned}$$

For another expression let $d_v = \sqrt{R^2+x^2}$ be the distance from P to the boundary of the disc, and $d_0 = |x|$ the distance from P to the origin. We have

$$X = 2\pi\sigma x(\frac{1}{d_v} - \frac{1}{d_0}).$$

b) Consider an $M = \pi R^2 \sigma$ point mass at the origin; the field generated at $(x, 0, 0)$ is

$$\left\{ \begin{array}{ll} -\frac{\pi R^2\sigma}{x^2} & \text{if } x > 0 \\ \frac{\pi R^2\sigma}{x^2} & \text{if } x < 0 \end{array} \right\} = -\frac{x}{|x|}\frac{\pi R^2\sigma}{x^2}$$

When P is far away we expect the field of the disc to be that of the particle; that is to say we expect that

$$\mathcal{X} = \lim_{x\to\infty} \frac{2\pi\sigma x(\frac{1}{\sqrt{R^2+x^2}} - \frac{1}{|x|})}{-\frac{x}{|x|}\frac{\pi R^2\sigma}{x^2}} = 1$$

Now

$$\mathcal{X} = -\frac{2}{R^2}\lim_{x\to\infty} x^2(\frac{|x|}{\sqrt{R^2+x^2}} - 1) = -\frac{2}{R^2}\lim_{x\to\infty} x^2(\frac{1}{\sqrt{(\frac{R}{x})^2+1}} - 1)$$

and as

$$\lim_{x\to\infty}\sqrt{(\frac{R}{x})^2+1} = 1$$

we have an $\infty \cdot 0$ indetermination that must be resolved; to be brief write $\sqrt{}$ instead of $\sqrt{(\frac{R}{x})^2+1}$:

$$\begin{aligned}
\mathcal{X} &= -\frac{2}{R^2}\lim_{x\to+\infty} x^2(\frac{1-\sqrt{}}{\sqrt{}}) = \\
&= -\frac{2}{R^2}\lim_{x\to+\infty} x^2(\frac{1-(\frac{R}{x})^2-1}{\sqrt{}(1+\sqrt{})}) = \\
&= -\frac{2}{R^2}\lim_{x\to+\infty} \frac{-R^2}{\sqrt{}(1+\sqrt{})} = \frac{2}{R^2}\frac{R^2}{2} = 1
\end{aligned}$$

c) We have

$$\lim_{x\to0^+} X = 2\pi\sigma \lim_{x\to0^+}(\frac{x}{\sqrt{R^2+x^2}}-1) = -2\pi\sigma$$

and analogously

$$\lim_{x\to0^-} X = 2\pi\sigma \lim_{x\to0^-}(\frac{x}{\sqrt{R^2+x^2}}+1) = 2\pi\sigma$$

both limits are finite.

d) The increment is $X(0^-) - X(0^+) = 2\pi\sigma - (-2\pi\sigma) = 4\pi\sigma$

□

Problem 126: Field of a disc with variable density.

Compute at a point P of the axis the field of a radius R disc centered at the origin with surface mass density $\sigma = f(r)$ (r is the distance from a point in the disc to the origin). If $\sigma = a + br^2$, see what happens when P crosses the disc.

Solution:

Let us use the same setting of the preceding problem. It is clear from the circular symmetry of σ that $\mathbf{g} = (X, 0, 0)$. Let's express X :

$$X = -2\pi x \int_0^R \frac{rf(r)}{(x^2+r^2)^{3/2}}dr$$

In the case $\sigma = a + br^2$ we have

$$X = -2\pi x \int_0^R \frac{r(a+br^2)}{(x^2+r^2)^{3/2}} dr$$

Integrating by parts

$$\begin{aligned}\int \frac{r(a+br^2)}{(x^2+r^2)^{3/2}} dr &= -(x^2+r^2)^{-1/2}(a+br^2) + \int (x^2+r^2)^{-1/2} 2br\, dr = \\ &= -(x^2+r^2)^{-1/2}(a+br^2) + 2b(x^2+r^2)^{1/2}\end{aligned}$$

and

$$X = -2\pi x(-(x^2+R^2)^{-1/2}(a+bR^2) + 2b(x^2+R^2)^{1/2} + \frac{a}{|x|} - 2b|x|)$$

Setting $d_v = \sqrt{R^2+x^2}$, $d_0 = |x|$ results in

$$X = -2\pi x(\frac{a}{d_0} - \frac{a+bR^2}{d_v} + 2b(d_v - d_0))$$

and taking into account that $d_v \to R$ and $d_0 \to 0$ when $x \to 0$,

$$\begin{aligned}\lim_{x\to 0^+} X &= -2\pi a \\ \lim_{x\to 0^-} X &= 2\pi a\end{aligned}$$

the change of sign being due to the sign of x/d_0. We see that crossing the surface from the upper space to the lower space results in an increment $4\pi a$ of the field.

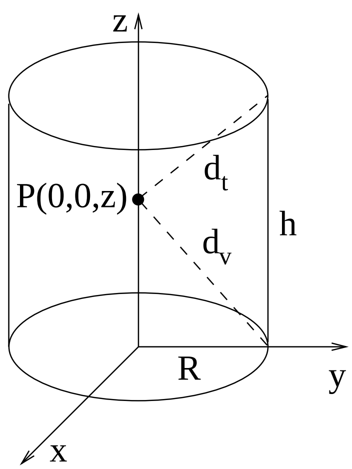
□

Problem 127: Field of a homogeneous cylinder.

Compute at a point P of the axis the field produced by a circular cylinder of radius R, height h and constant surface mass density σ.

Solution:

A figure:



From symmetry the field has the form $\mathbf{g} = (0, 0, Z)$ where

$$Z = -\sigma \int\int_S \frac{z-w}{r^3} dS,\ r = |(-u, -v, z-w)|$$

Using the cylinder's parametrization

$$\begin{aligned} u &= R\cos\theta, v = R\sin\theta, w = s \\ (\theta, s) &\in [0, 2\pi] \times [0, h],\ dS = R\, d\theta ds \end{aligned}$$

gives

$$\begin{aligned} Z &= -\sigma \int_0^{2\pi} \int_0^h \frac{z-s}{(R^2 + (z-s)^2)^{3/2}} R\, d\theta ds = \\ &= -2\pi R\sigma \int_0^h \frac{z-s}{(R^2 + (z-s)^2)^{3/2}} ds = \\ &= -2\pi R\sigma ((R^2 + (z-s)^2)^{-1/2})|_0^h = \\ &= 2\pi R\sigma (\frac{1}{\sqrt{R^2 + z^2}} - \frac{1}{\sqrt{R^2 + (z-h)^2}}) \end{aligned}$$

Let d_v be the distance from P to the boundary of the basis and d_t the distance from P to the boundary of the cover; then

$$Z = 2\pi R\sigma (\frac{1}{d_v} - \frac{1}{d_t})$$

We can test the result: when $z = h/2$ the field should vanish by symmetry. And this is so because $d_v = d_t$ and then $Z = 0$.

$\square$

Problem 128: Field of a homogeneous sphere and of a homogeneous ball.

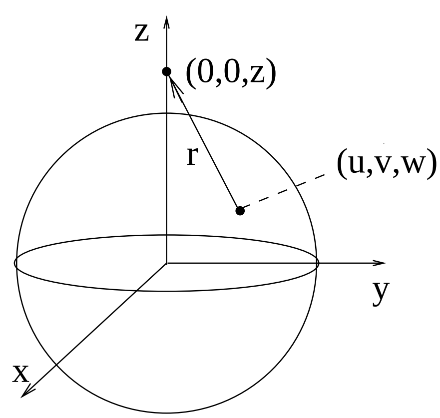
Let S be a material sphere of radius $R > 0$ centered at the origin and constant surface mass density σ. Let B be a material ball of radius $R > 0$ centered at the origin and constant volume mass density χ.

a) Compute at $P = (0,0,z)$, $z > 0, z \neq \pm R$ the gravitational field of S; separate the cases $z > R$ and $0 < z < R$.

b) Do the same for B.

Solution:

Position the axes as shown in the figure



a) From symmetry we see that the only nonvanishing component of the field is

$$Z(0,0,z) = -\sigma \int\int_S \frac{z-w}{r^3} dS, \; \mathbf{r} = (-u, -v, z-w)$$

If we pick-up the spherical parametrization of S,

$$u = R\sin\varphi\cos\theta, v = R\sin\varphi\sin\theta, w = R\cos\varphi$$

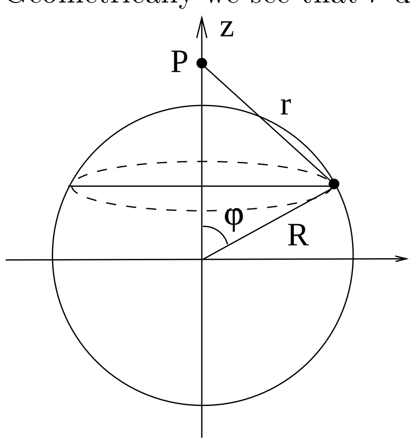
$$(\theta, \varphi) \in [0, \pi] \times [0, 2\pi],\ dS = R^2 \sin\varphi \, d\theta d\varphi,$$

we may compute r thus

$$r^2 = R^2 + z^2 - 2Rz\cos\varphi \, (*)$$

$$\begin{aligned} Z &= -\sigma \int_0^{2\pi} d\theta \int_0^{\pi} \frac{(z - R\cos\varphi)}{(R^2 + z^2 - 2Rz\cos\varphi)^{3/2}} R^2 \sin\varphi \, d\theta d\varphi = \\ &= -2\pi\sigma R^2 \int_0^{\pi} \frac{(z - R\cos\varphi)}{(R^2 + z^2 - 2Rz\cos\varphi)^{3/2}} \sin\varphi \, d\varphi \end{aligned}$$

Geometrically we see that r determines φ:



and to do the change of variables $\varphi = h(r)$ we differentiate $(*)$ thus

$$\begin{aligned} 2rdr &= 2Rz\sin\varphi d\varphi \Rightarrow \sin\varphi d\varphi = \frac{r}{Rz} dr \\ r_{\varphi=0} &= \sqrt{(R-z)^2} = |R - z| \\ r_{\varphi=\pi} &= \sqrt{(R+z)^2} = |R + z| \end{aligned}$$

From $(*)$ we obtain an expression of $z - R\cos\varphi$ in terms of r; doing the change of variables one obtains

$$\begin{aligned}
Z &= -2\pi\sigma R^2 \int_{|R-z|}^{|R+z|} \frac{r^2+z^2-R^2}{2zr^3}\frac{r}{Rz}dr = \\
&= -\frac{\pi\sigma R}{z^2}\int_{|R-z|}^{|R+z|}(\frac{z^2-R^2}{r^2}+1)dr = \\
&= \\
&= -\frac{\pi\sigma R}{z^2}(-\frac{z^2-R^2}{r}+r)|_{|R-z|}^{|R+z|} = \\
&= -\frac{\pi R\sigma}{z^2}(-\frac{z^2-R^2}{|R+z|}+|R+z|+\frac{z^2-R^2}{|R-z|}-|R-z|)
\end{aligned}$$

and as $z > 0$ we can write

$$\begin{aligned}
Z &= -\frac{\pi\sigma R}{z^2}(R-z+R+z++\frac{z^2-R^2}{|R-z|}-|R-z|) \\
&= -\frac{\pi\sigma R}{z^2}(2R+\frac{z^2-R^2}{|R-z|}-|R-z|)
\end{aligned}$$

Two cases appear:

$$\begin{aligned}
0 < z < R \;&\Rightarrow\; Z = 0 \\
0 < R < z \;&\Rightarrow\; Z = -\frac{4\pi\sigma R^2}{z^2} = -\frac{M}{z^2}
\end{aligned}$$

But we are free to choose the direction of the Oz axis and so the preceding value of the field holds in every direction. If, as usual, $\mathbf{r} = (x, y, z)$, $\mathbf{e}_r = \mathbf{r}/r$ we can write in vector form

$$\boxed{\mathbf{g}(\mathbf{r}) = \begin{cases} 0 & \text{if } P \text{ is interior} \\ -\dfrac{M}{r^2}\mathbf{e}_r & \text{if } P \text{ is exterior} \end{cases}}$$

Let us remark: the field of a spherical lamina vanishes in the interior, and in the exterior it is the field of point mass at the origin. Observe that the field on S (unknown up to now) can't possibly be continuous.

b) From symmetry the only nonvanishing component at $P = (0, 0, z)$, $z > 0, z \neq \pm R$ is

$$Z = -\chi \int\int\int_B \frac{z-w}{r^3}dV$$

i) Exterior points, $z > R$. Parametritze B by the spherical system:

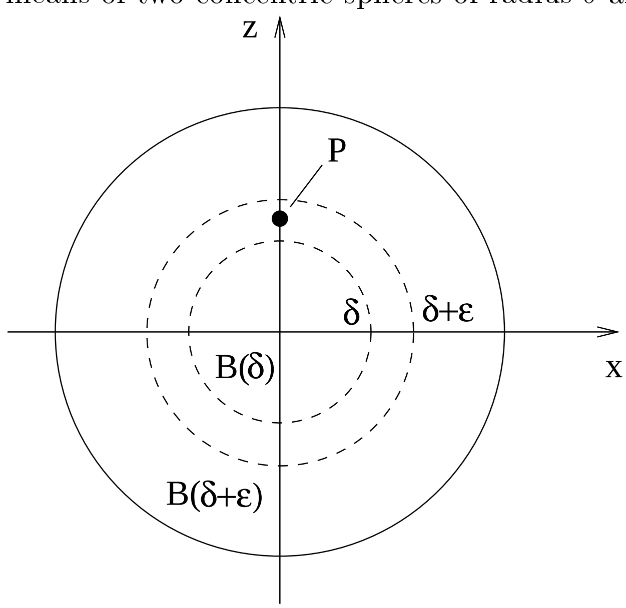
$$u = \rho \sin\varphi \cos\theta, v = \rho \sin\varphi \sin\theta, w = \rho \cos\varphi$$

$$(\rho, \varphi\theta) \in [0, R] \times [0, \pi] \times [0, 2\pi],\ dV = \rho^2 \sin\varphi\, d\theta ds$$

and using the computations in a) we have

$$\begin{aligned} Z &= -\chi \int_0^R \int_0^\pi \int_0^{2\pi} \frac{z - \rho\cos\varphi}{(\rho^2 + z^2 - 2\rho z \cos\varphi)^{3/2}} \rho^2 \sin\varphi d\rho d\varphi d\theta = \\ &= -\frac{4\pi\chi}{z^2} \int_0^R \rho^2 d\rho = -\frac{4\pi\chi}{z^2} \frac{R^3}{3} \\ &= -\frac{M}{z^2} \end{aligned}$$

ii) Interior points, $z < R$. The integral is improper (at $u = 0, v = 0, w = z$ because then $r = 0$), but one can see that it is convergent (see [Kell] p.18). To compute it isolate the point $P = (0, 0, z)$ by means of two concentric spheres of radius δ and $\delta + \epsilon$:



The spherical annulus of radius $\delta + \epsilon < r < R$ has a vanishing field at P which is an interior point. To see that, imagine the annulus as made of concentric spheres each one producing zero field. All the field at P is due to the 'internal' ball $B(\delta)$ with mass $M(\delta)$, whose value we know to be $g = -M(\delta)\frac{1}{z^2}$. Letting $\delta \to z$ and $\epsilon \to 0$ we arrive at:

$$Z = -M(z)\frac{1}{z^2}$$

Summing up, the field is

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -\frac{M}{r^2}\mathbf{e}_r & \text{if} \quad P \text{ is exterior} \\ -\frac{M(r)}{r^2}\mathbf{e}_r & \text{if} \quad P \text{ is interior} \end{cases}$$

That is for exterior points the sphere and the ball (of equal masses) create the same field, precisely that of a point mass at the origin. For interior points the field is null for the spherical lamina and the field is generated only by masses nearer than P of the center for the ball; the farther masses do not contribute.

Assume for a moment that the field at the exterior of the ball is not that of a point mass. If we went far away enough we would see the ball as a point mass, and it might happen to see two identical objects, two point masses, producing different fields.

□

Problem 129: Potential of a homogeneous sphere.

a) Show that the gravitational potencial of a homogeneous spherical lamina is that of a mass point of equal mass at the center (use the additivity of potentials).

b) Let $\mathbf{g}$ be the field and U a potential. Is U continuous? Is it true that $\mathbf{g} = \nabla U$? Are the derivatives of U continuous?

Solution:

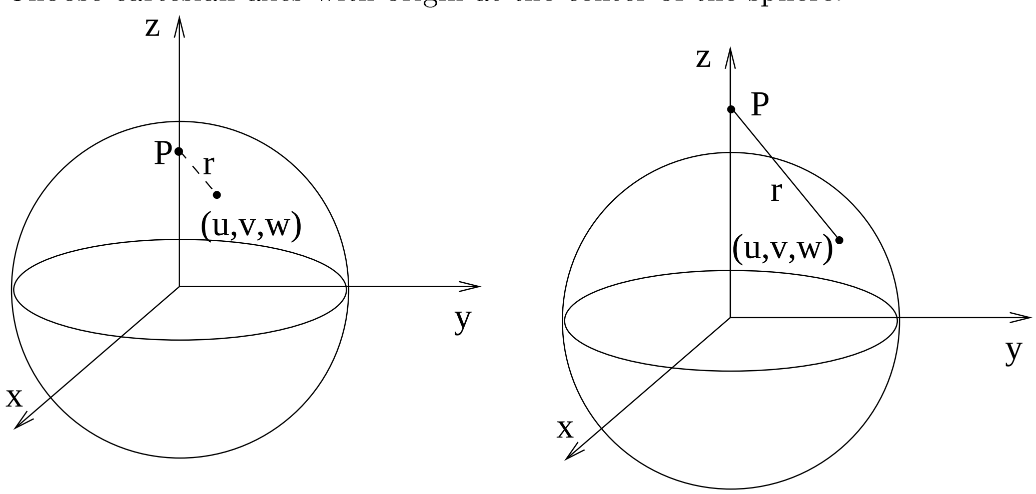
a) Two lines:

    i) The field of the lamina at exterior points is the same as that of a point mass at the center,$-M\frac{\mathbf{r}}{r^3}$. It will then have there a potential

$M\frac{1}{r}$. At interior points the field of the sphere vanishes and we pick-up a constant potential that we adjust to have it continuous on the sphere:

$$U(\mathbf{r}) = \begin{cases} M\frac{1}{r} & \text{if} \quad r \geq R \\ M\frac{1}{R} & \text{if} \quad r \leq R \end{cases}$$

ii) Choose cartesian axes with origin at the center of the sphere:



By symmetry it suffices to compute the potential at $P = (0,0,z), z > 0$. Let σ be the surface mass density; then

$$U(0,0,z) = \int\int_S \frac{\sigma}{r} dS, \; \mathbf{r} = (-u, -v, z-w), \; r = |\mathbf{r}|$$

The usual spherical parametrization gives

$$\begin{aligned} U(0,0,z) &= \int_0^{2\pi} d\theta \int_0^{\pi} \sigma \frac{1}{\sqrt{R^2+z^2-2zR\cos\varphi}} R^2 \sin\varphi d\varphi = \\ &= 2\pi\sigma R^2 \int_0^{\pi} \frac{1}{\sqrt{R^2+z^2-2zR\cos\varphi}} \sin\varphi d\varphi = \\ &= 2\pi\sigma R^2 \frac{1}{zR} \int_0^{\pi} \frac{zR}{\sqrt{R^2+z^2-2zR\cos\varphi}} \sin\varphi d\varphi = \\ &= 2\pi\sigma R \frac{1}{z} (\sqrt{R^2+z^2-2zR\cos\varphi}) \mid_{\varphi=0}^{\varphi=\pi} = \\ &= \frac{2\pi\sigma R}{z} (\mid R+z \mid - \mid R-z \mid) \end{aligned}$$

and

$$U(0,0,z) = \begin{cases} \frac{2\pi\sigma R}{z}(R+z-z+R) = \frac{4\pi R^2\sigma}{z} = \frac{M}{z} & \text{if } P \text{ is exterior} \\ \frac{2\pi\sigma R}{z}(R+z-R+z) = 4\pi R\sigma = \frac{M}{R} & \text{if } P \text{ is interior} \end{cases}$$

Using the symmetry we can write:

$$U(\mathbf{r}) = \begin{cases} \frac{M}{r} & \text{if } P \text{ is exterior} \\ \frac{M}{R} & \text{if } P \text{ is interior} \end{cases}$$

Let us define the potential to be $\frac{M}{R}$ at points on the sphere; then it will be continuous everywhere.

b) We have seen U to be everywhere continuous. It's derivatives for points not on the sphere are

$$\frac{\partial U}{\partial z}(0,0,z) = \begin{cases} -\frac{M}{z^2} & \text{if } P \text{ is exterior} \\ 0 & \text{if } P \text{ is interior} \end{cases}$$

which is the field of the sphere. They are discontinuous on S:

$$\begin{cases} \frac{\partial U}{\partial z}\mid_{z=R^-} = 0 \\ \frac{\partial U}{\partial z}\mid_{z=R^+} = -\frac{M}{R^2} \end{cases}$$

□

Problem 130: Potential of a homogeneous ball.

a) Find the potential of a material ball with constant volume density χ.

b) Show that the potential and its derivatives are everywhere continuous and that they give the field.

c) Show that $\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$ vanishes at exterior points and has the value $-4\pi\chi$ at interior points.

Solution:

a) With the same notations as in the preceding problem we have:

$$U(0,0,z) = \chi \int\int\int_V \frac{1}{r} dV, \mathbf{r} = (-u, -v, z - w)$$

In spherical coordinates (ρ, φ, θ) we have

$$\begin{aligned} U(0,0,z) &= \chi \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{\sqrt{\rho^2 + z^2 - 2z\rho\cos\varphi}} \rho^2 \sin\varphi d\rho d\varphi d\theta = \\ &= 2\pi\chi \int_0^R \left(\int_0^\pi \frac{1}{\sqrt{\rho^2 + z^2 - 2z\rho\cos\varphi}} \rho^2 \sin\varphi d\varphi\right) d\rho \end{aligned}$$

The innermost integral has been evaluated in the preceding problem and we obtain

$$U(0,0,z) = \frac{2\pi\chi}{z} \int_0^R \rho(\mid \rho + z \mid - \mid \rho - z \mid) d\rho$$

If $P = (0,0,z)$ is an exterior point $z > R$ and

$$U(0,0,z) = \frac{2\pi\chi}{z} \int_0^R 2\rho^2 d\rho = \frac{2\pi\chi}{z} \frac{2R^3}{3} = \frac{M}{z}$$

while if P is an interior point $z < R$ and

$$\begin{aligned} U(0,0,z) &= \frac{2\pi\chi}{z} \left(\int_0^z + \int_z^R\right) \rho(\mid \rho + z \mid - \mid \rho - z \mid) d\rho = \\ &= \frac{2\pi\chi}{z} \left(\int_0^z 2\rho^2 d\rho + \int_z^R 2z\rho d\rho\right) = \\ &= \frac{2\pi\chi}{z} \left(\frac{2}{3} z^3 + z(R^2 - z^2)\right) = \\ &= \frac{M(z)}{z} + 2\pi\chi(R^2 - z^2) \end{aligned}$$

$M(z)$ being the mass of $B(\mathbf{0}, z)$. Let's remind that at interior points the integral is improper at $(0,0,z)$ but nevertheless a convergent one. Using the symmetry we have:

$$U(\mathbf{r}) = \begin{cases} \frac{M}{r} & \text{at exterior points} \\ \frac{M(r)}{r} + 2\pi\chi(R^2 - r^2) & \text{at interior points} \end{cases}$$

b) Giving the value $\frac{M}{R}$ to the potential on the sphere, we have an everywhere continuous function. The derivative at exterior points is

$$\nabla U = -M\frac{\mathbf{r}}{r^3},$$

precisely the field of the ball. At interior points

$$\begin{aligned} U(\mathbf{r}) &= \frac{4}{3}\pi r^2\chi + 2\pi\chi(R^2 - r^2) \\ \frac{\partial U}{\partial x} &= (\frac{8}{3}\pi r - 4\pi r)\frac{x}{r}\chi = -\frac{4}{3}\pi x\chi = -M(r)\frac{x}{r^3} \end{aligned}$$

and the symmetry gives

$$\nabla U(\mathbf{r}) = -M(r)\frac{\mathbf{r}}{r^3},$$

that is, the field of a ball at interior points. On the boundary of the ball both expressions coincide and U is of class $\mathcal{C}^1$.

c) At exterior points

$$\nabla U(\mathbf{r}) = -M\frac{\mathbf{r}}{r^3} \Rightarrow \nabla^2 U(\mathbf{r}) = \text{div }(-M\frac{\mathbf{r}}{r^3}) = 0$$

and at interior points

$$\frac{\partial U}{\partial x} = -\frac{4}{3}\pi x\chi \Rightarrow \frac{\partial^2 U}{\partial x^2} = -\frac{4}{3}\pi\chi$$

and the same value for the other two derivatives. Then

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = -4\pi\chi$$

□

Problem 131: Logarithmic disc.

Let D be a homogeneous material disc of logarithmic particles; compute the potential U. The integration formula

$$\int_0^{2\pi} \log(1 - a\cos\theta)d\theta = 2\pi\log\frac{1+\sqrt{1-a^2}}{2}, 0 \le a < 1$$

may be useful.

Solution:

Let R be the radius of the disc and σ the surface density; the potential is

$$U(x,y) = \int\int_D \sigma \log(\frac{1}{r}) dS$$

A parametrization of the disc is:

$$\alpha(r,\theta) = (r\cos\theta, r\sin\theta), (r,\theta) \in [0,R] \times [0,2\pi]$$

It suffices to compute $U(x,0)$ when $x > 0$:

$$\begin{aligned}
U(x,0) &= \int\int_D \sigma \log(\frac{1}{\sqrt{(x-r\cos\theta)^2 + r^2\sin^2\theta}}) r dr d\theta = \\
&= -\frac{\sigma}{2}\int\int_D \log(x^2+r^2-2xr\cos\theta) r dr d\theta = \\
&= -\frac{\sigma}{2}\int\int_D \log((x^2+r^2)(1-\frac{2xr}{x^2+r^2}\cos\theta)) r dr d\theta = \\
&= -\sigma\pi\int_0^R \log(x^2+r^2) r dr - \frac{\sigma}{2}\int\int_D \log(1-\frac{2xr}{x^2+r^2}\cos\theta) r dr d\theta = \\
&= I_1 + I_2
\end{aligned}$$

a) For I_1, as a primitive of $\log x$ is $x(\log x - 1)$, we have:

$$I_1 = -\frac{\sigma\pi}{2}[(x^2+r^2)(\log(x^2+r^2)-1)]_0^R$$

b) To find I_2 put $a = \frac{2xr}{x^2+r^2}$ and use the integration formula to obtain

$$I_2 = -\frac{\sigma}{2}\int_0^R\int_0^{2\pi} \log(1-a\cos\theta) r dr d\theta = -\frac{\sigma}{2}\int_0^R 2\pi\log(\frac{1+\sqrt{1-a^2}}{2})\, r dr$$

Now compute the argument of log:

$$\begin{aligned}
1-a^2 &= 1 - \frac{4x^2r^2}{(x^2+r^2)^2} = \frac{(x^2-r^2)^2}{(x^2+r^2)^2} \\
\frac{1+\sqrt{1-a^2}}{2} &= \frac{1+\frac{|x^2-r^2|}{x^2+r^2}}{2} = \frac{1}{2}\frac{x^2+r^2+\mid x^2-r^2\mid}{x^2+r^2} = \\
&= \begin{cases} \frac{x^2}{x^2+r^2} & \text{if} \quad 0 < r < x \\ \frac{r^2}{x^2+r^2} & \text{if} \quad x < r \end{cases}
\end{aligned}$$

i) At exterior points $r < R < x$ and we obtain

$$\begin{aligned} I_2 &= -\sigma\pi \int_0^R \log(\frac{x^2}{x^2+r^2}) r dr = \\ &= -\sigma\pi \int_0^R \log x^2 r dr + \sigma\pi \int_0^R \log(x^2+r^2) r dr = \\ &= -\sigma\pi(\log x^2)\frac{R^2}{2} - I_1 \end{aligned}$$

So, for *exterior points*, letting M be the mass of the disc, $U(x,0) = -\frac{\sigma\pi R^2}{2}(\log x^2) = -M\log x$.

$$\boxed{U(x,0) = -M\log x}$$

ii) At interior points $x < R$ we have

$$I_2 = -\sigma\pi(\int_0^x \log(\frac{x^2}{x^2+r^2}) r dr + \int_x^R \log(\frac{r^2}{x^2+r^2}) r dr) = -\sigma\pi(J_1+J_2)$$

Compute separatedly both integrals

$$\begin{aligned} J_1 &= \int_0^x \log(\frac{x^2}{x^2+r^2}) r dr = \int_0^x \log(x^2) r dr - \int_0^x \log(x^2+r^2) r dr \\ J_2 &= \int_x^R \log(\frac{r^2}{x^2+r^2}) r dr = \int_x^R 2(\log r) r dr - \int_x^R \log(x^2+r^2) r dr \end{aligned}$$

Now we compute the three integrals involved:

$$\int_0^x \log(x^2) r dr = \frac{x^2}{2}\log(x^2)$$

$$\int_x^R 2(\log r) r dr = r^2(\log r - 1)|_{r=x}^{r=R}$$

$$\begin{aligned} -\int_0^x \log(x^2+r^2) r dr - \int_x^R \log(x^2+r^2) r dr &= \\ -\int_0^R \log(x^2+r^2) r dr &= \frac{1}{\sigma\pi} I_1 \end{aligned}$$

and then

$$\begin{aligned}
U &= I_1 - \sigma\pi(J_1 + J_2) = \\
&= I_1 - \sigma\pi(\frac{x^2}{2}\log(x^2) + r^2(\log r - 1)|_{r=x}^{r=R} + \frac{1}{\sigma\pi}I_1) = \\
&= -\sigma\pi(x^2\log x + R^2(\log R - 1) - x^2(\log x - 1)) = \\
&= -\sigma\pi(x^2 + R^2(\log R - 1)) = -M(x) - M(\log R - 1) = \\
&= -M(x)\log R
\end{aligned}$$

$M(x)$ being the mass of the disc of radius x.

$$U(x) = \begin{cases} -M\log x & \text{if} \quad (x,0) \text{ is exterior} \\ -M(x)\log R & \text{if} \quad P \text{ is interior} \end{cases},$$

a continuous function on the boundary of the disc because

$$\begin{aligned}
\lim_{x\to R^+} U(x) &= -M\log R \\
\lim_{r\to R^-} U(\mathbf{r}) &= -M\log R
\end{aligned}$$

□

Problem 132: Center of mass.

Let a material lamina S be that part of the sphere $x^2 + y^2 + z^2 = R^2$ in the first octant and let the surface mass density at each point be the square of the distance to the origin. Compute its center of mass.

Solution:

Parametrize S through spherical coordinates to obtain

$$\begin{aligned}
M &= \int\int_S \sigma dS = R^2 \int_0^{\pi/2}\int_0^{\pi/2} R\sin\varphi d\varphi d\theta = \\
&= R^3\frac{\pi}{2}\int_0^{\pi/2}\sin\varphi d\varphi = \frac{\pi R^3}{2}
\end{aligned}$$

and

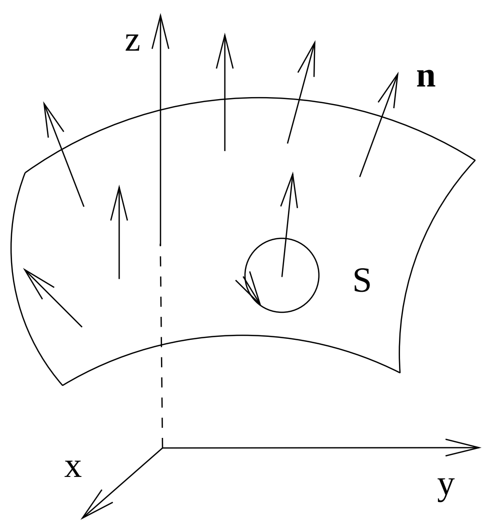
$$\begin{aligned}\int\int_S x\sigma dS &= R^2\int_0^{\pi/2}\int_0^{\pi/2} R\sin\varphi\cos\theta\, R\sin\varphi d\varphi d\theta = \\ &= R^4\int_0^{\pi/2}\sin^2\varphi d\varphi = \\ &= R^4\int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}d\varphi = \frac{\pi R^4}{4}\end{aligned}$$

$$\langle x\rangle = \frac{\frac{\pi R^4}{4}}{\frac{\pi R^3}{2}} = \frac{R}{2}$$

From the symmetry of the problem $\langle y\rangle = \langle z\rangle = \frac{R}{2}$.

5.4 Integration of vector fields

$\boxed{\mathbb{T}}$ An *orientation* of a surface S is a continuous, unit normal field:



This looks strange because after all our idea of orienting a plane has to do with turning clockwise or anticlockwise. But assume that the space has its proper orientation (maybe given through the screwdriver rule, or through the right hand rule or whatever). Then by means of **n** we can define on S a 'clockwise' sense of turning and an 'anticlockwise' sense, the clockwise one being obtained using the official rule so as to advance in the sense of $\mathbf{n}$, the unit normal vector.

A surface with an orientation is an *oriented surface.* A parametrization $\alpha : D \to \mathbb{R}^3$ preserves the orientation if $\mathbf{N} = \partial_{\mathbf{u}}\alpha \times \partial_{\mathbf{v}}\alpha$ has, at each point of S, the same direction as $\mathbf{n}$.

Let $\alpha : D \to \mathbb{R}^3$ be one such parametrization and $\mathbf{F}$ a continuous vector field on S. The integral of $\mathbf{F}$ on S is then

$$\boxed{\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_D \mathbf{F}(\alpha(u,v)) \cdot \mathbf{N}(u,v)dudv}$$

We know (see p.221) that whenever $\alpha : D \to \mathbb{R}^3$ and $\beta : D' \to \mathbb{R}^3$ are two equivalent parametrizations of a surface S through a diffeomorphism $h : D \to D'$, then the asociated normal vectors $\mathbf{N} = \partial_u\alpha \times \partial_v\alpha$ and $\mathbf{M} = \partial_u\beta \times \partial_v\beta$ satisfy

$$\mathbf{N} = (\det h')\mathbf{M}$$

If α preserves the orientation and $\det h' > 0$ then β preserves the orientation, because then $\mathbf{N}$ and $\mathbf{M}$ have the same sense. In this case

$$\int\int_\alpha \mathbf{F} \cdot d\mathbf{S} = \int\int_\beta \mathbf{F} \cdot d\mathbf{S}$$

and we see that our definition of the integral doesn't depend on the orientation preserving parametrization. If $\det h' < 0$ then $\mathbf{N}$ and $\mathbf{M}$ have have opposite directions and

$$\int\int_\alpha \mathbf{F} \cdot d\mathbf{S} = -\int\int_\beta \mathbf{F} \cdot d\mathbf{S}$$

The combination $d\mathbf{S} = \mathbf{n}dS = \mathbf{N}dudv$ is the *vector area element* (for full discussion of that concept see [Jan] p.169)

This integral of a field on an oriented surface is also called the *flux of the field through the surface* for the following reason. Look at the field as if it was the velocity field of a fluid; the integral adds up pieces as that in problem p.53 and we see that the flux is the volume of fluid that crosses S per unit time. The integral counts the volume crossed in the direction of $\mathbf{n}$ as a positive value and as a negative number in the opposite case.

□

Problem 133:

Let $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and consider the region of a cylinder parametrized by

$$\alpha(\theta,z) = (\cos\theta, \sin\theta, z), (\theta, z) \in D = (0,\pi) \times (0,1),$$

oriented through the normal vector associated to the parametrization. Compute $\int\int_\alpha \mathbf{F}\cdot d\mathbf{S}$.

Solution:

$$\begin{aligned}
\partial_\theta \alpha &= (-\sin\theta, \cos\theta, 0) \\
\partial_z \alpha &= (0, \quad 0, \quad 1) \\
\mathbf{N} &= (\cos\theta, \sin\theta, 0)
\end{aligned}$$

The integral is:

$$\begin{aligned}
\int\int_\alpha \mathbf{F}\cdot d\mathbf{S} &= \int_0^\pi \int_0^1 (\cos\theta, \sin\theta, z)\cdot(\cos\theta,\sin\theta,0)d\theta dz = \\
&= \int_0^\pi \int_0^1 d\theta dz = \pi
\end{aligned}$$

Problem 134:

Compute the flux of the vector field $\mathbf{F}(x,y,z) = (x, y+1, z)$ through the upper part of Viviani's vault V (see p.205) oriented by the exterior normal vector to the sphere.

Solution:

Let $D = \{(x,y,0) : x^2 + (y-\frac{1}{2})^2 \leq \frac{1}{4}, x > 0, y > 0\}$; being a radius 1 sphere we have $d\mathbf{S} = (x,y,z)dS$ and

$$\begin{aligned}
\phi &= \int\int_V \mathbf{F}\cdot d\mathbf{S} = \int\int_V (x, y+1, z)\cdot(x,y,z)dS = \\
&= \int\int_V (x^2+y^2+z^2+y)dS = \\
&= \int\int_V (1+y)dS = \{\text{cartesian parametrization}\} = \\
&= \int\int_D (1+y)\frac{1}{\sqrt{1-(x^2+y^2)}}dxdy = \{\text{polar coords}\} = \\
&= \int_0^{\pi/2} d\theta \int_0^{\sin\theta} (1+r\sin\theta)\frac{r}{\sqrt{1-r^2}}dr
\end{aligned}$$

As an exercise in integral calculus we have

$$\int \frac{r}{\sqrt{1-r^2}}dr = -\sqrt{1-r^2}$$

$$\int \frac{r^2}{\sqrt{1-r^2}}dr = -r\sqrt{1-r^2} + \int \sqrt{1-r^2}dr$$

The last integral is:

$$\int \sqrt{1-r^2}dr = \left\{ \begin{array}{ccc} r & = & \sin u \\ dr & = & \cos u du \end{array} \right\} = \int \sqrt{1-\sin^2 u}\cos u du = \int \cos^2 u du =$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} = \frac{\arcsin r}{2} + \frac{r\sqrt{1-r^2}}{2}$$

and:

$$\int (1+r\sin\theta)\frac{r}{\sqrt{1-r^2}}dr = -\sqrt{1-r^2} + \sin\theta(-r\sqrt{1-r^2} + \frac{\arcsin r}{2} + \frac{r\sqrt{1-r^2}}{2})$$

Evaluating this expression between $r=0$ and $r=\sin\theta$:

$$-\cos\theta + 1 + \sin\theta(-\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2})$$

Finally

$$\phi = \int_0^{\pi/2} (-\cos\theta + 1 + \sin\theta(-\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}))d\theta =$$

$$= (-\sin\theta + \theta - \frac{\sin^3\theta}{6} + \frac{1}{2}(\sin\theta - \theta\cos\theta))|_0^{\pi/2} = -1 + \frac{\pi}{2} - \frac{1}{6} + \frac{1}{2} = \frac{\pi}{2} - \frac{2}{3}$$

□

Problem 135:

Compute the flux of the vector field $\mathbf{F}(x, y, z) = (0, y, xyz)$ through the part of Viviani's cylinder (see p.208) in the first octant

$$E = \{(x, y, z) : x > 0, y > 0, z > 0\},$$

with the exterior normal orientation.

Solution:

We parametrize the region of Viviani's cylinder by

$$\alpha(\theta, z) = (R \sin 2\theta, 2R \sin^2 \theta, z),\ (\theta, z) \in [0, \pi/2] \times [0, 2R \cos \theta]$$

and obtain

$$d\mathbf{S} = (2R \sin 2\theta, -2R \cos 2\theta, 0)d\theta dz$$

Then

$$\begin{aligned}\phi &= \int_0^{\pi/2} \int_0^{2R\cos\theta} (0, 2R \sin^2 \theta, -) \cdot (2R \sin 2\theta, -2R \cos 2\theta, 0)dz d\theta = \\ &= \int_0^{\pi/2} (-4R^2 \sin^2 \theta \cos 2\theta) 2R \cos \theta d\theta = \\ &= -8R^3 \int_0^{\pi/2} \sin^2 \theta (\cos^2 \theta - \sin^2 \theta) \cos \theta d\theta = \\ &= -8R^3 \int_0^{\pi/2} (\sin^2 \theta (1 - \sin^2 \theta) \cos \theta - \sin^4 \theta \cos \theta) d\theta\end{aligned}$$

After a few calculations we obtain

$$\phi = -8R^3 \left(\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5}\right) \Big|_{\theta=0}^{\theta=\pi/2} = \frac{8}{15} R^3$$

$\square$

Problem 136: Heuristic of the flux.

Find without computations the flux of $\mathbf{F}$ through S when:

a) $\mathbf{F}(x, y, z) = (x, y, z)$, S the square $(0, 0, 0)$, $(b, 0, 0)$,$(b, b, 0)$, $(0, b, 0)$ and two similar squares in the planes $x = 0$ and $y = 0$.

b) $\mathbf{F}(x,y,z) = (x\mathbf{i} + y\mathbf{j})\log(x^2+y^2)$, S the lateral surface of the cylinder $\{(x,y,z) : x^2+y^2 = R^2, 0 \le z \le H\}$. What happens at the covers?

c) $\mathbf{F}(x,y,z) = (e^{r^2})(x,y,z)$, S the sphere with center at $\mathbf{0}$ and radius R.

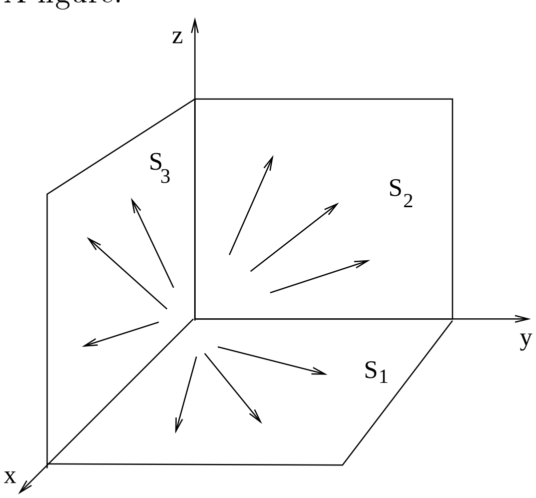
d) $\mathbf{F}(x,y,z) = (x^2, 3, xyz)$, S the triangle $(0,0,0)$,$(1,0,0)$,$(0,0,1)$.

e) $\mathbf{F}(x,y,z) = -\frac{\mathbf{r}}{r^3}$, S the sphere with center at $\mathbf{0}$ and radius R oriented by the unit exterior normal vector $\mathbf{n}$.

Solution:

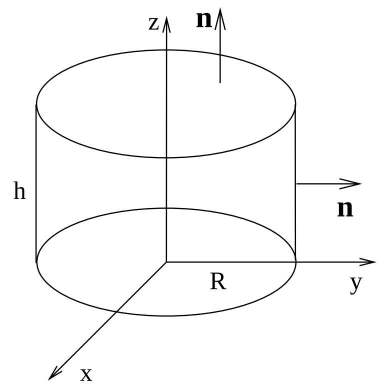
At each point of S we compute product of the normal component of $\mathbf{F}$ and the element of surface. The flux is the sum of those terms.

a) A figure:



At points of S the field is 'in S' and the normal component vanishes. The flux vanishes whatever orientation we give to the surface.

b) Let us give S the orientation of the exterior unit normal vector. On S the field is normal to the surface and its normal component is the module $R \mid \log R^2 \mid$.



Then the flux is obtained multiplying by the area:

$$\phi = R \mid \log R^2 \mid (2\pi RH)$$

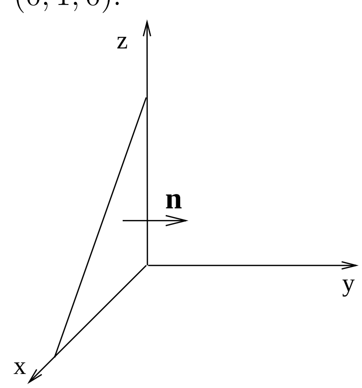
At the covers we see that the field is not defined at $(0,0,0)$ nor at $(0,0,H)$ because of the logarithm. But we can see that

$$\lim_{(x,y)\to(0,0)} \mid \mathbf{F} \mid = 0$$

and give the field the value zero at both points. Then the flux will vanish for the same reason that it vanishes on the lateral surface.

c) The normal component of the field is Re^{R^2} and the flux will be $\phi = (Re^{R^2})4\pi R^2$.

d) We first orientate the triangle through the constant vector field $\mathbf{n} = (0,1,0)$:



The normal component of the field is:

$$\mathbf{F} \cdot \mathbf{n} = (x^2, 3, xyz) \cdot (0,1,0) = 3$$

and the flux is

$$\phi = 3\frac{1}{2}$$

e) The unit normal exterior vector field is $\mathbf{n} = \frac{\mathbf{r}}{R}$ and the normal component of $\mathbf{F}$ is

$$-\frac{\mathbf{r}}{R^3} \cdot \frac{\mathbf{r}}{R} = -\frac{1}{R^2}$$

The flux is then

$$\phi = -\frac{1}{R^2} 4\pi R^2 = -4\pi$$

Notice this is the flux through the sphere of the gravitational field of a unit point mass at the origin. Should we have a point mass m the flux would be $\phi = -4\pi m$.

□

Problem 137:

Let $\mathbf{F}$ be a continuous vector field in an open set $U \subset \mathbb{R}^3$ and let $S \subset U$ be an oriented surface. Show that the flux through any region in S vanishes iff $\mathbf{F}$ is tangent to S at each point.

Solution:

If $\mathbf{F}$ is tangent to S, its normal component vanishes and so does the flux. Reciprocally if $\mathbf{F}$ is not tangent to S at the point $\mathbf{p}$ we have

$$\mathbf{F} \cdot \mathbf{n} \neq 0,$$

and, due to continuity, the product maintains the sign in a neighborhood U of $\mathbf{p}$. Choosing U as the region we would have

$$\int\int_U \mathbf{F} \cdot \mathbf{n} dS \neq 0$$

contradicting the hypothesis.

□

Problem 138:

Consider the paraboloid $S_1 = \{(x, y, z) : z = x^2 + y^2\}$, the plane $S_2 = \{(x, y, z) : 2x - z + 3 = 0\}$, and the vector field $\mathbf{F}(x, y, z) = (y, z, x - y)$. Compute:

a) The circulation of $\mathbf{F}$ along the intersection of S_1 and S_2.

b) The flux of $\mathbf{F}$ through S, the finite region of S_1 limited by S_2.

Solution:

a) Eliminating z we obtain the projecting cylinder

$$\begin{aligned} x^2 + y^2 &= 2x + 3 \\ x^2 - 2x + y^2 &= 3 \\ (x - 1)^2 + y^2 &= 4 \end{aligned}$$

A parametrization of the circumference $(x - 1)^2 + y^2 = 4, z = 0$ with center at $(1, 0)$ and radius $r = 2$ is:

$$\begin{aligned} x &= 1 + 2\cos\theta \\ y &= 2\sin\theta \end{aligned}$$

and 'climbing' to the paraboloid we obtain a parametrization of $C = S_1 \cap S_2$:

$$\begin{aligned} \gamma(\theta) &= (1 + 2\cos\theta, 2\sin\theta, 5 + 4\cos\theta), 0 \leq \theta \leq 2\pi \\ \gamma'(\theta) &= (-2\sin\theta, 2\cos\theta, -4\sin\theta) \end{aligned}$$

The circulation is

$$\int_C \mathbf{F} \cdot d\mathbf{l} =$$

$$= \int_0^{2\pi} (2\sin\theta, 5+4\cos\theta, 1+2\cos\theta-2\sin\theta)\cdot(-2\sin\theta, 2\cos\theta, -4\sin\theta)d\theta = 12\pi$$

where the orientation of the curve is that given by the parametrization.

b) A parametrization of the paraboloid is

$$\alpha(x,y) = (x, y, x^2 + y^2), (x,y) \in D$$

D being the disc limited by the circumference in a). The associated normal vector and its norm are

$$\begin{aligned} \alpha_x &= (1,0,2x) \\ \alpha_y &= (0,1,2y) \\ \mathbf{N} = \alpha_x \times \alpha_y &= (-2x,-2y,1) \end{aligned}$$

And the flux is:

$$\begin{aligned} \phi &= \int\int_D (y, x^2+y^2, x-y)\cdot(-2x,-2y,1)dxdy = \\ &= \left\{ \begin{array}{ccc} x &=& 1+\rho\cos\theta \\ y &=& \rho\sin\theta \end{array} \right\} = \\ &= \int_0^{2\pi}\int_0^2 (-2(\rho\sin\theta + \rho^2\sin\theta\cos\theta) - 2\rho\sin\theta(1+\rho\cos\theta)^2 \\ &\quad -2\rho^2\sin^3\theta + 1 + \theta\cos\theta - \theta\sin\theta)d\rho d\theta = \\ &= 2\pi\int_0^2 \rho d\rho = 4\pi \end{aligned}$$

This value is obtained choosing the orientation given by the associated normal vector; it points to the interior of the paraboloid. Had we choosen an exterior unit normal field to orient S we would have obtained -4π.

□

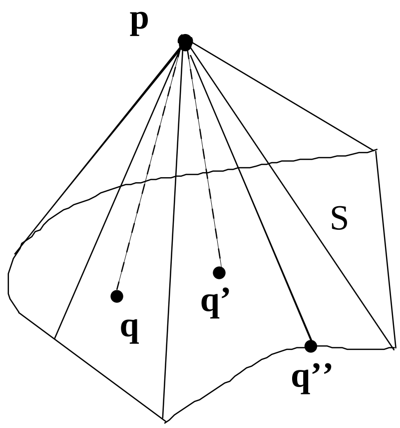
5.5 Solid angle

$\boxed{\mathbb{T}}$ Let $[\mathbf{p},\mathbf{q}\rangle$ be the half line emanating from $\mathbf{p}$ and passing through $\mathbf{q}$, and let $[\mathbf{p},\mathbf{q}]$ be the corresponding segment. Now if S is a surface in $\mathbb{R}^3$ and $\mathbf{p}$ an exterior point, the *solid angle of* S with vertex at $\mathbf{p}$ is:

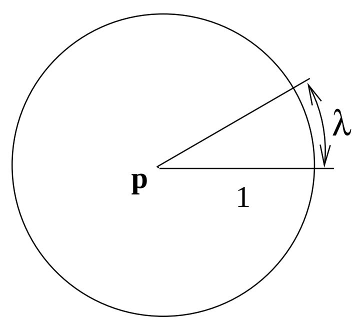
$$\mathbf{p} * S = \{[\mathbf{p},\mathbf{q}\rangle : \mathbf{q} \in S\}$$

and the *solid cone* generated by S with vertex at $\mathbf{p}$ is:

$$[\mathbf{p} * S] = \{[\mathbf{p},\mathbf{q}] : \mathbf{q} \in S\}$$

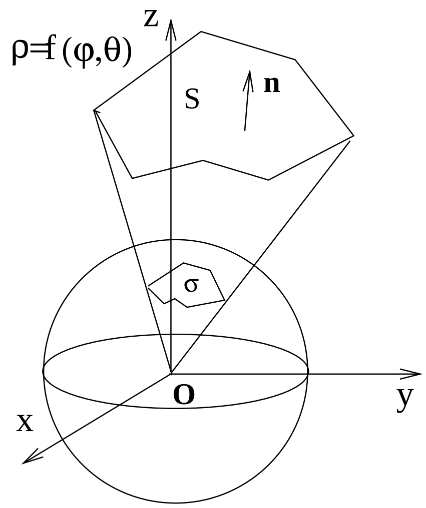


Let us remind that to obtain the measure in radians of *plane* angles we draw a radius 1 *circumference* with center at the vertex of the angle and then measure the *length* λ of the arc limited by the angle:



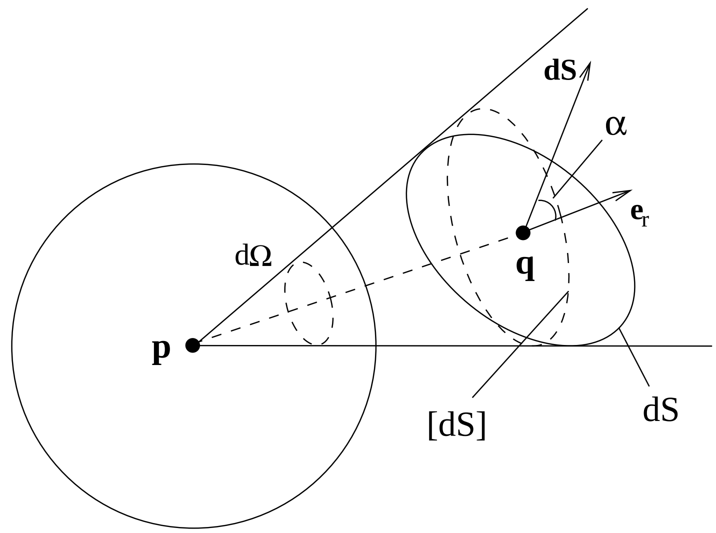
If the plane is oriented (anticlockwise usually) and we give an order to the sides of the angle we can give a sign to the measure.

Similarly to measure a *solid* angle $\mathbf{p}*S$ we draw a radius 1 *sphere* centered at $\mathbf{p}$ and measure the *area* of the region σ cut on the sphere by the solid angle:



In the case of of a plane angle, to obtain a signed measure we had to assign a positive sense of turning on the circumference (where things happened) as well as give an order to the sides of the angle. Now to obtain a *signed solid angle* we orient the sphere (where things will happen) by the unit normal exterior field; the orientation of the solid angle is an orientation of S.

We search a *formula* (et.: small form) to compute the measure and the sign of oriented solid angles. We divide the problem in small problems (Arquimedes, Descartes) breaking down the surface in surface elements dS which we may think as small pieces of plane tangent to S. To introduce the tangency we use the vector surface element $d\mathbf{S} = \mathbf{n}dS$, $\mathbf{n}$ being the unit normal field giving the orientation of S:



In the figure we project dS, the tilted form in continuous line, onto a plane through $\mathbf{q}$ orthogonal to $\mathbf{e}_r$ the unit position vector (the position vector being $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{pq}}$). We obtain the form in dashed line whose area is

$$(\cos\alpha)dS = \mathbf{e}_r \cdot \mathbf{n}dS = \mathbf{e}_r \cdot d\mathbf{S}$$

and the area of the projection of this area onto the unit sphere is the element of solid angle

$$d\Omega = \frac{1}{r^2}\mathbf{e}_r \cdot d\mathbf{S}$$

We obtain the measure of the solid angle vith vertex at $\mathbf{p}$ adding the contributions $d\Omega$:

$$\boxed{\Omega = \int\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S}}$$

The integral is evaluated on S; if $\mathbf{p} = (a, b, c)$, $\mathbf{q} = (u, v, w)$ then

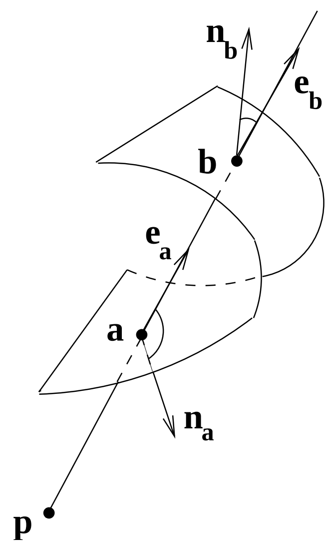
$$\mathbf{r} = (u-a, v-b, w-c), r = \sqrt{(u-a)^2 + (v-a)^2 + (w-c)^2}$$

and we have the formula

$$\Omega = \int\int_S \frac{(u-a, v-b, w-c)}{((u-a)^2 + (v-b)^2 + (w-c)^2)^{3/2}} \cdot d\mathbf{S}$$

Now $d\Omega$ is positive if $\mathbf{e}_r$ and $\mathbf{n}$ point to the same side of S and negative if they point to different sides. The sign of Ω will depend on the orientation of S. For instance take for S a semisphere centered at $\mathbf{0}$ oriented by the exterior normal; the solid angle with vertex at the origin measures 2π, but if we orient S by the interior normal the measure will be -2π.

In the following figure the contribution at point $\mathbf{b}$ is positive, and that at point $\mathbf{a}$ it is negative. Try to see geometrically that the solid angle of a sphere from an exterior point vanishes.



Notice that $\frac{\mathbf{r}}{r^3}$ is the gravitational field of a -1 mass (!) at the origin (see p.56) and then the *solid angle* is the flux of this field through S. If the minus sign associated to a mass is disturbing we can think in the electric field $\frac{\mathbf{r}}{r^3}$ generated by a point charge $+1$ at the origin (see p.61)

□

Problem 139:

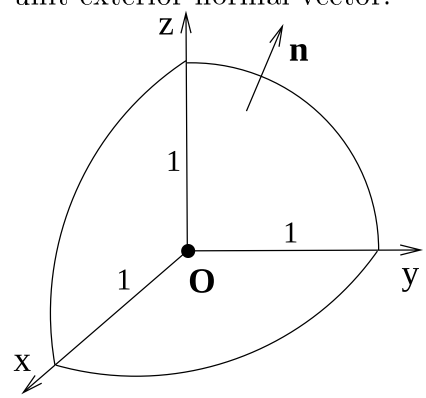
Find the measure of the following solid angles expressing them as $\mathbf{p} * S$ and giving an orientation to S.

a) $C = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$.

b) $D = \{(x, y, z) : x^2 + y^2 \leq az^2,\ z \geq 0, a > 0\}$.

Solution:

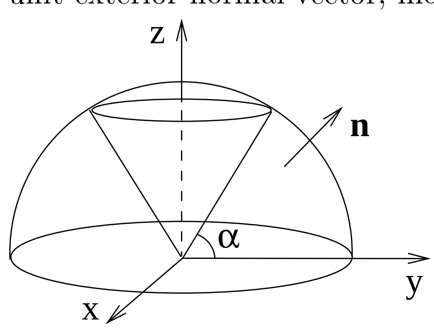
a) It is clear that the first octant of the unit sphere centered at the origin is the S we must choose. Orient this piece of the sphere through the unit exterior normal vector:



We see what the region on the unit sphere measures:

$$\Omega = \frac{4\pi}{8} = \frac{\pi}{2}$$

b) The solid angle given is generated by the region A in the upper unit sphere that lies outside the cone. We orient this region through the unit exterior normal vector; moreover we have $\tan\alpha = \sqrt{a}$.



Let β be the spherical parametrization of S^2; the exterior normal is

$\mathbf{n} = (u, v, w)$ and put $\mathbf{r} = (u, v, w), r = \mid \mathbf{r} \mid$. The solid angle is

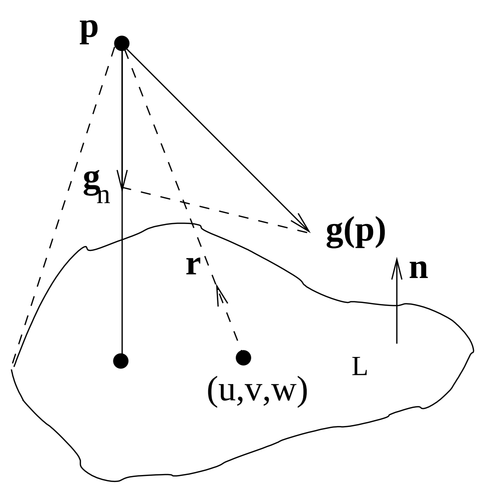
$$\begin{aligned}
\Omega &= \int\int_A \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = \\
&= \int\int_A (u, v, w) \cdot (u, v, w) dS = \\
&= \int\int_A (u^2 + v^2 + w^2) dS = \int\int_A 1 dS = \\
&= \int_0^{2\pi} d\theta \int_{\pi/2 - \arctan\sqrt{a}}^{\pi/2} \sin\varphi d\varphi = 2\pi(-\cos\varphi)_{\pi/2-\arctan\sqrt{a}}^{\pi/2} = \\
&= 2\pi(\sin(\arctan\sqrt{a})) = 2\pi \frac{\sqrt{a}}{\sqrt{1+a}}
\end{aligned}$$

□

Problem 140: Solid angle and attraction.

Let $\mathbf{g}$ be the gravitational field created by a plane, bounded, material lamina L with constant superficial density σ, that we assume oriented by a normal field $\mathbf{n}$. Show that the normal component of the field is $\sigma\Omega$, Ω being the measure of the solid angle $\mathbf{p} * L$.

Solution:



When computing the gravitational field at $\mathbf{p}$ we follow the rule 'from the source (the lamina) to the point' (see p.55) and if $\mathbf{p} = (x, y, z)$ we have

$$\mathbf{g}(\mathbf{p}) = -\sigma \int\int_L \frac{\mathbf{r}}{r^3} dS, \ \mathbf{r} = (x-u, y-v, z-w), r = |\mathbf{r}|$$

The normal component is

$$g_n(\mathbf{p}) = \mathbf{g}(\mathbf{p}) \cdot \mathbf{n} = -\sigma \int\int_L \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = -\sigma \int\int_L \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S}$$

While to find the solid angle of L with vertex at $\mathbf{p}$ we use $\mathtt{r} = (u-x, v-y, w-z)$ that satisfyes $\mathbf{r} = -\mathbf{r}$ and compute

$$\Omega = \int\int_L \frac{\mathtt{r}}{|\ \mathtt{r}\ |^3} \cdot d\mathbf{S} = -\int\int_L \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S}$$

Finally

$$g_n(\mathbf{p}) = \sigma\Omega$$

□

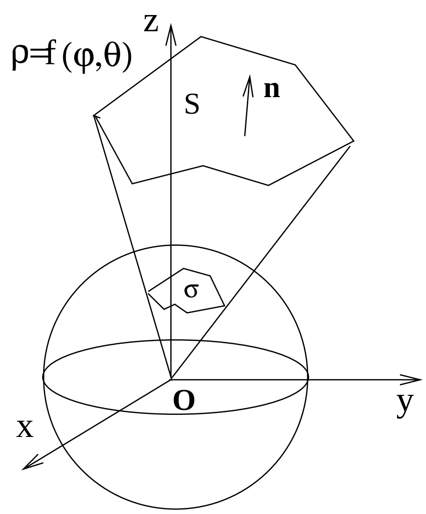
Problem 141: Solid angle and attraction.

Let V be the region of a solid angle with vertex at the origin limited by a surface S whose equation in spherical coordinates is $\rho = f(\varphi, \theta)$, $(\varphi, \theta) \in D$ and assume in V a mass density χ. Show that the z component of the gravitational field of the body at the origin is

$$Z = \int\int_S (\int_0^{f(\varphi,\theta)} \chi d\rho) \cos\varphi d\Omega$$

Solution:

Observe the figure:



To compute the field, using the rule 'from source to point' , we write $\mathbf{r} = (-u, -v, -w)$, $r = (u^2 + v^2 + w^2)^{1/2}$; the component $Z(0, 0, 0)$ is simply

$$\begin{aligned} Z(\mathbf{0}) &= -\int\int\int_V \frac{-w}{r^3}\chi dV = \{\text{spherical coords}\} = \\ &= \int\int_D (\int_0^{f(\varphi,\theta)} \frac{\rho\cos\varphi}{\rho^3}\chi\rho^2 \sin\varphi d\rho) d\varphi d\theta = \\ &= \int\int_D (\int_0^{f(\varphi,\theta)} \chi d\rho) \cos\varphi \sin\varphi d\varphi d\theta \end{aligned}$$

While to compute the solid angle we use $\mathbf{e}_r = \frac{(u,v,w)}{r}$ and $d\Omega = \frac{\mathbf{e}_r}{r^2} \cdot d\mathbf{S}$; we have the parametrization of S

$$\alpha(\varphi, \theta) = f(\varphi, \theta)(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = f(\varphi, \theta)\mathbf{e}_r,\ (\varphi, \theta) \in D$$

and we let the normal vector associated to it orient S . Then :

$$\begin{aligned} \partial_\varphi\alpha &= f_\varphi\mathbf{e}_r + f\,\partial_\varphi\mathbf{e}_r \\ \partial_\theta\alpha &= f_\theta\mathbf{e}_r + f\,\partial_\theta\mathbf{e}_r \\ \mathbf{N} &= (f_\varphi\mathbf{e}_r + f\,\partial_\varphi\mathbf{e}_r) \times (f_\theta\mathbf{e}_r + f\,\partial_\theta\mathbf{e}_r) = \\ &= f f_\varphi(\mathbf{e}_r \times \partial_\theta\mathbf{e}_r) + f f_\theta(\partial_\varphi\mathbf{e}_r \times \mathbf{e}_r) + f^2(\partial_\varphi\mathbf{e}_r \times \partial_\theta\mathbf{e}_r) \\ d\Omega &= \frac{\mathbf{e}_r}{r^2} \cdot d\mathbf{S} = \frac{\mathbf{e}_r}{r^2} \cdot f^2(\partial_\varphi\mathbf{e}_r \times \partial_\theta\mathbf{e}_r) d\varphi d\theta \end{aligned}$$

The partial derivatives of $\mathbf{e}_r$ are:

$$\begin{aligned}
\partial_\varphi \mathbf{e}_r &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \\
\partial_\theta \mathbf{e}_r &= (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0) \\
\partial_\varphi \mathbf{e}_r \times \partial_\theta \mathbf{e}_r &= (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)
\end{aligned}$$

$$\begin{aligned}
d\Omega &= \frac{f^2}{r^2}\mathbf{e}_r \cdot (\partial_\varphi \mathbf{e}_r \times \partial_\theta \mathbf{e}_r)\, d\varphi d\theta = \\
&= \mathbf{e}_r \cdot (\partial_\varphi \mathbf{e}_r \times \partial_\theta \mathbf{e}_r)\, d\varphi d\theta = \\
&= (\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi) d\varphi d\theta = \\
&= (\sin^3\varphi + \sin\varphi\cos^2\varphi)\, d\varphi d\theta = \\
&= \sin\varphi\, d\varphi d\theta
\end{aligned}$$

and the attraction is

$$\begin{aligned}
Z(\mathbf{0}) &= \int\int_D (\int_0^{f(\varphi,\theta)} \chi d\rho) \cos\varphi \sin\varphi\, d\varphi d\theta = \\
&= \int\int_S (\int_0^{f(\varphi,\theta)} \chi d\rho) \cos\varphi\, d\Omega
\end{aligned}$$

□

Chapter 6

Integral theorems

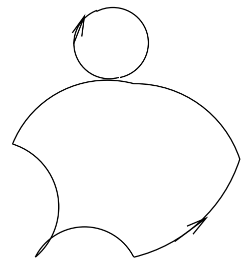
6.1 Green's Theorem

$\boxed{\mathbb{T}}$ Let $U \subset \mathbb{R}^2$ be the bounded region limited by a piecewise $\mathcal{C}^1$ simple closed curve C, positively oriented (i.e.: the region U lies to the left of the traversed curve) and $\mathbf{F} = (P, Q) \in \mathcal{C}^1(C \cup U)$. Then:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{l} = \int\int_U (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy}$$

(The theorem remains true under the weaker hypotheses that $\mathbf{F}$ be differentiable and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ continuous. See J.Bruna, J.Cufí, ANÀLISI COMPLEXA, Manuals 49. Universitat Autònoma Barcelona. 2008.)
In the following figure we can see from left to right curves that are: simple $\mathcal{C}^1$, non simple $\mathcal{C}^1$, simple piecewise $\mathcal{C}^1$, non simple piecewise $\mathcal{C}^1$.

Notice that if the curve is not simple it may be impossible to leave the bounded region on the left:



and we see as well that several bounded regions may exist in this case.

□

Problem 142:

Let $a = (0,0), b = (2,3)$ and C the boundary of the rectangle $R(a,b)$ traversed in anticlockwise sense. Compute $\int_C \mathbf{F} \cdot d\mathbf{l}$ using Green's theorem in the following cases:

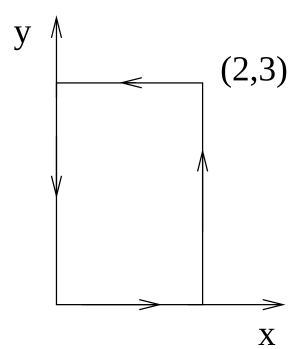
a) $\mathbf{F}(x,y) = (0,x)$

b) $\mathbf{F}(x,y) = (x+y, y^2)$

c) $\mathbf{F}(x,y) = (xy^2, 2x - y)$

d) $\mathbf{F}(x,y) = (\sin(\frac{\pi}{2}xy), 2x)$

Solution:



a) $\int_C \mathbf{F} \cdot d\mathbf{l} = \int\int_R 1 dxdy = \text{Area}(R) = 6.$

b) $\int_C \mathbf{F} \cdot d\mathbf{l} = \int\int_R -1 dxdy = -\text{Area}(R) = -6.$

c)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= \int\int_R (2-2xy)dxdy = 2\int\int_R 1dxdy - 2\int\int_R xydxdy = \\ &= 12 - 2(\int_0^2 xdx \int_0^3 ydy) = 12 - 18 = -6.\end{aligned}$$

d)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= \int\int_R (2 - \frac{\pi}{2}x\cos(\frac{\pi}{2}xy))dxdy = \\ &= 12 - \int_0^2 dx \int_0^3 \frac{\pi}{2}x\cos(\frac{\pi}{2}xy))dy = 12 - \int_0^2 x\sin(\frac{\pi}{2}xy)|_{y=0}^{y=3}dx = \\ &= 12 - \int_0^2 \sin(\frac{3\pi}{2}xy)dx = 12 - \frac{4}{3\pi}\end{aligned}$$

□

Problem 143: Area computations through line integrals.

a) Let U be the bounded region of C, a piecewise $\mathcal{C}^1$ simple closed curve, positively oriented. Show that

$$\text{Area}(U) = -\int_C ydx = \int_C xdy = \frac{1}{2}\int_C(-ydx + xdy)$$

b) Find

  i) The area of an arc of a cycloid.

  ii) The area of one leaf of the four-leaf clover given in polar coordinates by $r = 3\sin 2\theta$.

Solution:

a) We obtain the three results applying Green's theorem to the region U and to the fields

$$\mathbf{F}(x,y) = (-y,0), \mathbf{F}(x,y) = (0,x), \mathbf{F}(x,y) = \frac{1}{2}(-y,x),$$

respectively. For instance

$$\begin{aligned}-\int_C ydx &= \int_C (-y,0)\cdot d\mathbf{l} = \int\int_D [\partial_x(0)-\partial_y(-y)]dxdy\\ &= \int\int_D 1dxdy = \text{Area}(U).\end{aligned}$$

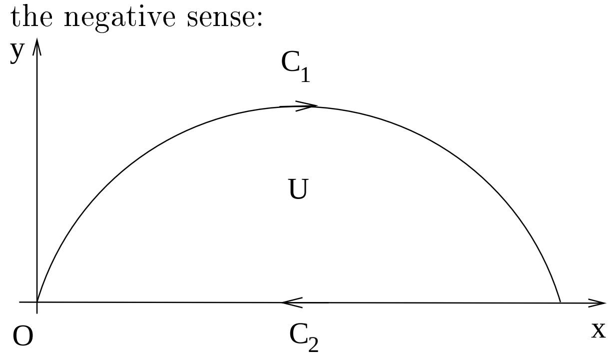
Observation: There is an analogous method for volume computation that uses surface integrals (see p.331).

b) In each case we use the appropiate formula:

i) Remind the parametrization of C_1, one arc of the cycloid:

$$\left.\begin{aligned}x &= R(u-\sin u)\\ y &= R(1-\cos u)\end{aligned}\right\},\ 0\le u\le 2\pi$$

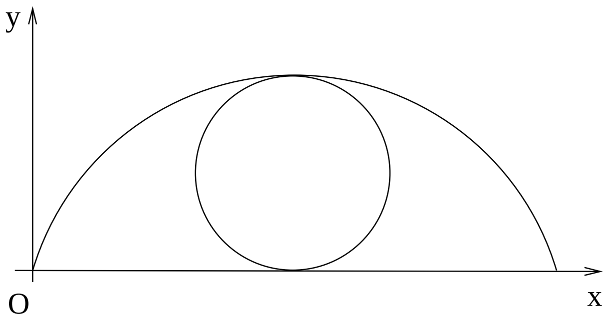
Let U be the bounded region of the closed curve C that consists of C_1 and the segment $C_2=[2\pi R,0]$ on the Ox axis traversed in the negative sense:



The curve C is piecewise C^1, simple and closed but is negatively oriented. That is why we have

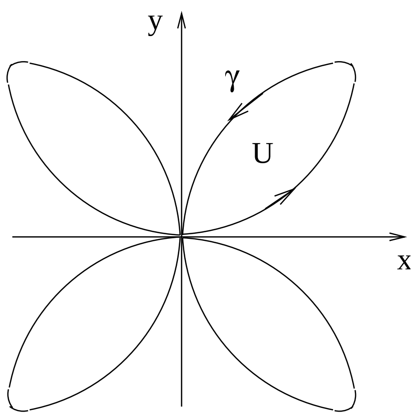
$$\begin{aligned}\text{Area}(U) &= -\int_{C_1\cup C_2} xdy = -\int_{C_1} xdy - \int_{C_2} xdy =\\ &= -\int_0^{2\pi} R(u-\sin u)R\sin udu + 0 =\\ &= R^2(-\int_0^{2\pi} u\sin udu + \int_0^{2\pi}\sin^2 udu) =\\ &= R^2(u\cos u|_0^{2\pi} + \int_0^{2\pi}\cos udu) + R^2\int_0^{2\pi}\frac{1-\cos 2u}{2}du =\\ &= R^2(2\pi+\pi) = 3\pi R^2\end{aligned}$$

The area is thrice the area of the generating wheel. We can visualize this result: the area of both regions next to the central disc in the figure is the same as the area of the disc.



ii) The cartesian parametrization of one leaf is

$$\gamma(\theta) = (3\sin 2\theta\cos\theta, 3\sin 2\theta\sin\theta),\ 0 \leq \theta \leq \pi/2.$$



Using the second formula in a) we have

$$\begin{aligned}
\text{Area}(U) &= \int_C x dy = \int_0^{\pi/2} 3\sin 2\theta\cos\theta \frac{d}{d\theta}(3\sin 2\theta\sin\theta)d\theta = \\
&= \int_0^{\pi/2} 3\sin 2\theta\cos\theta(6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta) = \\
&= \int_0^{\pi/2} (18\sin 2\theta\cos 2\theta\sin\theta\cos\theta + 9\sin^2 2\theta\cos^2\theta)d\theta = \\
&= \int_0^{\pi/2} (\frac{9}{2}\sin 4\theta\sin 2\theta + 9\sin^2 2\theta\frac{1+\cos 2\theta}{2})d\theta
\end{aligned}$$

Let us compute separatedly each integral; we use the trigonometric formula $\sin a \sin b = \frac{\cos(a-b)-\cos(a+b)}{2}$:

$$\begin{aligned}\int_0^{\pi/2} 9\sin 4\theta \sin 2\theta d\theta &= \frac{9}{2}\int_0^{\pi/2} \frac{1}{2}(\cos 2\theta - \cos 6\theta)d\theta = \\ &= \frac{9}{4}(\frac{\sin 2\theta}{2} - \frac{\sin 6\theta}{6})|_0^{\pi/2} = 0\end{aligned}$$

$$\begin{aligned}&\int_0^{\pi/2} 9\sin^2 2\theta \frac{1+\cos 2\theta}{2} d\theta = \\ &= 9\int_0^{\pi/2}(\frac{\sin^2 2\theta}{2} + \frac{\sin^2 2\theta \cos 2\theta}{2})d\theta = \\ &= 9\int_0^{\pi/2}(\frac{1-\cos 4\theta}{4} + \frac{\sin^2 2\theta \cos 2\theta}{2})d\theta = \\ &= 9(\frac{\theta}{2} - \frac{\sin 4\theta}{} + \frac{\sin^3 2\theta}{6})|_0^{\pi/2} = \frac{9}{8}\pi\end{aligned}$$

Finally the area is

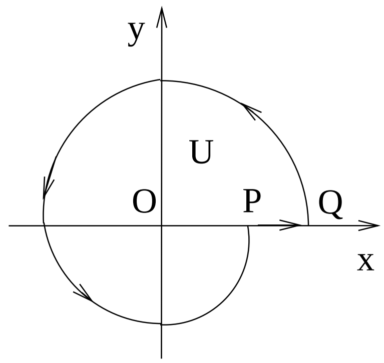
$$\text{Area}(U) = \frac{9}{8}\pi$$

□

Problem 144:

The parametrized curve $\gamma(t) = (e^{-t}\cos t, e^{-t}\sin t), t \in [0, 2\pi]$ (a spiral) joined to a segment in the Ox axis forms a closed curve. Find the area of U, the bounded region of C.

Solution:



We use the formula Area $(U) = \int_C xdy$ because doing so we won't have to integrate along the segment since there we have $dy = 0$.

$$\begin{aligned} \text{Area } (U) &= \int_0^{2\pi} e^{-t} \cos t(-e^{-t} \sin t + e^{-t} \cos t)dt \\ &= \int_0^{2\pi} e^{-2t}(\cos^2 t - \sin t \cos t)dt \end{aligned}$$

Integrating by parts

$$\begin{aligned} \int_0^{2\pi} e^{-2t} \cos^2 t\, dt &= -\frac{1}{2}e^{-2t} \cos^2 t \,|_0^{2\pi} - \int_0^{2\pi} -\frac{1}{2}e^{-2t}(2\cos t(-\sin t))dt = \\ &= -\frac{1}{2}e^{-4\pi} + \frac{1}{2} - \int_0^{2\pi} e^{-2t} \sin t \cos t\, dt = \\ &= \frac{1}{2}(1 - e^{-4\pi}) - \int_0^{2\pi} e^{-2t} \sin t \cos t\, dt \end{aligned}$$

and then

$$\begin{aligned} \text{Area } (U) &= \frac{1}{2}(1 - e^{-4\pi}) - 2\int_0^{2\pi} e^{-2t} \sin t \cos t\, dt = \\ &= \frac{1}{2}(1 - e^{-4\pi}) - \int_0^{2\pi} e^{-2t} \sin 2t\, dt \end{aligned}$$

This last integral is

$$\begin{aligned} I &= \int_0^{2\pi} e^{-2t}\sin 2t\,dt = \\ &= -\frac{1}{2}e^{-2t}\sin 2t\,|_0^{2\pi} + \int_0^{2\pi} e^{-2t}\cos 2t\,dt = \\ &= \int_0^{2\pi} e^{-2t}\cos 2t\,dt = J \end{aligned}$$

and

$$\begin{aligned} J &= \int_0^{2\pi} e^{-2t}\cos 2t\,dt = \\ &= -\frac{1}{2}e^{-2t}\cos 2t\,|_0^{2\pi} - \int_0^{2\pi} e^{-2t}\sin 2t\,dt = \\ &= -\frac{1}{2}e^{-4\pi} + \frac{1}{2} - I = \\ &= \frac{1}{2}(1-e^{-4\pi}) - I \end{aligned}$$

Then

$$\begin{aligned} 2I &= \frac{1}{2}(1-e^{-4\pi}) \\ I &= \frac{1}{4}(1-e^{-4\pi}) \end{aligned}$$

and

$$\text{Area }(U) = \frac{1}{4}(1-e^{-4\pi})$$

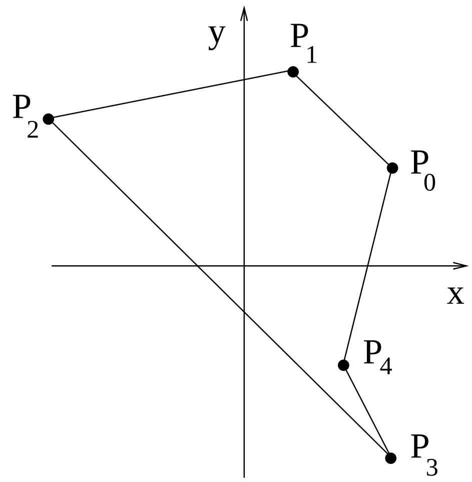
□

Problem 145: Area enclosed by a polygonal line: a computing science problem.

Compute the area of the region U enclosed by the simple closed polygonal line C with vertices at

$$P_0 = (3,2), P_1 = (1,4), P_2 = (-4,3), P_3 = (3,-4), P_4 = (2,-2).$$

Solution:



We use the line integral method; from the symmetric formula we have:

$$\text{Area}(U) = \frac{1}{2}\int_C (-ydx + xdy)$$

Let $P_i = (a_i, b_i)$, $i = 1, \ldots, 4$ and let us work first on the segment $[P_0, P_1]$:

$$\begin{aligned} X(t) &= (1-t)\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + t\begin{pmatrix} a_1 \\ b_1 \end{pmatrix},\ t \in [0,1] \\ X'(t) &= \begin{pmatrix} a_1 - a_0 \\ b_1 - b_0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \int_{[P_0,P_1]} xdy &= \int_0^1 ((1-t)a_0 + ta_1)(b_1 - b_0)dt = \\ &= (b_1 - b_0)(-\frac{(1-t)^2}{2}a_0 + \frac{t^2}{2}a_1)|_0^1 = \\ &= \frac{a_0 + a_1}{2}(b_1 - b_0) \end{aligned}$$

and by symmetry

$$\int_{[P_0,P_1]} ydx = \frac{b_0 + b_1}{2}(a_1 - a_0)$$

so

$$\frac{1}{2}\int_{[P_0,P_1]} -ydx + xdy = \frac{1}{2}(a_0 b_1 - a_1 b_0)$$

and the area is

$$\text{Area}(U) = \frac{1}{2}\sum_{i=0}^{3}(a_i b_{i+1} - a_{i+1}b_i) + \frac{1}{2}(a_4 b_0 - a_0 b_4)$$

To do the computation it is useful to put the data in a table

i	a_i	b_i
0	3	2
1	1	4
2	−4	3
3	3	−4
4	2	−2
0	3	2

and then calculate the successive minors 2×2 :

$$\frac{1}{2}(10 + 19 + 7 + 2 + 10) = 24$$

One sees the usefulness of this result when applied to a list of vertices with several hundreds of items; a trivial program will allow us to obtain the area.

□

Problem 146:

Consider the ellipse

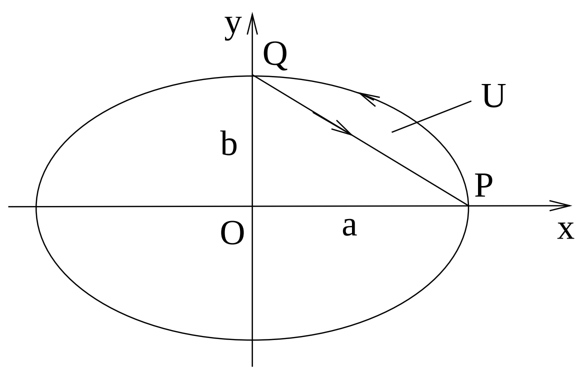
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the field $\mathbf{F}(x, y) = (-y, x)$ and the points $P = (a, 0), Q = (0, b)$.

a) Compute $\int_C \mathbf{F} \cdot d\mathbf{l}$, C being the curve consisting of the segment $[Q, P]$ and the arc PQ of the ellipse traversed in the positive sense.

b) Compute directly the area of the region U limited by C and check the result of a) using Green's theorem.

Solution:



a) The integral along the segment is

$$\begin{aligned} \gamma(t) &= (ta, (1-t)b),\ t \in [0,1] \\ \gamma'(t) &= (a, -b) \end{aligned}$$

$$\begin{aligned} \int_{[Q,P]} \mathbf{F} \cdot d\mathbf{l} &= \int_0^1 (-(1-t)b, ta) \cdot (a, -b) dt = \\ &= \int_0^1 -ab dt = -ab \end{aligned}$$

and the integral along the arc of ellipse is

$$\begin{aligned} \gamma(t) &= (a \cos t, b \sin t),\ t \in [0, \pi/2] \\ \gamma'(t) &= (-a \sin t, b \cos t) \end{aligned}$$

$$\begin{aligned} \int_{\widehat{PQ}} \mathbf{F} \cdot d\mathbf{l} &= \int_0^{\pi/2} (-b \sin t, a \cos t) \cdot (-a \sin t, b \cos t) dt = \\ &= \int_0^{\pi/2} ab\, dt = \frac{\pi}{2} ab \end{aligned}$$

and

$$\int_C \mathbf{F} \cdot d\mathbf{l} = (\frac{\pi}{2} - 1)ab$$

b) The area seeked is obtained subtracting from the area of a quarter of the ellipse the area of the triangle OPQ , that is $\frac{\pi ab}{4} - \frac{ab}{2} = (\frac{\pi}{4} - \frac{1}{2})ab$. On another hand the symmetric formula for area calculation tells us that it should be half the line integral, and so it happens. The result in a) is checked.

□

Problem 147:

Green's theorem is valid in some more general situations than the one described above. For instance:

a) Let C and C' be two piecewise $\mathcal{C}^1$ simple closed curves. Assume C' contained in the bounded region of C. Show that if $\mathbf{F} \in \mathcal{C}^1(D \cup \partial D)$, D being the region in between C' and C, then we can apply Green's theorem to D; explain the orientations.

b) Let C be the ellipse $\gamma(t) = (2\cos t, 3\sin t)$, $t \in [0, 2\pi]$; compute:

$$\int_C (\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2})dy$$

c) Let C be a closed curve; the index of C respect to $\mathbf{0} = (0,0)$ is:

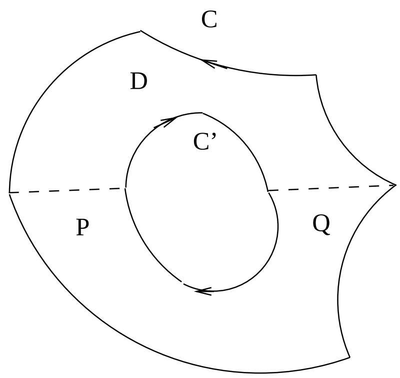
$$n(C;\mathbf{0}) = \frac{1}{2\pi}\int_C (\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2})dy$$

Show that if C is a simple closed curve positively oriented and D its bounded region then:

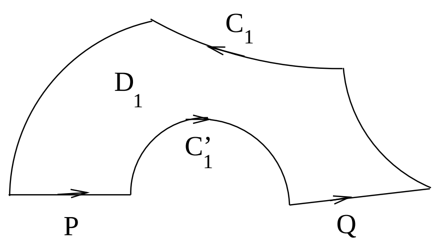
$$n(C;\mathbf{0}) = \begin{cases} 1 & \text{if } \mathbf{0} \in D \\ 0 & \text{if } \mathbf{0} \notin D \cup \partial D \end{cases}$$

Solution:

a) Orient both curves as in the figure, that is so that D is on the left of both curves:



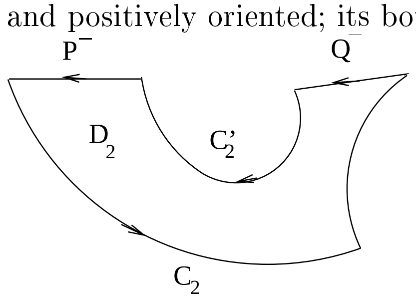
Consider the cuts P and Q. Now the curve $C_1+P+C_1'+Q$ is piecewise $\mathcal{C}^1$, simple, closed and positively oriented; its bounded region D_1 lies on the left:



Applying Green's theorem we obtain:

$$\int_{C_1+P+C_1'+Q} \mathbf{F}\cdot d\mathbf{l} = \int\int_{D_1} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy$$

Analogously the curve $C_2+Q^-+C_2'+P^-$ is piecewise $\mathcal{C}^1$, simple, closed and positively oriented; its bounded region D_2 lies on the left:



Apply again Green's theorem to obtain

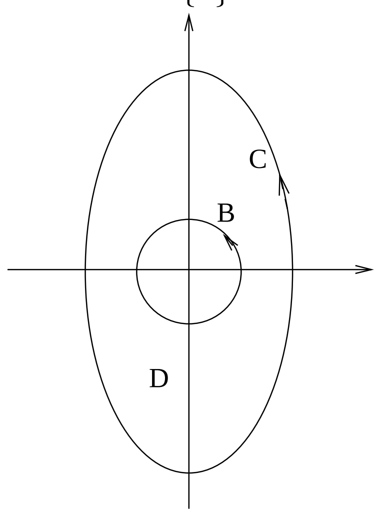
$$\int_{C_2+Q^-+C_2'+P^-} \mathbf{F}\cdot d\mathbf{l} = \int\int_{D_2} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy$$

Adding term by term the two preceding equalities and taking into account that the line integral vanishes when integrated on two opposed curves (that is: along the cuts), we obtain the following version of Green's theorem:

$$\int_{C+C'} \mathbf{F} \cdot d\mathbf{l} = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) dxdy$$

Remark that the orientation assigned to C' leaves the region D to the left.

b) The field $\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ satisfies the mixed derivatives condition in $\mathbb{R}^2 - \{\mathbf{0}\}$. Let B be the unit circumference positively oriented:



Applying Green's theorem to the region D:

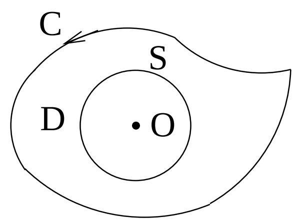
$$\int_{C+B^-} \mathbf{F} \cdot d\mathbf{l} = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) dxdy = 0 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{l} = \int_B \mathbf{F} \cdot d\mathbf{l}$$

but the integral on the unit circumference B is easy

$$\begin{aligned} \int_B \mathbf{F} \cdot d\mathbf{l} &= \int_0^{2\pi} (\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}\cos t) dt = \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi \end{aligned}$$

Notice we have been able to change the integration path because in the region D the mixed derivatives condition is satisfied.

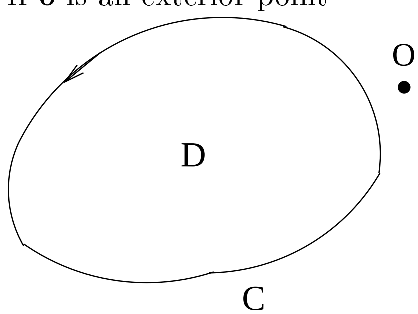
c) If $\mathbf{0} \in D$ let S be a circumference with center at $\mathbf{0}$ and contained in D:



By the same reason we had in b) we can change the integration path:

$$\frac{1}{2\pi}\int_C (\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy) = \frac{1}{2\pi}\int_S (\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy) = 1$$

If $\mathbf{0}$ is an exterior point



we can apply Green's theorem to the region D, because the field $\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ is $\mathcal{C}^1$ in D. We have

$$\int_C (\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy) = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy = \int\int_D 0dxdy = 0$$

Obviously we can define the index of C respect to a point $P = (a,b)$ thus

$$n(C;P) = \frac{1}{2\pi}\int_C \frac{-(y-b)dx + (x-a)dy}{(x-a)^2 + (y-b)^2}$$

□

Problem 148:

a) Compute the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}, a > 0$.

b) Compute the area enclosed.

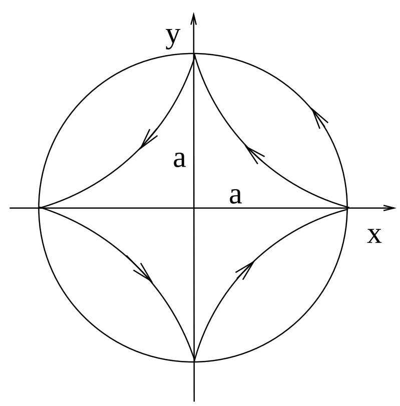
c) Let $\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$; show that the integrals of $\mathbf{F}$ along the astroid and along the circumference

$$x^2 + y^2 = a^2$$

traversed in the same sense coincide. What is the common value?

Solution:

A figure:



a) A parametrization of the astroid is

$$\begin{aligned} \gamma(t) &= (a\cos^3 t, a\sin^3 t), 0 \le t \le 2\pi \\ \gamma'(t) &= (-3a\cos^2 t \sin t, 3a\sin^2 t \cos t) \\ \mid \gamma'(t) \mid &= 3a\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} = \\ &= 3a \mid \cos t \sin t \mid \end{aligned}$$

and its length is:

$$\begin{aligned} L &= 3a\int_0^{2\pi} \mid \cos t \sin t \mid dt = 4 \cdot 3a\int_0^{\pi/2} \cos t \sin t dt = \\ &= 12a\frac{\sin^2 t}{2} \mid_0^{\pi/2} = 6a \end{aligned}$$

b) We use the symmetric formula (see p.265):

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_\gamma (-ydx + xdy) = \\
&= \int_0^{2\pi} (-a\sin^3 t)(-3a\cos^2 t\sin t) + a\cos^3 t\, 3a\sin^2 t\cos t dt = \\
&= 4\cdot 3a^2 \int_0^{\pi/2} (\sin^4 t\cos^2 t + \cos^4 t\sin^2 t)dt = \\
&= 12a^2 \int_0^{\pi/2} \sin^2 t\cos^2 t dt = \\
&= 12a^2 \int_0^{\pi/2} \frac{1-\cos 2t}{2}\cdot\frac{1+\cos 2t}{2} dt = \\
&= 3a^2 \int_0^{\pi/2} (1-\cos^2 2t)dt = \\
&= 3a^2(\pi/2 - \int_0^{\pi/2} \frac{1+\cos 4t}{2} dt) = 3a^2(\pi/2 - \pi/4) = \\
&= \frac{3\pi}{4}a^2
\end{aligned}$$

But we can as well use one of the other formulae:

$$\begin{aligned}
\text{Area}(U) &= \int_\gamma xdy \\
&= \int_0^{2\pi} a\cos^3 t(3a\sin^2 t\cos t)dt = 3a^2 \int_0^{2\pi} \cos^4 t\sin^2 t dt = \\
&= 3a^2 \int_0^{2\pi} \frac{(1+\cos 2t)^2}{4}\frac{1-\cos 2t}{2} dt = \\
&= \frac{3a^2}{8}\int_0^{2\pi} (1-\cos^2 2t)(1+\cos 2t)dt = \\
&= \frac{3a^2}{8}\int_0^{2\pi} (1 - \frac{1+\cos 4t}{2} + \cos 2t - (1-\sin^2 2t)\cos 2t)dt = \\
&= \frac{3\pi}{8}a^2
\end{aligned}$$

c) As the mixed derivatives condition is satisfied both line integrals coincide by the same reason we had in b) of the preceding problem. Then we can integrate along a circumference, let's say the unit one:

$$\begin{aligned}\int_\gamma \mathbf{F}\cdot d\mathbf{l} &= \int_C \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\cdot d\mathbf{l} = \\ &= \int_0^{2\pi} (-\sin\theta, \cos\theta)\cdot(-\sin\theta,\cos\theta)d\theta = \\ &= \int_0^{2\pi} 1 d\theta = 2\pi\end{aligned}$$

□

Problem 149:

Let $D \subset \mathbb{R}^2$ be the bounded region of a piecewise $\mathcal{C}^1$, simple, regular closed curve C positively oriented. If $\mathbf{F} = (P, Q) \in \mathcal{C}^1(D \cup \partial D)$ (where $C = \partial D$, the boundary of D) and $\mathcal{F} = (P, Q, 0)$, prove the following equalities:

a)
$$\int_{\partial D} \mathbf{F}\cdot d\mathbf{l} = \int\int_D (\partial_x Q - \partial_y P) dxdy$$

b)
$$\int_{\partial D} \mathbf{F}\cdot d\mathbf{l} = \int\int_D (\text{rot } \mathcal{F})_z dxdy$$

c)
$$\int_{\partial D} \mathbf{F}\cdot \mathbf{n} dl = \int\int_D \text{div } \mathbf{F}\, dxdy$$

d)
$$\int_C \mathbf{r}\cdot\mathbf{n} dl = 2\text{Area } (D)$$

Solution:

a) This is Green's theorem.

b) $(\text{rot}\mathcal{F})_z = \partial_x Q - \partial_y P$ and it is again Green's theorem.

c) Call $\mathbf{n}$ the normal vector to C pointing to the exterior of D. If

$$\gamma : [a, b] \to \mathbb{R}^2,\ \gamma(s) = (x(s), y(s))$$

is an arc-length parametrization of C, that normal vector must be $\mathbf{n} = (y', -x')$ or $\mathbf{n} = (-y', x')$. We have to choose the one that makes $(\mathbf{n}, \gamma')$ a positive basis. As we have

$$\det \begin{pmatrix} y' & x' \\ -x' & y' \end{pmatrix} = (y')^2 + (x')^2 > 0$$

we see that $\mathbf{n} = (y', -x')$. Then

$$\begin{aligned} \int_{\partial D} \mathbf{F} \cdot \mathbf{n} dl &= \int_a^b (P, Q) \cdot (y', -x') ds = \int_a^b (-Qx' + Py') ds = \\ &= \int_{\partial D} (-Q, P) \cdot d\mathbf{l} = \int \int_D (\partial_x P - \partial_y(-Q)) dx dy = \\ &= \int \int_D \text{div } \mathbf{F} \, dx dy \end{aligned}$$

d)

$$\int_C \mathbf{r} \cdot \mathbf{n} dl = \int \int_D \text{div } \mathbf{r} \, dx dy = \int \int_D 2 dx dy = 2\text{Area } (D)$$

The result in b) is Stokes's theorem in the plane and the result in c) is the divergence theorem in the plane.

□

6.2 Stokes theorem

Surfaces with boundary

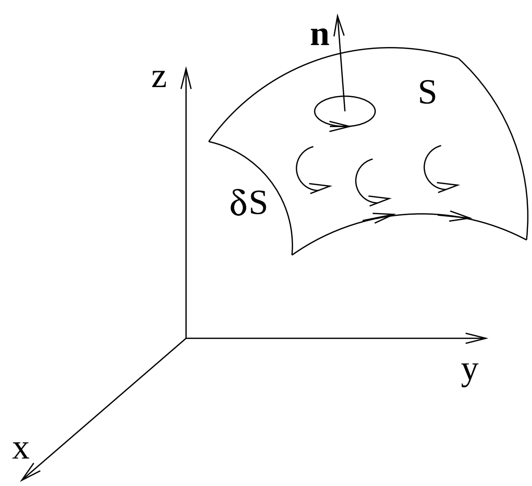
$\boxed{\mathbb{T}}$ Let S be a surface that has as boundary a curve C; we write ∂S for the boundary. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field in an open set $U \subset \mathbb{R}^3$ that contains S and ∂S. Then

$$\boxed{\int \int_S \text{rot } \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{l}}$$

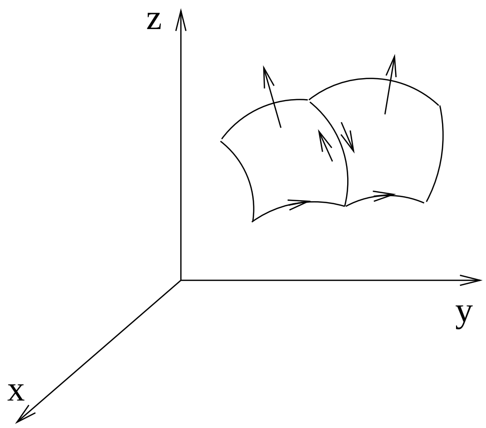
In words: the flux of **F**'s rotational through S equals the circulation of **F** along the boundary.

There is an important point about orientations since both integrals depend on them; the orientation of the boundary has to match that of the surface. Let us give some intuitive rules about that.

Assume S oriented by a continuous unit normal field **n** that gives a positive sense of rotation in S through an orientation of the whole space (see p.244). Imagine small dust like particles distributed on S and ∂S; on S they are dragged by the positive turning sense. In ∂S the wind produced in S moves the particles thus fixing an orientation for ∂S; it is called the induced orientation. A figure:



We can as well apply the theorem to surfaces glued through their boundaries. But one has to be careful with orientations in the sense just explained. For instance:



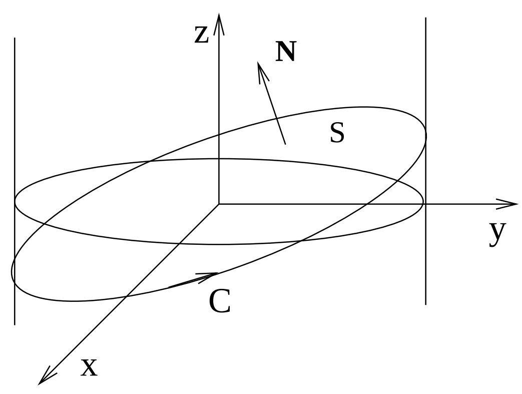
□

Problem 150: Checking.

Check Stokes theorem applied to the field $\mathbf{F}(x, y, z) = (x, x + y, x + y + z)$ and the closed curve $C = \{(x, y, z) : x^2 + y^2 = R^2, x + y = z, R > 0\}$.

Solution:

C is the intersection of a cylinder and a plane; we see geometrically that it can be the boundary of many surfaces. According to the theorem any one of them will do; we choose as S the region of the plane $x + y = z$ limited by C:



A parametrization of S and its associated normal vector are

$$\alpha(r, \theta) = (r \cos\theta, r \sin\theta, r(\cos\theta + \sin\theta)), (r, \theta) \in (0, R) \times (0, 2\pi)$$

$$\begin{aligned}\partial_r\alpha &= (\cos\theta, \sin\theta, \cos\theta + \sin\theta)\\ \partial_\theta\alpha &= (-r\sin\theta, r\cos\theta, r(-\sin\theta + \cos\theta))\\ \mathbf{N} &= (-r, -r, r)\end{aligned}$$

Orient S with that normal field. To check the theorem we need a parametrization of $\partial S = C$ such as

$$\begin{aligned}\gamma(\theta) = \alpha(R,\theta) &= (R\cos\theta, R\sin\theta, R(\cos\theta + \sin\theta)), \theta \in (0, 2\pi)\\ \gamma'(\theta) &= (-R\sin\theta, R\cos\theta, R(\cos\theta - \sin\theta))\end{aligned}$$

that has the induced orientation. We can see this geometrically: the screwdriver rule tells us that the normal generated by the way C 'turns around' is in the $\mathbf{N}$ direction. Algebraically this amounts to see that

$$\gamma(\theta) \times \gamma'(\theta) \cdot \mathbf{N} > 0$$

wich is easily checked. Or we can say that γ is the restriction of α and $\gamma' = \partial_\theta\alpha$; then γ is already well oriented because:

$$\gamma(\theta) \times \gamma'(\theta) \cdot \mathbf{N} = (\alpha(R,\theta) \times \partial_\theta\alpha) \cdot \mathbf{N} = (R\partial_r\alpha \times \partial_\theta\alpha) \cdot \mathbf{N} = RN^2 > 0$$

- Flux of rot$\mathbf{F}$ through S:

$$\begin{aligned}\text{rot}\mathbf{F} &= (1, -1, 1)\\ \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^R \int_0^{2\pi} (1, -1, 1) \cdot (-r, -r, r) dr d\theta = \int_0^R \int_0^{2\pi} r dr d\theta = \pi R^2\end{aligned}$$

- Circulation of $\mathbf{F}$ along C:

$$\int_0^{2\pi} R(\cos\theta, \cos\theta + \sin\theta, 2(\cos\theta + \sin\theta)) \cdot R(-\sin\theta, \cos\theta, \cos\theta - \sin\theta) d\theta \;=$$

$$\begin{aligned}&= R^2 \int_0^{2\pi} (3\cos^2\theta - 2\sin^2\theta) d\theta =\\ &= R^2 \int_0^{2\pi} (3\frac{1 + \cos 2\theta}{2} - 2\frac{1 - \cos 2\theta}{2}) d\theta =\\ &= R^2(3\pi - 2\pi) = \pi R^2\end{aligned}$$

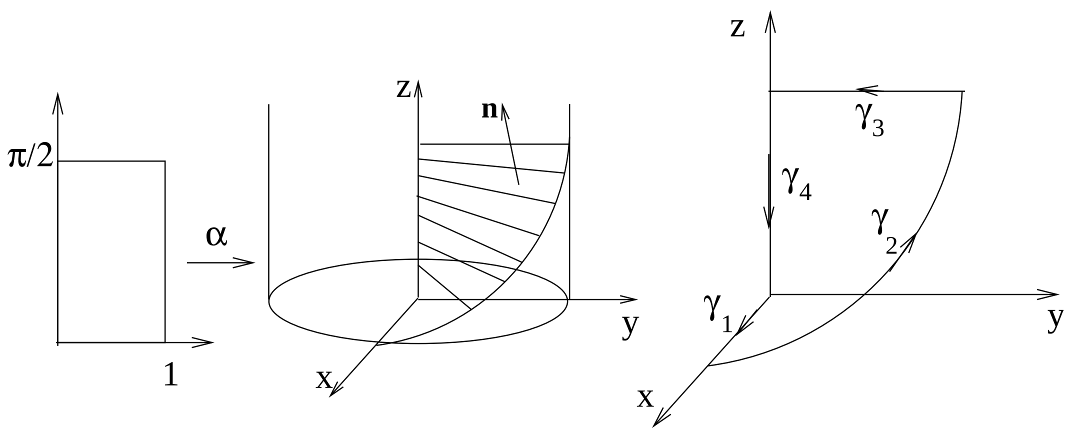
□

Problem 151:

Check Stokes theorem applied to the field $\mathbf{F}(x,y,z) = (x,y,z)$ and the surface S parametrized by $\alpha(r,\theta) = (r\cos\theta, r\sin\theta, \theta), (r,\theta) \in (0,1) \times (0,\pi/2)$ (helicoidal ramp).

Solution:

A figure:



- We orient S by the associated normal vector:

$$\begin{aligned} \partial_r\alpha &= (\cos\theta, \sin\theta, 0) \\ \partial_\theta\alpha &= (-r\sin\theta, r\cos\theta, 1) \\ \mathbf{N} &= (\sin\theta, -\cos\theta, r) \end{aligned}$$

  The flux of the rotational through the surface is

$$\text{rot}\mathbf{F} = 0 \Rightarrow \int\int_S \text{rot}\mathbf{F} \cdot d\mathbf{S} = 0$$

- Circulation of $\mathbf{F}$ along the boundary:
  To parametrize the boundary according to the orientation of S observe that the normal vector points towards the upper half space of the plane.

We see in the figure the induced orientation in the boundary. Then

$$\begin{aligned}
\gamma_1(t) &= (t,0,0), 0 \le t \le 1 \\
\gamma_2(\theta) &= (\cos\theta, \sin\theta, \theta), 0 \le \theta \le \pi/2 \\
\gamma_3(t) &= (0, 1-t, \pi/2), 0 \le t \le 1 \\
\gamma_4(\theta) &= (0,0,\pi/2-\theta), 0 \le \theta \le \pi/2
\end{aligned}$$

The circulation is the sum of the contributions along each piece:

$$\begin{aligned}
\int_{\gamma_1} \mathbf{F}\cdot d\mathbf{l} &= \int_0^1 (t,0,0)\cdot(1,0,0)dt = \int_0^1 tdt = \frac{1}{2} \\
\int_{\gamma_2} \mathbf{F}\cdot d\mathbf{l} &= \int_0^{\pi/2} (\cos\theta, \sin\theta, \theta)\cdot(-\sin\theta, \cos\theta, 1)d\theta = \int_0^{\pi/2} \theta d\theta = \frac{\pi^2}{8} \\
\int_{\gamma_3} \mathbf{F}\cdot d\mathbf{l} &= \int_0^1 (0, 1-t, \pi/2)\cdot(0,-1,0)dt = \int_0^1 (t-1)dt = -\frac{1}{2} \\
\int_{\gamma_4} \mathbf{F}\cdot d\mathbf{l} &= \int_0^{\pi/2} (0,0,\pi/2-\theta)\cdot(0,0,-1)d\theta = \\
&= \int_0^{\pi/2} (\theta - \frac{\pi}{2})d\theta = \frac{\pi^2}{8} - \frac{\pi^2}{4} = -\frac{\pi^2}{8}
\end{aligned}$$

We obtain

$$\int_\gamma \mathbf{F}\cdot d\mathbf{l} = 0$$

and we have checked Stokes theorem.

□

Problem 152:

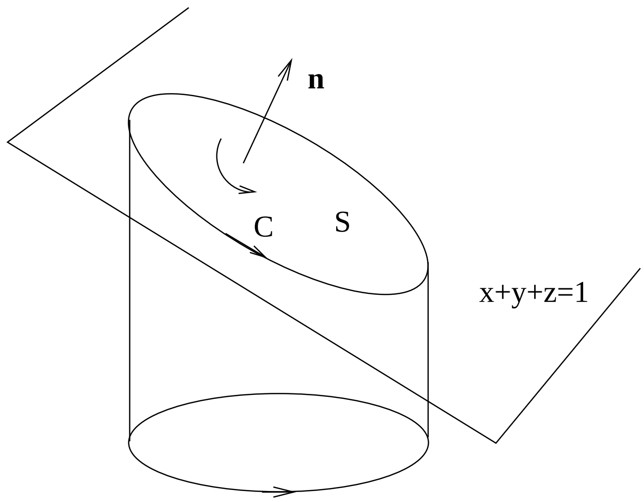
Evaluate the following integral using Stokes theorem

$$\int_C -y^3dx + x^3dy - z^3dz$$

C being the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, with an orientation such that its projection on $z=0$ turns anticlockwise (usual positive sense) as seen from $z>0$.

Solution:

A figure:



Let S be the region of the plane limited by the cylinder; parametrize S by

$$\begin{array}{rcl} \alpha : D = D(\mathbf{0};1) & \to & \mathbb{R}^3 \\ (x,y) & \mapsto & (x,y,1-x-y) \end{array}$$

that has as associated normal vector $\mathbf{N} = (1,1,1)$. We see geometrically that ∂S has the induced orientation and we can apply Stokes theorem. Let $\mathbf{F}(x,y,z) = (-y^3, x^3, -z^3)$ that has rot $\mathbf{F} = (0,0,3(x^2+y^2))$. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{l} &= \int\int_{D(\mathbf{0};1)} (0,0,3(x^2+y^2)) \cdot (1,1,1) dxdy = \\ &= 3\int_0^1 \int_0^{2\pi} r^2 r dr d\theta = 3\frac{1}{4}2\pi = \frac{3}{2}\pi \end{aligned}$$

□

Problem 153:

Let C be the curve intersection of the surfaces

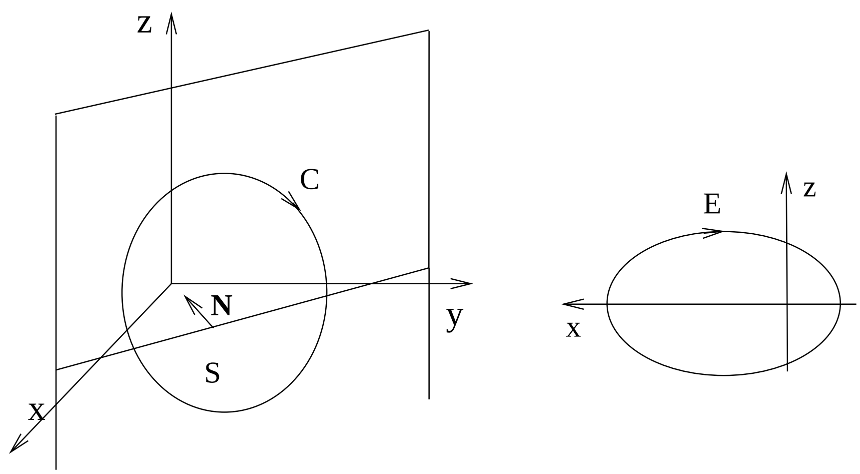
$$x+y=2b,\ x^2+y^2+z^2 = 2b(x+y), b>0$$

oriented in the anticlockwise sense when seen from the origin. Use Stokes theorem to evaluate

$$I = \int_C y dx + z dy + x dz$$

Solution:

C is the circumference intersection of the plane parallel to the Oz axis $x+y = 2b$ and the sphere $x^2 + y^2 + z^2 = 2b(x+y)$ with center $(b, b, 0)$ and radius $\sqrt{2}b$. In the figure we can see C with the given orientation, and its projection on the plane $y = 0$:



Eliminating y from the system giving the intersection we obtain the projecting cylinder on the the plane $y = 0$:

$$\left.\begin{array}{rcl} x+y & = & 2b \\ x^2+y^2+z^2 & = & 2b(x+y) \end{array}\right\} \Rightarrow \begin{array}{l} x^2 + (2b-x)^2 + z^2 = 4b^2 \\ 2x^2 - 4bx + z^2 = 0 \end{array}$$

and completing squares we obtain the ellipse E

$$\frac{(x-b)^2}{b^2} + \frac{z^2}{(\sqrt{2}b)^2} = 1, y = 0$$

If S is the region in the plane that has C as boundary we have the parametrization

$$\begin{aligned} \alpha(x,z) &= (x, 2b-x, z), (x,z) \in \text{interior } (E) \\ \mathbf{N} &= (-1,-1,0) \end{aligned}$$

$\mathbf{N}$ gives S the right orientation and we can apply Stokes theorem to the field $\mathbf{F}(x,y,z) = (y,z,x)$ whose rotational is rot $\mathbf{F} = (-1,-1,-1)$; let $\mathcal{R}(E)$ be the region enclosed by E:

$$\begin{aligned} I &= \int\int_S \text{rot } \mathbf{F} \cdot d\mathbf{S} = \\ &= \int\int_{\mathcal{R}(E)} (-1,-1,-1)\cdot(-1,-1,0)dxdy = \\ &= 2\int\int_{\mathcal{R}(E)} dxdy = 2\cdot \text{Area } \mathcal{R}(E) = 2\cdot \pi b\sqrt{2}b = 2\sqrt{2}\pi b^2 \end{aligned}$$

□

Problem 154:

Let $f, g \in \mathcal{C}^1(\mathbb{R}^2), h \in \mathcal{C}^1(\mathbb{R}^3)$ and $\mathbf{F}(x,y,z) = (f(x,z)+ay, g(y,z)+bx, h(x,y,z))$. Let C be a simple closed curve contained in $z=0$ and oriented by leaving its bounded region U to the left (the plane$z=0$ being oriented by $(0,0,1)$). Show:

$$\int_C \mathbf{F}\cdot d\mathbf{l} = (b-a)\text{Area } (U)$$

Solution:

Apply Stokes theorem to U oriented by $(0,0,1)$:

$$\begin{aligned} \text{rot } \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f(x,z)+ay & g(y,z)+bx & h(x,y,z) \end{pmatrix} = \\ &= (\partial_y h - \partial_z g, \partial_z f - \partial_x h, b-a) \end{aligned}$$

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{l} &= \int\int_U \text{rot } \mathbf{F}\cdot d\mathbf{S} = \int\int_U \text{rot } \mathbf{F}\cdot(0,0,1)dS = \\ &= (b-a)\int\int_U dS = (b-a)\text{Area } (U) \end{aligned}$$

□

Problem 155:

Let S be a surface with boundary the curve C and assume both oriented in a compatible form. For each $\mathbf{p} = (a, b, c) \notin S \cup C$ consider the gravitational field of a unit mass point at $\mathbf{p}$

$$\mathbf{g}(x, y, z) = -\frac{1}{r^3}(x-a, y-b, z-c), r = \mid (x-a, y-b, z-c) \mid,$$

and the flux through S as a function of $\mathbf{p}$

$$\phi(\mathbf{p}) = \int\int_S \mathbf{g} \cdot d\mathbf{S}.$$

Show that

$$\nabla\phi = \int_C \frac{(\mathbf{x} - \mathbf{p}) \times d\mathbf{x}}{r^3}$$

Solution:

The first component of the gradient is

$$\begin{aligned}(\nabla\phi)_1 &= \tfrac{\partial\phi}{\partial a} = \frac{\partial}{\partial a}(\int\int_S -\frac{1}{r^3}(x-a, y-b, z-c) \cdot d\mathbf{S}) = \\ &= -\int\int_S (\frac{\partial}{\partial a}(\frac{x-a}{r^3}), \frac{\partial}{\partial a}(\frac{y-b}{r^3}), \frac{\partial}{\partial a}(\frac{z-a}{r^3})) \cdot d\mathbf{S}\end{aligned}$$

We compute the derivatives

$$\frac{\partial}{\partial a}(\frac{x-a}{r^3}) = \frac{-r^3 - 3r^2\frac{a-x}{r}(x-a)}{r^6} = \frac{-r^2 + 3(x-a)^2}{r^5}$$

$$\frac{\partial}{\partial a}(\frac{y-b}{r^3}) = -\frac{3r^2\frac{a-x}{r}(y-b)}{r^6} = \frac{3(x-a)(y-b)}{r^5}$$

$$\frac{\partial}{\partial a}(\frac{z-c}{r^3}) = \frac{3(x-a)(z-c)}{r^5}$$

and the first component of the gradient can be written

$$(\nabla\phi)_1 = -\int\int_S \frac{1}{r^5}(-r^2 + 3(x-a)^2, 3(x-a)(y-b), 3(x-a)(z-c)) \cdot d\mathbf{S}$$

On the other side

$$(\mathbf{x}-\mathbf{p})\times d\mathbf{x} = ((y-b)dz-(z-c)dy,(z-c)dx-(x-a)dz,(x-a)dy-(y-b)dx)$$

$$(\int_C \frac{(\mathbf{x}-\mathbf{p})\times d\mathbf{x}}{r^3})_1 = \int_C \frac{1}{r^3}((y-b)dz-(z-c)dy)$$

Now we use Stokes theorem applied to the field $\mathbf{F} = \frac{1}{r^3}(0,-(z-c),y-b,)$ whose rotational is

$$\text{rot } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & -\frac{z-c}{r^3} & \frac{y-b}{r^3} \end{pmatrix} =$$

$$= (\frac{2r^2-3(y-b)^2-3(z-c)^2}{r^5}, \frac{3(x-a)(y-b)}{r^5}, \frac{3(x-a)(z-c)}{r^5}) =$$

$$= (\frac{-r^2+3(x-a)^2}{r^5}, \frac{3(x-a)(y-b)}{r^5}, \frac{3(x-a)(z-c)}{r^5})$$

and

$$(\int_C \frac{(\mathbf{x}-\mathbf{p})\times d\mathbf{x}}{r^3})_1 = \int\int_S \frac{1}{r^5}(-r^2+3(x-a)^2, 3(x-a)(y-b), 3(x-a)(z-c))\cdot d\mathbf{S}$$

that coincides with the first component of $\nabla\phi$. The other components are checked in the same way.

□

Problem 156:

a) Compute directly the flux of the field $\mathbf{F}(x,y,z) = (-x,0,z)$ through the surface $S = \{(x,y,z) : x^2+y^2+z^2 = 1, z > \frac{1}{2}\}$.

b) Find a vector potential for $\mathbf{F}$ of the form $\mathbf{A}(x,y,z) = (X,0,Z)$.

c) Check the result in a) by means of Stokes theorem.

Solution:

a) Parametrize S:

$$\alpha(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), 0 \le \varphi \le \pi/3, 0 \le \theta \le 2\pi$$

with associated normal vector

$$\begin{aligned}
\frac{\partial\alpha}{\partial\varphi} &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \\
\frac{\partial\alpha}{\partial\theta} &= (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, \cos\varphi) \\
\mathbf{N} &= (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)
\end{aligned}$$

We orient S by $\mathbf{N}$. The flux is:

$$\phi = \int_0^{2\pi}\int_0^{\pi/3}(-\sin\varphi\cos\theta, 0, \cos\varphi)\cdot(\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)d\varphi d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/3}(-\sin^3\varphi\cos^2\theta + \sin\varphi\cos^2\varphi)d\varphi d\theta =$$

$$= -\pi\int_0^{\pi/3}\sin^3\varphi d\varphi + 2\pi\int_0^{\pi/3}\sin\varphi\cos^2\varphi d\varphi = \frac{3}{8}\pi$$

b) $\mathbf{F}$ may have a vector potential:

$$\text{div } \mathbf{F} = \frac{\partial(-x)}{\partial x} + \frac{\partial z}{\partial z} = -1 + 1 = 0$$

$\mathbf{A} = (X, Y, Z)$ must satisfy (see p.159)

$$\text{rot } \mathbf{A} = \mathbf{F} : \begin{cases} \partial_y Z &= -x \\ \partial_z X - \partial_x Z &= 0 \\ -\partial_y X &= z \end{cases}$$

from first and third equations

$$\begin{aligned}
Z &= -xy + \varphi(x,z) \\
X &= -yz + \psi(x,z)
\end{aligned}$$

and substituting into the second

$$-y + \frac{\partial \psi}{\partial z} = -y + \frac{\partial \varphi}{\partial x}$$

We choose $\varphi = \psi = 0$ and obtain

$$\mathbf{A} = (-yz, 0, -xy)$$

c) Using $\mathbf{A}$ we have

$$\phi = \int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S \text{rot } \mathbf{A} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{A} \cdot d\mathbf{l}$$

The vector $\mathbf{N}$ points to the exterior of the sphere and we orient ∂S accordingly. The following parametrization gives the right orientation:

$$\begin{aligned} \gamma(\theta) &= (\sqrt{3}/2 \cos\theta, \sqrt{3}/2 \sin\theta, 1/2) \\ \gamma'(\theta) &= (-\sqrt{3}/2 \sin\theta, \sqrt{3}/2 \cos\theta, 0) \end{aligned}$$

and then

$$\phi = \int_{\partial S} \mathbf{A} \cdot d\mathbf{l} =$$

$$\begin{aligned} &= \int_0^{2\pi} (-\frac{1}{2}\frac{\sqrt{3}}{2} \sin\theta, 0, -\frac{1}{2}\frac{\sqrt{3}}{2} \cos\theta) \cdot (-\sqrt{3}/2 \sin\theta, \sqrt{3}/2 \cos\theta, 0) d\theta = \\ &= \int_0^{2\pi} \frac{3}{8} \sin^2\theta d\theta = \frac{3}{8}\pi \end{aligned}$$

□

Problem 157: Flux and circulation.

a) Compute the flux of the field $\mathbf{F}(x, y, z) = (x^2, 0, -2xz)$ through the surface

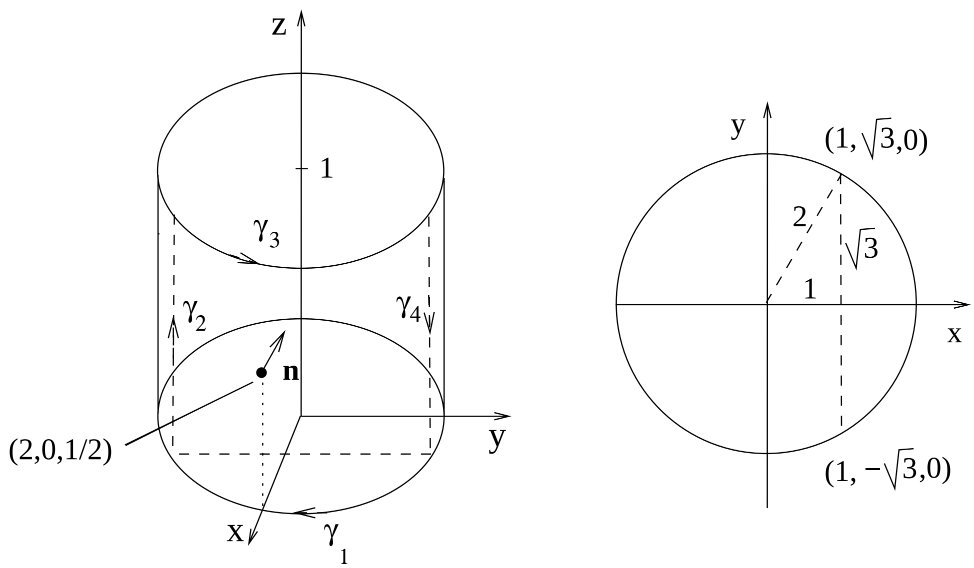
$$S = \{(x, y, z) : x^2 + y^2 = 4, x \geq 1, 0 \leq z \leq 1\},$$

oriented in such a form that the normal at point $P = (2, 0, 1/2)$ is $\mathbf{n} = (-1, 0, 0)$.

b) Using Stokes theorem check the result in a).

Solution:

A figure



a) Parametrize S by means of cylindrical coordinates

$$\begin{aligned} \alpha(\theta, z) &= (2\cos\theta, 2\sin\theta, z), \\ (\theta, z) &\in D = [-\pi/3, \pi/3] \times [0, 1] \end{aligned}$$

The associated normal vector is $\mathbf{N} = 2(\cos\theta, \sin\theta, 0)$; now the point $P = (2, 0, 1/2)$ has the parameters $(0, 1/2)$ and the normal vector there is $\mathbf{N}(0, 1/2) = 2(1, 0, 0)$ which is opposed to $\mathbf{n}$. We can proceed with the parametrization we have and change the sign at the end or reparametrize by

$$\beta(z, \theta) = \alpha(\theta, z)$$

that has an associated normal vector satifying $\mathbf{N}_\beta = -\mathbf{N}_\alpha$ and will have the good orientation. We use the first procedure:

$$\begin{aligned}
\int\int_D \mathbf{F}\cdot\mathbf{N}_\alpha d\theta dz &= \int\int_D (4\cos^2\theta, 0, -4(\cos\theta)z)\cdot 2(\cos\theta,\sin\theta,0)d\theta dz = \\
&= 8\int\int_D \cos^3\theta d\theta dz = 8\int\int_D (1-\sin^2\theta)\cos\theta d\theta dz = \\
&= 8\quad (\sin\theta - \frac{\sin^3\theta}{3})|_{-\pi/3}^{\pi/3} = 8(2\frac{\sqrt{3}}{2} - 2\frac{3\frac{\sqrt{3}}{8}}{3}) = 6\sqrt{3}
\end{aligned}$$

So with the given orientation we have

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = -6\sqrt{3}$$

b) $\mathbf{F}$ has a vector potential since div $\mathbf{F} = 2x - 2x = 0$. Using the method of p.156 we have

$$\begin{aligned}
\mathbf{A}(x,y,z) &= \int_0^1 \mathbf{F}(t\mathbf{x})\times t\mathbf{x}\,dt = \\
&= \int_0^1 (t^2x^2, 0, -2t^2xz)\cdot(tx,ty,tz)dt = \\
&= (2xyz, -3x^2z, x^2y)\int_0^1 t^3 dt = \\
&= \frac{1}{4}(2xyz, -3x^2z, x^2y)
\end{aligned}$$

To check the result obtained in a) we shall integrate $\mathbf{A}$ along the four arcs in the figure that have the orientation induced by that of the surface. Then:

$$\begin{aligned}
\gamma_1(\theta) &= (2\cos\theta, -2\sin\theta, 0), \theta\in[-\pi/3,\pi/3] \\
\gamma_1'(\theta) &= (-2\sin\theta, -2\cos\theta, 0)
\end{aligned}$$

$$\int_{\gamma_1}\mathbf{A}\cdot d\mathbf{l} = \int_{-\pi/3}^{\pi/3}\frac{1}{4}(0,0,-8\cos^2\theta\sin\theta)\cdot(-2\sin\theta,-2\cos\theta,0)d\theta = 0$$

$$\begin{aligned}
\gamma_2(t) &= (1, -\sqrt{3}, t), t\in[0,1] \\
\gamma_2'(t) &= (0,0,1)
\end{aligned}$$

$$\int_{\gamma_2} \mathbf{A}\cdot d\mathbf{l} = \int_0^1 \frac{1}{4}(\ldots,\ldots,-\sqrt{3})\cdot(0,0,1)dt = -\frac{\sqrt{3}}{4}$$

$$\begin{aligned}\gamma_3(\theta) &= (2\cos\theta, 2\sin\theta, 1), \theta\in[-\pi/3,\pi/3]\\ \gamma_3'(\theta) &= (-2\sin\theta, 2\cos\theta, 0)\end{aligned}$$

$$\int_{\gamma_3} \mathbf{A}\cdot d\mathbf{l} =$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{4}(8\sin\theta\cos\theta, -12\cos^2\theta, -8\cos^2\theta\sin\theta)\cdot(-2\sin\theta, 2\cos\theta, 0)d\theta =$$

$$\begin{aligned}&= \frac{1}{4}\int_{-\pi/3}^{\pi/3}(-16\sin^2\theta\cos\theta - 24\cos^3\theta)d\theta =\\ &= -4\frac{\sin^3\theta}{3}\,|_{-\pi/3}^{\pi/3} -6(\sin\theta - \frac{\sin^3\theta}{3})\,|_{-\pi/3}^{\pi/3} =\\ &= 2\frac{\sin^3\theta}{3}\,|_{-\pi/3}^{\pi/3} -6\sin\theta\,|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{2} - 6\sqrt{3} = -\frac{11\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\gamma_4(t) &= (1, \sqrt{3}, 1-t), t\in[0,1]\\ \gamma_4'(t) &= (0,0,-1)\end{aligned}$$

$$\int_{\gamma_4} \mathbf{A}\cdot d\mathbf{l} = \int_0^1 \frac{1}{4}(\ldots,\ldots,\sqrt{3})\cdot(0,0,-1)dt = -\frac{\sqrt{3}}{4}$$

We obtain

$$\int_{\partial S} \mathbf{A}\cdot d\mathbf{l} = -2\frac{\sqrt{3}}{4} - \frac{11\sqrt{3}}{2} = -6\sqrt{3}$$

□

Problem 158:

Let $u, v \in \mathcal{C}^2(\mathbb{R}^3)$ scalar fields.

a) Show that $\nabla u \times \nabla v$ has a vector potential.

b) Is any of the following fields such a vector potential?

   i) $\nabla(uv)$.

   ii) $u\nabla v$.

   iii) $v\nabla u$.

c) If $u(x,y,z) = x^3 - y^3 + z^2$ and $v(x,y,z) = x + y + z$, compute

$$I = \int\int_S \nabla u \times \nabla v \cdot d\mathbf{S},$$

S being the upper hemisphere of the unit sphere oriented through the exterior noisrmal.

Solution:

a) Let us show that $\nabla u \times \nabla v$ has a vanishing divergence

$$\begin{aligned} \text{div }(\nabla u \times \nabla v) &= \text{div }(u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x) = \\ &= (u_{yx} v_z + u_y v_{zx} - u_{zx} v_y - u_z v_{yx}) + \\ &+ (u_{zy} v_x + u_z v_{xy} - u_{xy} v_z - u_x v_{zy}) + \\ &+ (u_{xz} v_y + u_x v_{yz} - u_{yz} v_x - u_y v_{xz}) \end{aligned}$$

and the terms vanish in pairs because of the equality of cross derivatives. Being u, v defined in all of $\mathbb{R}^3$ there is a vector potential.

b)

   i)

$$\text{rot } \nabla(uv) = 0$$

   and it is not a vector potential.

   ii) rot $(u\nabla v) = \nabla u \times \nabla v$ and we see that $u\nabla v$ is a vector potential.

   iii) rot $(v\nabla u) = \nabla v \times \nabla u$ and we see that $-v\nabla u$ is a vector potential.

c) We know that $u\nabla v = (x^3 - y^3 + z^2)(1,1,1)$ is a vector potential and Stokes theorem gives

$$\begin{aligned} \int\int_S \nabla u \times \nabla v \cdot d\mathbf{S} &= \int\int_S \text{rot }((x^3 - y^3 + z^2)(1,1,1)) \cdot \mathbf{n} dS = \\ &= \int_{\partial S} (x^3 - y^3 + z^2)(1,1,1) d\mathbf{l} \end{aligned}$$

Parametrizing ∂S by

$$\begin{aligned} \gamma(t) &= (\cos t, \sin t, 0), t \in [0, 2\pi] \\ \gamma'(t) &= (-\sin t, \cos t, 0) \end{aligned}$$

we obtain

$$\begin{aligned} I &= \int_0^{2\pi} (\cos^3 t - \sin^3 t)(-\sin t + \cos t) dt = \\ &= \int_0^{2\pi} (\cos^4 t - \cos^3 t \sin t + \sin^4 t - \sin^3 t \cos t) dt = \\ &= \int_0^{2\pi} (\cos^4 t + \sin^4 t) dt = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2} \end{aligned}$$

□

Problem 159:

Let S be a surface oriented by the unit normal field $\mathbf{n}$, and $u, v \in \mathcal{C}^2$ (in an open set containing S) numerical functions. Show that if ∇u is orthogonal to $\nabla v \times \mathbf{n}$ on S, then

$$\int_{\partial S} u \nabla v \cdot d\mathbf{l} = 0$$

Solution:

We use Stokes theorem and the property of the mixed product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$:

$$\begin{aligned} \int_{\partial S} u \nabla v \cdot d\mathbf{l} &= \int\int_S \text{rot }(u \nabla v) \cdot d\mathbf{S} = \int\int_S (\nabla u \times \nabla v) \cdot \mathbf{n} dS = \\ &= \int\int_S (\nabla v \times \mathbf{n}) \cdot \nabla u dS = \int\int_S 0 dS = 0 \end{aligned}$$

□

Problem 160: Independence of the path.

a) Let $S \subset \mathbb{R}^3$ an oriented surface such that any simple closed curve in S is the boundary of a region $\mathcal{R} \subset S$, and let $\mathbf{F}(x, y, z)$ be a vector field in $\mathbb{R}^3$. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{l} = 0$$

for every simple closed curve $C \subset S$ iff rot $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of S ($\mathbf{n}$ is the normal field giving the orientation). If this condition is fulfilled then the line integral depends only on the endpoints.

b) Let $S = \{(x, y, z) : x^2 + y^2 = 4, 1 \leq x, 0 \leq z \leq 1\}$ and $\mathbf{F}(x, y, z) = (-y, x, 1)$; is the line integral of $\mathbf{F}$ along curves in S independent of the path? If this is so compute the integral of $\mathbf{F}$ between $(1, \sqrt{3}, 0)$ and $(1, -\sqrt{3}, 1)$. If that is not so compute the circulation of $\mathbf{F}$ along the boundary of S

Solution:

a)

i) If $(\text{rot } \mathbf{F}) \cdot \mathbf{n} = 0$ then rot $\mathbf{F}$ is tangent to S and its flux through any region $\mathcal{R} \subset S$ vanishes. Then let $C \subset S$ be a simple closed curve and $\mathcal{R} \subset S$ the region such that $\partial\mathcal{R} = C$. Giving C the orientation induced by that of S we can apply Stokes theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int\int_{\mathcal{R}} \text{rot}\mathbf{F} \cdot d\mathbf{S} = 0$$

ii) If $\int_C \mathbf{F} \cdot d\mathbf{l} = 0$ for every simple closed curve $C \subset S$ for every region such that $\partial\mathcal{R} = C$ Stokes theorem gives

$$\int\int_{\mathcal{R}} (\text{rot } \mathbf{F}) \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot d\mathbf{l} = 0$$

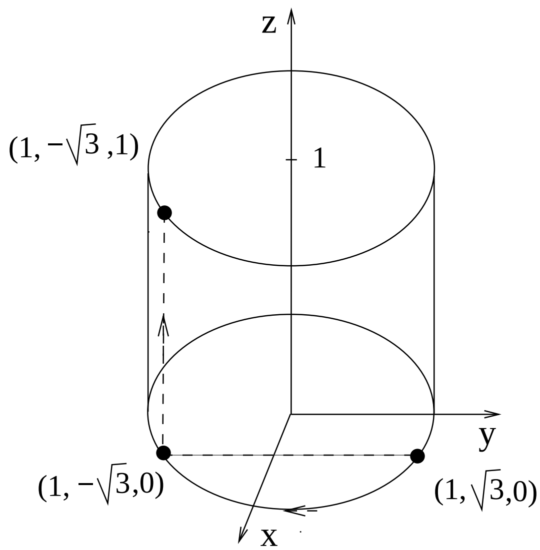
If we had $(\text{rot}\mathbf{F} \cdot \mathbf{n})_{\mathbf{p}} > 0$ at a point $\mathbf{p} \in S$ it would remain positive in a neighborhood $\mathcal{V}$ of $\mathbf{p}$ by continuity; for any simple closed curve

$C_{\mathcal{V}} \subset \mathcal{V}$ we would have

$$\int_{C_{\mathcal{V}}} \mathbf{F} \cdot d\mathbf{l} = \int\int_{\mathcal{V}} \text{rot } \mathbf{F} \cdot \mathbf{n} dS > 0,$$

a contradiction that shows that it must be $((\text{rot } \mathbf{F}) \cdot \mathbf{n})_{\mathbf{p}} = 0, \forall \mathbf{p} \in S$.

b) Remind the figure (see p.293):



$$\text{rot } \mathbf{F}(x, y, z) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 1 \end{pmatrix} = (0, 0, 2)$$

and as $\mathbf{n} = (x, y, 0)$ is normal to the surface and $(\text{rot } \mathbf{F}) \cdot \mathbf{n} = (0, 0, 2) \cdot (x, y, 0) = 0$, $\mathbf{F}$ satisfies the condition of point a). This means that we can choose any path joining $(1, \sqrt{3}, 0)$ and $(1, -\sqrt{3}, 1)$. Consider first the segment $[(1, \sqrt{3}, 0), (1, -\sqrt{3}, 0)]$:

$$\begin{aligned} \gamma(t) &= (1, -t, 0),\ \theta \in [-\sqrt{3}, \sqrt{3}] \\ \gamma'(\theta) &= (0, -1, 0) \end{aligned}$$

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{l} &= \int_{-\pi/3}^{\pi/3} (t, 1, 1) \cdot (0, -1, 0) d\theta = \\ &= \int_{-\pi/3}^{\pi/3} -1 d\theta = -\frac{2\pi}{3} \end{aligned}$$

And now we 'climb' along a cylinder's generatrix from $(1, -\sqrt{3}, 0)$ to $(1, -\sqrt{3}, 1)$:

$$\begin{aligned}\Gamma(z) &= (1, -\sqrt{3}, z),\ z \in [0,1] \\ \Gamma'(z) &= (0,0,1)\end{aligned}$$

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{l} = \int_0^1 (\sqrt{3}, 1, 1) \cdot (0,0,1) dz = 1$$

and the integral is

$$I = -\frac{2\pi}{3} + 1.$$

□

6.3 Gauss theorem

Let $U \subset \mathbb{R}^3$ be a bounded open set with a closed surface as boundary ∂U (or several surfaces glued along common boundaries making a closed figure), oriented by the *exterior* normal field and $\mathbf{F} \in \mathcal{C}^1(U \cup \partial U)$. Then

$$\boxed{\int\int\int_U \text{div } \mathbf{F}\, dV = \int\int_{\partial U} \mathbf{F} \cdot d\mathbf{S}}$$

Gauss theorem is also called the *divergence theorem.*

□

6.3.1 Gauss theorem

Problem 161: Checking.

Check Gauss theorem when

a) $\mathbf{F}(x, y, z) = (2x, y^2, z^2)$ and U the unit ball.

b) $\mathbf{F}(x, y, z) = (x, y, z)$ and the cube $[0, 1] \times [0, 1] \times [0, 1]$.

Solution:

a) On on hand div $\mathbf{F} = 2 + 2y + 2z$ and

$$\int\int\int_U \text{div } \mathbf{F}\, dV = \int\int\int_U (2 + 2y + 2z)dV = 2\frac{4}{3}\pi = \frac{8\pi}{3}$$

because by symmetry $\int\int\int_U ydV = \int\int\int_U zdV = 0$.
On another hand if we parametrize S^2 with spherical coordinates and remind that $\mathbf{n} = (x, y, z)$ is an exterior normal vector, we have:

$$\begin{aligned}
\int\int_{\partial U} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{\partial U} (2x, y^2, z^2) \cdot (x, y, z)dS = \\
&= \int\int_{\partial U} (2x^2 + y^3 + z^3)dS = \\
&= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\varphi\cos^2\theta + \sin^3\varphi\sin^3\theta + \cos^3\varphi)\sin\varphi d\varphi d\theta
\end{aligned}$$

Some terms have a vanishing integral:

$$\begin{aligned}
\int_0^{\pi} \cos^3\varphi\sin\varphi d\varphi &= 0 \\
\int_0^{2\pi} \sin^4\varphi\sin^3\theta d\theta &= 0,
\end{aligned}$$

The first vanishes because $\cos\varphi$ is an odd function respect to $\pi/2$. The second vanishes because $\sin\theta$ is an odd function respect to π. It remains to compute

$$\int_0^{2\pi}\int_0^{\pi} 2\sin^3\varphi\cos^2\theta d\varphi d\theta$$

We have

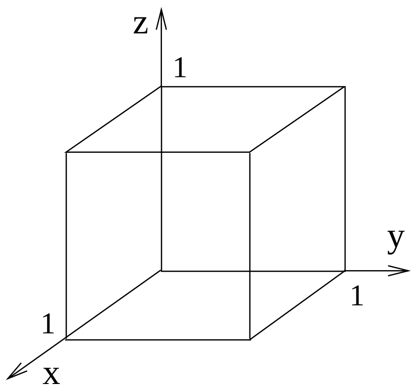
$$\begin{aligned}
\int_0^{2\pi} 2\sin^3\varphi\frac{1+\cos 2\theta}{2}d\theta &= 2\pi\sin^3\varphi \\
\int_0^{\pi} 2\pi\sin^3\varphi d\varphi &= 2\pi\int_0^{\pi}(1-\cos^2\varphi)\sin\varphi d\varphi = \\
&= 2\pi(2 + \frac{\cos^3\varphi}{3}|_0^{\pi}) = \frac{8\pi}{3}
\end{aligned}$$

We have seen that

$$\int\int_{\partial U} \mathbf{F} \cdot d\mathbf{S} = \frac{8\pi}{3}$$

and so we have checked the divergence theorem.

b) A figure



On one hand $\text{div}\,\mathbf{F} = 1 + 1 + 1 = 3$ and

$$\int\int\int_U \text{div}\ \mathbf{F}\,dV = \int\int\int_U 3dV = 3$$

On another hand we must find the flux crossing the six faces of the cube oriented by the exterior normal vector:

- $\mathbf{n} = (1, 0, 0)$ is the normal to the face $(1, y, z)$ and we have

$$\mathbf{F} \cdot \mathbf{n} = (1, y, z) \cdot (1, 0, 0) = 1 \Rightarrow \phi_1 = \text{Area(face)} = 1$$

- $\mathbf{n} = (-1, 0, 0)$ is the normal to the face $(0, y, z)$ and we have

$$\mathbf{F} \cdot \mathbf{n} = (0, y, z) \cdot (-1, 0, 0) = 0 \Rightarrow \phi_{-1} = 0$$

- $\mathbf{n} = (0, 1, 0)$ is the normal to the face $(x, 1, z)$ and we have

$$\mathbf{F} \cdot \mathbf{n} = (x, 1, z) \cdot (0, 1, 0) = 1 \Rightarrow \phi_2 = 1$$

- $\mathbf{n} = (0, -1, 0)$ is the normal to the face $(x, 0, z)$ and we have

$$\mathbf{F} \cdot \mathbf{n} = (x, 0, z) \cdot (0, -1, 0) = 0 \Rightarrow \phi_{-2} = 0$$

and a similar result for the last pair of faces. Summing up:

$$\int\int_{\partial U} \mathbf{F} \cdot d\mathbf{S} = 3$$

□

Problem 162: Computation of a flux.

Using the theorem of the divergence compute the flux of $\mathbf{F}$ through the unit sphere S^2 oriented by the exterior normal when:

a) $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$

b) $\mathbf{F}(x, y, z) = (xz^2, 0, z^3)$

Solution:

a) div $\mathbf{F} = 2(x + y + z)$ and the divergence theorem gives:

$$\int\int_{S^2} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{B^2} 2(x + y + z)dxdydz = 0,$$

the symmetry being taken into account.

b) div $\mathbf{F} = z^2 + 3z^2 = 4z^2$ and we have

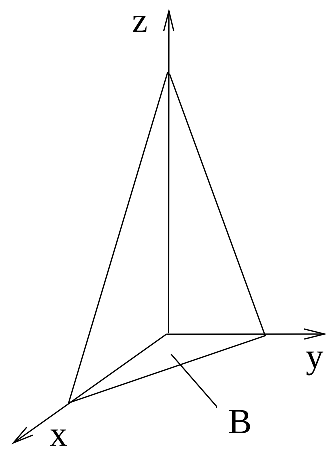
$$\begin{aligned}\int\int_{S^2} \mathbf{F} \cdot d\mathbf{S} &= \int\int\int_{B^2} 4z^2 dxdydz = \{\text{spherical coords}\} = \\ &= 4\int_0^1 dr \int_0^\pi d\varphi \int_0^{2\pi} r^2 \cos^2\varphi r^2 \sin\varphi d\theta \\ &= 4 \cdot 2\pi(\frac{r^5}{5})|_{r=0}^{r=1} \int_0^\pi \cos^2\varphi \sin\varphi d\varphi = \\ &=\tfrac{4}{5}\ 2\pi(-\frac{\cos^3\varphi}{3})|_{\varphi=0}^{\varphi=\pi} = \frac{16}{15}\pi\end{aligned}$$

□

Problem 163: Computation of a flux.

Compute the flux of the vector field $\mathbf{F}(x, y, z) = (yz, zx, xy)$ across the lateral surface L of a pyramid with vertex at $(0, 0, 2)$ and basis the triangle B with vertices at $(0, 0, 0), (2, 0, 0), (0, 1, 0)$.

Solution:



Let Φ be the flux through all the faces of the pyramid if they are given the exterior normal orientation; the theorem of the divergence applies and we have

$$\Phi = \Phi_L + \Phi_B = \int\int\int_V \text{div } \mathbf{F}\, dV = 0$$

and

$$\begin{aligned}
\Phi_B = \int\int_B \mathbf{F}\cdot d\mathbf{S} &= \int\int_B (0,0,xy)\cdot(0,0,-1)dS = \int\int_B -xydS = \\
&= \int_0^2 dx \int_0^{-x/2+1} xydy = -\int_0^2 x\frac{y^2}{2} \Big|_{y=0}^{y=-x/2+1} dx = \\
&= -\int_0^2 (\frac{x^3}{8} - \frac{x^2}{2} + \frac{x^2}{2})dx = -(\frac{x^4}{32} - \frac{x^3}{6} + \frac{x^2}{4}) \Big|_{x=0}^{x=2} = \\
&= -\frac{1}{6}
\end{aligned}$$

so

$$\Phi_L = \int\int_L \mathbf{F}\cdot d\mathbf{S} = \frac{1}{6}$$

□

Problem 164:

Consider that part V of the solid cylinder

$$U = \{(x, y, z) : x^2 + y^2 \leq 4, z \geq 0\}$$

limited by the plane $y + z = 10$. Find:

a) The volume of V.

b) The area of the upper cover.

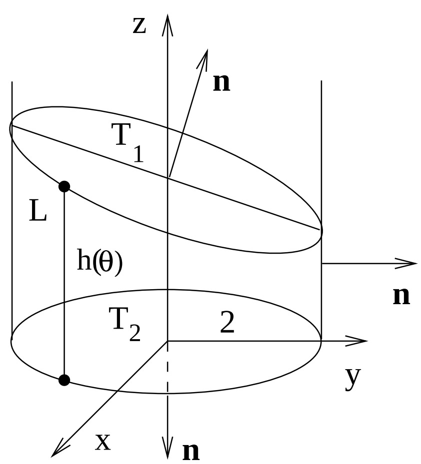
c) The lateral area.

d) The flux of the field $\mathbf{F}(x, y, z) = (0, 0, z)$ across the upper cover.

e) The line integral $\int_C \mathbf{F} \cdot d\mathbf{l}$ where $C = \partial(U \cap P)$, P being the plane $y + z = a, 2 \leq a \leq 10$.

Solution:

We are free to choose the orientations. First at all a figure:



a) V is the region under the graph of the function $f(x, y) = 10 - y$ and its volume is

$$\begin{aligned}
\text{Vol }(V) &= \int\int_{D(\mathbf{0};2)} (10-y)dxdy = \{\text{polar coords}\} \\
&= \int_0^2 dr \int_0^{2\pi} (10 - r\sin\theta) r dr d\theta = \\
&= \int_0^2 (10r\theta + r^2\cos\theta)\mid_0^{2\pi} dr = \\
&= \int_0^2 20\pi r dr = 40\pi
\end{aligned}$$

Let us do a check of this result using Gauss theorem. To that end let S be the surface wich is the boundary of V, oriented by the exterior normal. Think about it as decomposed into two covers T_1, T_2 and a lateral surface L. Then

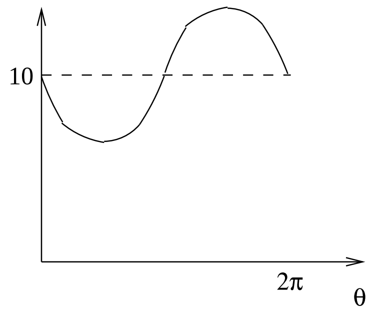
$$\begin{aligned}
\text{Vol }(V) &= \int\int\int_V 1 dxdydz = \int\int\int_V \text{div }(x,0,0)dxdydz = \\
&= \int\int_S (x,0,0)\cdot d\mathbf{S}
\end{aligned}$$

From $y+z=10$ we obtain the exterior normal to T_1: $\mathbf{n} = \frac{1}{\sqrt{2}}(0,1,1)$. And the exterior normal to T_2 is $\mathbf{n} = (0,0,-1)$. Then

$$\begin{aligned}
\text{Vol }(V) &= \int\int_{T_1} (x,0,0)\cdot \frac{1}{\sqrt{2}}(0,1,1)dS + \\
&+ \int\int_{T_2} (x,0,0)\cdot(0,0,-1)dS + \int\int_L (x,0,0)\cdot d\mathbf{S} = \\
&= 0 + 0 + \int\int_L (x,0,0)\cdot d\mathbf{S}
\end{aligned}$$

Parametrize L by:

$$\alpha(\theta, z) = (2\cos\theta, 2\sin\theta, z), \theta\in[0,2\pi], z\in[0,10-2\sin\theta]$$



$$\begin{aligned}\partial_\theta\alpha &= (-2\sin\theta, 2\cos\theta, 0)\\ \partial_z\alpha &= (0,0,1)\\ \mathbf{N} &= (2\cos\theta, 2\sin\theta, 0)\end{aligned}$$

and the integral is

$$\begin{aligned}\int\int_L (x,0,0)\cdot d\mathbf{S} &= \int_0^{2\pi} d\theta \int_0^{10-2\sin\theta} 4\cos^2\theta dr =\\ &= 4\int_0^{2\pi}(10-2\sin\theta)\cos^2\theta d\theta =\\ &= 40\int_0^{2\pi}\cos^2\theta d\theta - 80\int_0^{2\pi}\sin\theta\cos^2\theta d\theta =\\ &= 40\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta + 80\frac{\cos^3\theta}{3}\mid_0^{2\pi} =\\ &= 40\pi\end{aligned}$$

b) We can see geometrically that we are computing the area of an ellipse with semiaxes 2 and $2\sqrt{2}$ (because the plane $y+z=10$ has turned $\pi/4$ respect to the plane $z=0$) and has an area

$$\text{Area } = 4\sqrt{2}\pi$$

We can check that result parametrizing the surface of the ellipse; to do so we simply 'climb' from the disc $D(\mathbf{0};2)$ to the plane $y+z=10$:

$$\begin{aligned}\beta(r,\theta) &= (r\cos\theta, r\sin\theta, 10-r\sin\theta), (r,\theta)\in[0,2]\times[0,2\pi]\\ \partial_r\beta &= (\cos\theta, \sin\theta, -\sin\theta)\\ \partial_r\beta &= (-r\sin\theta, r\cos\theta, -r\cos\theta)\\ \mathbf{N} &= (0, r(\sin^2\theta+\cos^2\theta), r) = (0,r,r)\\ \mid\mathbf{N}\mid &= \sqrt{2}r\end{aligned}$$

and its area is:

$$\text{Area } = \int\int_{T_1} dS = \int\int_D r\sqrt{2} dr d\theta = 2\pi\sqrt{2}\frac{4}{2} = 4\sqrt{2}\pi$$

c) The area of L is the sum of those bars $h(\theta)$ in the first figure:

$$\text{Area } (L) = \int_0^{2\pi} (10 - 2\sin\theta) d\theta = 20\pi$$

Alternatively we may use the parametrization of L we had in a); as $|\mathbf{N}| = 2$ we have

$$\begin{aligned} \text{Area } (L) &= \int_0^{2\pi} d\theta \int_0^{10-2\sin\theta} 2dz = \\ &= \int_0^{2\pi} 2(10 - 2\sin\theta) d\theta = 20\pi \end{aligned}$$

d) Using the parametrization β of T_1:

$$\begin{aligned} \phi_{T_1} &= \int\int_{T_1} (0,0,z) \cdot d\mathbf{S} = \\ &= \int_0^{2\pi} \int_0^2 (0,0,10 - 2\sin\theta) \cdot (0,r,r) d\theta dr = \\ &= \int_0^{2\pi} \int_0^2 (10 - 2\sin\theta) r d\theta dr = 2\int_0^{2\pi} (10 - 2\sin\theta) d\theta = 40\pi \end{aligned}$$

Alternatively we may use Gauss theorem applied to the field $(0,0,z)$. This field is tangent to L and gives no flux across. On T_2 the field vanishes and so does ϕ_{T_2}. Then

$$\int\int\int_V \text{div } (0,0,z) dV = \phi_{T_1} + \phi_{T_2} + \phi_L = \phi_{T_1}$$

and

$$\phi_{T_1} = \int\int\int_V \text{div } (0,0,z) dV = \int\int\int_V 1 dV = \text{Vol } (V) = 40\pi$$

e) Let us use Stokes theorem; let T_a be the upper cover of the body (the intersection of $y+z=a$ with the solid cylinder) and taking into account that rot $\mathbf{F}=0$ we obtain:

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int\int_{T_a} \text{rot } \mathbf{F}\cdot d\mathbf{S} = 0$$

As a check we parametrize C

$$\begin{aligned} \gamma(\theta) &= (2\cos\theta, 2\sin\theta, a-2\sin\theta), \theta\in[0,2\pi] \\ \gamma'(\theta) &= (-2\sin\theta, 2\cos\theta, -2\cos\theta) \end{aligned}$$

and

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{l} &= \int_0^{2\pi} (0,0,a-2\sin\theta)\cdot(-2\sin\theta, 2\cos\theta, -2\cos\theta)d\theta = \\ &= -2a\int_0^{2\pi}\cos\theta + 4\int_0^{2\pi}\sin\theta\cos\theta d\theta = \\ &= 0 + 4\frac{\sin^2\theta}{2} \,|_0^{2\pi} = 0 \end{aligned}$$

□

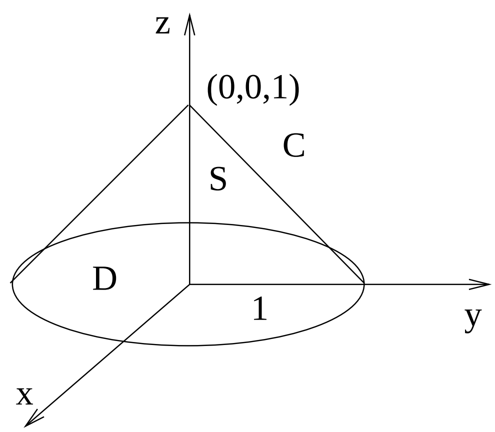
Problem 165:

Let C be the solid cone with vertex at $(0,0,1)$ and basis the disc $D=\{(x,y,0) : x^2+y^2\leq 1\}$; call S the lateral surface of C oriented by the exterior normal. Let $\mathbf{F}(x,y,z)=(x^2,0,y^2)$ and compute $\int\int_S \mathbf{F}\cdot d\mathbf{S}$:

a) Parametrizing S.

b) Using the divergence theorem.

Solution:



a) Parametrize S:

$$\begin{aligned}\alpha(r,\theta) &= (r\cos\theta, r\sin\theta, 1-r), r\in[0,1], \theta\in[0,2\pi]\\ \partial_r\alpha &= (\cos\theta,\sin\theta,-1)\\ \partial_\theta\alpha &= (-r\sin\theta, r\cos\theta, 0)\\ \mathbf{N} &= (r\cos\theta, r\sin\theta, r)\end{aligned}$$

and notice that $\mathbf{N}$ points to the exterior. We have

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^{2\pi}(r^2\cos^2\theta, 0, r^2\sin^2\theta)\cdot(r\cos\theta, r\sin\theta, r)drd\theta =\\ &= \int_0^1\int_0^{2\pi}(r^3\cos^3\theta + r^3\sin^2\theta)drd\theta =\\ &= \frac{1}{4}\int_0^{2\pi}(\cos\theta(1-\sin^2\theta) + \frac{1+\cos 2\theta}{2})d\theta =\\ &= \frac{1}{4}(0+\frac{2\pi}{2}) = \frac{\pi}{4}\end{aligned}$$

b) The theorem of the divergence states that

$$\int\int\int_C \text{div } \mathbf{F}dV = \int\int_S \mathbf{F}\cdot d\mathbf{S} + \int\int_D \mathbf{F}\cdot d\mathbf{S}$$

Orienting D by $\mathbf{n} = (0,0,-1)$ we have

$$\begin{aligned}\int\int_D \mathbf{F}\cdot d\mathbf{S} &= \int\int_D (x^2,0,y^2)\cdot(0,0,-1)dxdy = \\ &= \int\int_D -y^2 dxdy = \{\text{polar coords}\} = \\ &= -\int_0^1\int_0^{2\pi} r^2\sin^2\theta\, r dr d\theta = \\ &= -\frac{1}{4}\int_0^{2\pi}\sin^2\theta d\theta = -\frac{\pi}{4}\end{aligned}$$

and

$$\int\int\int_C \text{div } \mathbf{F} dV = \int\int\int_C 2x dxdydz = 0$$

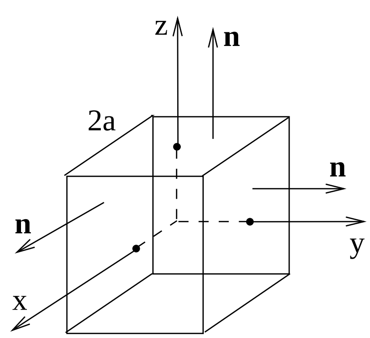
The integral vanishes because there are 'so many' positive x as negative ones (or we can make the change of variables $x = r\cos\theta, y = r\sin\theta, z = z,\ r\in[0,1], \theta\in[0,2\pi], z\in[0,1-r]$ that has a jacobian determinant $r>0$). The divergence theorem ckecks the result in a).

□

Problem 166:

Compute the flux of $\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{r^3}$ across S, the surface of a cube C with sides $2a$ centered at $\mathbf{0}$, oriented by the exterior normal.

Solution:



The field is not defined at $\mathbf{0}$ and we *cannot* use the divergence theorem. Nevertheless delete from C the closed ball $\overline{B}(\mathbf{0}; a/2) = \{\mathbf{x} : \mid \mathbf{x} \mid \leq a/2\}$ and call U the open set left. Then $\partial U = S \cup S^2_{a/2}$; we orient the sphere by the interior normal. Apply the divergence theorem to U to obtain:

$$0 = \int\int\int_U \operatorname{div} \frac{\mathbf{r}}{r^3}\, dV = \int\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} + \int\int_{S_{a/2}} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S}$$

We know that with the exterior normal orientation

$$\int\int_{S_{a/2}} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} \quad = \quad 4\pi,$$

and with the interior normal orientation we have $\int\int_{S_{a/2}} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = -4\pi$. Then

$$\int\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = 4\pi$$

Alternatively we may use the solid angle concept (see p.253) and obtain the same result just by noticing that the projection of the cube on the unit sphere S^2 is the whole sphere.

□

Problem 167:

a) Using the divergence theorem compute

$$\int\int_S (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS$$

S being the surface $x^2 + y^2 + z^2 = 2az, a > 0$ and α, β, γ the angles of the exterior normal to S with the coordinate axis.

b) Same question if S is the surface of the cube $[0, a] \times [0, a] \times [0, a]$.

Solution:

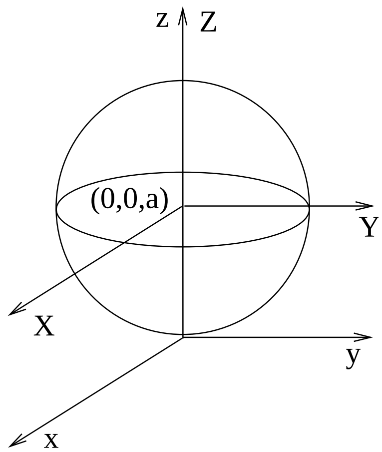
Let us transform the integral to one that is a flux

$$\begin{aligned}\int\int_S (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS \quad &= \quad \int\int_S (x^2, y^2, z^2) \cdot \mathbf{n}\, dS = \\ &= \quad \int\int_S (x^2, y^2, z^2) \cdot d\mathbf{S}\end{aligned}$$

which is the flux of the field $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$, and now we can use the divergence theorem.

a) The equation $x^2+y^2+z^2=2az$ is equivalent to $x^2+y^2+(z-a)^2=a^2$ and thus it is a sphere with center at $(0,0,a)$ and radius $r=a$. Choose new coordinate axes with origin at $(0,0,a)$, that is make the change of coordinates

$$X=x, Y=y, Z=z-a$$



The sphere's equation is now

$$X^2+Y^2+Z^2=a^2$$

The field in the new coordinates is:

$$\mathbf{F}(X,Y,Z)=(X^2,Y^2,(Z+a)^2)$$

$$I=\int\int_S (X^2,Y^2,(Z+a)^2)\cdot \mathbf{n}dS$$

And the theorem of the divergence gives:

$$\begin{aligned}\int\int_S (X^2,Y^2,(Z+a)^2)\cdot \mathbf{n}dS &= \int\int\int_V \operatorname{div}(X^2,Y^2,(Z+a)^2)dV = \\ &= \int\int\int_V (2X+2Y+2(Z+a))dV = \\ &= 2a\int\int\int_V 1dV = \\ &= 2a\frac{4}{3}\pi a^3 = \frac{8\pi a^4}{3}\end{aligned}$$

We have used the fact that, $\int\int\int_V XdV = \int\int\int_V YdV = \int\int\int_V ZdV = 0$ by symmetry.

b) Now we have:

$$\int\int_S (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)dS \quad = \quad \int\int\int_V 2(x+y+z)dV$$

and we compute each coordinate separatedly

$$\int\int\int_V xdxdydz = (\int_0^a xdx)(\int\int_{[0,a]^2} dydz) = \frac{a^2}{2}a^2 = \frac{a^4}{2}$$

and the same result is obtained for the other coordinates; finally

$$\int\int_S (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)dS = 2\cdot 3\cdot\frac{a^4}{2} = 3a^4$$

□

Problem 168:

Consider the cylinder $D = \{(x,y,z) : x^2+y^2 = 1\}$, the hiperbolic paraboloid $H = \{(x,y,z) : z = xy\}$, the intersection curve $C = D\cap H$ and the field $\mathbf{F}(x,y,z) = (0, x, \frac{y^2-x^2}{2})$.

a) Compute directly $\int_C \mathbf{F}\cdot d\mathbf{l}$

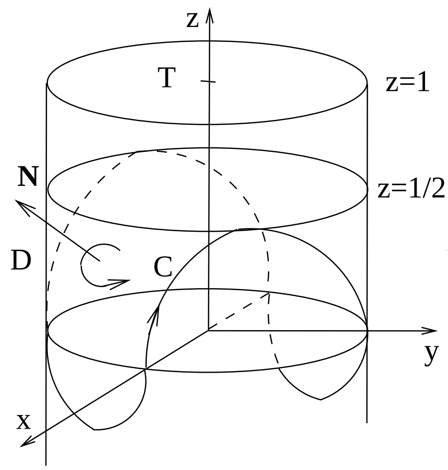
b) Check the result of a) using Stokes theorem.

c) Compute $\int_C \mathrm{rot}\mathbf{F}\cdot d\mathbf{l}$.

d) Let L be the region of D limited by the curve C and by $z = 1$; compute $\int\int_L \mathbf{F}\cdot d\mathbf{S}$ using the divergence theorem.

Solution:

A figure:



a) To parametrize C it suffices to 'climb' from the basis circumference to the paraboloid (see p.28):

$$\begin{aligned}\gamma(\theta) &= (\cos\theta, \sin\theta, \cos\theta\sin\theta)\\ &= (\cos\theta, \sin\theta, \frac{\sin 2\theta}{2})\\ \gamma'(\theta) &= (-\sin\theta, \cos\theta, \cos 2\theta)\end{aligned}$$

In the figure we can see the orientation of C; we compute the line integral with this orientation:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{l} &= \int_0^{2\pi} (0, \cos\theta, \frac{\sin^2\theta - \cos^2\theta}{2})\cdot(-\sin\theta, \cos\theta, \cos 2\theta)d\theta = \\ &= \int_0^{2\pi} (\cos^2\theta + \frac{1}{2}(\sin^2\theta - \cos^2\theta)(\cos^2\theta - \sin^2\theta))d\theta = \\ &= \int_0^{2\pi} (\cos^2\theta - \frac{1}{2}(\cos^4\theta + \sin^4\theta) + \sin^2\theta\cos^2\theta)d\theta\end{aligned}$$

and using many times the formulae $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we obtain

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \frac{\pi}{2}$$

b) To use Stokes theorem we fill the curve with the surface S given by:

$$\begin{aligned}\beta(x,y) &= (x,y,xy), (x,y) \in D_1 = \{(x,y) : x^2+y^2 \leq 1\} \\ \beta_x &= (1,0,y) \\ \beta_y &= (0,1,x) \\ \mathbf{N} &= (-y,-x,1)\end{aligned}$$

The third component of $\mathbf{N}$ is positive and the vector points up, as we can see in the figure. This orientation of S induces in C the orientation used in a). As rot $\mathbf{F} = (y,x,1)$, we have:

$$\begin{aligned}\int\int_S \text{rot}\mathbf{F}\cdot d\mathbf{S} &= \int\int_{D_1} (y,x,1)\cdot(-y,-x,1)dxdy \\ &= \int\int_{D_1}(1-x^2-y^2)dxdy = \\ &= \{\text{polar coords}\} = \int_0^{2\pi}\int_0^1 (1-r^2)rdrd\theta = \\ &= 2\pi(\frac{1}{2}-\frac{1}{4}) = \frac{\pi}{2}\end{aligned}$$

and Stokes theorem gives

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int\int_S \text{rot }\mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2}$$

c) To compute $\int_C \text{rot}\mathbf{F}\cdot d\mathbf{l}$ we apply Stokes theorem again:

$$\int_C \text{rot }\mathbf{F}\cdot d\mathbf{l} = \int\int_S \text{rot }(\text{rot }\mathbf{F})\cdot d\mathbf{S} = \int\int_S \mathbf{0}\cdot d\mathbf{S} = 0$$

d) Let T be the upper cover of the region in the cylinder and U the volume enclosed by the solid cylinder between S and T; the divergence theorem gives:

$$\begin{aligned}0 &= \int\int\int_U \text{div }(0,x,\frac{y^2-x^2}{2})dV = \\ &= \int\int_L \mathbf{F}\cdot d\mathbf{S} + \int\int_S \mathbf{F}\cdot d\mathbf{S} + \int\int_T \mathbf{F}\cdot d\mathbf{S}\end{aligned}$$

and

$$\int\int_L \mathbf{F}\cdot d\mathbf{S} = -\int\int_S \mathbf{F}\cdot d\mathbf{S} - \int\int_T \mathbf{F}\cdot d\mathbf{S}$$

Notice that L, S, T have to be oriented by the exterior normal. The parametrization β we had in b) has the associated normal vector pointing to the interior of U; so we must change the sign:

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= -\int\int_{D_1} (0, x, \frac{y^2 - x^2}{2}) \cdot (-y, -x, 1)dxdy = \\
&= -\int\int_{D_1} (-\frac{3}{2}x^2 + \frac{1}{2}y^2)dxdy = \{\text{polar coords}\} = \\
&= -\int_0^1 \int_0^{2\pi} r^2(-\frac{3}{2}\cos^2\theta + \frac{1}{2}\sin^2\theta)rdrd\theta = \\
&= -\frac{1}{4}(-\frac{3}{2}\pi + \frac{1}{2}\pi) = \frac{\pi}{4}
\end{aligned}$$

Then the flux with the orientation given by the exterior normal will be $-\pi/4$.
A parametrization of T with the associated normal pointing to the exterior of u is:

$$\begin{aligned}
\beta(x, y) &= (x, y, 1), (x, y) \in D_1 = \{(x, y) : x^2 + y^2 \leq 1\} \\
\mathbf{N} &= (0, 0, 1) \\
\int\int_T \mathbf{F}\cdot d\mathbf{S} &= \int\int_{D_1} (0, x, \frac{y^2 - x^2}{2}) \cdot (0, 0, 1)dxdy = \{\text{polar coords}\} = \\
&= \int_0^1 \int_0^{2\pi} \frac{1}{2}r^2(\sin^2\theta - \cos^2\theta)rdrd\theta = 0
\end{aligned}$$

Finally

$$\int\int_L \mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{4}$$

□

Problem 169:

A solid of revolution respect to the Oz axis is limited on the floor by the disc D, $x^2 + y^2 \leq 1, z = 0$, on the side by the piece of a cylinder C with equations $x^2 + y^2 = 1, 0 \leq z \leq 1 + a$ and on the roof by the paraboloid P: $z = 1+a(x^2+y^2), a > -1$. Let us orient all these surfaces in an anticlockwise sense when seen from the exterior. Consider the field $\mathbf{F}(x, y, z) = (x, y, 0)$.

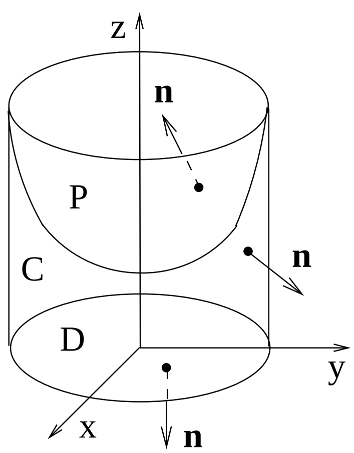
a) Compute the integral of $\mathbf{F}$ on the parabolic cover.

b) Compute the integral of $\mathbf{F}$ on the cylindrical side.

c) Use Gauss theorem to compute the volume in terms of the results obtained in a), b).

Solution:

A figure:



a) Parametrize the cover P:

$$\begin{aligned}
\alpha(x, y) &= (x, y, 1 + a(x^2 + y^2)), (x, y) \in D \\
\partial_x \alpha &= (1, 0, 2ax) \\
\partial_y \alpha &= (0, 1, 2ay) \\
\mathbf{N} &= (-2ax, -2ay, 1)
\end{aligned}$$

and the integral of $\mathbf{F}$ is:

$$\begin{aligned}\int\int_P (x,y,0)\cdot d\mathbf{S} &= \int\int_D (x,y,0)\cdot(-2ax,-2ay,1)dxdy = \\ &= -2a\int\int_D (x^2+y^2)dxdy = \{\text{polar coords}\} = \\ &= -2a\int_0^1\int_0^{2\pi} r^2 r dr d\theta = -4\pi a\frac{1}{4} = -\pi a\end{aligned}$$

b) The side C we parametrize by

$$\begin{aligned}\beta(\theta,z) &= (\cos\theta,\sin\theta,z), \theta\in[0,2\pi], z\in[0,1+a] \\ \partial_\theta\beta &= (-\sin\theta,\cos\theta,0) \\ \partial_y\beta &= (0,0,1) \\ \mathbf{N} &= (\cos\theta,\sin\theta,0)\end{aligned}$$

and the integral of $\mathbf{F}$ is:

$$\begin{aligned}\int\int_C (x,y,0)\cdot d\mathbf{S} &= \int\int_C (\cos\theta,\sin\theta,0)\cdot(\cos\theta,\sin\theta,0) = \\ &= \int_0^{2\pi}\int_0^{1+a} d\theta dz = 2\pi(1+a)\end{aligned}$$

c) Notice that the associated normal vectors to the parametrizations α,β point to the exterior and we can use the divergence theorem to obtain:

$$\begin{aligned}\int\int\int_V \text{div } \mathbf{F}dV &= \int\int_p \mathbf{F}\cdot d\mathbf{S} + \int\int_C \mathbf{F}\cdot d\mathbf{S} + \int\int_D \mathbf{F}\cdot d\mathbf{S} = \\ &= -\pi a + 2\pi(1+a) + 0 = \pi(2+a)\end{aligned}$$

and as

$$\int\int\int_V \text{div } \mathbf{F}dV = \int\int\int_V 2dV = 2\text{Vol }(V)$$

we get the result

$$\text{Vol }(V) = \frac{\pi}{2}(2+a)$$

□

Problem 170:

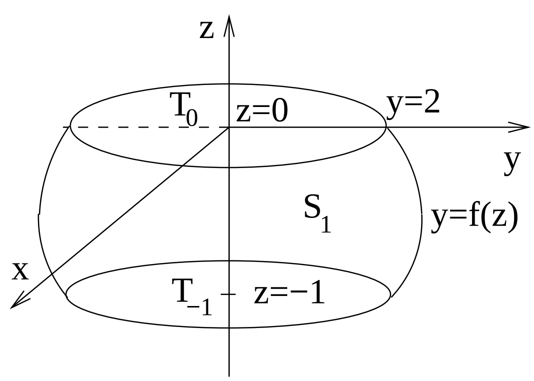
Let f be a positive differentiable function such that $f(0) = f(-1) = 2$. Let S be the surface of revolution obtained revolving the curve $y = f(z), x = 0$ around the Oz axis. Consider the vector field $\mathbf{F}(x, y, z) = (x, y, -2z)$.

a) Let S_1 be that part of S limited by the planes $z = -1, z = 0$; compute directly the flux of $\mathbf{F}$ across S_1.

b) Check the result in a) using the divergence theorem.

Solution:

A figure:



a) A parametrization of S_1 is

$$\alpha(\theta, z) = (f(z)\cos\theta, f(z)\sin\theta, z),\ (\theta, z) \in [0, 2\pi] \times [-1, 0]$$

with associated normal vector

$$\mathbf{N} = \alpha_\theta \times \alpha_z = (f\cos\theta, f\sin\theta, -ff')$$

Thus

$$\begin{aligned} \int_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_{-1}^{0}\int_{0}^{2\pi} (f(z)\cos\theta, f(z)\sin\theta, -2z) \cdot (f\cos\theta, f\sin\theta, -ff')d\theta dz = \\ &= \int_{-1}^{0}\int_{0}^{2\pi} (f^2 + 2ff'z)d\theta dz = 2\pi \int_{-1}^{0} (f^2 + 2ff'z)dz = \\ &= 2\pi(f^2 z)|_{z=-1}^{z=0} = 8\pi \end{aligned}$$

Notice that as $f > 0$ the normal vector points to the exterior.

b) Consider the region $U \subset \mathbb{R}^3$ limited by S_1 and by the covers

$$\begin{aligned} T_{-1} &= \{(x,y,z) : x^2+y^2 \leq f(-1)^2 = 4, z=-1\} \\ T_0 &= \{(x,y,z) : x^2+y^2 \leq f(0)^2 = 4, z=0\} \end{aligned}$$

As div $\mathbf{F} = 0$ the divergence theorem gives:

$$\int\int_{T_{-1}} \mathbf{F}\cdot d\mathbf{S} + \int\int_{T_0} \mathbf{F}\cdot d\mathbf{S} + \int\int_{S_1} \mathbf{F}\cdot d\mathbf{S} = 0$$

where the surfaces must be oriented by the exterior normal; this is so for the parametrization of a). On T_0 the flux vanishes, because

$$\mathbf{F}\cdot\mathbf{n} = (x,y,0)\cdot(0,0,1) = 0$$

Parametrizing T_{-1} by

$$\alpha(\theta,\rho) = (\rho\cos\theta, \rho\sin\theta, -1),\ (\theta,z) \in [0,2\pi]\times[0,2],$$

we have an associated normal vector pointing to the exterior:

$$\alpha_\theta \times \alpha_\rho = (0,0,-\rho)$$

The flux is

$$\begin{aligned} \int\int_{T_{-1}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2 (\rho\cos\theta, \rho\sin\theta, 2)\cdot(0,0,-\rho)d\theta d\rho = \\ &= \int_0^{2\pi}\int_0^2 -2\rho d\theta d\rho = -8\pi \end{aligned}$$

and from the theorem we obtain

$$\int\int_{S_1} \mathbf{F}\cdot d\mathbf{S} = -\int\int_{T_{-1}} \mathbf{F}\cdot d\mathbf{S} = -(-8\pi) = 8\pi$$

We have checked the result of a).

□

Problem 171:

Consider the paraboloid $P = \{(x, y, z) : z = x^2 + y^2\}$, the plane $\pi = \{(x, y, z) : x + y + z = 1\}$ and the vector fields

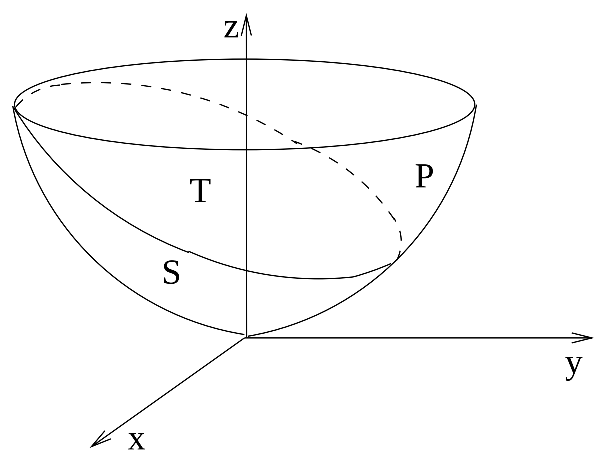
$$\mathbf{F}(x, y, z) = (y - z, z - x, x - y), \mathbf{G}(x, y, z) = (y, -x, 0).$$

a) Show that $\mathbf{F}$ is tangent to π and that $\mathbf{G}$ is tangent to P.

b) Compute the flux of $\mathbf{F}$ across S the region of P limited by π.

c) Compute the flux of $\mathbf{G}$ across T the region of π limited by P.

Solution:



a) A perpendicular vector to π is $\mathbf{N} = (1, 1, 1)$. $\mathbf{F}$ satisfies

$$\mathbf{F} \cdot \mathbf{N} = (y - z, z - x, x - y) \cdot (1, 1, 1) = 0$$

and is tangent to π; then $\mathbf{F}$ has a vanishing flux through any region in π.
On another hand, if $f(x, y, z) = x^2 + y^2 - z$, a vector fiel normal to P is $\mathbf{N} = \nabla f = (2x, 2y, -1)$ and $\mathbf{G}$ satisfies

$$\mathbf{G} \cdot \mathbf{N} = (y, -x, 0) \cdot (2x, 2y, -1) = 2xy - 2xy = 0$$

that is, $\mathbf{G}$ is tangent to P and has a vanishing flux across any region in P.

b) Let U be the region of $\mathbb{R}^3$ limited by the paraboloid and the plane; as $\nabla \cdot \mathbf{F} = 0$, using the divergence theorem we obtain:

$$0 = \int\int\int_U \nabla \cdot \mathbf{F} dV = \int\int_S \mathbf{F} \cdot d\mathbf{S} + \int\int_T \mathbf{F} \cdot d\mathbf{S}$$

and as $\mathbf{F}$ has a vanishing flux across T

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0$$

c) Analogously $\nabla \cdot \mathbf{G} = 0$ and

$$0 = \int\int\int_U \nabla \cdot \mathbf{G} dV = \int\int_S \mathbf{G} \cdot d\mathbf{S} + \int\int_T \mathbf{G} \cdot d\mathbf{S}$$

and as $\mathbf{G}$ has a vanishing flux across S

$$\int\int_T \mathbf{G} \cdot d\mathbf{S} = -\int\int_S \mathbf{G} \cdot d\mathbf{S} = 0$$

□

Problem 172:

Consider the fields

$$\begin{aligned} \mathbf{G}(x,y,z) &= (1-r^2)(y,-x,e^{z^2}) \\ \mathbf{H}(x,y,z) &= \frac{\mathbf{r}}{r^3} \\ \mathbf{F}(x,y,z) &= \mathbf{H} + \text{rot } \mathbf{G} \end{aligned}$$

a) Compute div $\mathbf{F}$.

b) Compute the flux of $\mathbf{F}$ across S, the upper unit semisphere.

c) Compute the flux of $\mathbf{G}$ across S, the upper unit semisphere.

Solution:

a) div $\mathbf{F} =$ div $\mathbf{H} +$ div (rot $\mathbf{G}$) $= 0$ because $\mathbf{H}$ is a gravitational field of a point mass that we know to have vanishing divergence.

b) We cannot close the semisphere by a lower cover to apply the divergence theorem because $\mathbf{H}$ is undefined at $\mathbf{0}$; so let us proceed directly. Orienting the semisphere by the exterior normal we have:

$$\int\int_S \mathbf{H}\cdot d\mathbf{S} = \int\int_S \frac{\mathbf{r}}{r^3}\cdot \mathbf{r}dS = \int\int_S \frac{1}{r}dS = \{r=1\} = \int\int_S dS = 2\pi$$

Or we can use the concept of solid angle to arrive at the same value (see p.*253*).
For the flux of rot $\mathbf{G}$ we close S with the disc $T = \{(x,y,z) : x^2+y^2 \leq 1, z=0\}$ oriented by the exterior normal $\mathbf{n} = (0,0,-1)$. The theorem of the divergence applies and we obtain

$$\int\int_S \text{rot } \mathbf{G}\cdot d\mathbf{S} + \int\int_T \text{rot } \mathbf{G}\cdot d\mathbf{S} = \int\int\int_U \text{div (rot } \mathbf{G})dV = 0$$

and using Stokes theorem

$$\int\int_S \text{rot } \mathbf{G}\cdot d\mathbf{S} = -\int\int_T \text{rot } \mathbf{G}\cdot d\mathbf{S} = -\int_{\partial T} \mathbf{G}\cdot d\mathbf{l} = 0$$

due to the fact that $\mathbf{G}$ vanishes on ∂T, because $r = 1$. Finally

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = 2\pi$$

c) On the unit sphere $r = 1$ and we see that $\mathbf{G}$ vanishes there; so

$$\int\int_S \mathbf{G}\cdot d\mathbf{S} = 0$$

□

Problem 173:

Let S be the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

oriented by the exterior unit normal $\mathbf{n}$ and let $d(x,y,z)$ be the distance from the origin to the tangent plane to S at the point (x,y,z). Show:

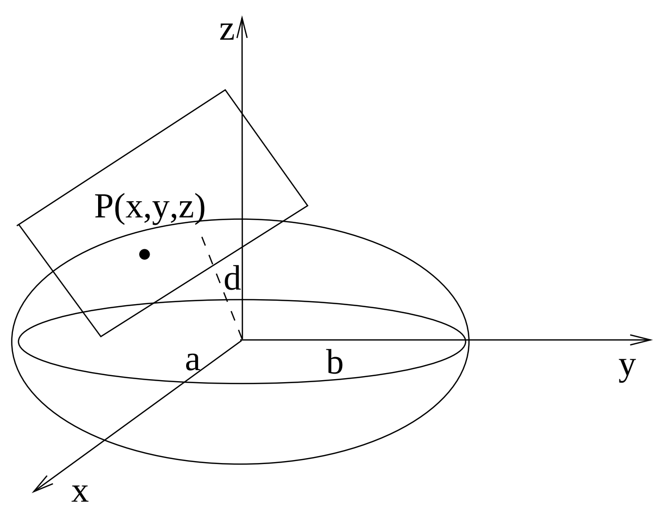
a) If $\mathbf{F}(x,y,z) = (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$ then $\mathbf{F} \cdot \mathbf{n} = \frac{1}{d}$.

b) $\int\int_S \frac{1}{d} dS = \frac{4}{3}\pi(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c})$.

c) $\int\int_S d\, dS = 4\pi abc$.

d) Compute the flux across S of the field $\mathbf{G} = (\frac{xz}{a^2}, \frac{yz}{b^2}, \frac{z^2}{c^2})$.

Solution:



a) S is the level 1 set of the function $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$; an exterior normal vector is

$$\nabla f = 2(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) = 2\mathbf{F}$$

and

$$\mathbf{n} = \frac{\mathbf{F}}{|\mathbf{F}|}$$

thus

$$\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot\frac{\mathbf{F}}{|\mathbf{F}|} = |\mathbf{F}| = \sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

The tangent plane to S at the point $\mathbf{p}_0 = (x_0, y_0, z_0) \in S$ is $\nabla f(\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0) = 0$:

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0) = 0$$

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z = \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2} = 1$$

The distance from the origin is:

$$d = \frac{1}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}$$

and

$$\frac{1}{d} = \sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}} = |\mathbf{F}| = \mathbf{F}\cdot\mathbf{n}$$

b) Using the preceding result and the divergence theorem

$$\int\int_S \frac{1}{d}\,dS = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_V \nabla\cdot\mathbf{F}dV =$$
$$= \int\int\int_V (\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})dV = (\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})\int\int\int_V 1dV =$$
$$= \frac{4}{3}\pi abc(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}) = \frac{4}{3}\pi(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c})$$

c) As it has been so comfortable, we try to copy the line in b); we would like a field $\mathbf{H}$ such that $\mathbf{H}\cdot\mathbf{n} = d \; on \; S$ that is

$$\mathbf{H}\cdot\mathbf{n} = \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

But we know $\mathbf{n}$ to be

$$\mathbf{n} = \frac{(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2})}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}},$$

and we see that we should choose $\mathbf{H}(x,y,z) = (x,y,z)$; then

$$\int\int_S d\,dS = \int\int_S \mathbf{H}\cdot\mathbf{n}\,dS = \int\int_S \mathbf{H}\cdot d\mathbf{S} = \int\int\int_V \nabla\cdot\mathbf{H}dV =$$

$$= 3\int\int\int_V 1dV = 3\mathrm{Vol}(V) = 3\frac{4}{3}\pi abc = 4\pi abc$$

d) We simply apply the divergence theorem:

$$\int\int_S \mathbf{G}\cdot d\mathbf{S} = \int\int\int_V \nabla\cdot\mathbf{G}dV = \int\int\int_V (\frac{z}{a^2}+\frac{z}{b^2}+\frac{2z}{c^2})dV =$$

$$= (\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{c^2})\int\int\int_V zdV = 0,$$

taking into account the symmetry.

□

Problem 174:

Let S be a closed surface enclosing a region U, $\mathbf{n}$ the unit exterior normal, and $\mathbf{v}$ a fixed vector. Show:

a) $\int\int_S \cos(\mathbf{v},\mathbf{n})dS = 0$.

b) $\int\int\int_U \mathrm{div}\ \mathbf{n}\,dV = \text{Area }(S)$.

Solution:

a) We convert the integral into a flux integral

$$I = \int\int_S \cos(\mathbf{v},\mathbf{n})dS = \int\int_S \frac{1}{\mid\mathbf{v}\mid}\mathbf{v}\cdot\mathbf{n}dS = \frac{1}{\mid\mathbf{v}\mid}\int\int_S \mathbf{v}\cdot d\mathbf{S}$$

Now the divergence theorem shows that

$$I = \frac{1}{\mid\mathbf{v}\mid}\int\int\int_U \mathrm{div}\ \mathbf{v}\,dV = 0$$

b) The divergence theorem gives

$$\int\int\int_U \mathrm{div}\ \mathbf{n}\,dV = \int\int_S \mathbf{n}\cdot d\mathbf{S} = \int\int_S \mathbf{n}\cdot\mathbf{n}dS = \int\int_S 1dS = \text{Area }(S)$$

□

Problem 175:

Let $\mathbf{F} \in \mathcal{C}^2(\mathbb{R}^3)$ be a vector field such that $\nabla^2 \mathbf{F} = 0$ and let U be a region to which we can apply the divergence theorem. Show that

$$\int\int\int_U |\text{ rot } \mathbf{F}|^2 \, dV = \int\int_{\partial U} (\mathbf{F} \times \text{rot } \mathbf{F}) \cdot d\mathbf{S} + \int\int\int_U \mathbf{F} \cdot \text{grad } (\text{div } \mathbf{F}) dV$$

Solution:

Equivalently we want to prove that

$$\int\int_{\partial U} (\mathbf{F} \times \text{rot } \mathbf{F}) \cdot d\mathbf{S} = \int\int\int_U (|\text{ rot } \mathbf{F}|^2 - \mathbf{F} \cdot \text{grad } (\text{div } \mathbf{F})) dV$$

Apply the formula (see p.72)

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

to $\mathbf{G} = \text{rot } \mathbf{F}$ and obtain

$$\begin{aligned} \text{div } (\mathbf{F} \times \text{rot } \mathbf{F}) &= \text{rot } \mathbf{F} \cdot \text{rot } \mathbf{F} - \mathbf{F} \cdot \text{rot } (\text{rot } \mathbf{F}) = \\ &= |\text{ rot } \mathbf{F}|^2 - \mathbf{F} \cdot \text{rot } (\text{rot } \mathbf{F}) \end{aligned}$$

and by the divergence theorem it suffices to show that rot (rot $\mathbf{F}$) = grad (div $\mathbf{F}$). We have:

$$\text{rot } (\text{rot } \mathbf{F}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (\partial_y F_3 - \partial_z F_2) & (\partial_z F_1 - \partial_x F_3) & (\partial_x F_2 - \partial_y F_1) \end{pmatrix} =$$

$$= (\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x \partial z}, \ldots, \ldots)$$

And taking into account the condition $\nabla^2 \mathbf{F} = 0$ we may write

$$\text{rot } (\text{rot } \mathbf{F}) = (\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}, \ldots, \ldots)$$

On the other hand

$$\begin{aligned} \text{grad } (\text{div } \mathbf{F}) &= \text{grad } (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) = \\ &= (\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x}, \ldots, \ldots) \end{aligned}$$

We have proved the equality for the first component; the equality of the other components is seen in a similar manner.

□

Problem 176: Let $U \subset \mathbb{R}^3$ a simply connected open set with boundary $S = \partial U$, a closed surface oriented by the unit exterior normal vector $\mathbf{n}$. Let $\mathbf{F}, \mathbf{G} \in \mathcal{C}^1(U \cup S)$ vector fields such that in U

$$\begin{aligned} \text{rot } \mathbf{F} &= \text{rot } \mathbf{G} \\ \text{div } \mathbf{F} &= \text{div } \mathbf{G} \end{aligned}$$

and on S

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{G} \cdot \mathbf{n}$$

Show that $\mathbf{F} = \mathbf{G}$ in U. Hint: show that $\mathbf{X} = \mathbf{F} - \mathbf{G}$ has a potential φ and apply the divergence theorem to $\varphi \mathbf{X}$.

Solution: In U

$$\begin{aligned} \text{rot } \mathbf{X} &= \text{rot } \mathbf{F} - \text{rot } \mathbf{G} = \mathbf{0} \\ \text{div } \mathbf{X} &= \text{div } \mathbf{F} - \text{div } \mathbf{G} = 0 \end{aligned}$$

and taking into account that U is simply connected, the field $\mathbf{X}$ has a potential: $\mathbf{X} = \nabla \varphi$. Apply the divergence theorem to $\varphi \mathbf{X}$:

$$\int \int_{\partial U} \varphi \mathbf{X} \cdot d\mathbf{S} = \int \int \int_U \text{div}(\varphi \mathbf{X}) dV$$

The left hand integral vanishes because $\mathbf{X} \cdot \mathbf{n} = 0$ on ∂U. For the right hand integral we have

$$\text{div}(\varphi \mathbf{X}) = \nabla \varphi \cdot \mathbf{X} + \varphi \, \text{div} \mathbf{X} = \nabla \varphi \cdot \mathbf{X} = \mathbf{X} \cdot \mathbf{X} = |\mathbf{X}|^2$$

Thus

$$\int \int \int_U |\mathbf{X}|^2 dV = 0$$

and as $|\mathbf{X}|^2 \geq 0$ and $\mathbf{X}$ is continuous we have $|\mathbf{X}|^2 = 0$ that is $\mathbf{X} = 0$.

□

Problem 177: A nowhere zero scalar field u satisfies

$$|\nabla u|^2 = 4u,\ \nabla \cdot (u\nabla u) = 10u$$

Let S^2 be the unit sphere oriented by the exterior normal. Compute

$$\int\int_{S^2} \frac{\partial u}{\partial \mathbf{n}} dS$$

Solution: Let B be the unit ball; we have

$$\int\int_{S^2} \frac{\partial u}{\partial \mathbf{n}} dS = \int\int_{S^2} \nabla u \cdot \mathbf{n} dS = \int\int\int_B \mathrm{div}(\nabla u) dV = \int\int\int_B \nabla^2 u dV$$

but

$$\begin{aligned} \mathrm{div}(u\nabla u) &= \nabla u \cdot \nabla u + u\nabla^2 u \\ 10u &= 4u + u\nabla^2 u \\ \nabla^2 u &= 6 \end{aligned}$$

Then

$$\int\int_{S^2} \frac{\partial u}{\partial \mathbf{n}} dS = \int\int\int_B 6 dV = 6\frac{4}{3}\pi = 8\pi$$

□

6.3.2 Volume calculation

Problem 178 : Volume calculations using surface integrals. Volume of cones.

We know how to compute areas through line integrals (see p.265); in a similar way we can compute volumes by means of surface integrals.

a) Let $U \in \mathbb{R}^3$ be a region to which we can apply the divergence theorem. Show:

$$\begin{aligned} \mathrm{Vol}(U) &= \int\int_{\partial U} (x,0,0) \cdot d\mathbf{S} = \int\int_{\partial U} (0,y,0) \cdot d\mathbf{S} = \int\int_{\partial U} (0,0,z) \cdot d\mathbf{S} = \\ &= \frac{1}{3} \int\int_{\partial U} (x,y,z) \cdot d\mathbf{S} \end{aligned}$$

∂U being oriented by the exterior normal.

b) Application: volume of a general cone. Let C be a simple closed curve contained in a plane π at a distance d from the origin. Let $\mathcal{R}$ be the bounded region of C and let V be the solid cone with base $\mathcal{R}$ and vertex at the origin. Show that

$$\boxed{\text{Vol }(V) = \frac{1}{3}\text{Area}(\mathcal{R})d}$$

Assume we can apply the divergence theorem to V.

c) To use the preceding formula we must know how to compute Area$(\mathcal{R})$. Show that the area enclosed by a simple closed curve C contained in the plane $\pi : ax + by + cz = p$, where $a^2 + b^2 + c^2 = 1$ is

$$A = \frac{1}{2}\int_C ((bz - cy)dx + (cx - az)dy + (ay - bx)dz)$$

C being oriented leaving $\mathcal{R}$ to the left as seen from the side to which the normal vector (a, b, c) points.

d) That is the formula, but where does it come from? To see it remind (see p.280) that if C is a $\mathcal{C}^1$ regular, simple, closed plane curve, positively oriented with bounded region $\mathcal{R}$ and $\mathbf{n}$ is the exterior unit normal vector to C then:

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_C \mathbf{r}\cdot\mathbf{n}\,dl$$

Use this formula to prove the one in c).

Solution:

a) Applying the divergence theorem to each of the fields

$$\mathbf{F}(x, y, z) = (x, 0, 0), \mathbf{F}(x, y, z) = (0, y, 0), \mathbf{F}(x, y, z) = (0, 0, z),$$

we obtain the first three results; the fourth is the average of the others. Comment: We have seen that the volume of U can be computed using the formula

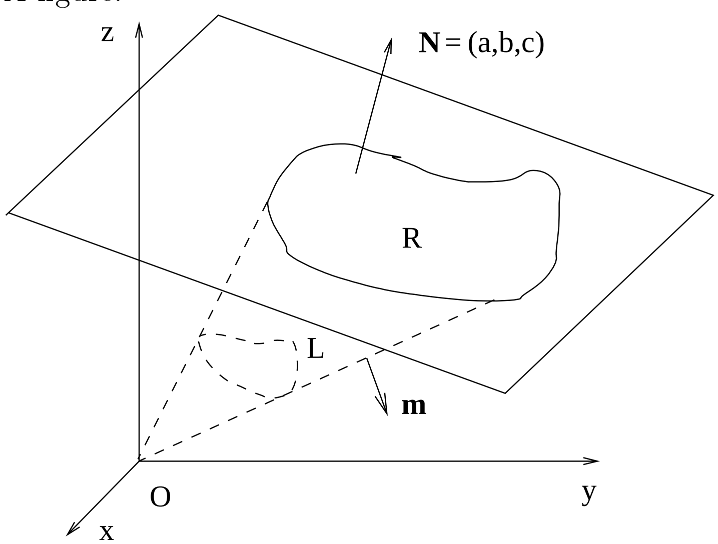
$$\frac{1}{3}\int\int_{\partial U}(x, y, z)\cdot d\mathbf{S} = \frac{1}{3}\int\int_{\partial U}\mathbf{r}\cdot\mathbf{n}dS$$

which is the flux of $\mathbf{r}$ across the boundary of U. Compare with the formula in d) that gives the area of a plane curve C

$$\frac{1}{2}\int_C \mathbf{r}\cdot\mathbf{n}dl$$

and notice that $\mathbf{n}$ being the normal vector to the curve, we are again computing the flux of the field $\mathbf{r}$ across C, the boundary of $\mathcal{R}$.

b) A figure:



Using the last formula in a) and taking into account that $\partial V = \mathcal{R}\cup L$ where L is the lateral surface of the cone we have:

$$\begin{aligned}\text{Vol}(V) &= \frac{1}{3}\int\int_{\partial V}(x,y,z)\cdot d\mathbf{S} = \\ &= \frac{1}{3}\int\int_{\mathcal{R}}(x,y,z)\cdot d\mathbf{S} + \frac{1}{3}\int\int_{L}(x,y,z)\cdot d\mathbf{S}\end{aligned}$$

But $\int\int_L (x,y,z)\cdot d\mathbf{S} = 0$ because the field (x,y,z) is perpendicular to $\mathbf{m}$, the exterior normal vector to L.
On another hand let (a,b,c) be the unit vector orthogonal to the plane π pointing to the exterior of V (this is so iff $(a,b,c)\cdot(x,y,z) > 0$ at points (x,y,z) in the plane); then the equation of the plane is

$$ax + by + cz = p$$

with $p \geq 0$ and

$$d = \frac{|a \cdot 0 + b \cdot 0 + c \cdot 0 - p|}{\sqrt{a^2 + b^2 + c^2}} = |-p| = p$$

We obtain:

$$\text{Vol}(V) = \frac{1}{3} \int \int_{\mathcal{R}} (x, y, z) \cdot d\mathbf{S} = \frac{1}{3} \int \int_{\mathcal{R}} (x, y, z) \cdot (a, b, c) dS =$$

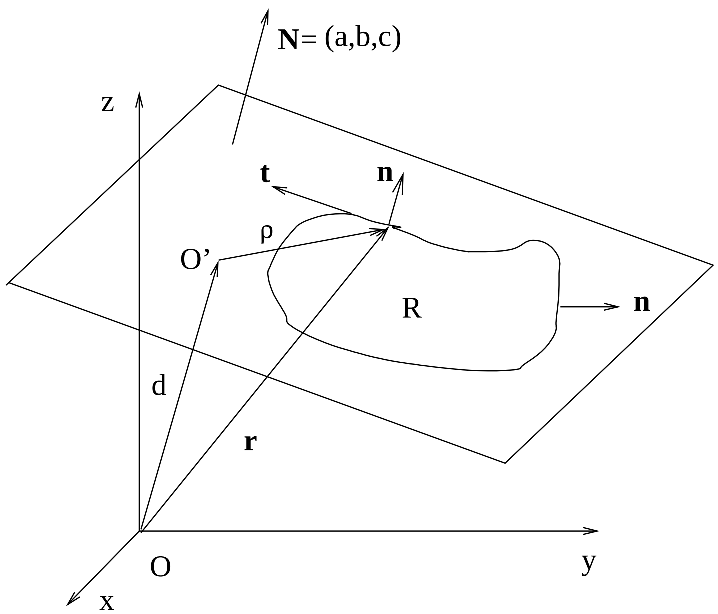
$$= \frac{1}{3} \int \int_{\mathcal{R}} (ax + by + cz) dS = \frac{1}{3} d \int \int_{\mathcal{R}} dS = \frac{1}{3} \text{Area}(\mathcal{R}) d$$

Notice that this result generalizes the usual one for cones with circular basis: $\frac{1}{3}$(Area basis)$\times$(height). Now we know, for instance, that the volume of an elliptic cone with basis semiaxes a, b and height d is $\frac{1}{3}\pi abd$. Note as well that a pyramid is a cone.

c) Orient the plane by the unit normal vector $\mathbf{n} = (a, b, c)$ and the curve C according to this choice; this allows the use of Stokes theorem

$$\begin{aligned}
\frac{1}{2} \int_C ((bz - cy)dx + (cx - az)dy + (ay - bx)dz) &= \\
= \frac{1}{2} \int_C (bz - cy, cx - az, ay - bx) \cdot d\mathbf{l} &= \\
= \frac{1}{2} \int \int_R \text{rot}(bz - cy, cx - az, ay - bx) \cdot d\mathbf{S} &= \\
= \frac{1}{2} \int \int_R 2(a, b, c) \cdot \mathbf{n} dS &= \\
= \int \int_{\mathcal{R}} \mathbf{n} \cdot \mathbf{n} dS = \int \int_{\mathcal{R}} 1 dS &= \text{Area}(\mathcal{R})
\end{aligned}$$

d) A figure:



We should adapt the above mentioned formula $\frac{1}{2}\int_C \mathbf{r}\cdot\mathbf{n}dl$ to do computations in $\mathbb{R}^3$. Orient the plane $\pi : ax + by + cz = p, a^2 + b^2 + c^2 = 1$ by the vector $\mathbf{N} = (a, b, c)$. Then we may orient the curve C leaving the bounded region $\mathcal{R}$ to the left (as seen from that side of the plane containing the curve pointed by $\mathbf{N}$). Let d be the distance from the origin to the plane and choose in π a new origin $O' = d\,\mathbf{N}$. Let $\rho = \mathbf{r} - d\,\mathbf{N}$ the radial field from O'; the formula gives

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_C \rho\cdot\mathbf{n}dl$$

If $\mathbf{t}$ is the unit tangent vector to C we have:

$$\mathbf{n} = \mathbf{t}\times\mathbf{N} \Rightarrow \rho\cdot\mathbf{n} = \rho\cdot(\mathbf{t}\times\mathbf{N}) = \mathbf{t}\cdot(\mathbf{N}\times\rho)$$

and

$$\mathbf{N}\times\rho = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x-da & y-db & z-dc \end{vmatrix} = (bz - cy, cx - az, ay - bx)$$

Substituting into the formula for the area gives:

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_C (\mathbf{N}\times\rho)\cdot\mathbf{t}dl = \frac{1}{2}\int_C (\mathbf{N}\times\rho)d\mathbf{l} =$$

$$= \frac{1}{2} \int_C (bz - cy, cx - az, ay - bx) d\mathbf{l} =$$

$$= \frac{1}{2} \int_C (bz - cy) dx + (cx - az) dy + (ay - bx) dz$$

□

Problem 179: Second Pappus-Guldin theorem.

Let C be a simple closed curve in the semiplane $y = 0, x \geq 0$; let $\mathcal{R}$ the bounded region of C. Name U the solid body generated by revolving $\mathcal{R}$ around the Oz axis. Show that

$$\text{Vol}(U) = \text{Area}(\mathcal{R}) \cdot 2\pi \langle x \rangle$$

$\langle x \rangle$ being the average of x on $\mathcal{R}$. Application: find the volume of

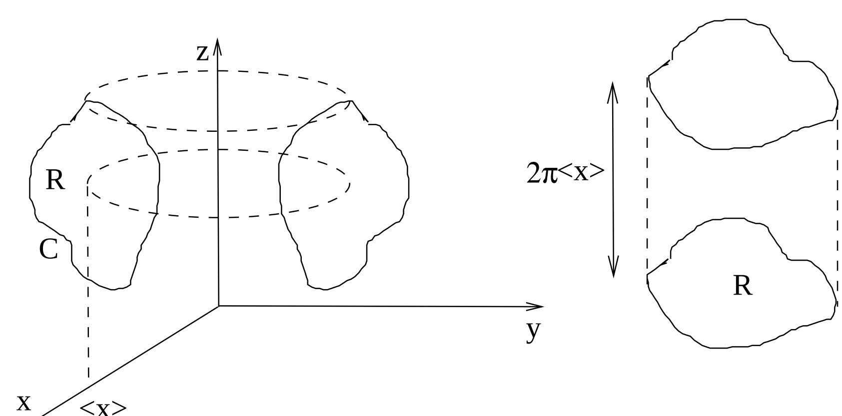
a) A circular straight cone.

b) A ball.

c) A torus.

Solution:

A figure:



We want to use the formula $\mathrm{Vol}(U) = \frac{1}{3}\int\int_{\partial U}(x, y, z)\cdot d\mathbf{S}$ and to that end we parametrize ∂U as a surface of revolution. Start from a parametrization of C:

$$\gamma(t) = (x(t), z(t)),\ t \in [a, b]$$

to obtain

$$\begin{aligned}
\alpha(\theta, t) &= (x(t)\cos\theta, x(t)\sin\theta, z(t)),\ (\theta, t) \in [0, 2\pi] \times [a, b] \\
\partial_\theta \alpha &= (-x\sin\theta, x\cos\theta, 0) \\
\partial_t \alpha &= (x'\cos\theta, x'\sin\theta, z') \\
\partial_\theta \alpha \times \partial_t \alpha &= (xz'\cos\theta, xz'\sin\theta, -xx')
\end{aligned}$$

Then

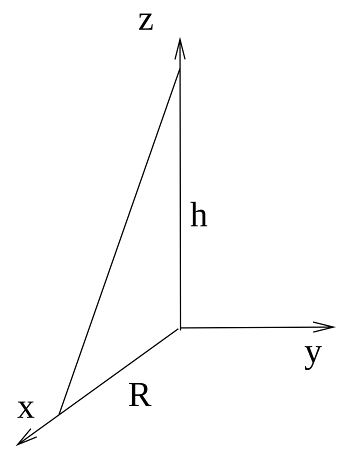
$$\begin{aligned}
\mathrm{Vol}(U) &= \frac{1}{3}\int\int_{\partial U}(x, y, z)\cdot d\mathbf{S} = \\
&= \frac{1}{3}\int_0^{2\pi}\int_a^b (x\cos\theta, x\sin\theta, z)\cdot(xz'\cos\theta, xz'\sin\theta, -xx')d\theta dt = \\
&= \frac{2\pi}{3}\int_a^b (x^2 z' - xx'z)dt
\end{aligned}$$

Now we have to relate this to the area of $\mathcal{R}$; Green's theorem does that:

$$\begin{aligned}
\mathrm{Vol}(U) &= \frac{2\pi}{3}\int_a^b (-xz, x^2)\cdot(x', z')dt = \\
&= \frac{2\pi}{3}\int_C (-xz, x^2)\cdot d\mathbf{l} = \\
&= \frac{2\pi}{3}\int\int_{\mathcal{R}} (2x - (-x))dxdz = \\
&= \frac{2\pi}{3} 3\,\langle x\rangle\,\mathrm{Area}(\mathcal{R}) = 2\pi\langle x\rangle\,\mathrm{Area}(\mathcal{R})
\end{aligned}$$

Applications:

a) Consider a right cone of height h and circular basis of radius R, generated by revolving the triangle in the figure around the Oz axis:



Parametrize the segment by $z = -\frac{h}{R}(x-R)$, $y=0$; to use the Pappus-Guldin theorem we compute the average of x on $\mathcal{R}$:

$$\begin{aligned} \int\int_{\mathcal{R}} x dx dz &= \int_0^R (\int_0^{-\frac{h}{R}(x-R)} x dz) dx = \int_0^R (x(-\frac{h}{R}(x-R))) dx = \\ &= -\frac{h}{R}(\frac{x^3}{3} - \frac{Rx^2}{2})|_0^R = \frac{hR^2}{6} \end{aligned}$$

and the average of x is:

$$\langle x \rangle = \frac{\frac{hR^2}{6}}{\frac{hR}{2}} = \frac{R}{3}$$

The theorem gives:

$$\mathrm{Vol}(U) = \frac{hR}{2} 2\pi \frac{R}{3} = \frac{1}{3}\pi R^2 h$$

Of course it's quicker if we think the figure as a right circular cone and apply the corresponding formula.

b) A ball of radius R is obtained revolving around the Oz axis the half disc

$$D = \{(x,y,z) : 0 \le x \le \sqrt{R^2 - z^2}, y = 0, -R \le z \le R\}$$

. We have:

$$\begin{aligned}\int\int_D x\,dxdz &= \int_{-R}^{R}(\int_0^{\sqrt{R^2-z^2}} xdx)dz = \\ &= \frac{1}{2}\int_{-R}^{R}(R^2-z^2)dz = \int_0^R (R^2-z^2)dz = \\ &= (R^2z-\frac{z^3}{3})|_0^R = \frac{2R^3}{3},\end{aligned}$$

and the average value of x is:

$$\langle x\rangle = \frac{\frac{2R^3}{3}}{\frac{\pi R^2}{2}} = \frac{4R}{3\pi}$$

Pappus-Guldin theorem gives

$$\text{Vol}(U) = \frac{\pi R^2}{2}2\pi\frac{4R}{3\pi} = \frac{4}{3}\pi R^3$$

c) Consider the solid torus obtained by revolving around the Oz axis the disc $D = \{(x,y,z) : (x-a)^2+z^2 \le b^2, y=0\}$. It is geometrically clear that $\langle x\rangle = a$. Then

$$\text{Vol}(U) = \pi b^2 2\pi a = 2\pi^2 ab^2$$

symmetry which is the same as the volume of a cylinder of height $2\pi a$ and basis a disc of area πb^2.

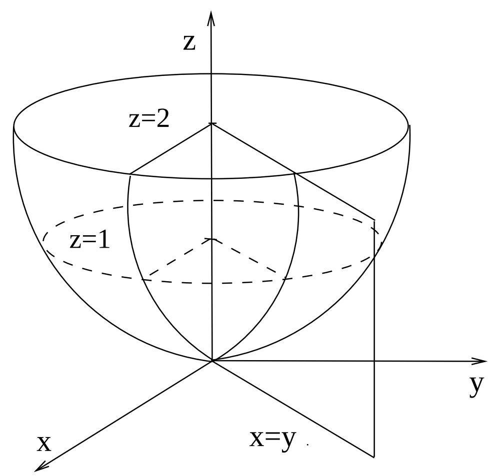
□

Problem 180:

Find the volume of the region U limited by the surfaces

$$z = x^2+y^2,\ z=1,\ z=2,\ y=0,\ y=x.$$

Solution: In the figure



we can see that $\mathrm{Vol}(U) = \frac{1}{8}\mathrm{Vol}(U')$, U' being the region of the paraboloid limited by $z = 1, z = 2$. Applying the divergence theorem to the field $\mathbf{F} = (x, y, 0)$ and to the region U' we have

$$\begin{aligned}\mathrm{Vol}(U') &= \int\int\int_{U'} 1dV = \frac{1}{2}\int\int\int_{U'} \mathrm{div}(x, y, 0)\, dV = \\ &= \frac{1}{2}\int\int_{\partial U'} (x, y, 0) \cdot d\mathbf{S}\end{aligned}$$

At the upper cover $\partial U'$ the exterior unit normal is $\mathbf{n} = (0, 0, 1)$ and $\mathbf{F} \cdot \mathbf{n} = 0$; the same is true in the lower cover. It suffices to calculate the flux across S, the lateral surface of the paraboloid that we parametrize by

$$\begin{aligned}\alpha(\theta, \rho) &= (\rho\cos\theta, \rho\sin\theta, \rho^2),\ (\theta, \rho) \in [0, 2\pi] \times [1, \sqrt{2}] \\ \mathbf{N}(\theta, \rho) &= (2\rho^2\cos\theta, 2\rho^2\sin\theta, -\rho)\end{aligned}$$

with $\mathbf{N}$ pointing to the exterior. The flux is

$$\begin{aligned}\int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_1^{\sqrt{2}} (\rho\cos\theta, \rho\sin\theta, 0) \cdot (2\rho^2\cos\theta, 2\rho^2\sin\theta, -\rho) d\theta d\rho = \\ &= \int_0^{2\pi}\int_1^{\sqrt{2}} 2\rho^3 d\theta d\rho = 4\pi\end{aligned}$$

the volume of U' is $\frac{1}{2}4\pi = 2\pi$ and that of U is

$$\text{Vol}(U) = \frac{2\pi}{8} = \frac{\pi}{4}$$

□

Problem 181:

a) Compute in terms of d :

i) A, the area the skullcap

$$C_d = \{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 < d < z < 1\}$$

ii) V, the volume of the region

$$U_d = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, 0 < d < z < 1\}$$

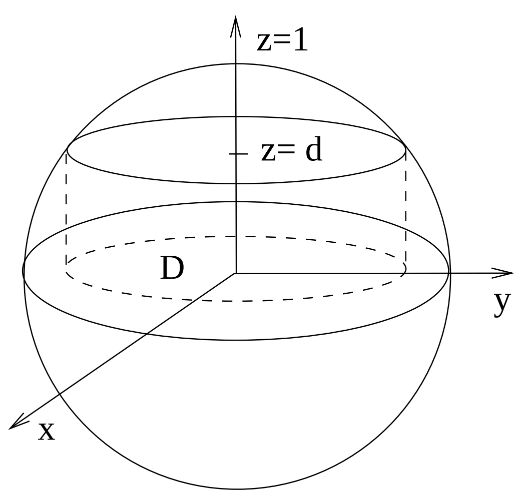
b) Using the theorem of the divergence compute the flux of the field $\mathbf{F}(x, y, z) = (x, 2y, z)$ across the upper unit semisphere

$$S^2_+ = \{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 < z\}.$$

c) Compute the flux of $\mathbf{F}$ across the surface

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, 0 < z < \frac{1}{2}\}.$$

Solution:



a) If φ means colatitude and φ_d is the colatitude corresponding to a height d we have

i)

$$A = \int\int_{C_d} 1 dS = \int_0^{2\pi} d\theta \int_0^{\varphi_d} \sin\varphi d\varphi =$$

$$= -2\pi(\cos\varphi)|_0^{\varphi_d} = 2\pi(1-d)$$

Check: when $d = 0$ we have a complete semisphere with area 2π.

ii) Closing the skullcap C_d with the disc $D_d = \{(x, y, d) : x^2 + y^2 \leq 1 - d^2\}$ we can use the formula

$$V = \frac{1}{3}(\int\int_{C_d} \mathbf{r} \cdot d\mathbf{S} + \int\int_{D_d} \mathbf{r} \cdot d\mathbf{S})$$

- On one hand

$$\int\int_{C_d} \mathbf{r} \cdot d\mathbf{S} = \int\int_{C_d} (x, y, z) \cdot (x, y, z) dS = \int\int_{C_d} 1 dS = 2\pi(1-d),$$

as we have seen in a) i).

- On another hand

$$\int\int_{D_d} \mathbf{r} \cdot d\mathbf{S} = \int\int_{D_d} (x, y, d) \cdot (0, 0, -1) dS =$$

$$= \int\int_{\{x^2+y^2 \leq 1-d^2\}} -d\, dxdy = -d\pi(1-d^2)$$

Finally

$$V = \frac{2\pi}{3}(1-d) - d\frac{\pi}{3}(1-d^2) = \frac{\pi}{3}d^3 - \pi d + \frac{2\pi}{3} = \frac{\pi}{3}(d^3 - 3d + 2)$$

b) To use the divergence theorem we close the surface S^2_+ by means of the unit disc D so establishing a closed surface M that we orient by the exterior normal; let V be the enclosed region. If ϕ is the flux across M then:

$$\phi = \int\int_M \mathbf{F} \cdot d\mathbf{S} = \int\int_{S^2_+} \mathbf{F} \cdot d\mathbf{S} + \int\int_D \mathbf{F} \cdot d\mathbf{S}$$

The divergence theorem gives

$$\int\int_M \mathbf{F}\cdot d\mathbf{S} = \int\int\int_V \nabla\cdot\mathbf{F}dV = 4\int\int\int_V 1dV = 4\frac{1}{2}\frac{4}{3}\pi = \frac{8}{3}\pi$$

On another hand as the unit exterior normal vector in D is $(0,0,-1)$ we obtain:

$$\int\int_D \mathbf{F}\cdot d\mathbf{S} = \int\int_D (x,2y,0)\cdot(0,0,-1)dxdy = \int\int_D 0dxdy = 0$$

Then

$$\int\int_{S^2_+} \mathbf{F}\cdot d\mathbf{S} = \frac{8}{3}\pi$$

c) Now the flux across S is (flux across S^2_+)-(flux across $C_{1/2}$). Using again the divergence theorem

$$\int\int_{C_{1/2}} \mathbf{F}\cdot d\mathbf{S} + \int\int_{D_{1/2}} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_{U_{1/2}} \nabla\cdot\mathbf{F}dV =$$

$$= 4\int\int\int_{U_{1/2}} dV = 4\frac{\pi}{3}(d^3 - 3d + 2)|_{d=1/2} = \frac{5\pi}{6}$$

and taking into account that

$$\int\int_{D_{1/2}} \mathbf{F}\cdot d\mathbf{S} = \int\int_{\{x^2+y^2\leq 3/4\}} (x,2y,\frac{1}{2})\cdot(0,0,-1)dxdy =$$

$$= -\int\int_{\{x^2+y^2\leq 3/4\}} \frac{1}{2}dxdy = -\frac{1}{2}\frac{3}{4}\pi$$

we have

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \frac{5\pi}{6} - (-\frac{3}{8}\pi) = \frac{29}{24}\pi$$

□

Problem 182:

Let $U \subset \mathbb{R}^3$ a region to wich we can apply the divergence theorem; show that

$$\int\int\int_U r^2 dV = \frac{1}{5}\int\int_{\partial U} r^2\mathbf{r}\cdot d\mathbf{S}$$

Solution:

Apply the divergence theorem; to do this we first have

$$\begin{aligned} \text{div } (r^2\mathbf{r}) &= \text{div } (r^2(x,y,z)) = \\ &= 2r\frac{x}{r}x + r^2 + 2r\frac{y}{r}x + r^2 + 2r\frac{z}{r}x + r^2 = \\ &= 2r^2 + 3r^2 = 5r^2 \end{aligned}$$

and then

$$\int\int_{\partial U} r^2\mathbf{r}\cdot d\mathbf{S} = \int\int\int_U 5r^2 dV$$

□

6.3.3 Green's formulae

Problem 183: Towards Green's identities.

a) Express the divergence theorem when $\mathbf{F}$ is a gradient: $\mathbf{F} = \nabla u$, u a scalar field.

b) What do we obtain when u is harmonic $(:=\nabla^2 u = 0)$?

c) What when u is a second degree polynomial ?

Solution:

a) If $\mathbf{F} = \nabla u$

$$\text{div}\mathbf{F} = \text{div}(\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u$$

and the divergence theorem is:

$$\int\int\int_U \nabla^2 u\, dV = \int\int_{\partial U} \nabla u\cdot d\mathbf{S}$$

Taking into account that $\int\int_{\partial U} \nabla u\cdot d\mathbf{S} = \int\int_{\partial U} \nabla u\cdot\mathbf{n}\, dS = \int\int_{\partial U} \frac{\partial u}{\partial \mathbf{n}} dS$ we obtain the following expression:

$$\int\int\int_U \nabla^2 u\, dV = \int\int_{\partial U} \frac{\partial u}{\partial \mathbf{n}} dS$$

b) If u is harmonic, $\nabla^2 u \equiv 0$ in U and then

$$\int\int_{\partial U} \frac{\partial u}{\partial \mathbf{n}} dS = 0$$

c) If $u = ax^2 + by^2 + cz^2 + dxy + exz + fyz + \cdots$ (the $\cdots$ are the terms of degree zero or one) we have

$$\nabla^2 u = 2a + 2b + 2c$$

and

$$2(a+b+c)\int\int\int_U dV = 2(a+b+c)\text{Vol}(U) = \int\int_{\partial U} \frac{\partial u}{\partial \mathbf{n}} dS$$

and we see that the normal derivative is proportional to the volume.

□

Problem 184. Green's identities.

a) Express the divergence theorem when $\mathbf{F}$ is a weighted gradient, $\mathbf{F} = f\nabla g$ where $f, g \in C^1(U \cup \partial U)$ to obtain *Green's first identity*:

$$\boxed{\int\int\int_U (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \int\int_{\partial U} f\frac{\partial g}{\partial \mathbf{n}} dS} \text{ Green 1}$$

b) Changing f and g in the first Green identity and subtracting, obtain *Green's second identity*:

$$\boxed{\int\int\int_U (f\nabla^2 g - g\nabla^2 f) dV = \int\int_{\partial U} (f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}) dS} \text{ Green 2}$$

Solution:

a) In this case the divergence theorem is

$$\int\int\int_U \text{div } (f\nabla g) dV = \int\int_{\partial U} f\nabla g \cdot d\mathbf{S}$$

We compute both integrands:

$$\mathrm{div}(f\nabla g) = \nabla f \cdot \nabla g + f\mathrm{div}(\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g$$

$$\nabla g \cdot d\mathbf{S} = \nabla g \cdot \mathbf{n} dS = \frac{\partial g}{\partial \mathbf{n}} dS$$

Substituting we obtain Green's first identity:

$$\int\int\int_U (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \int\int_{\partial U} f\frac{\partial g}{\partial \mathbf{n}} dS$$

b) Green's first identity for f and g is

$$\int\int\int_U (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \int\int_{\partial U} f\frac{\partial g}{\partial \mathbf{n}} dS$$

and the same identity for g and f is:

$$\int\int\int_U (g\nabla^2 f + \nabla f \cdot \nabla g) dV = \int\int_{\partial U} g\frac{\partial f}{\partial \mathbf{n}} dS$$

Subtracting the second from the first we arrive at the second of Green's identities:

$$\int\int_U (f\nabla^2 g - g\nabla^2 f) dV = \int\int_{\partial U} (f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}) dS$$

□

Problem 185:

Let f be a harmonic function in $\overline{B_R} = \{\mathbf{x} \in \mathbb{R}^3 :\mid \mathbf{x} \mid\leq R\}$; show that the average of f on the boundary of B_R satisfies

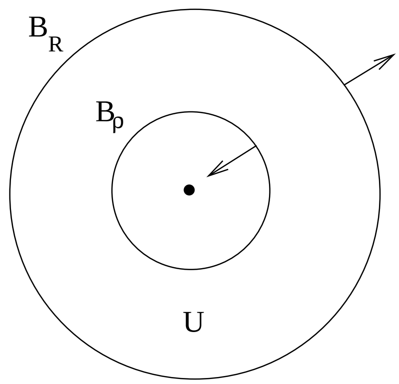
$$\langle f \rangle_{S_R} = f(\mathbf{0})$$

symmetry Hint: apply Green 2 to f and $\frac{1}{r}$.

Solution:

We may use Green's formulae in an open set where the divergence theorem applies, but $\frac{1}{r}$ has a singularity at $\mathbf{0}$. We remove a whole small ball $\overline{B_\rho} \subset B_R$ whose boundary we orient by the normal vector pointing to the interior as shown in the figure. Consider $U = B_R \setminus \overline{B_\rho}$ and write Green 2 applied to it:

$$\int\int\int_U (f\nabla^2 g - g\nabla^2 f)dV = \int\int_{\partial U} (f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}})dS$$



The functions $f = f, g = \frac{1}{r}$ are harmonic in U and the left term integral will vanish. The right term integral is:

$$\begin{aligned}\int\int_{S_R} (f\frac{\partial(\frac{1}{r})}{\partial \mathbf{n}} - \frac{1}{r}\frac{\partial f}{\partial \mathbf{n}})dS &= \int\int_{S_R} (f(-\frac{\mathbf{r}}{r^3} \cdot \mathbf{n}) - \frac{1}{r}\nabla f \cdot \mathbf{n})dS = \\ &= \int\int_{S_R} (f(-\frac{1}{R^2}) - \frac{1}{R}\nabla f \cdot \mathbf{n})dS\end{aligned}$$

and by the divergence theorem

$$\begin{aligned}\int\int_{S_R} -\frac{1}{R}\nabla f \cdot \mathbf{n} dS &= -\frac{1}{R}\int\int_{S_R} \nabla f \cdot d\mathbf{S} = \\ &= -\frac{1}{R}\int\int\int_{B_R} \text{div (grad } f)dV = \\ &= -\frac{1}{R}\int\int\int_{B_R} \nabla^2 f dV = 0\end{aligned}$$

because of the harmonicity of f in B_R.
Analogously, reminding the orientation given to S_ρ, and using again the

divergence theorem we obtain:

$$\begin{aligned}\int\int_{S_\rho}(f\frac{\partial(\frac{1}{r})}{\partial \mathbf{n}}-\frac{1}{r}\frac{\partial f}{\partial \mathbf{n}})dS &= \int\int_{S_\rho}(f(-\frac{\mathbf{r}}{r^3}\cdot\mathbf{n})-\frac{1}{r}\nabla f\cdot\mathbf{n})dS = \\ &= \int\int_{S_\rho}(f\frac{1}{\rho^2}-\frac{1}{\rho}\nabla f\cdot\mathbf{n})dS = \\ &= \int\int_{S_\rho} f\frac{1}{\rho^2}dS\end{aligned}$$

Summing up those calculations

$$\begin{aligned}0 &= -\frac{1}{R^2}\int\int_{S_R} f dS + \frac{1}{\rho^2}\int\int_{S_\rho} f dS \\ \frac{1}{R^2}\int\int_{S_R} f dS = 4\pi\langle f\rangle_{S_R} &= \frac{1}{\rho^2}\int\int_{S_\rho} f dS = 4\pi\langle f\rangle_{S_\rho} \to 4\pi f(0)\end{aligned}$$

We have obtained

$$\langle f\rangle_{S_R} = f(0)$$

□

Green's identities are used to prove the uniqueness of solutions for the problems of Dirichlet and Neumann.

Problem 186: Dirichlet and Neumann problems.

Let $U \subset \mathbb{R}^3$ be an open set to which we can apply the divergence theorem.

a) The *Dirichlet problem* is to find a function u harmonic in U and $u \in C^1(\overline{U})$, such that $u(\mathbf{p}) = \varphi(\mathbf{p})$ in ∂U, φ being a given continuous function in ∂U. Briefly

$$\begin{cases}\nabla^2 u = 0 \text{ in } U \\ u = \varphi \text{ in } \partial U\end{cases}$$

Show that if a solution of this problem exists then it is unique. Hint: apply Green 1 to the difference of solutions to show that it must be zero.

b) The *Neumann problem* is to find a function u harmonic in U and $u \in C^1(\overline{U})$, such that $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{p}) = \varphi(\mathbf{p})$ in ∂U, φ being a given continuous function in ∂U. Briefly

$$\begin{cases} \nabla^2 u &= 0 \text{ a } U \\ \frac{\partial u}{\partial \mathbf{n}} &= \varphi \text{ a } \partial U \end{cases}$$

Show that solutions differ in a constant.

Solution:

a) Assume U is connected. Let u_1 and u_2 be solutions of the Dirichlet problem. Applying the first Green identity to $f = g = u_1 - u_2$

$$\int\int\int_U ((u_1 - u_2)\nabla^2(u_1 - u_2) + |\nabla(u_1 - u_2)|^2)dV =$$

$$= \int\int_{\partial U} (u_1 - u_2)\frac{\partial(u_1 - u_2)}{\partial \mathbf{n}} dS$$

On ∂U the integrand is $(u_1 - u_2)(\mathbf{p}) = \varphi(\mathbf{p}) - \varphi(\mathbf{p}) = 0$ and the right hand term vanishes. In the left hand term $\nabla^2(u_1 - u_2) = 0$ because both functions are harmonic in U . Then

$$\int\int\int_U |\nabla(u_1 - u_2)|^2 dV = 0$$

Being $|\nabla(u_1 - u_2)|^2 \geq 0$ a nonnegative continuous function with vanishing integral we must have $\nabla(u_1 - u_2) = 0$. As U is a connected set in $\mathbb{R}^3$ we can join any two points with a polygonal line without leaving U. Applying to each side of the polygonal the mean value theorem we may conclude that $u_1 - u_2 = \text{const}$ in U. But $u_1 - u_2 = 0$ on ∂U and $u_1 - u_2$ is continuous in $\bar{U}$; then the constant must be zero and $u_1 = u_2$ in $\bar{U}$.

Notice that we have shown that if two harmonic functions coincide on the boundary then they coincide as well in U.

b) Let u_1 and u_2 solutions of the Neumann problem; applying the first Green identity to $f = g = u_1 - u_2$ the right hand term $\int\int_{\partial U}(u_1 - u_2)\frac{\partial(u_1 - u_2)}{\partial \mathbf{n}} dS$ vanishes, now because we are assuming that $\frac{\partial(u_1 - u_2)}{\partial \mathbf{n}} = 0$ on ∂U. Now we may conclude that $u_1 - u_2 = \text{const}$ in $\bar{U}$.

Notice we have proved that whenever two harmonic functions have the same normal derivative on the boudary, they differ in a constant.

□

Problem 187:

Let $u \in C^1(\overline{U}) \cap C^2(U)$ be a nontrivial solution of

$$\begin{cases} \nabla^2 u + \lambda u &= 0 \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{cases}$$

Prove that $\lambda \geq 0$.

Solution:

Taking $f = g = u$ in Green's first identity we have:

$$\int\int\int_U (u\nabla^2 u + \nabla u \cdot \nabla u) dV = \int\int_{\partial U} u \frac{\partial u}{\partial \mathbf{n}} dS$$

In the left hand side term $\nabla^2 u = -\lambda u$ and in the right hand side $u = 0$ because the integration is on ∂U. Then:

$$\int\int\int_U (-\lambda u^2 + |\nabla u|^2) dV = 0$$

$$\lambda \int\int\int_U u^2 dV = \int\int\int_U |\nabla u|^2 dV$$

and we see that $\lambda \geq 0$.

□

Problem 188: Heat equation and energy.

Let $U \subset \mathbb{R}^3$ be an open set to which we can apply the divergence theorem and $u(x, y, z, t) \in \mathcal{C}^2$ a solution of heat's equation

$$k\nabla^2 u = \partial_t u, \; k > 0$$

such that $\frac{\partial u}{\partial \mathbf{n}} = 0$ on ∂U. Show that $E(t) = \frac{1}{2} \int\int\int_U u^2 dV$ (the energy in U) is a nonincreasing function. Hint: compute $\nabla \cdot (u\nabla u)$. (u may be thought as the temperature at points of the body U)

Solution:

$$\frac{dE}{dt} = \frac{1}{2}\int\int\int_U 2u\frac{\partial u}{\partial t}dV = \int\int\int_U uk\nabla^2 u\, dV$$

We are not integrating a divergence; to tranform the integrand compute

$$\nabla \cdot (u\nabla u) = \nabla \cdot (u\partial_x u, u\partial_y u, u\partial_z u) =$$

$$= (\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2 + u\nabla^2 u =$$

$$= |\nabla u|^2 + u\nabla^2 u$$

and then

$$\frac{dE}{dt} = k\int\int\int_U (\nabla \cdot (u\nabla u) - |\nabla u|^2)dV =$$

now using the divergence theorem

$$\int\int\int_U \nabla \cdot (u\nabla u)dV = \int\int_{\partial U} (u\nabla u) \cdot \mathbf{n}dS = \int\int_{\partial U} u\frac{\partial u}{\partial \mathbf{n}}dS = 0$$

Finally

$$\frac{dE}{dt} = k\int\int\int_U -|\nabla u|^2 dV \leq 0$$

and $E(t)$ is nonincreasing.

□

6.3.4 Gauss integral theorem

Problem 189: Gauss integral theorem.

Let $U \subset \mathbb{R}^3$ be an open set to which we can apply the divergence theorem and $\partial U = S$, a closed surface oriented by unit exterior normal $\mathbf{n}$. Show that

$$\int\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = \begin{cases} 4\pi & \text{if } \mathbf{0} \text{ interior to } S \\ 0 & \text{if } \mathbf{0} \text{ exterior to } S \end{cases}$$

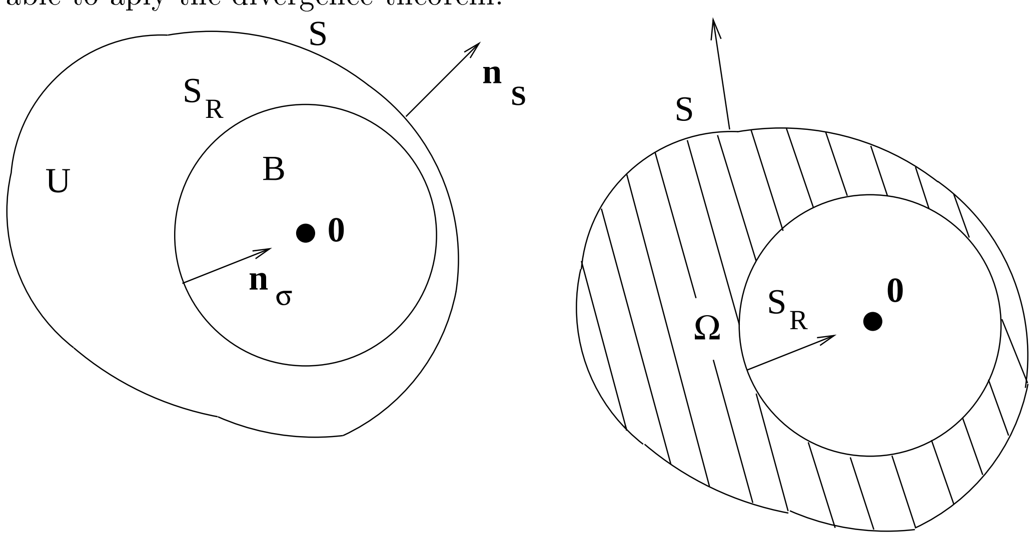
Solution:

The result says that the solid angle of a closed surface oriented by the exterior normal is 4π if the vertex is interior to the surface and 0 if it is exterior. We can say as well that the electric field generated by a positive unit charge has a flux 4π if the charge is an interior one and is 0 if it is exterior. The gravitational field of a unit mass has a flux -4π if the mass is interior and a flux 0 if it is exterior.

a) If $\mathbf{0}$ is exterior to S the field $\frac{\mathbf{r}}{r^3}$ has no singularity in $U \cup S$ and, moreover, it has zero divergence; the divergence theorem gives:

$$\int\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = \int\int\int_U 0 dV = 0$$

b) If $\mathbf{0}$ is interior to S we cannot apply the divergence theorem because the field has a singularity at $\mathbf{0}$. Choose a ball $B = \{\mathbf{r} :\mid \mathbf{r} \mid \leq R\} \subset U$ and remove it from U. The region left $\Omega = U \setminus B$ has a boundary $\partial\Omega = S \cup S_R$; orient this boundary as shown in the figure so as to be able to aply the divergence theorem.



On S_R

$$\int\int_{S_R} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = \int\int_{S_R} \frac{1}{R^3}(x,y,z) \cdot (-\frac{1}{R}(x,y,z))dS = -\int\int_{S_R} \frac{1}{R^2} dS = -4\pi$$

In Ω the field $\frac{\mathbf{r}}{r^3}$ has zero divergence and the divergence theorem gives:

$$\begin{aligned} 0 &= \int\int\int_\Omega \text{div}\,(\frac{\mathbf{r}}{r^3})dV = \int\int_S \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S} + \int\int_{S_R} \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S} \\ \int\int_S \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S} &= -\int\int_{S_R} \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S} = 4\pi \end{aligned}$$

□

Problem 190:

Gauss integral theorem shows that if m is a point mass and $\mathbf{g}$ its gravitational field, the flux across any closed surface S oriented by the exterior normal is $-4\pi m$ if m is an interior point of S and 0 if m is exterior.

Show that:

a) If $m_1, \dots, m_n$ are point masses and $\mathbf{g}$ their gravitational field, the flux of $\mathbf{g}$ across any closed surface S that doesn't pass through any of the masses and oriented by the exterior normal is $-4\pi M$, M being the sum of the interior masses.

b) The result in a) is true for a continuous distribution and a closed surface S that doesn't cut the distribution.

Solution:

a) According to Gauss integral theorem the flux due to exterior masses vanishes; let $\mathbf{g}_1, \dots, \mathbf{g}_k$ be the fields generated by the interior masses $m_1, \dots, m_k$. Then

$$\begin{aligned} \int\int_S \mathbf{g}\cdot d\mathbf{S} &= \int\int_S \mathbf{g}_1\cdot d\mathbf{S} + \cdots + \int\int_S \mathbf{g}_k\cdot d\mathbf{S} = \\ &= -4\pi(m_1 + \cdots + m_k) = -4\pi M \end{aligned}$$

b) The field generated by a continuous mass distribution $\chi(x,y,z)$ located in a region V is:

$$\mathbf{g}(x,y,z) = \int\int\int_V -\chi\frac{\mathbf{r}}{r^3}dV,\ \mathbf{r} = (x-u, y-v, z-w), (u,v,w) \in V$$

Let a closed surface S be given and call V_1 that part of the distribution which is outside S and V_2 the interior part.

i) Let $\mathbf{g}_1$ the field created by the masses in V_1. As there are no masses of V_1 inside of S r doesn't vanish and the integrand has no singularity. Moreover $\frac{\mathbf{r}}{r^3}$ has continuous derivatives and we can differentiate the integral thus

$$\text{div } \mathbf{g}_1 = \int\int\int_{V_1} -\chi(u,v,w)(\frac{\partial}{\partial x}(\frac{x-u}{r^3})+\frac{\partial}{\partial y}(\frac{y-v}{r^3})+\frac{\partial}{\partial z}(\frac{z-w}{r^3}))dV$$

but the integrand satisfies

$$\frac{\partial}{\partial x}(\frac{x-u}{r^3}) + \frac{\partial}{\partial y}(\frac{y-v}{r^3}) + \frac{\partial}{\partial z}(\frac{z-c}{r^3}) = \text{div } \frac{\mathbf{r}}{r^3} = 0,$$

and div $\mathbf{g}_1 = 0$ in U, the interior region of S. By the divergence theorem its flux through S is

$$\int\int_S \mathbf{g}_1 \cdot d\mathbf{S} = \int\int\int_U \text{div } \mathbf{g}_1 dV = 0$$

Summing up: the flux of the field from exterior masses is zero.

ii) Let $\mathbf{g}_2$ be the field created by interior masses; its flux across S is

$$\int\int_S \mathbf{g}_2 \cdot d\mathbf{S} = \int\int_S (\int\int\int_{V_2} -\chi\frac{\mathbf{r}}{r^3}dV) \cdot d\mathbf{S}$$

As S doesn't cut the distribution r doesn't vanish on S and, being the integrand continuous, we can invert the order of integration:

$$\int\int_S \mathbf{g} \cdot d\mathbf{S} = \int\int\int_{V_2} (\int\int_S -\chi\frac{\mathbf{r}}{r^3} \cdot d\mathbf{S})dV$$

But from Gauss integral theorem, for interior points

$$\int\int_S -\frac{\mathbf{r}}{r^3} \cdot d\mathbf{S} = -4\pi$$

Then

$$\int\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi \int\int\int_{V_2} \chi dV = -4\pi M$$

Under special conditions on χ the result is true even if S cuts the distribution (see [Kell], p. 73).

□

Problem 191: The divergence teorem in the plane.

Guess a plane version of the divergence theorem and derive it from the usual divergence theorem.

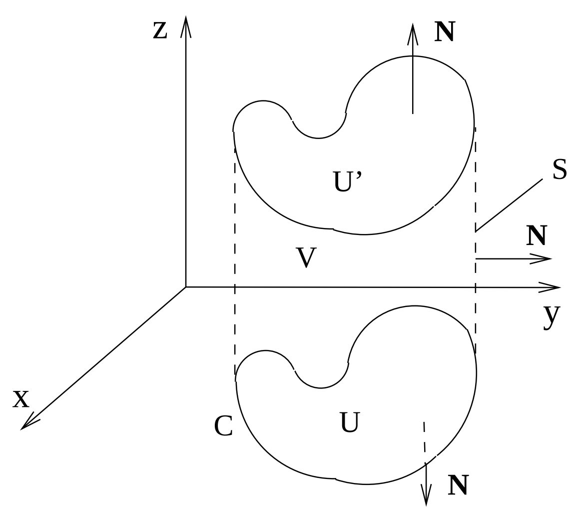
Solution:

Let $\mathbf{F}(x,y) = (X(x,y), Y(x,y))$ be a field in $\mathbb{R}^2$ and $C \subset \mathbb{R}^2$ a simple closed curve that has U as bounded region. Let $\mathbf{n}$ be the exterior normal vector to C; we would like to show that

$$\boxed{\int\int_U \text{div }\mathbf{F}\,dxdy = \int_{\partial U} \mathbf{F}\cdot\mathbf{n}\,dl}$$

To use Gauss theorem we convert our two dimensional problem into a three dimensional one. To this end define a field in $\mathbb{R}^3$ and a cylindrical volume:

$$\begin{aligned}\mathbf{F}_E(x,y,z) &= (X(x,y), Y(x,y), 0)\\ V &= U \times [0,1]\end{aligned}$$



Orient ∂V by the exterior normal $\mathbf{N}$ as shown in the figure and apply the divergence theorem:

$$\int\int\int_V \text{div }\mathbf{F}_E dV = \int\int_{\partial V} \mathbf{F}_E \cdot d\mathbf{S}$$

a) The left hand term is

$$\int\int\int_V \text{div } \mathbf{F}_E dV = \int\int\int_V (\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}) dV = $$
$$= \int_0^1 (\int\int_U \text{div } \mathbf{F} dxdy) dz = \int\int_U \text{div } \mathbf{F} dxdy$$

b) As to the right hand term notice first that the boundary ∂V is the union of two covers $U \simeq U \times \{0\}, U' \simeq U \times \{1\}$ and a lateral surface S. The flux is null across the covers, for $\mathbf{F}_E \cdot \mathbf{N} = (X, Y, 0) \cdot (0, 0, \pm 1) = 0$. To compute the flux across S we parametrize it by

$$\alpha(t, s) = (x(t), y(t), s), (t, s) \in [a, b] \times [0, 1]$$

$(x(t), y(t))$ being a parametrization of C traversing it in the positive sense (leaving U to the left).
Then the associated normal vector that points to the exterior of C is the one that makes $(\mathbf{n}, \mathbf{t})$ a positive basis. It is $\mathbf{n} = (y', -x')$, for

$$\det \begin{pmatrix} y' & x' \\ -x' & y' \end{pmatrix} > 0$$

The associated normal vector of the parametrization is

$$\begin{aligned} \alpha_t &= (x'(t), y'(t), 0) \\ \alpha_s &= (0, 0, 1) \\ \alpha_t \times \alpha_s &= (y', -x', 0) \end{aligned}$$

that points to the exterior. Now we may compute the flux:

$$\begin{aligned} \int\int_{\partial V} \mathbf{F}_E \cdot d\mathbf{S} &= \int\int_S (X, Y, 0) \cdot (y', -x', 0) dS = \\ &= \int_0^1 (\int_a^b (Xy' - Yx') dt) ds = \\ &= \int_a^b (Xy' - Yx') dt = \int_a^b (X, Y) \cdot (y', -x') dt = \\ &= \int_a^b (X, Y) \cdot \frac{(y', -x')}{\mid (y', -x') \mid} \mid (y', -x') \mid dt = \\ &= \int_{\partial U} \mathbf{F} \cdot \mathbf{n} \mid (x', y') \mid dt = \int_{\partial U} \mathbf{F} \cdot \mathbf{n} dl \end{aligned}$$

□

Problem 192: Gauss integral theorem in the plane.

Guess and prove a Gauss integral theorem in the plane.

Solution:

Let C be a simple closed curve, positively oriented let $\mathbf{n}$ be its exterior normal and let U be the bounded region determined by C. We guess that

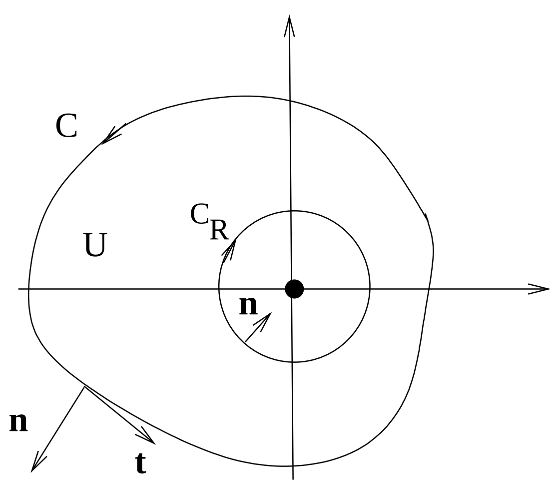
$$\int_C \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl = \begin{cases} 0 & \text{if} \quad \mathbf{0} \text{ is exterior to } C \\ ? & \text{if} \quad \mathbf{0} \text{ is interior to} C \end{cases}$$

a) If $\mathbf{0}$ is exterior to U then r is never zero and reminding that div $\frac{\mathbf{r}}{r^2} = 0$ (see p.116) we can apply the divergence theorem in the plane:

$$\int_C \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl = \int\int_U \text{div } \frac{\mathbf{r}}{r^2} \, dxdy = 0$$

b) If $\mathbf{0}$ is interior we cannot apply the divergence theorem in the plane since $\mathbf{0}$ is a singularity of the field, but we can proceed as in the proof of Gauss integral theorem. Remove a small disc $D = \{\mathbf{r} :| \mathbf{r} |\leq R\} \subset U$ from U to obtain $\Omega = U \setminus B$; then orient $\partial\Omega = C \cup C_R$ as in the figure and apply the theorem:

$$0 = \int\int_\Omega \text{div } \frac{\mathbf{r}}{r^2} \, dxdy = \int_C \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl + \int_{C_R} \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl$$



To compute the last integral parametrize C_R by

$$\gamma(t) = (R\cos t, -R\sin t), | \gamma'(t) |= R$$

and take into account that the unit interior normal vector is $\mathbf{n} = (-\cos t, \sin t)$; then:

$$\int_{C_R} \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl = \int_0^{2\pi} \frac{(R\cos t, -R\sin t)}{R^2} \cdot (-\cos t, \sin t) R dt = -2\pi$$

and by the divergence theorem

$$\int_C \frac{\mathbf{r}}{r^2} \cdot \mathbf{n} dl = 2\pi$$

□

6.3.5 Continuity equation; energy conservation.

$\boxed{\mathbb{T}}$ Consider a fluid with velocity field $\mathbf{v}(x, y, z)$ and density $\rho(x, y, z, t)$. The vector $\mathbf{J} = \rho\mathbf{v}$ is the current density vector and, in a similar way as $\mathbf{v} \cdot \mathbf{n}\Delta S$ measures the *volume* of fluid crossing ΔS in a unit of time, $\mathbf{J} \cdot \mathbf{n} dS$ measures the *mass* of fluid that crosses dS in a unit of time.

□

Problem 193: Continuity equation.

Let U be a region in $\mathbb{R}^3$ and S its boundary, a closed surface oriented by the exterior normal. Evaluating in two ways the mass exiting from U find the continuity equation

$$\boxed{\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0}$$

Solution:

The mass in U is $\int\int\int_U \rho dV$ and the mass exiting per unit time is

$$-\frac{d}{dt}\int\int\int_U \rho dV$$

On another side S the mass exiting per unit time is

$$\int\int_S \mathbf{J} \cdot d\mathbf{S}$$

Now express the conservation of mass:

$$\int\int_S \mathbf{J}\cdot d\mathbf{S} = -\frac{d}{dt}\int\int\int_U \rho dV$$

Transforming the left hand term using the divergence theorem and differentiating the right hand term we have:

$$\begin{aligned}\int\int\int_U \text{div } \mathbf{J}\, dV &= -\int\int\int_U \frac{\partial\rho}{\partial t} dV \\ \int\int\int_U (\text{div } \mathbf{J} + \frac{\partial\rho}{\partial t}) dV &= 0\end{aligned}$$

As this is true in any region of the fluid we obtain:

$$\text{div } \mathbf{J} + \frac{\partial\rho}{\partial t} = 0$$

□

$\boxed{\mathbb{T}}$ Analogous arguments will give expressions for the conservation of the energy and the conservation of the charge.

If $T(x, y, z)$ gives the temperature distribution in a body, the vector $\mathbf{h}$ that has the direction given by $-\nabla T$ and module the amount of energy crossing in a unit time a unit surface perpendicular to ∇T is called the density of energy flux. Newton's law of cooling is $\mathbf{h} = -k\nabla T$; if ρ is the energy density then $\rho = c\rho_0 T$ where c is the specific heat and ρ_0 a constant.

□

Problem 194: Energy conservation.

Find an expression for the energy conservation. Using Newton's law and the relation between ρ and T derive the heat equation.

Solution:

Let U be a region in $\mathbb{R}^3$ and S its boundary, a closed surface oriented by the exterior normal. The energy in U is $\int\int\int_U \rho dV$ and the energy exiting per unit time is

$$-\frac{d}{dt}\int\int\int_U \rho dV$$

Another expression for the exiting energy is $\int\int_S \mathbf{h}\cdot d\mathbf{S}$; an expression for the energy conservation is

$$\int\int_S \mathbf{h}\cdot d\mathbf{S} = -\frac{d}{dt}\int\int\int_U \rho dV$$

Transforming the left hand term using the divergence theorem, differentiating the right hand term and taking into account that the equality obtained is true for any region in U we have:

$$\boxed{\text{div } \mathbf{h} + \frac{\partial \rho}{\partial t} = 0}$$

Assuming k, c, ρ_0 constants and using Newton's law we obtain

$$\begin{aligned} \text{div } \mathbf{h} &= \text{div } (-k\nabla T) = -k\text{div } \nabla T = -k\nabla^2 T \\ \frac{\partial \rho}{\partial t} &= c\rho_0 \frac{\partial T}{\partial t} \end{aligned}$$

Now substituting into the energy conservation equation we obtain

$$\boxed{\nabla^2 T = a\frac{\partial T}{\partial t}}$$

that is, the heat equation.

□

Chapter 7

Electromagnetism

7.1 Maxwell equations

$\boxed{\mathbb{T}}$ When there are charges present the space acquires a 'state of electromagnetic tension' that Michael Faraday (1791-1867) described by means of the idea of a field. James Clerk Maxwell (1831-1875) was the first to establish the complete equations of the electromagnetism (see [Feyn] vol. II):

Maxwell equations

$$\boxed{\begin{array}{ll} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, & \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \\ \nabla \cdot \mathbf{B} = 0, & c^2 \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \dfrac{\mathbf{j}}{\epsilon_0} \end{array}}$$

Two vector fields, the *electric field* $\mathbf{E}(x, y, z, t)$ and the *magnetic field* $\mathbf{B}(x, y, z, t)$, describe that 'state of electromagnetic tension' at point (x, y, z) at the instant t. The sources of that space tension are the electric charges, described by means of a scalar function, the *electric charge density* $\rho(x, y, z, t)$, and the electric currents, described by means of a vector function, the *electric current density* $\mathbf{j}(x, y, z, t)$.

Given the sources Maxwell's equations allow (at least in principle) the computation of the electric field $\mathbf{E} = (E_x, E_y, E_z)$ and the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$. Then, once the fields are known, we can compute the force they exert on a charge q that moves with a velocity $\mathbf{v}$:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Then Newton's equation allows the computation of the movement of the charge.

Maxwell equations unified electricity, magnetism and light; moreover they predicted the existence of electromagnetic waves. They were as well the starting point for the discovery of special relativity and were an inspiration for the general relativity, both theories presented by A. Einstein in 1905 and 1915 (see [Feyn], vol I). They have also been a model for contemporary physics (see [Ba-Mu]).

□

Problem 195: Integral form of Maxwell's equations.

Apply Stokes's theorem or Gauss's theorem as needed and give the equations an integral form. Application: an electrostatic field is given by $\mathbf{E}(x, y, z, t) = (yz, zx, xy)$; find the charge contained in the unit sphere (assume the surface doesn't cut the distribution).

Solution:

a) Apply the divergence theorem to a region $U \subset \mathbb{R}^3$ with boundary S, a closed surface, and use the first of Maxwell equations $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$:

$$\int\int_S \mathbf{E} \cdot d\mathbf{S} = \int\int\int_U \nabla \cdot \mathbf{E}\, dV = \int\int\int_U \frac{\rho}{\epsilon_0} dV = \frac{q_{\text{int}}}{\epsilon_0}$$

In words: the flux of an electric field equals the charge in the interior/ϵ_0 (Gauss law).

b) Apply Stokes theorem to a surface $S \subset \mathbb{R}^3$ with boundary a curve C and use the equation $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$:

$$\int_C \mathbf{E} \cdot d\mathbf{l} = \int\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\partial_t (\int\int \mathbf{B} \cdot d\mathbf{S})$$

In words: the circulation of the electric field is minus the time derivative of the flux of magnetic field (Faraday's law).

c) Apply the divergence theorem to a region $U \subset \mathbb{R}^3$ with boundary S, a closed surface, and use the Maxwell equation $\nabla \cdot \mathbf{B} = 0$:

$$\int\int_S \mathbf{B} \cdot d\mathbf{S} = \int\int\int_U \nabla \cdot \mathbf{B}\, dV = 0$$

In words: the flux of a magnetic field across a closed surface vanishes.

d) Apply Stokes theorem to a surface $S \subset \mathbb{R}^3$ with boundary a curve C and use the equation $c^2 \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \frac{\mathbf{j}}{\epsilon_0}$:

$$c^2 \int_C \mathbf{B} \cdot d\mathbf{l} = c^2 \int\int_S \nabla \times \mathbf{B} d\mathbf{S} = \partial_t (\int\int_S \mathbf{E} \cdot d\mathbf{S}) + \int\int_S \frac{\mathbf{j}}{\epsilon_0} \cdot d\mathbf{S}$$

In words: the circulation of the magnetic field is the time variation of the electric flux+current across S.

e) Applying Gauss law:

$$q = \epsilon_0 \int\int_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \int\int\int_U \nabla \cdot \mathbf{E}\, dV = \epsilon_0 \int\int\int_U 0 dV = 0$$

□

Problem 196: Gauss law in electrostatics.

Use Gauss law and symmetry arguments to

a) Show that the electric field in the exterior of a uniformly charged sphere is the same as the field of a point charge (with the charge of the sphere) at the center of the sphere. Is the result valid if we consider a uniformly charged ball?

b) Find the electric field at interior points of a uniformly charged sphere.

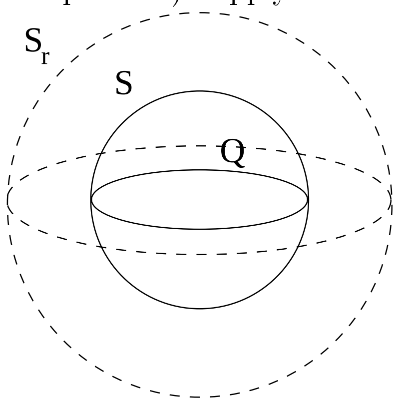
c) Find the electric field at interior points of a uniformly charged ball.

d) Find the electric field created by a uniformly charged straight wire.

e) Find the electric field created by a uniformly charged plane.

Solution:

a) Assume the sphere is S_R^2 and has a total charge q; from the symmetry we see that the exterior field $\mathbf{E}(\mathbf{r}, t)$ is central and stationary (time independent). Apply Gauss law to the sphere S_r^2 with radius $r > R$.

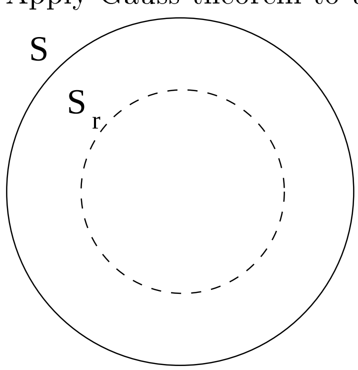


Let $E(r) = \mid \mathbf{E}(r) \mid$; then

$$\frac{q}{\epsilon_0} = \int\int_{S_r} \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E(r) \Rightarrow E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

that is the same as the field of a point charge q at the origin. This is true as well for a charged ball.

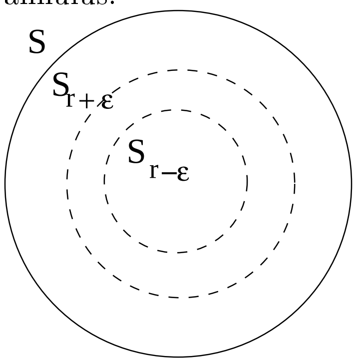
b) The field is central; let $\mid \mathbf{E}(r) \mid = E(r)$ and σ the surface charge density. Apply Gauss theorem to the sphere S_r^2 with $r > R$:



$$0 = \int\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) 4\pi r^2 \Rightarrow E(r) = 0$$

If there is no charge in the interior the field vanishes there; this shows that we can avoid the fields inside of the sphere.

c) We cannot use the argument in b) because then S^2_r would cut the distribution of charge. Instead decompose the ball B_R in three regions $B_R = B_{r-\epsilon} \cup C_{r-\epsilon,r+\epsilon} \cup C_{r+\epsilon,R}$, an innermost ball and two spherical annulus:



$B_{r-\epsilon}$ and $C_{r+\epsilon,R}$ are assumed charged and the region $C_{r-\epsilon,r+\epsilon}$ empty. Now compute the flux across S^2_r due to the charged parts. The field generated there by the interior sphere is $E'(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$, q being the charge of the sphere:

$$q = \int\int\int_{B_{r-\epsilon}} \chi dV = \chi\frac{4}{3}\pi(r-\epsilon)^3$$

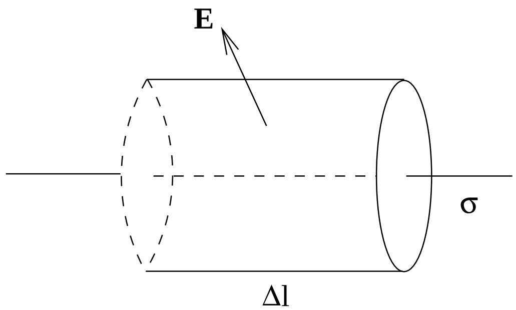
so the field is

$$E'(r) = \frac{\chi}{3\epsilon_0}\frac{(r-\epsilon)^3}{r^2}$$

The field generated by the region $C_{r+\epsilon,R}$ vanishes on S^2_r. Letting $\epsilon \to 0$ we will obtain the field on S^2_r:

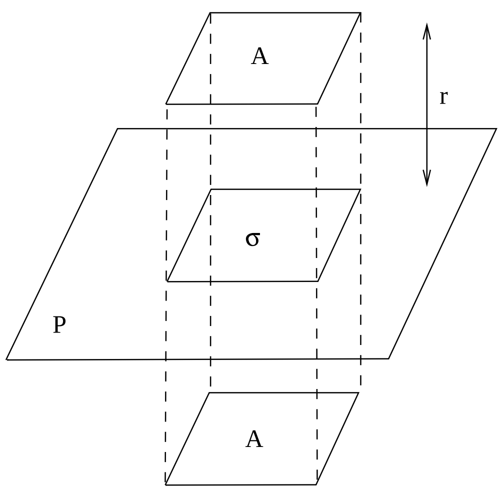
$$E(r) = \frac{\chi}{3\epsilon_0}r$$

d) Let λ be the linear charge density. From the symmetry the field depends only on the distance to the wire; let $E(r)$ be its modulus. Aply Gauss law to a cylindrical surface as in the figure:



$$\lambda \Delta l = \epsilon_0 \int\int_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 E(r) 2\pi r \Delta l \Rightarrow E(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

e) Let σ be the surface charge density. The symmetry implies that the field is perpendicular to the plane; let $E(r)$ its module. Apply Gauss theorem to a cubic surface as in the figure:



$$\sigma A = \epsilon_0 \int\int_S \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 2AE(r) \Rightarrow E(r) = \frac{\sigma}{2\epsilon_0}$$

□

Electrostatics and magnetostatics

$\boxed{\mathbb{T}}$ We say that the equations

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ c^2 \nabla \times \mathbf{B} &= \partial_t \mathbf{E} + \frac{\mathbf{j}}{\epsilon_0} \end{aligned}$$

are coupled because both fields appear in both equations. But assume the state of the sources is independent of time; then so will be **E** and **B** and the equations uncouple:

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ c^2 \nabla \times \mathbf{B} &= 0 \end{aligned}$$

a) Electrostatics deals with the equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \nabla \times \mathbf{E} = 0$$

b) Magnetostatics does with

$$\nabla \cdot \mathbf{B} = 0,\ c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0}$$

Both problems are, loosely speaking, dual:

a) Find an irrotational field with a given divergence.

b) Find a field with zero divergence and a given rotational.

□

7.2 Electrostatics

$\boxed{\mathbb{T}}$ The datum is $\rho(x, y, z)$ and we seek $\mathbf{E}(x, y, z)$.

The equations are $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \mathbf{E} = 0$.

We know that under good topological conditions if $\nabla \times \mathbf{E} = 0$ then $\mathbf{E} = -\nabla \phi$ where ϕ is a potential function now called the *electrostatic potential.* The potential produced by a charge distribution of density ρ in a region V is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\int\int_V \frac{\rho(u, v, w)}{r} dudvdw, r = \mid (x - u, y - v, z - w) \mid$$

It satisfies $E = -\nabla \phi$ and substituting into the first Maxwell equation we see that ϕ is a solution of *Poisson's equation*

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

that now we know how to solve. From ϕ we have the field $\mathbf{E} = -\nabla \phi$, the force on a charge $\mathbf{F} = -q\nabla \phi$ and we can write the equations of motion $q\mathbf{E} = m\ddot{\mathbf{r}}$

□

7.3 Magnetostatics

$\boxed{\mathbb{T}}$ The datum is $\mathbf{j}(x, y, z)$ and we seek $\mathbf{B}(x, y, z)$.

The equations are $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0}$.

We know that under good topological conditions if $\nabla \cdot \mathbf{B} = 0$ then $\mathbf{B} = \nabla \times \mathbf{A}$ for a certain field $\mathbf{A}$ called the vector potential of $\mathbf{B}$.

Remind that $\mathbf{A}' = \mathbf{A} + \nabla\psi$ (ψ an arbitrary function) has the same rotacional (this is called a gauge freedom). Notice that we can find ψ from

$$\text{div } \mathbf{A} + \nabla^2\psi = 0,$$

solving three Poisson's equations. Then div $\mathbf{A}' = 0$; we do this choice in magnetostatics (a different one is made in the resolution of the complete equations).

□

Problem 197: Solution of the magnetostatics problem.

From the magnetostatics equations show that the vector potential satisfies a Poisson equation.

Solution:

Substituting $\mathbf{B} = \nabla \times \mathbf{A}$ in Maxwell equation $\nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0 c^2}$ we obtain

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{\mathbf{j}}{\epsilon_0 c^2}$$

Now, from problem **34**,c), recall the formula

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

Using it and reminding that $\nabla \cdot \mathbf{A} = 0$ was our choice, we have

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A}$$

and $\mathbf{A}$ satisfies the Poisson vector equation

$$\nabla^2\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2}$$

whose solution is

$$\mathbf{A}(x,y,z) = \frac{1}{4\pi\epsilon_0 c^2}\int\int\int \frac{\mathbf{j}(u,v,w)}{r}dudvdw$$

We have solved the magnetostatics problem: being given the current density $\mathbf{j}$ we can find the potential vector $\mathbf{A}$ and then the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

□

7.4 Complete Maxwell equations

$\boxed{\mathbb{T}}$ Consider again the complete set of Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}$$
$$\nabla \cdot \mathbf{B} = 0 \quad , \quad c^2 \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \frac{\mathbf{j}}{\epsilon_0}$$

Charge conservation

The *electric current density* $\mathbf{j}$ is the amount of charge that crosses per unit time a unit area surface perpendicular to the movement of the charges. The flux of $\mathbf{j}$ across a surface is the *electric current.*

□

Problem 198:

From Maxwell's equations deduce the equation of charge conservation $\nabla \cdot \mathbf{j} + \partial_t \rho = 0$.

Solution:

Let the divergence operate on both sides of the equation $c^2 \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \frac{\mathbf{j}}{\epsilon_0}$:

$$0 = \partial_t(\nabla \cdot \mathbf{E}) + \frac{\nabla \cdot \mathbf{j}}{\epsilon_0} = \frac{1}{\epsilon_0}(\partial_t \rho + \nabla \cdot \mathbf{j}) \Rightarrow \nabla \cdot \mathbf{j} + \partial_t \rho = 0$$

□

Wave equation

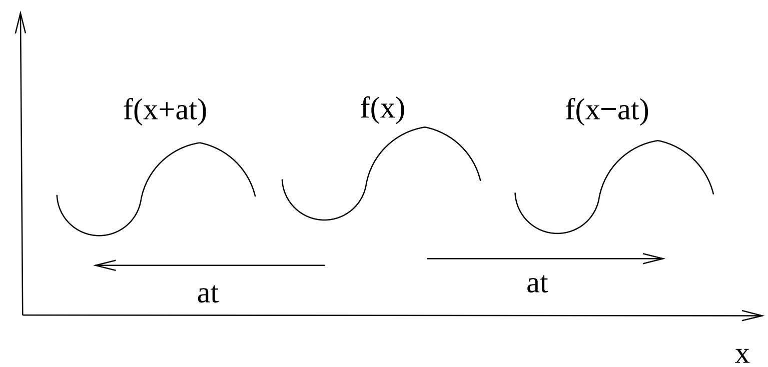
$\boxed{\mathbb{T}}$ Consider a string along the Ox axis and let $u(x,t)$ be the transversal displacement at point x and at the instant t. Then the function

$$u(x,t) = f(x - at)$$

is a waveform travelling to the right with a speed $a > 0$ (f is an arbitrary function), while

$$u(x,t) = g(x + at)$$

is a waveform travelling to the left with a speed $a > 0$ (g is an arbitrary function).



The wave superposition

$$u(x,t) = f(x - at) + g(x + at)$$

is the general form of a wave in the string. Eliminating the arbitrary functions f, g by differentiation we obtain the differential equation these waves satisfy:

$$\partial_x u = f'(x - at) + g'(x + at) \qquad \partial_t u = -af'(x - at) + ag'(x + at)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - at) + g''(x + at) \qquad \frac{\partial^2 u}{\partial t^2} = a^2 f''(x - at) + a^2 g''(x + at)$$

and we see the one dimensional *wave equation* is:

$$\boxed{a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}}$$

Analogously in two dimensions (think in a drumhead) we would have

$$a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = \frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \nabla^2 u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

and in dimension three

$$a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = \frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \nabla^2 u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

□

Problem 199:

a) From Maxwell's equations show there is a function ϕ (the electric potential) and a vector function $\mathbf{A}$ (the vector potential) such that

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

b) Let ψ an arbitrary function; show that if we take $\mathbf{A}_1 = \mathbf{A} + \nabla\psi$, $\phi_1 = \phi - \partial_t\psi$ we obtain the same fields $\mathbf{E}, \mathbf{B}$.

c) Show that we can choose $\phi, \mathbf{A}$ so as to satisfy the equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{\mathbf{j}}{\epsilon_0 c^2}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

that relate the sources of the field to the *potentials*.

Solution:

a) From $\nabla \cdot \mathbf{B} = 0$ we know that $\mathbf{B}$ has a vector potential, a field $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$. Substituting in the second of Maxwell's equations $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, we obtain

$$\nabla \times \mathbf{E} = -\partial_t(\nabla \times \mathbf{A}) = -\nabla \times \partial_t \mathbf{A}$$

or

$$\nabla \times (\mathbf{E} + \partial_t \mathbf{A}) = 0$$

The field $\mathbf{E} + \partial_t \mathbf{A}$ has zero rotational and so has a potential ϕ:

$$\mathbf{E} + \partial_t \mathbf{A} = -\nabla\phi \Rightarrow \mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$$

b) It's a computation:

$$\nabla \times \mathbf{A}_1 = \nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A} = \mathbf{B}$$

And

$$-\nabla\phi_1 - \partial_t \mathbf{A}_1 = -\nabla\phi + \nabla\partial_t\psi - \partial_t\mathbf{A} - \partial_t\nabla\psi = -\nabla\phi - \partial_t\mathbf{A} = \mathbf{E}$$

c)

i) Substituting $\mathbf{B} = \nabla\times\mathbf{A}$ in Maxwell's equation $c^2\nabla\times\mathbf{B} = \partial_t\mathbf{E} + \frac{\mathbf{j}}{\epsilon_0}$ we have

$$c^2\nabla \times (\nabla \times \mathbf{A}) = \partial_t\mathbf{E} + \frac{\mathbf{j}}{\epsilon_0}$$

or, applying the operator identity in problem **34**,c) and isolating the source,

$$c^2\nabla(\nabla \cdot \mathbf{A}) - c^2\nabla^2\mathbf{A} + \partial_t\nabla\phi + \frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{\mathbf{j}}{\epsilon_0}$$

Assume that $\mathbf{A}$ and ϕ are already known. To simplify the preceding equation we introduce new potentials

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{A} + \nabla\psi \\ \phi_1 &= \phi - \partial_t\psi \end{aligned}$$

They satisfy

$$c^2\nabla(\nabla \cdot \mathbf{A}_1) - c^2\nabla^2\mathbf{A}_1 + \partial_t\nabla\phi_1 + \frac{\partial^2\mathbf{A}_1}{\partial t^2} = \frac{\mathbf{j}}{\epsilon_0} \quad (*)$$

We choose ψsuch that $\nabla \cdot \mathbf{A}_1 = -\frac{1}{c^2}\frac{\partial\phi_1}{\partial t}$;

$$\nabla \cdot \mathbf{A}_1 = \nabla \cdot \mathbf{A} + \nabla^2\psi = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

that is, ψ must satisfy the equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\nabla \cdot \mathbf{A} - \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

a wave equation. With this choice (*) becomes

$$\nabla^2\mathbf{A}_1 - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_1}{\partial t^2} = -\frac{\mathbf{j}}{\epsilon_0 c^2}$$

ii) Substituting $\mathbf{E} = -\nabla\phi_1 - \partial_t \mathbf{A}_1$ into the first Maxwell equation we obtain $\nabla \cdot (-\nabla\phi_1 - \partial_t \mathbf{A}_1) = \frac{\rho}{\epsilon_0}$ or

$$\nabla^2\phi_1 + \partial_t(\nabla \cdot \mathbf{A}_1) = -\frac{\rho}{\epsilon_0} \ (*)$$

and the choice $\nabla \cdot \mathbf{A}_1 = -\frac{1}{c^2}\frac{\partial\phi_1}{\partial t}$ gives

$$\nabla^2\phi_1 - \frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

iii) In free space (:=where there are no charges nor currents) we have:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0,$$

and we see that ϕ and $\mathbf{A}$ satisfy the wave equation.

□

Problem 200: Show that **E** and **B** satisfy in free space the wave equation.

Solution:

a) Applying the rotational to the wave equation for **A** we have:

$$0 = \nabla\times(\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}) = \nabla^2(\nabla\times\mathbf{A}) - \frac{1}{c}\frac{\partial^2}{\partial t^2}(\nabla\times\mathbf{A}) = \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

that is to say **B** satisfies the wave equation with speed c.

b) To obtain a similar result for **E** we start from the equality:

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$$

Differentiating respect to t and taking into account the wave equation for ϕ we have:

$$\frac{\partial\mathbf{E}}{\partial t} = -\nabla\frac{\partial\phi}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2} = -\nabla\frac{\partial\phi}{\partial t} - c^2\nabla^2\mathbf{A}$$

Differentiate again with respect to t to obtain

$$\begin{aligned}\frac{\partial^2 \mathbf{E}}{\partial t^2} = & -\nabla \tfrac{\partial^2 \phi}{\partial t^2} - c^2 \partial_t \nabla^2 \mathbf{A} = & -\nabla (c^2 \nabla^2 \phi) - c^2 \partial_t \nabla^2 \mathbf{A} = \\ = & \quad c^2 \nabla^2 (-\nabla \phi \partial_t \mathbf{A}) = & \quad c^2 \nabla^2 \mathbf{E}\end{aligned}$$

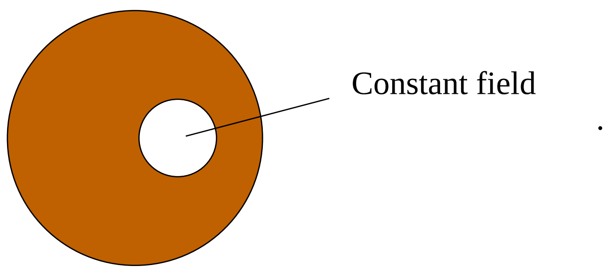
as we wanted to see.

□

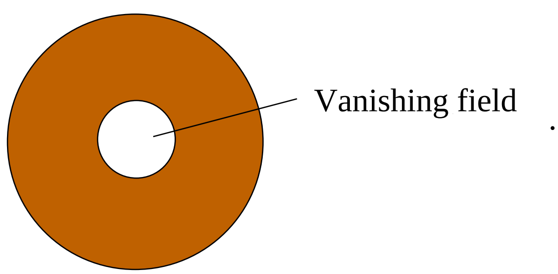
Epilogue

Two results and a problem.

The gravitational field of a uniform (:= constant density) mass sphere is constant (in modulus as well as in direction) in the interior of an interior hollow sphere:

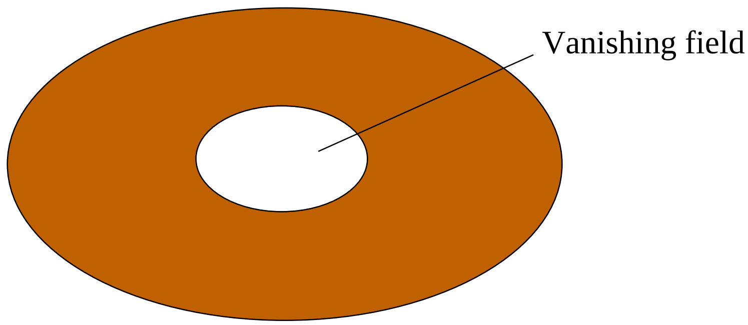


The gravitational field of a uniform mass sphere vanishes in the interior of an interior concentric hollow sphere:



May be you are thinking about why Jules Verne did not use those facts in his well known novel *A journey to the earth's center.* Well, the point is that both preceding results are true if there were no other bodies in the Universe, but as that is not so, the gravitational fields from other bodies still create in the interior of the hollow spheres a certain gravitational field.

Newton proved that the second result holds true in the interior of an ellipsoidal body (see [Kell] p.22):



The problem is to find out if there are other shapes for which this fact is still true. The answer is that there are no other such figures but, despite having enough space in the margin, I can't find out the reference (Hint: the solution is in a book about partial differential equations).

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