SYNCHRONIZATION TECHNIQUES FOR DIGITAL DEMODULATION

SIGNAL MODEL

For a given modulation, the signal model is 'a priori' known:

$r(t) = s(t;\underline{\phi}) + n(t)$ where $n(t)$ is an AWGN term

and $\underline{\phi} = (\tau_o, \theta_o, f_o, A_o, \{c_n\})$

τ_o is the 'timing' error
θ_o is the 'carrier phase' error
f_o is the 'carrier frequency' error
A_o is the 'signal amplitude'
$\{c_n\}$ is the 'information symbols' sequence

Further on, we will assume a modulated digital PAM signal:

$$s(t;\underline{\phi}) = A_o \left[\sum_n c_n h(t - nT - \tau_o) \right] e^{j\theta_o(t)} \quad where \quad \theta_o(t) = \theta_o + 2\pi f_o t$$

where all the conventional synchronization parameters are reflected:

MCRB FOR SYNCHRONIZATION

$$MCRB(\lambda) = \frac{1}{E_{r,\mathbf{u}}\left\{\left(\frac{\partial \ln f(r \mid \mathbf{u}, \lambda)}{\partial \lambda}\right)^2\right\}}$$

$$f(r \mid \mathbf{u}, \lambda) = \exp\left[-\frac{1}{2N_0}\int_{T_o} \left|r(t) - s(t)\right|^2 dt\right]$$

$$MCRB(\lambda) = \frac{N_0}{E_{\mathbf{u}}\left\{\int_{T_o} \left|\frac{\partial s(t)}{\partial \lambda}\right|^2 dt\right\}}$$

A. Andrea, U. Mengali, “The Modified Cramer-Rao Bound and Its Application to Synchronization Problems”, IEEE Trans. on Communications, vol. 42, No. 2/3/4, 1994.

MCRB FOR FREQUENCY ESTIMATION

$$\int_{T_o} \left| \frac{\partial s(t)}{\partial f_0} \right|^2 dt = 4\pi^2 \int_{T_o} t^2 \left| \sum_n c_n h(t - nT - \tau_o) \right|^2 dt$$

$$E_{c,\tau_o} \left\{ \left| \sum_n c_n h(t - nT - \tau_o) \right|^2 \right\} = E_s$$

$$E_{\mathbf{u}_f} \left\{ \int_{T_o} \left| \frac{\partial s(t)}{\partial f_0} \right|^2 dt \right\} = \frac{2\pi^2 E_s}{3T} T_0^3$$

$$\boxed{MCRB(f_0) = \frac{3T}{2\pi^2 T_0^3} \frac{1}{E_s / N_0}}$$

MCRB FOR PHASE ESTIMATION

$$MCRB(\theta_0) = \frac{N_0}{E_{\mathbf{u}_\theta}\left\{\int_{T_o}\left|\frac{\partial s(t)}{\partial \theta}\right|^2 dt\right\}}$$

$$\int_{T_o}\left|\frac{\partial s(t)}{\partial \theta_0}\right|^2 dt = \int_{T_o}\left|\sum_n c_n h(t - nT - \tau_o)\right|^2 dt$$

$$E_{\mathbf{u}_\theta}\left\{\int_{T_o}\left|\frac{\partial s(t)}{\partial \theta_0}\right|^2 dt\right\} = 2E_s L \qquad L = \frac{T_0}{T}$$

$$\boxed{MCRB(\theta_0) = \frac{1}{2L}\frac{1}{E_s / N_0}}$$

MCRB FOR TIMING ESTIMATION

$$MCRB(\tau_0) = \frac{1}{8\pi^2 \xi L} \frac{1}{E_s / N_0}$$

$$\xi = T^2 \frac{\int_{-\infty}^{\infty} f^2 \left| H(f) \right|^2 df}{\int_{-\infty}^{\infty} \left| H(f) \right|^2 df}$$

ξ is the normalized mean square bandwidth of H(f)

Timing estimation should be easier with wideband signals.

MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Joint ML estimation/detection rule: $\left(\hat{c},\hat{\theta}\right)=\arg\left[\max_{c,\theta} f\left(r\,|\,c,\theta\right)\right]$

- Data-aided (DA): using a preamble of known symbols.

$$\hat{\theta}\left(c_0\right)=\arg\left[\max_{\theta} f\left(r\,|\,c=c_0,\theta\right)\right]$$

- Decision-Directed (DD): using segments of decoded data as if it where the true symbols.

$$\hat{\theta}\left(c_L\right)=\arg\left[\max_{\theta} f\left(r\,|\,c=\hat{c}_L,\theta\right)\right]$$

- Non-Data-Aided (NDA): Performing the averaging operation to remove the data dependency.

$$f\left(r\,|\,\theta\right)=\sum_{\substack{all\ possible\\ sequences\ c}} f\left(r\,|\,c,\theta\right)P\left(c\right)$$

$$\hat{\theta}=\arg\left[\max_{\theta} f\left(r\,|\,\theta\right)\right]$$

MAXIMUM LIKELIHOOD ESTIMATION (II)

- Under the AWGN assumption, the ML optimal parameter estimation corresponds to the conventional Minimum Mean Square Error (M.M.S.E.), that is, the received data and the signal model fitting:

$$\min_{\phi} \varepsilon^2 = \min_{\phi} \int_{T_o} \left| r(t) - s(t;\phi) \right|^2 dt$$

- Thus:

$$\min_{\phi} \varepsilon^2 = \min_{\phi} \left[\int_{T_o} \left| r(t) \right|^2 dt + \int_{T_o} \left| s(t;\phi) \right|^2 dt - 2\,\mathrm{Re}\left\{ \int_{T_o} s^*(t;\phi) r(t) dt \right\} \right]$$

(cross-correlator)

?

(constant envelope signals)

- And for ‘constant envelope signals’, it is necessary to perform the multi-dimensional search as follows:

$$\min_{\phi} \varepsilon^2 \equiv \max_{\phi} \mathrm{Re}\left\{ \int_{T_o} s^*(t;\phi) r(t) dt \right\}$$

DA ESTIMATION

- As we saw, the optimal parameters estimation becomes from maximizing:

$$\min_{\phi} \varepsilon^2 \equiv \max_{\phi} \mathrm{Re}\left\{ \int_{T_o} s^*(t;\phi) r(t) dt \right\}$$

- Thus, we have to maximize:

$$\max_{\theta_o} \mathrm{Re}\left\{ \sum_n c_n^* e^{-j\hat{\theta}_o} \underbrace{\int_{T_o} r(t) h^*(t - nT - \tau_o) dt}_{x(n)} \right\} = \max_{\theta_o} \mathrm{Re}\left\{ e^{-j\hat{\theta}_o} \sum_n c_n^* x(n) \right\}$$

$$\boxed{\hat{\theta}_o = \arg\left\{ \sum_n c_n^* x(n) \right\}}$$

DD ESTIMATION

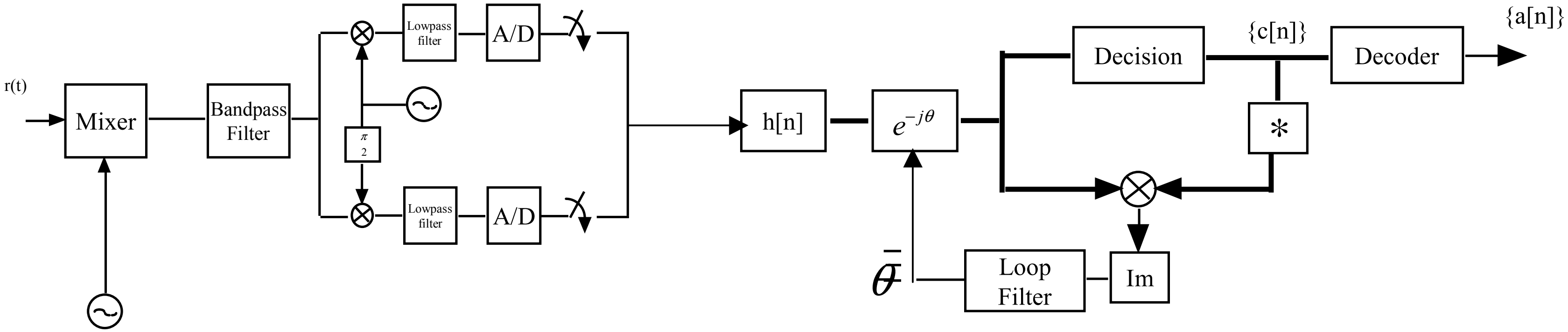
$$\max_{\theta_o} \mu^2 = \max_{\theta_o} \operatorname{Re}\left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_o} r(t) h^*(t - nT - \tau_o) dt \right\}$$

$$\frac{\partial}{\partial \hat{\theta}_o} \mu^2 = 0 \quad \text{where} \quad \mathrm{u}_\theta \equiv \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = \frac{\partial}{\partial \hat{\theta}_o} \operatorname{Re}\left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_o} r(t) h^*(t - nT - \tau_o) dt \right\}$$

ML Estimation: $$\mathrm{u}_\theta \equiv \operatorname{Im}\left\{ \sum_n \hat{c}_n^* e^{-j\hat{\theta}_o} \int_{T_o} r(t) h^*(t - nT - \tau_o) dt \right\}$$

Loop equations: $$\boxed{\hat{\theta}(n+1) = \hat{\theta}(n) + k_\theta u_\theta(n)}$$

CLOSED LOOP SCHEME



DD Schemes: BPSK/QPSK

Sampled Matched Filter output:

$$I_k + jQ_k = \int_{T_o} r(t)h^*(t-nT)dt \Big|_{t=kT}$$

$$u_\theta \equiv \frac{\partial}{\partial\hat{\theta}_o}\mu^2 = \mathrm{Im}\left\{\hat{c}_n^* e^{-j\hat{\theta}_o}(I_k + jQ_k)\right\}$$

$$(I_k' + jQ_k') = e^{-j\hat{\theta}_o}(I_k + jQ_k)$$

$$\hat{c}_n = \begin{cases} BPSK: & sign\left[I_k'\right] \\ QPSK: & sign\left[I_k'\right] + j\,sign\left[Q_k'\right] \end{cases}$$

BPSK

$$u_\theta = \mathrm{Im}\left\{sign\left[I_k'\right]\left(I_k' + jQ_k'\right)\right\}$$

$$\boxed{u_\theta = sign\left[I_k'\right] Q_k'}$$

QPSK

$$u_\theta = \mathrm{Im}\left\{\left(sign\left[I_k'\right] - j\,sign\left[Q_k'\right]\right)\left(I_k' + jQ_k'\right)\right\}$$

$$\boxed{u_\theta = sign\left[I_k'\right] Q_k' - sign\left[Q_k'\right] I_k'}$$

NDA Schemes: BPSK/QPSK

Sampled Matched Filter output:

$$I_k + jQ_k = \left. \int_{T_o} r(t) h^*(t - nT) dt \right|_{t=kT}$$

$$u_\theta \equiv \frac{\partial}{\partial \hat{\theta}_o} \mu^2 = \mathrm{Im}\left\{ \hat{c}_n^* e^{-j\hat{\theta}_o} (I_k + jQ_k) \right\}$$

$$(I_k' + jQ_k') = e^{-j\hat{\theta}_o} (I_k + jQ_k)$$

USE OF A NON-LINEARITY →

$$(I_k'' + jQ_k'') = \left[I_k' + jQ_k' \right]^M \text{ (for MPSK)}$$

$$c = e^{j2\pi l / M} \rightarrow c^M = 1$$

$$\max_{\theta_o} \mathrm{Re}\left\{ \sum_n c_n^* e^{-j\hat{\theta}_o} \int_{T_o} r(t) h^*(t - nT - \tau_o) dt \right\} = \max_{\theta_o} \mathrm{Re}\left\{ e^{-j\hat{\theta}_o} \sum_n c_n^* x(n) \right\}$$

$$\max_{\theta_o} \mathrm{Re}\left\{ e^{-j\hat{\theta}_o M} \sum_n x(n)^M \right\} \rightarrow e^{-j\hat{\theta}_o M} = \exp\left[j \arg \sum_n x(n)^M \right]$$

Ambiguity: if $\hat{\theta}$ is a solution so is $\hat{\theta}+\frac{2\pi l}{M}$ *for* $l=2,...M-1$

$$e^{j\left(\hat{\theta}+\frac{2\pi l}{M}\right)M}=e^{j\hat{\theta}M}$$

BPSK

$$\mathrm{u}_\theta=\mathrm{Im}\{I_k''+jQ_k''\}$$

$$\boxed{\mathrm{u}_\theta=Q_k''}$$

- Phase ambiguity: π
- Frequency error: $2f_e$

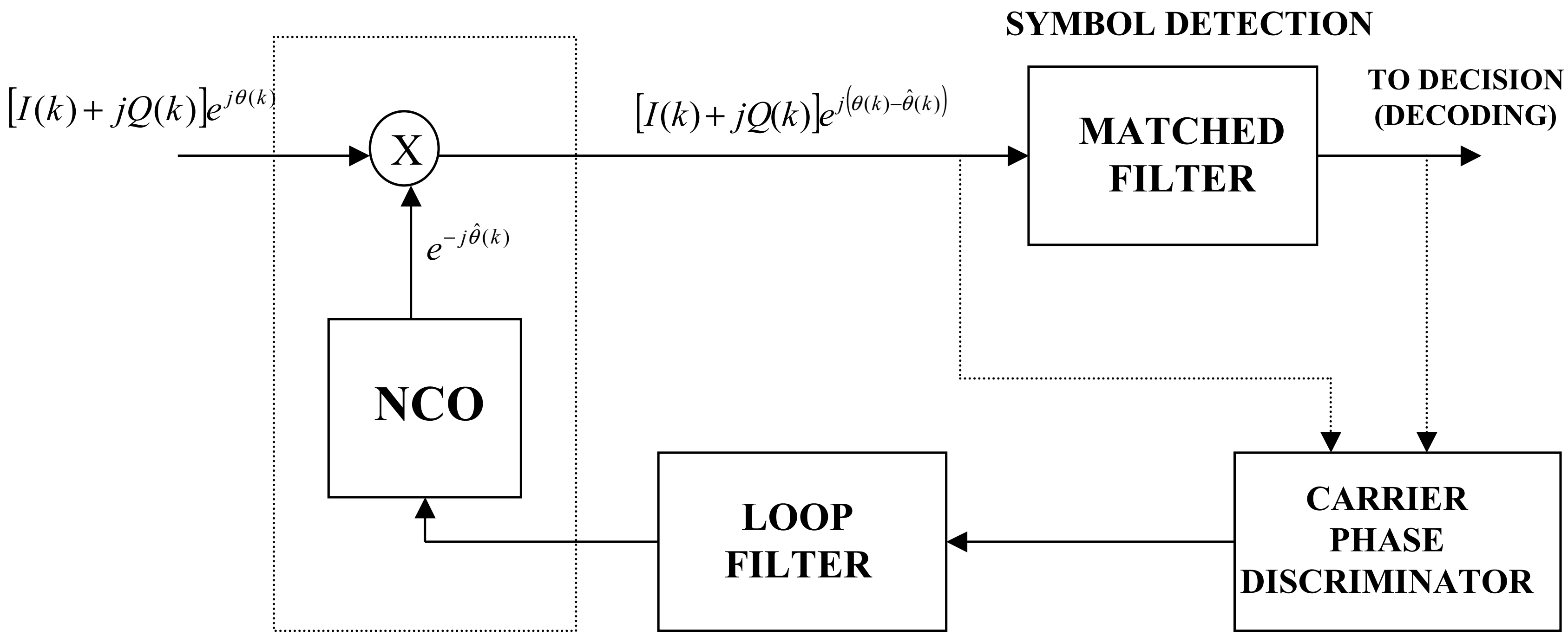
QPSK

$$\mathrm{u}_\theta=\mathrm{Im}\{I_k''+jQ_k''\}$$

$$\boxed{\mathrm{u}_\theta=Q_k''}$$

- Phase ambiguity: $\frac{\pi}{2}$
- Frequency error: $4f_e$

CARRIER ACQUISITION AND TRACKING



SIGNAL MODEL

- Let's consider an I&Q pure sinusoid:

$$r(t) = A_o e^{j(2\pi(f_o+f_e)t+\theta_0)} + n(t)$$

f_0 is the carrier frequency
f_e is the frequency error (or doppler error)
θ_0 is the carrier phase

- The In-phase and the Quadrature components of the sampled signal becomes:

$$I(k) = \frac{A_0}{2}\cos(\theta(k) - \hat{\theta}(k)) + \frac{1}{2}n_i(t)$$

$$Q(k) = \frac{A_0}{2}sin(\theta(k) - \hat{\theta}(k)) + \frac{1}{2}n_q(t)$$

$\omega_0 = 2\pi f_0 T_s$ is the normalized carrier frequency
$\omega_e = 2\pi f_e T_s$ is the normalized frequency error
$\theta(k) = (\omega_0 + \omega_e)k + \theta_0$ is the carrier phase evolution
$\hat{\theta}(k)$ is the carrier phase estimate

- The ***phase error*** is defined by:

$$\psi(k) = \theta(k) - \hat{\theta}(k)$$

- A typical ***carrier phase discriminator*** is (among others)

$$e(k) = g[\psi(k)] = arctan\left(\frac{Q(k)}{I(k)}\right) \quad |e(k)| \le \pi$$

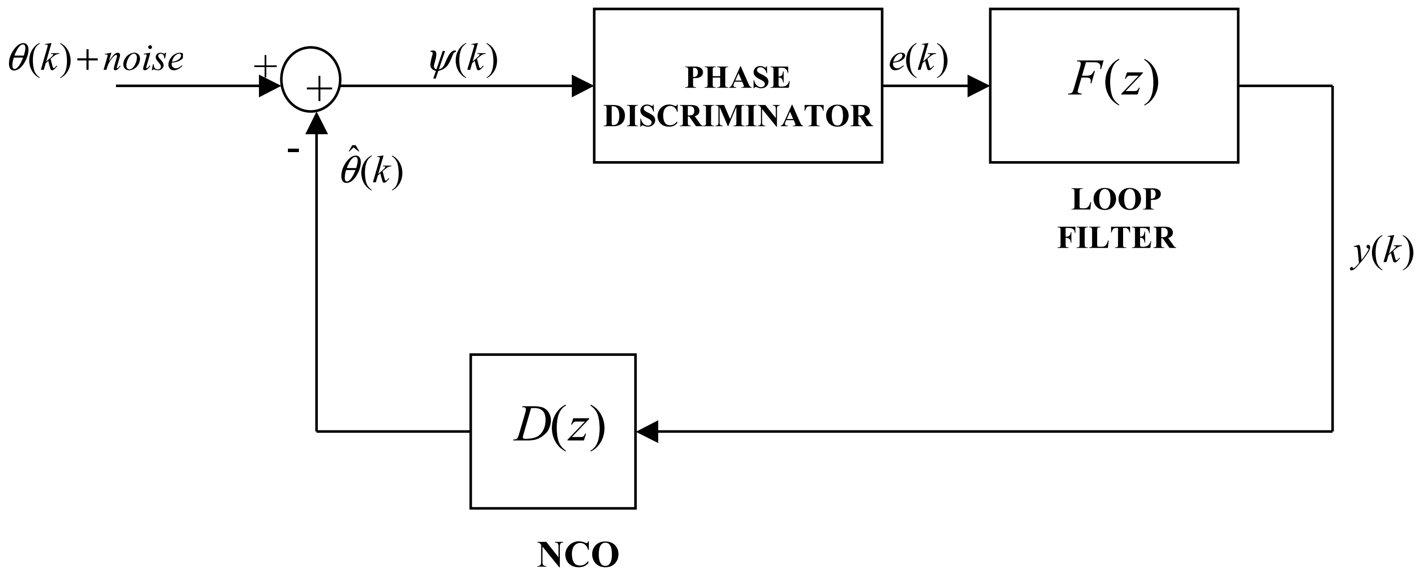
CARRIER DISCRIMINATORS (I)

- A typical ***carrier phase discriminator*** is:

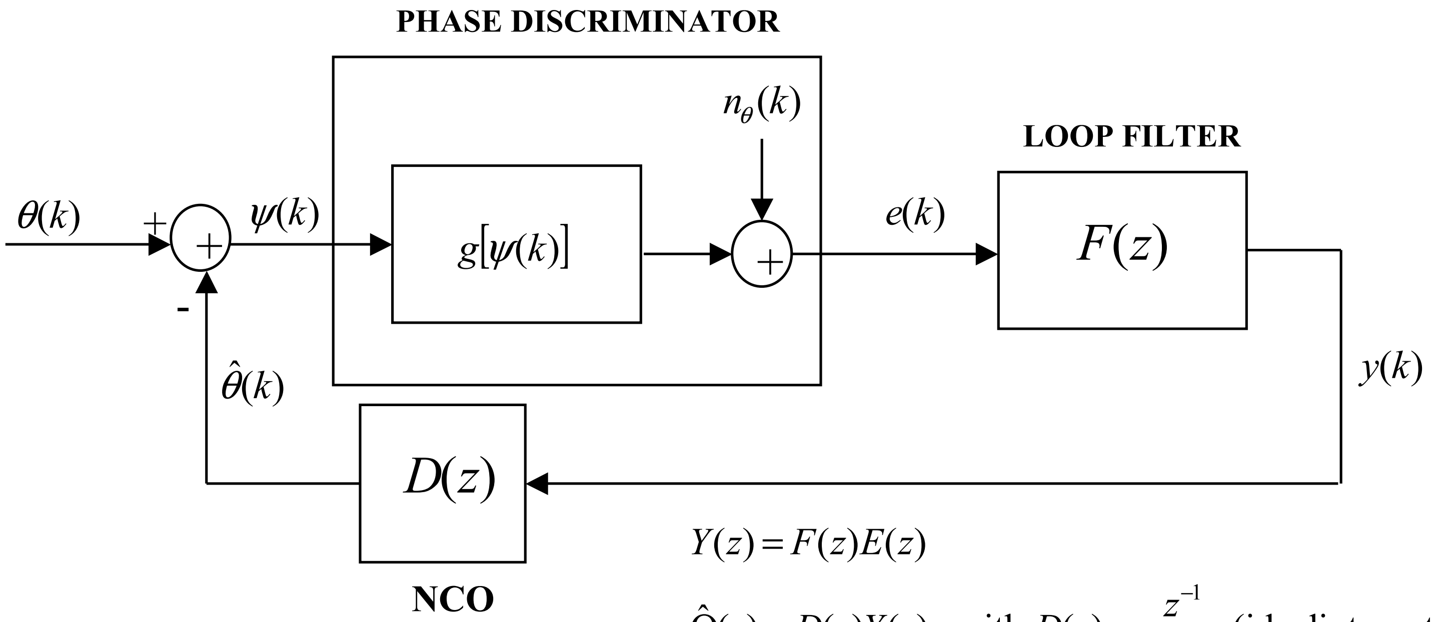
$$e(k) = g[\psi(k)] = arctan\left(\frac{Q(k)}{I(k)}\right) \quad |e(k)| \leq \pi$$

or:

$$e(k) = g[\psi(k)] = imag[I(k) + jQ(k)] = Q(k)$$



CARRIER DISCRIMINATORS (II)



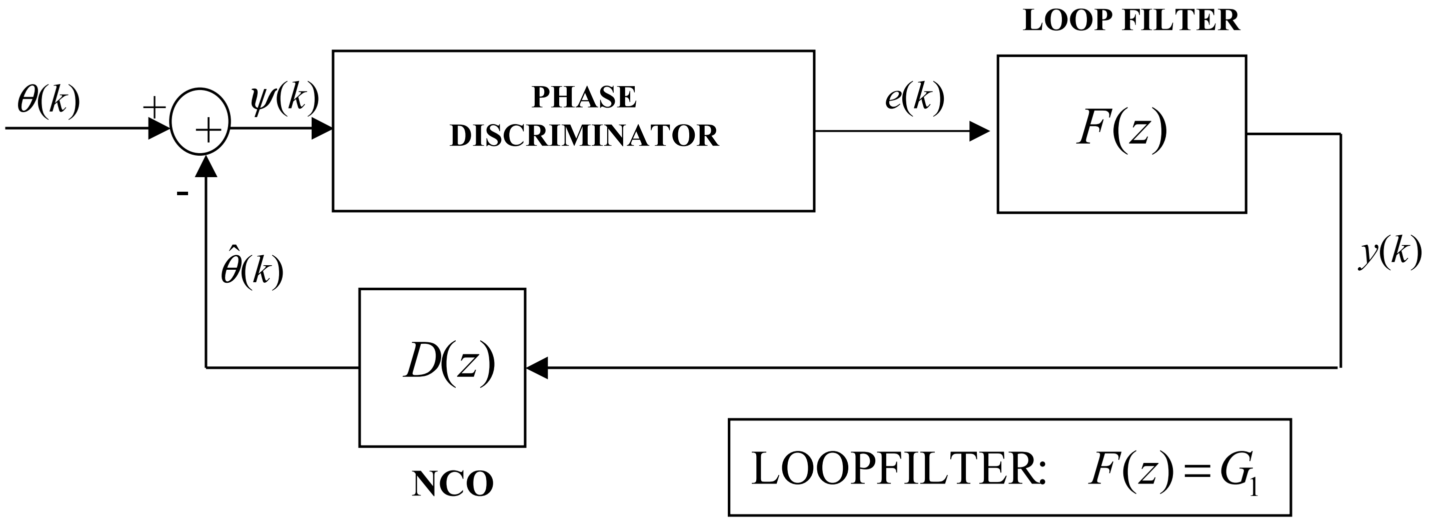
$Y(z) = F(z)E(z)$

$\hat{\Theta}(z) = D(z)Y(z)$ with $D(z) = \frac{z^{-1}}{1-z^{-1}}$ (idealintegrator)

→ NOISEFREE: $e(k) = g[\psi(k)] = \psi(k) = \theta(k) - \hat{\theta}(k) \Rightarrow |e(k)| = |\psi(k)| \leq \pi$

→ WITHNOISE: $e(k) = g[\psi(k)] + n_\theta(k)$ $|e(k)| \leq \pi$

FIRST ORDER LOOP FILTER (NOISE FREE)



LOOPFILTER: $F(z) = G_1$

$$\Psi(z) = \frac{1}{1 + F(z)D(z)}\Theta(z)$$

$$D(z) = \frac{z^{-1}}{1 - z^{-1}} \quad \text{(NCO)}$$

$$\psi(k) = (1 - G_1)\psi(k-1) + \left[\theta(k) - \theta(k-1)\right]$$

FIRST ORDER: TRACKING ERROR

$$\boxed{\psi(k) = (1 - G_1)\psi(k-1) + [\theta(k) - \theta(k-1)]}$$

Filtrado/integración   Error/innovación

- We will consider a linear phase error evolution (carrier phase and frequency errors):

$$\theta(k) = a_1 k + a_0$$

a_0 → Phase error

a_1 → Frequency error

- The basic difference equation becomes:

$$\psi(k) = (1 - G_1)\psi(k-1) + a_1$$

which defines the system transient response for a given initial $\psi(0) = \psi_0$

- Under steady-state conditions:

$$\psi(k) = \psi(k-1) = \psi_{ss} \quad \Rightarrow \quad \psi_{ss} = (1 - G_1)\psi_{ss} + a_1 \quad \Rightarrow \quad \boxed{\psi_{ss} = \frac{a_1}{G_1}}$$

TRACKING ERROR

FIRST ORDER: LOCK-IN

$$\psi(k) = (1 - G_1)\psi(k-1) + a_1$$

- To ensure the system Lock-In, it is necessary to avoid any phase aliasing (wrapping) in the loop:

$$|\psi(k)| \leq \pi \quad \forall k$$

- The worst cases are the following:

$\rightarrow a_1 > 0$ (positive frequency error) with $\psi(k\text{-}1) = \pm\pi$

$$\psi(k) = a_1 + (1 - G_1)\pi < \pi \quad \Rightarrow \quad G_1 > \frac{a_1}{\pi}$$

$$\psi(k) = a_1 - (1 - G_1)\pi > -\pi \quad \Rightarrow \quad G_1 < 2 - \frac{a_1}{\pi}$$

$\rightarrow a_1 < 0$ (negative frequency error) with $\psi(k\text{-}1) = \pm\pi$

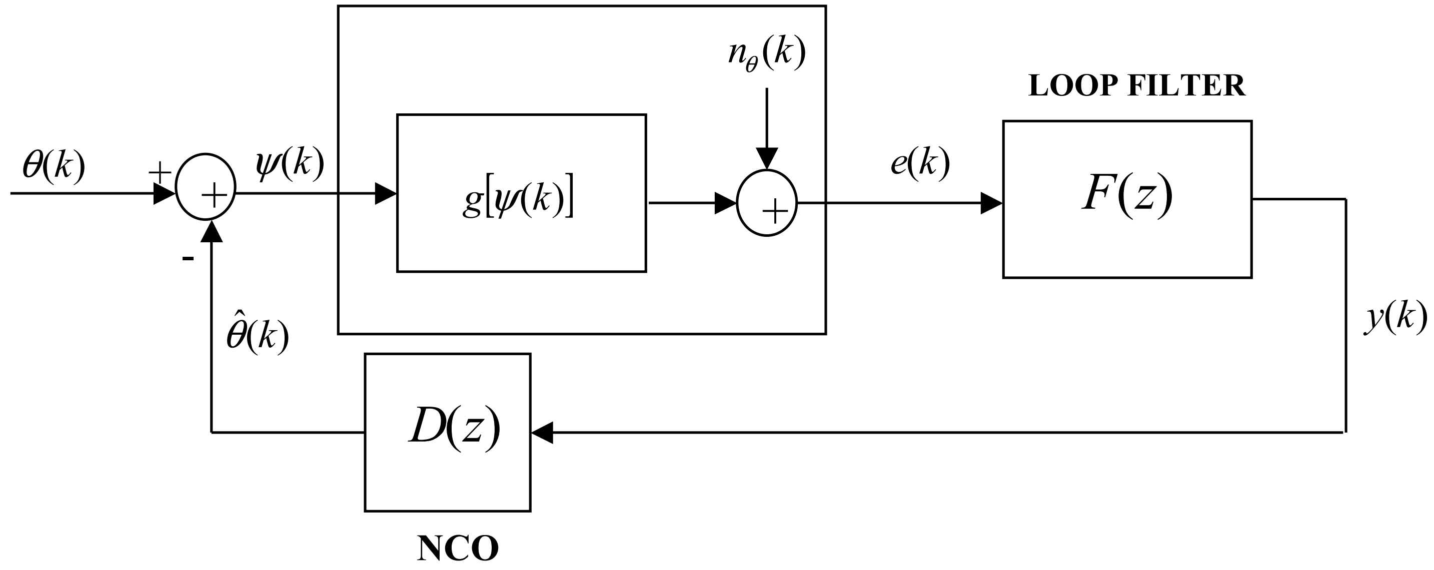
$$G_1 > \frac{-a_1}{\pi}$$

$$G_1 < 2 + \frac{a_1}{\pi}$$

LOCK-IN: $\dfrac{|a_1|}{\pi} < G_1 < 2 - \dfrac{|a_1|}{\pi} \Rightarrow |a_1| < \pi \quad (Nyquist)$

$$|1 - G_1| < 1$$

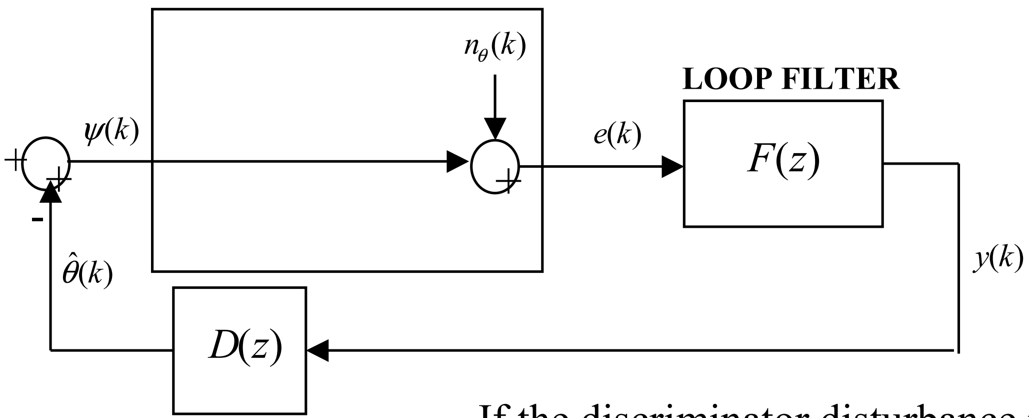
FIRST ORDER LOOP FILTER (WITH NOISE)



$$\left.\begin{aligned} \psi(k) &= \theta(k) - \hat{\theta}(k) \\ e(k) &= g\left[\psi(k)\right] + n_\theta(k) \\ y(k) &= G_1 e(k) \\ \hat{\theta}(k) &= \hat{\theta}(k-1) + y(k-1) \end{aligned}\right\} \Rightarrow \boxed{\psi(k) = \psi(k-1) - G_1 g\left[\psi(k-1)\right] + a_1 - G_1 n_\theta(k-1)}$$

FIRST ORDER MARKOV PROCESS

PRAGMATIC APPROACH



$$\Psi(z) = \frac{D(z)F(z)}{1+D(z)F(z)} N_\theta(z)$$

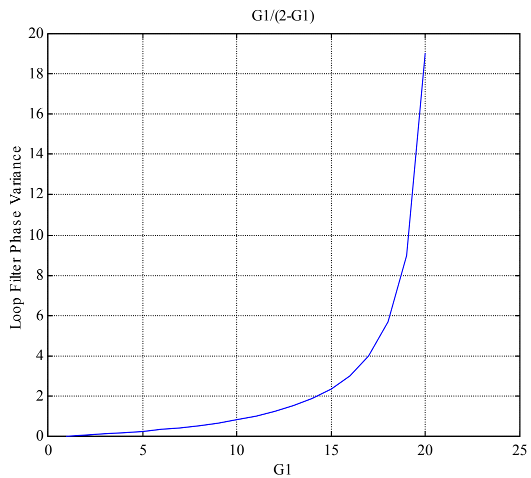
- If the discriminator disturbance noise is white: $S_{n_\theta n_\theta}(f) = \frac{\sigma^2_{n_\theta}}{f_s}$

$$\rightarrow \quad E[\psi_{ss}] \approx \frac{a_1}{G_1}$$

$$\rightarrow \quad E[\psi^2_{ss}] \approx \frac{T_s}{2\pi} \int_{-\pi}^{+\pi} \left| \frac{D(z)F(z)}{1+D(z)F(z)} \right|^2_{z=e^{j2\pi f T_s}} \frac{\sigma^2_{n_\theta}}{f_s} df \quad \text{with} \quad T_s = \frac{1}{f_s}$$

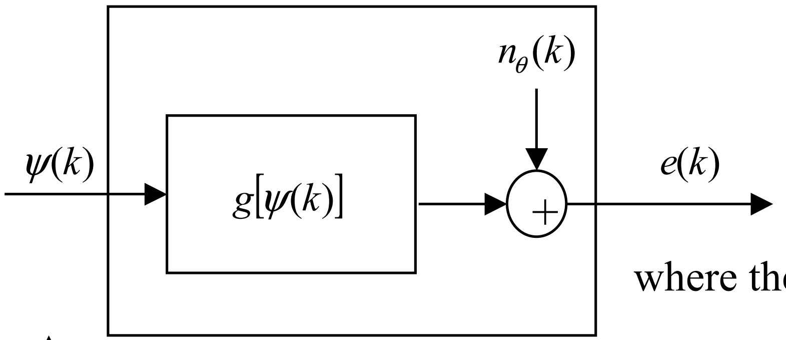
$$\mathrm{var}[\psi_{ss}] = E[\psi^2_{ss}] - E^2[\psi_{ss}]$$

$$\mathrm{var}[\psi_{ss}] \approx \frac{G_1}{2-G_1} \sigma^2_{n_\theta}$$



PHASE DISTURBANCE MEAN POWER

PHASE DISCRIMINATOR

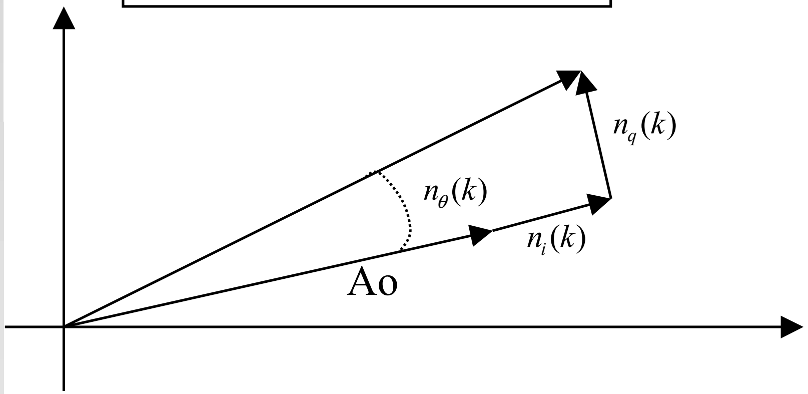


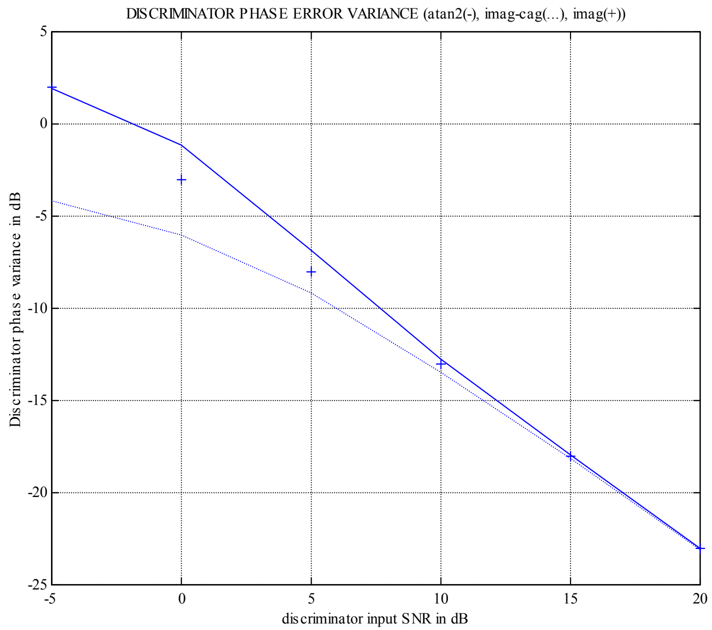
$$n_\theta(k) = \arctan\left[\frac{n_q(k)}{A_0 + n_i(k)}\right] \quad \text{for} \quad k = 1, 2, \ldots, N$$

$$\downarrow$$

$$\hat{\sigma}^2_{n_\theta} = \frac{1}{N}\sum_{k=1}^{N} \left|n_\theta(k)\right|^2$$

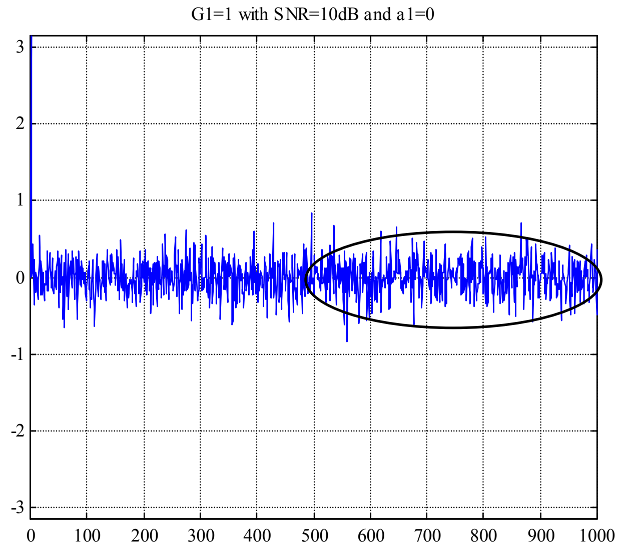
where the noise complex samples are zero mean and gaussian

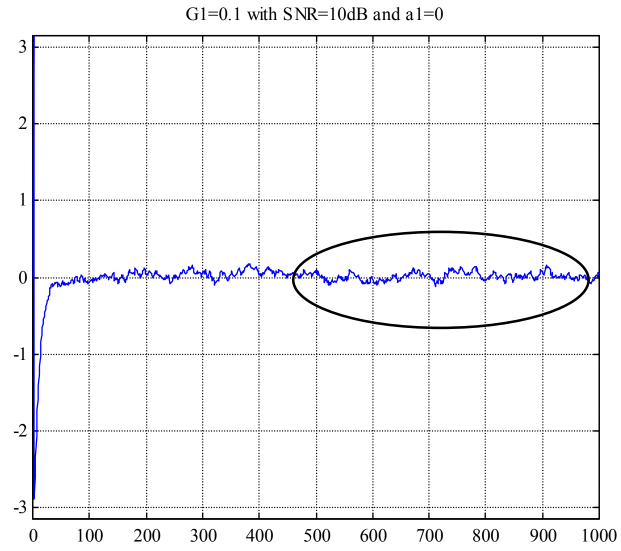


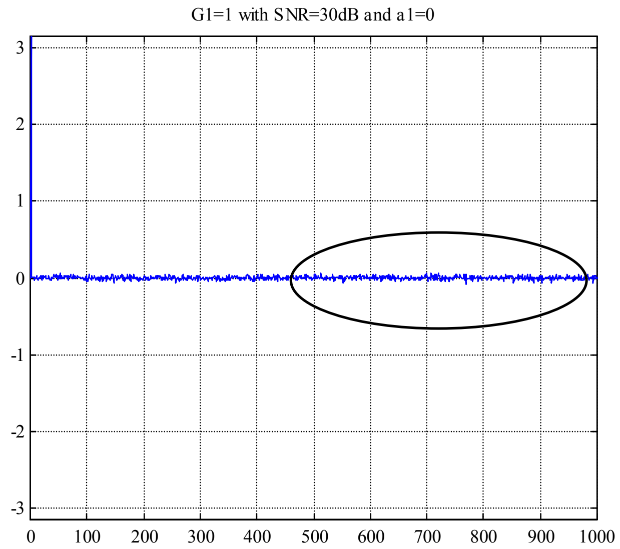


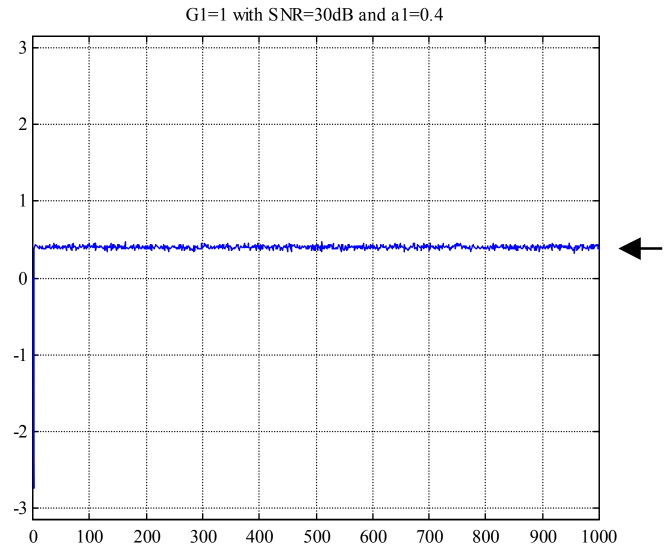
The discriminator noise disturbance has to be generated at the working SNR, that is:

$$SNR = \frac{S}{N} = \frac{1}{E\left[n_i^2\right] + E\left[n_q^2\right]} = \frac{1}{2E\left[n_i^2\right]} = \frac{1}{2E\left[n_q^2\right]}$$


G1=1 with SNR=10dB and a1=0
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=0.1 with SNR=10dB and a1=0
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=1 with SNR=30dB and a1=0
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=1 with SNR=30dB and a1=0.4
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000

SECOND ORDER LOOP FILTER (NOISE FREE)

LOOPFILTER: $F(z) = G_1 + \frac{G_2}{1 - z^{-1}} \quad r = G_1 + \frac{G_2}{G_1}$

$$\Psi(z) = \frac{1}{1 + F(z)D(z)}\Theta(z)$$

$$D(z) = \frac{z^{-1}}{1 - z^{-1}} \quad \text{(NCO)}$$

$$\begin{aligned}\psi(k) = {} & 2\psi(k-1) - \psi(k-2) + \\ & + [\theta(k) - 2\theta(k-1) + \theta(k-2)] - \\ & - (G_1 + G_2)\psi(k-1) + G_1\psi(k-2)\end{aligned}$$

- We will consider a ***quadratic*** phase error evolution:

$$\theta(k) = a_2 k^2 + a_1 k + a_0$$

a_0: Phase error

a_2: Frequency Rate error

a_1: Frequency error

TRACKING ERROR : $\psi_{ss} = \frac{2a_2}{G_2} \quad \forall a_1$!!!!

LOCK - IN : $\left\{\begin{array}{l} 2 + \left|\frac{2a_2}{\pi}\right| < G_1(r+1) < 4 - \left|\frac{2a_2}{\pi}\right| \\ \left|\frac{2a_2}{\pi}\right| < G_1(r-1) < 2 - \left|\frac{2a_2}{\pi}\right| \end{array}\right\rangle \quad \begin{array}{c} |a_2| < \frac{\pi}{2} \\ r > 1 \end{array}$

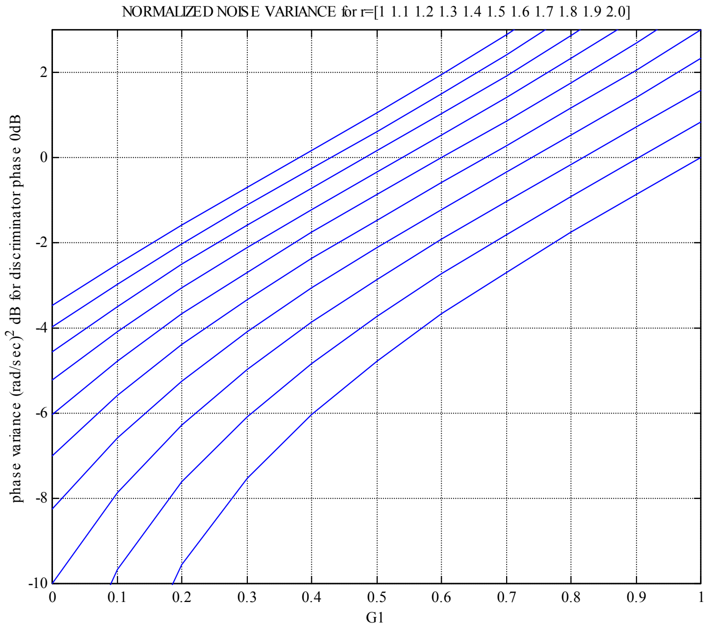
SECOND ORDER LOOP FILTER (WITH NOISE)

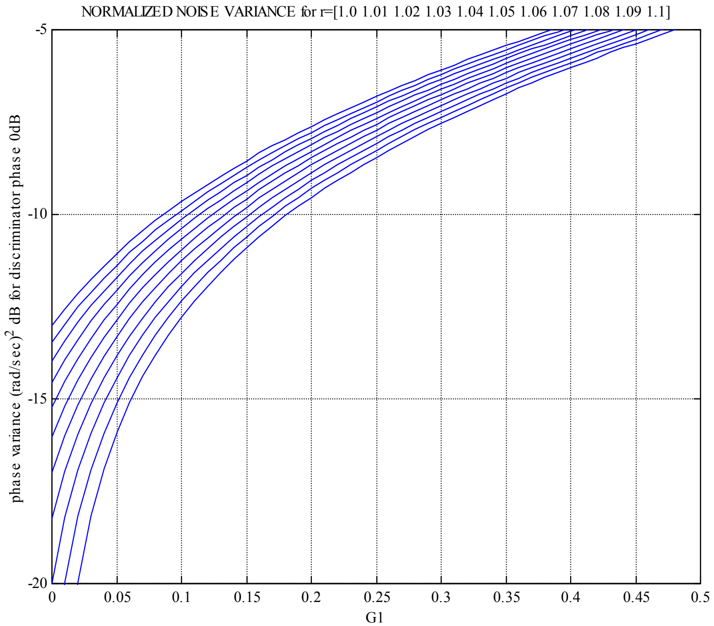
$\rightarrow \quad E[\psi_{ss}] \approx \dfrac{2a_2}{G_2}$

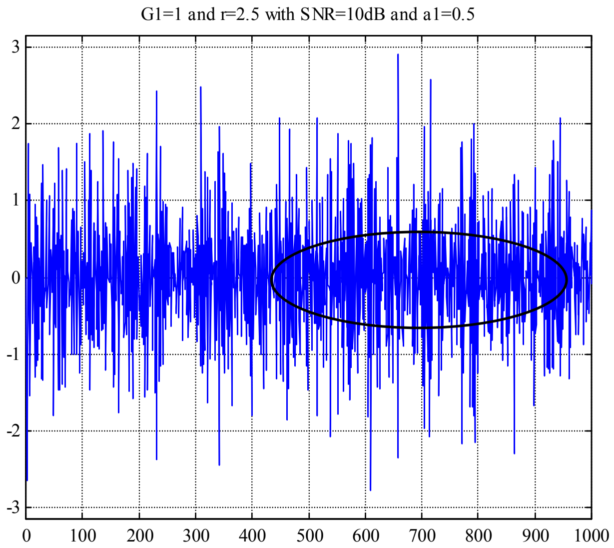
$\rightarrow \quad E[\psi_{ss}^2] \approx \dfrac{T_s}{2\pi}\int_{-\pi}^{+\pi}\left|\dfrac{D(z)F(z)}{1+D(z)F(z)}\right|^2_{z=e^{j2\pi f T_s}} \dfrac{\sigma_{n_\theta}^2}{f_s}\,df \quad \text{with} \quad T_s = \dfrac{1}{f_s}$

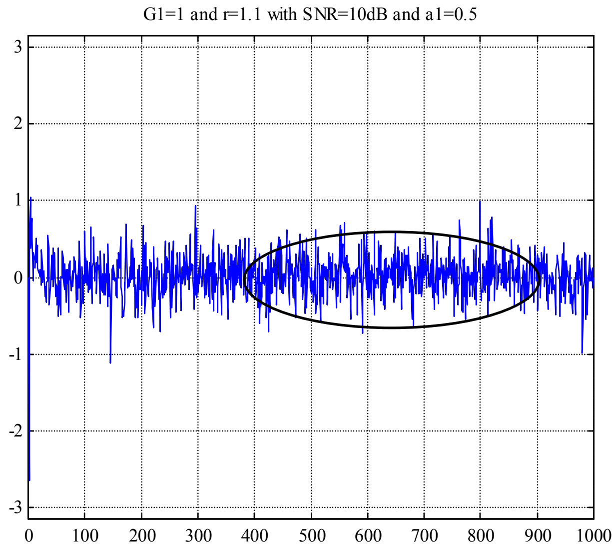
$$\operatorname{var}[\psi_{ss}] = E[\psi_{ss}^2] - E^2[\psi_{ss}]$$

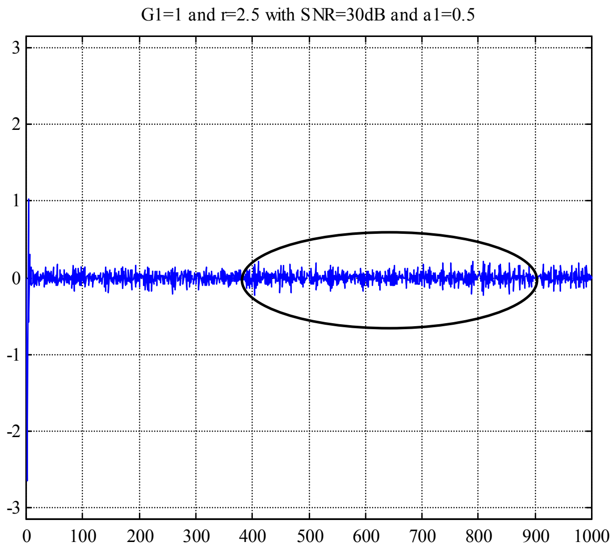
$$\operatorname{var}[\psi_{ss}] \approx \frac{2(r-1)+G_1(r+1)}{4-G_1(r+1)}\sigma_{n_\theta}^2$$

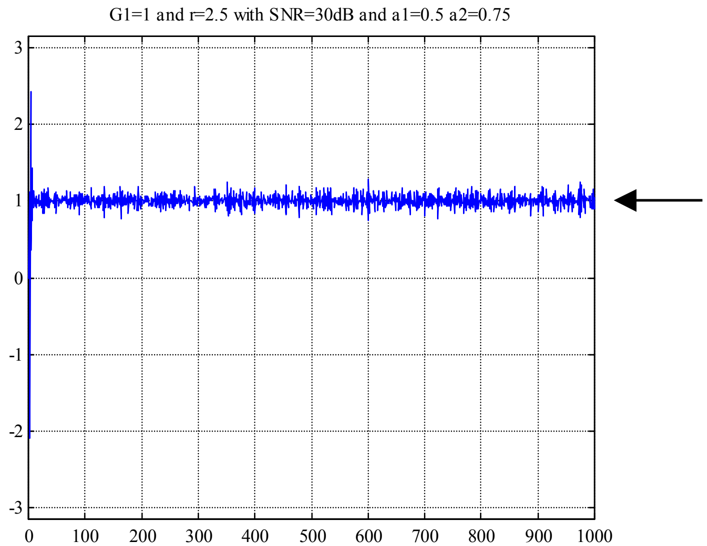





G1=1 and r=2.5 with SNR=10dB and a1=0.5
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=1 and r=1.1 with SNR=10dB and a1=0.5
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=1 and r=2.5 with SNR=30dB and a1=0.5
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000


G1=1 and r=2.5 with SNR=30dB and a1=0.5 a2=0.75
3
2
1
0
-1
-2
-3
0 100 200 300 400 500 600 700 800 900 1000