A NEW DOUBLE CURVED ELEMENT FOR TECHNICAL TEXTILE ANALYSIS WITH BENDING RESISTANCE

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Summary. A new double curved spatial element was developed to analyze textile structures. It has a bending resistance beside the in-plane stiffness. The element is based on a 8-node double curved membrane element[1]. To find the equilibrium solution the Total Lagrange strategy is used with the dynamic relaxation method (DRM). Deformations are calculated with a continuum mechanical method.

1 INTRODUCTION

For the analysis of textile structures membrane elements with in-plane stiffness are widely used. To analyze the shape of wrinkles or any not tensioned stage we need bending resistance.

The new method of present paper is based on a double curved membrane element[1]. The strategy of the analysis is the following:

- The numerical description is based on a double-curved finite element with a parametric surface coordinate system.
- Deformations are calculated between the initial (deformation free) and the actual geometry (Total Lagrange Method).
- Deformations are calculated by tensor analysis (Nonlinear Continuum Mechanics).
- The equilibrium shape is searched by dynamic relaxation.

The original method[1] used C0 finite elements. It was sufficient to describe the proper geometry of a surface for membrane analysis. For bendable shell analysis a C1 element is needed and the strategy of the analysis is presented here.

In our work we apply the thin shell theorem, namely the out of plane shear stresses are calculated from the bending moments[2].

2 THE DISCRETE MESH

9-node quadrilateral double curved, C1 continuous elements are chosen[3]. All nodes have freedom to move spatially and rotate except the internal node; it has only rotational freedom.

By taking into account large distortions of the geometry the side nodes and the internal nodes of the elements leave the middle position. For a linear transformation (and proper integration) between the parametric coordinate system of the element and the 3D global coordinate system we must control the movement of the internal nodes. By splitting the geometry of the surface and the geometry of the material it is possible to control the difficulty mentioned above. For this an advanced 8-node membrane element[4] was used in the membrane analysis[1]. That element is C0 continuous.

In practice the difference between the surface and the material geometry is small. The computational cost of using the C0 advanced 8-node element is acceptable. But if we want to use an advanced finite element technique for a C1 element, the computation becomes even more complicated and it requires an extreme computational cost. So in the technique presented here we neglect the difference between the surface and the material geometry.

3 CALCULATION OF THE DEFORMATIONS

3.1 Membrane deformations

Calculation of the membrane deformations are well described in [1]. The most important steps are shown here. The deformation is calculated from the change of the geometry between the free initial geometry (described by the $\bar{r}^0$ position vector) and the actual geometry (described by $\bar{r}$). The bases of the two stages are:

$$\bar{g}_k^0 = \frac{\partial \bar{r}^0}{\partial x^{0k}}, \tag{1}$$

$$\bar{g}_p = \frac{\partial \bar{r}}{\partial x^p}, \tag{2}$$

where $\bar{g}_k^0$ and $\bar{g}_p$ are the basis vectors, *k* and *p* can take the value ξ, η and *3* according to the surface coordinate system (*3* is the normal direction)[5]. The expression of the strain based on large elongations is the following:

$$\varepsilon_e = \frac{ds}{ds^0} - 1 = \sqrt{C_{kl} e^{0k} e^{0l}} - 1, \tag{3}$$

$$\frac{1}{2}\sin\gamma_{I,II} = \frac{H_{kl} e_I^{0k} e_{II}^{0l}}{(1+\varepsilon_I)(1+\varepsilon_{II})}, \tag{4}$$

where ε_e and $\gamma_{I,II}$ are the strains in the $\bar{e}$ directions and the distortion between the $\bar{e}_I$ and $\bar{e}_{II}$ directions, e^{0k} are the contravariant components of the directions in the initial stage (the material law is known usually in the initial stage, so the best is to calculate the deformations to the initial stage), C_{kl} are the covariant components of the Green-deformation tensor (it describes the transformation of the metric tensor of the two stages based on the initial stage), and H_{kl} are the covariant components of the Lagrangian-deformation tensor (it describes the difference between the scalar product of the differential lines of the two stage). The $\bar{e}_I$ and $\bar{e}_{II}$ directions are ordinary directions, in practice they are parallel with the directions of the yarns of the technical textiles.

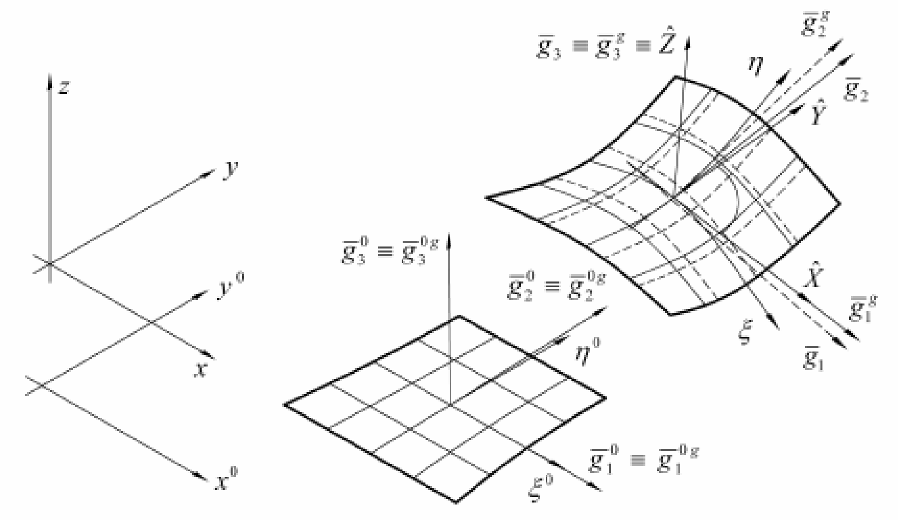

Figure 1: The basis system of the initial and the actual geometry.

The calculation of the Green-deformation tensors is the following:

$$g_{pq} = \bar{g}_k^0 \cdot \bar{\bar{F}}^T \cdot \bar{\bar{F}} \cdot \bar{g}_l^0 = \bar{g}_k^0 \cdot \bar{\bar{C}} \cdot \bar{g}_l^0 . \tag{5}$$

$$\bar{\bar{C}} = \bar{\bar{F}}^T \cdot \bar{\bar{F}} = \bar{g}^{0k} \otimes \bar{g}_p \cdot \bar{g}_p \otimes \bar{g}^{0k} , \tag{6}$$

where $\bar{\bar{F}}$ is the deformation gradient and g_{pq} is the metric tensor of the actual stage. The expression of the Lagrange-deformation tensor is the following:

$$\begin{aligned} &dr_I \cdot dr_{II} - dr_I^0 \cdot dr_{II}^0 = dr_I^0 \cdot \bar{\bar{F}}^T \cdot \bar{\bar{F}} \cdot dr_{II}^0 - dr_I^0 \cdot dr_{II}^0 = \\ &dr_I^0 \cdot \left(\bar{\bar{F}}^T \cdot \bar{\bar{F}} - \bar{\bar{I}}\right) \cdot dr_{II}^0 = dr_I^0 \cdot 2\bar{\bar{H}} \cdot dr_{II}^0 \end{aligned} , \tag{7}$$

$$\bar{\bar{H}} = \frac{1}{2}\left(\bar{\bar{F}}^T \cdot \bar{\bar{F}} - \bar{\bar{I}}\right) = \frac{1}{2}\left(\bar{\bar{C}} - \bar{\bar{I}}\right), \tag{8}$$

where dr_I, dr_{II}, dr_I^0 and dr_{II}^0 are line elements in the surface. The deformation gradient expressed by the base vectors of the two stages can be obtained as:

$$\overline{\overline{F}} = \overline{g}_p \otimes \overline{g}^{0k} . \tag{9}$$

For small elongations the Lagrangian-deformation tensor gives a good approximation of the strain. But when elongations are large, the (3-4) formulas must be used.

3.2 Changing of curvatures

The bending moments can be calculated from the varying curvatures. The curvature of a surface can be obtained from the gradient of the normal vector of the surface[2]:

$$grad\,\overline{n} = \frac{\partial \overline{n}}{\partial \xi^k} \otimes g^k \text{ and,} \tag{10}$$

$$\overline{n} = \overline{g}_3 = \frac{\overline{g}_\xi \otimes \overline{g}_\eta}{\left|\overline{g}_\xi \otimes \overline{g}_\eta\right|} \tag{11}$$

where $\overline{n}$ is the normal vector of the surface (equivalent with $\overline{g}_3$). The gradient of a surface is calculated by the simplified formula:

$$grad\,\overline{n} = -\begin{bmatrix} \overline{n}\dfrac{\partial \overline{g}_1}{\partial \xi} & \overline{n}\dfrac{\partial \overline{g}_2}{\partial \xi} & 0 \\ \overline{n}\dfrac{\partial \overline{g}_1}{\partial \eta} & \overline{n}\dfrac{\partial \overline{g}_2}{\partial \eta} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{12}$$

The curvature of the surface can be obtained by the element vector of any direction:

$$\omega_e = grad\,\overline{n}_{kl}\, e^k e^l , \tag{13}$$

where ω_e is the curvature in the direction of the e vector.

In practice, the initial geometry is flat for textile membranes (i.e. the flat pattern is the initial geometry), so the variation of the curvature is in the gradient of the actual geometry. If the initial geometry is not flat, the difference between the initial and the actual curvatures must be taken into account:

$$\Delta\omega_e = grad\,\overline{n}_{kl}\, e^k e^l - grad\,\overline{n}^0_{kl}\, e^{0k} e^{0l} . \tag{14}$$

4 CALCULATION OF THE INTERNAL FORCES

4.1 Membrane forces

The membrane forces can be calculated from the membrane strains. To study the equilibrium of the actual stage the Cauchy stress tensor is needed (all the geometry and directions are described in the actual stage). Traditionally during the measurements the material law is connected to the initial geometry, that is why the deformation tensors of the initial stage are used in this paper. In ordinary nonlinear problems (large displacements with small enlargement) the Second-Piola-Kirchoff (SPK) stress tensor is used: the stress directions are in the initial coordinate system and the geometry of the structure is described in the initial geometry. But the last statement is not exactly true: during the transformation not only the area of the elementary surface is changed but the stress vector, too:

$$dA = dA^0 \cdot \overline{\overline{F}}^T \cdot \frac{1}{\left|\overline{\overline{F}}\right|} \text{ and} \tag{15}$$

$$\overline{\overline{T}} = \overline{\overline{F}} \cdot \overline{\overline{S}} \cdot \overline{\overline{F}}^T \cdot \frac{1}{\left|\overline{\overline{F}}\right|}, \tag{16}$$

where $\overline{\overline{T}}$ is the Cauchy stress tensor (the real stress tensor of the actual geometry), $\overline{\overline{S}}$ is the SPK stress tensor (the stress tensor of the initial geometry) and dA is the elementary area. In equation (15) there is a similar transformation of the area to equation (16), but to change the direction from the initial to the actual state there is a second transformation (multiplication by the gradient tensor $\overline{\overline{F}}$), which transformation scales the size of the vector beside changing the direction. It is generally a negligible side-effect if there are no large elongations.

The material-laws give back not the SPK stress tensor but the Biot stress tensor. To get the Cauchy stress tensor from the Biot stress tensor we need a little bit more complication in the transformation process:

$$\overline{\overline{T}} = \overline{\overline{R}} \cdot \overline{\overline{T_0}} \cdot \overline{\overline{F}}^T \cdot \frac{1}{\left|\overline{\overline{F}}\right|}, \tag{17}$$

where $\overline{\overline{T_0}}$ is the Biot stress tensor and $\overline{\overline{R}}$ is the rotation tensor. Compared to the usage of the SPK stress tensor the rotation tensor must be used. That can be obtained by the following:

$$\overline{\overline{R}} = \overline{\tilde{g}}_p \otimes \overline{g}^{0k}, \tag{18}$$

where $\overline{\tilde{g}}_p$ is the non stretched basis vector system, and can be obtained from the basis vectors of the initial stage:

$$\overline{\tilde{g}}_k \cdot \overline{\tilde{g}}_l = \overline{g}^0{}_k \cdot \overline{g}^0{}_l , \tag{19}$$

if the normal vector of the actual stage is used as $\overline{\tilde{g}}_3$ and $\overline{\tilde{g}}_\xi$ has the direction of $\overline{g}_\xi$ and the length of $\overline{g}_\xi^0$[5].

4.2 Bending moments

The moments of the structure can be calculated from the above presented equation (13):

$$\overline{\overline{M}}_0 = \overline{\overline{D}} \cdot \overline{\varpi} , \tag{20}$$

where $\overline{\overline{M}}$ is the vector of the moment, $\overline{\overline{D}}$ is the matrix of the constitutive law and the $\overline{\varpi}$ is the vector of the rotations and the torsion of the sections. The rotation vector is invariant so the constitutive law can be used in the initial coordinate system, and then the values of the moment vector can be transformed to the actual stage like the membrane stresses:

$$\overline{\overline{M}} = \overline{\overline{R}} \cdot \overline{\overline{M_0}} \cdot \overline{\overline{F}}^T \cdot \frac{1}{\left|\overline{\overline{F}}\right|} . \tag{21}$$

4.3 Out of plane shear forces

In the thin shell theorem the out-of-plane shear forces derived from the bending moments[2] are the following:

$$\overline{V} = \begin{bmatrix} \frac{\partial}{\partial \xi} & 0 & \frac{\partial}{\partial \eta} \\ 0 & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \xi} \end{bmatrix} \begin{bmatrix} M_\xi \\ M_\eta \\ M_{\xi\eta} \end{bmatrix} . \tag{22}$$

where $\overline{V}$ is the vector consists of the outof-plane shear forces and the moments are presented in vector format instead of tensor format.

5 THE NODAL EQUILIBRIUM

To find the equilibrium of the structure we seek the nodal equilibrium. Having a nonlinear problem an iterative process is needed. The dynamic relaxation method (DRM) is our choice. The DRM uses fictive dynamic analysis[6] with appropriate dumping to find the equilibrium configuration. In membrane analysis the nodal movement is followed. Due to the bending resistance the rotation of the tangent plane of the surface must be used, too. With C1 finite elements it is possible to carry out this process.

5.1 External loads

By ordinary finite element transformations it is possible to reduce the surface loads to the nodes:

$$\bar{f}_{ex} = \int_A \overline{\overline{N}}^T \cdot \bar{q}\, dA\,, \tag{24}$$

where $\bar{f}_{ex}$ is the nodal force vector from the external loads, $\overline{\overline{N}}$ is the matrix with the shape functions and $\bar{q}$ is the surface load vector.

5.2 Internal forces

$$\bar{f}_{in} = \int_A \overline{\overline{B}}_\sigma^T \cdot \bar{\sigma}\, dA + \int_A \overline{\overline{B}}_\tau^T \cdot \bar{\tau}\, dA + \int_A \overline{\overline{B}}_M^T \cdot \overline{M}\, dA\,, \tag{25}$$

where $\bar{f}_{in}$ is the nodal force vector from the internal forces, $\overline{\overline{B}}_\sigma$, $\overline{\overline{B}}_\tau$ and $\overline{\overline{B}}_M$ are the transformation matrices of the internal forces, $\bar{\sigma}$ is the vector of the membrane forces, $\bar{\tau}$ is the vector of the out-of-surface shear forces and $\overline{M}$ is the vector of the bending moments.

To get the $\overline{\overline{B}}$ matrices the operator matrices of the deformations are needed[2]:

$$\bar{\varepsilon}_{(\xi\eta)}^{\hat{X}\hat{Y}} = \overline{\overline{L}}_\varepsilon \cdot \hat{u}_{(\xi\eta)} = \begin{bmatrix} \frac{\partial}{\partial \hat{X}} & 0 \\ 0 & \frac{\partial}{\partial \hat{Y}} \\ \frac{\partial}{\partial \hat{Y}} & \frac{\partial}{\partial \hat{X}} \end{bmatrix} \begin{bmatrix} u_{(\xi\eta)}^{\hat{X}} \\ u_{(\xi\eta)}^{\hat{Y}} \end{bmatrix}, \tag{26}$$

where $\hat{X}$ and $\hat{Y}$ are the coordinates of the invariant surface coordinate system used for the integration[1] $\bar{\varepsilon}_{(\xi\eta)}^{\hat{X}\hat{Y}}$ gives back the membrane strains with the parametric coordinates of the invariable coordinate system of the surface, $\overline{\overline{L}}_\varepsilon$ is the operator matrix, $\bar{u}_{(\xi\eta)}$ is the in-plain displacement vector according to the parametric coordinate system. (Details of the transformations can be found in [1])

The expression of the shear deformation is the following:

$$\bar{\gamma} = \overline{\overline{L}}_\gamma \cdot \bar{u}_{(\xi\eta)} = \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} [u_3], \tag{27}$$

where $\bar{\gamma}$ is the out of plain strain vector, $\overline{\overline{L}}_\gamma$ is the operator matrix, $\bar{u}_{(\xi\eta)}$ is the out of plain movement vector according to the parametric coordinate system. Strains perpendicular to the surface are neglected. The following formula shows the calculation of the curvatures:

$$\overline{\omega} = \overline{\overline{L}}_{\omega} \cdot \overline{\Theta}_{(\xi\eta)} = \begin{bmatrix} \frac{\partial}{\partial \xi} & 0 \\ 0 & \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} \Theta_{\xi} \\ \Theta_{\eta} \end{bmatrix}, \tag{28}$$

where $\overline{\omega}$ is the vector of the change of the curvature of the surface, $\overline{\overline{L}}_{\omega}$ is the operator matrix, $\overline{\Theta}_{(\xi\eta)}$ is the rotation according to the parametric coordinate system.

The operators of the membrane strains and the curvature variation are identical. It means, the $\overline{\overline{B}}$ matrices are identical too. The $\overline{\overline{B}}$ matrix of [1] can be used, the detailed development of the formula can be found in the original paper[1]. Here just the final formula is expressed:

$$\overline{\overline{B}} = \frac{1}{|J|} \begin{bmatrix} \hat{X} \cdot \left(\overline{g}_2^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \xi} - \overline{g}_1^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) \\ \hat{Y} \cdot \left(-\overline{g}_2^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \xi} + \overline{g}_1^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) \\ \hat{Y} \cdot \left(\overline{g}_2^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \xi} - \overline{g}_1^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) + \hat{X} \cdot \left(-\overline{g}_2^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \xi} + \overline{g}_1^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) \end{bmatrix}, \tag{29}$$

where $|J|$ is the Jacobian determinant. Finally, $\overline{\overline{B}}_{\tau}$ is:

$$\overline{\overline{B}}_{\tau} = \frac{1}{|J|} \begin{bmatrix} \hat{X} \cdot \left(\overline{g}_2^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \xi} - \overline{g}_1^g \cdot \hat{Y} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) \\ \hat{Y} \cdot \left(-\overline{g}_2^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \xi} + \overline{g}_1^g \cdot \hat{X} \frac{\partial \overline{\overline{N}}}{\partial \eta} \right) \end{bmatrix}. \tag{30}$$

5.3 Dynamic Relaxation Method

The difference of the external and the internal nodal forces accelerate the fictitious mass of the nodes:

$$a_i = \frac{f_{ex,i} - f_{in,i}}{m_i}, \tag{31}$$

where a_i is the acceleration of the i^{th} freedom, and m_i is the mass of the i^{th} freedom. The freedom can be a movement or a rotation. To handle movements and rotations, we assume a (fictive) masses and (fictive) moment of inertia at each node, respectively. Both viscous dumping and kinetic dumping can be used[1] to find the equilibrium position of the system.

6 CONCLUSIONS

A new method was developed for analysis of thin shell structures. It extends the membrane analysis technology[1] with bending stiffness. The calculation of the deformations by tensor analysis gives the chance to omit any approximation. Only the quality of the discrete mesh is a limitation of the accuracy of the analysis.

The new method gives an effective tool for analyzing thin shells and wrinkling of textiles and textile membranes.

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