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SHEAR DEFORMATIONS IN INFLATED CYLINDRICAL BEAMS: AN OLD MODEL REVISITED

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Summary. The paper considers the effects of the shear deformations on the load-displacement response of pressurized thin-walled cylindrical beams of circular shape, subject to lateral loads. In order to schematize the nonlinear behavior of partly wrinkled beams under simultaneous bending and shear, use is made of some classical structural models which account for the inability of the wall material of sustaining compressive stresses. A particular attention is posed on the correct determination of the shear stiffness within the wrinkled zones of the beam. The system of non linear equations that govern the equilibrium of the inflated beams after the onset of the post-critical phase, when wrinkling of the cross-sections still remains small or moderate, is suitable to be numerically solved by standard incremental-iterative algorithms.

1 INTRODUCTION

A number of efficient structural and continuum models are currently available to describe the mechanical response of pressurized beams in bending until their final collapse. Obviously, each approach, structural or continuum, has strengths and weaknesses since their validity fields as well as their goals are often very different. Structural models[1, 2] are more common and easier to use compared to the continuous models (either analitical[3] or numerical[4]) but cannot provide those local information that very often are necessary and that are obtained more quickly with the continuous models. By contrast, analyses that use continuous models are always slow and problematic since they require a more precise and detailed description of the mechanical problem. For example, it is easy to assign a concentrated lateral load in a generic section of the beam while it is extremely difficult to specify an equivalent load condition to the continuous model. Things are even more difficult when describing the constraints. In many cases the results offered by the two models for the same mechanical problem may differ unexpectedly and to be hard to compare each other.

Fortunately, a third opportunity is offered by the bridging models[5, 6], definable also as advanced structural models, by which it is possible to get the required local information, usually the averaged values of some quantities, without suffering the complications of the continuum models.

With reference to the cylindrical inflatable beams of circular cross-section in bending, the founder of such models is undoubtedly the one proposed by Comer and Levy[5] to study the mechanical response of a cantilever beam subjected to a concentrated or a uniformly distributed lateral load, along the wrinkled phase which precedes the collapse. Because of its simplicity and capability of giving the correct response in many practical situations, this model has been intensively used in the following by many other authors[7, 8] as a starting point in order to incorporate some typical properties of the materials used to realize the cylindrical wall, in particular, anisotropy or, very often, other special non-linear constitutive laws characterizing the tissues composing the structural membranes.

A point of the above model that, in our opinion, has not yet been sufficiently considered so far concerns the effects of the shear deformations, usually disregarded, on the load-displacements response of the partly wrinkled inflated beams. Although this simplification be legitimate when considering the ordinary slender beams in bending, this argument is no more valid for the inflated beams: in fact, because of their peculiar small bending stiffness, to make acceptable the values of the lateral displacements under possible transversal loads of appreciable magnitude, they need using sizes of the cross-sections no longer negligible compared to their span. In addition, this topic becomes particularly important during the wrinkled phase since, contrary to the bending stiffness which decreases slowly for increasing wrinkling, the shear stiffness of the wrinkled cross-sections diminishes very rapidly.

For this reason, in this paper, we analyze the kinematical effects of the shear deformations arising in inflated cylindrical beams, partly wrinkled, subjected to bending, focusing on the still reagents parts of their cross-sections. The system of nonlinear equations governing the equilibrium of the inflated beams subject to simultaneous bending and shear appears suitable to be numerically solved by standard incremental-iterative algorithms, so that the mechanical response of the beams during the loading phase that follows the onset of wrinkling can be accurately monitored.

2 THE MECHANICAL MODEL AND THE STATE "0"

We consider a cylindrical thin-walled beam of circular shape, subjected to an internal pressure p. Let r be the radius of its mean surface and t the wall thickness. This may be the real value in the case of a very thin shell or an equivalent fictitious one if the wall is made of a structural tissue. In any case, we admit the thickness be so small that the wall does not possesses any bending or torsional stiffness, so that it may sustain only membrane states of stress; moreover, we admit that the material is sufficiently soft when it is contracted, so that no compressive stress may be engendered, but at the same time, it is stiff enough in tension to avoid appreciable variations of the radius r.

With reference to a rectangular coordinate system (O, x, y, z), in the initial configuration, chosen as reference, the axis of the cylinder lies along the z-axis. The origin of the reference system is placed at the centroid of one of its bases.

For what concerns the kinematical aspects, we assume that the hypotheses of small displacements/rotations and small deformations hold; moreover, we assume that during the considered loading phase the internal pressure be sufficient to maintains circular the shape of the cross-sections. Finally, the usual Navier-Bernoulli hypothesis on the plane sections holds for all the cross-sections of the cylinder, whether they belong to taut or partly wrinkled regions. With regard to the constitutive law, we assume the material be linear elastic, homogeneous and isotropic when it is subject to elongations. In case of contraction, instead, the material does not react in any way.

Before the application of any external load, the inflated beam lies in its "*0*" state. Here, if we admit the effects of the dead load may be neglected, the state of stress within the wall is uniform and characterized by the principal stresses

$$\sigma_1 = \sigma_c = p\,r/t \;\text{ and }\; \sigma_2 = \sigma_l = p\,r/2\,t\,, \tag{1}$$

where σ_c denotes the circumferential stress and σ_l the axial one. The distribution of the axial stress is represented in Fig. 1a.

3 THE UNWRINKLED PHASE

For small or moderate values of the bending moment *M*, the neutral axis $n-n$ does not intersect the cross-section, so the state of stress appears as in Figure 1b.

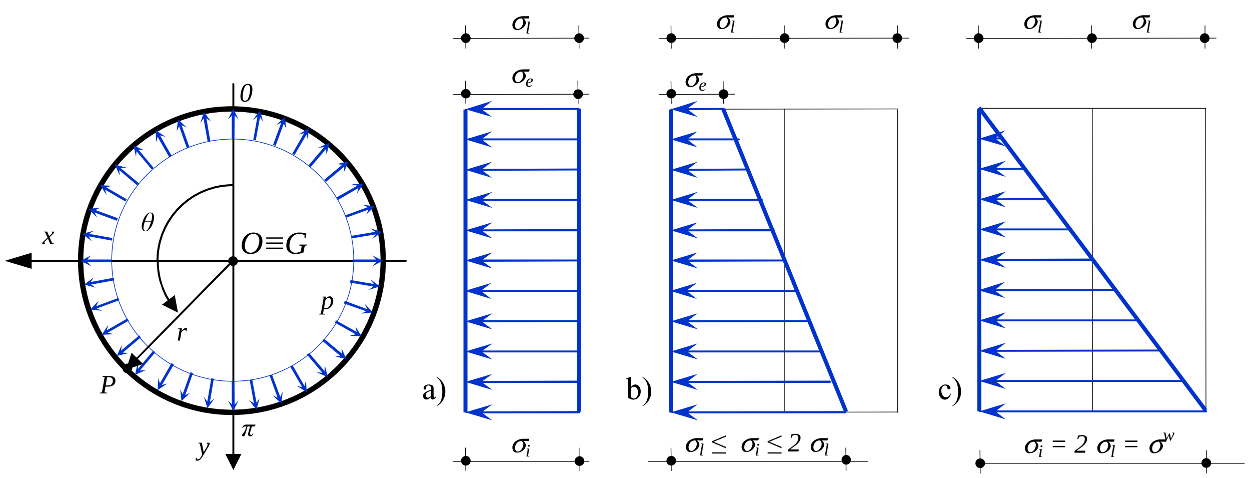

Figure 1: Stress distributions: a) at the state 0, b) during the unwrinkled phase, c) at the onset of wrinkling.

Because of the Navier-Bernoulli hypothesis, the normal stress at point *P(r,θ)* is

$$\sigma_z(\theta) = \sigma_e\,(1+\cos\theta)/2 + \sigma_i\,(1-\cos\theta)/2 \qquad \text{for} \quad 0 \le \theta \le \pi\,, \tag{2}$$

where θ denotes the angular position of the point P and σ_i and σ_e denote the unknown normal stresses at the intrados and at the extrados of the cylinder, respectively.
In the absence of axial force, $N = 0$, the equilibrium of the beam along the *z*-axis

$$2\int_0^{\pi} \sigma_z(\theta)\, r\, t\, d\theta - p\,\pi\, r^2 = 0\,, \tag{3}$$

gives the first static equivalence relation

$$\sigma_i + \sigma_e = p\ r/t\,, \tag{4}$$

while the rotational equilibrium about the *x*-axis

$$2\int_0^{\pi} \sigma_z(\theta)\, r^2 \cos(\pi-\theta)\, t\, d\theta = M\,, \tag{5}$$

leads to the second static equivalence relation

$$(\sigma_i - \sigma_e)\, \pi\, r^2\, t/2 = M\,. \tag{6}$$

Putting together (4) and (6), we have

$$\sigma_i = \frac{p\, r}{2\, t} + \frac{M}{\pi\, r^2\, t} \quad \text{and} \quad \sigma_e = \frac{p\, r}{2\, t} - \frac{M}{\pi\, r^2\, t}\,. \tag{7}$$

At the onset of wrinkling $\sigma_e = 0$, thus $M = M^w = p\, \pi\, r^3/2$ and $\sigma_i = \sigma^w = p\ r/t$.

As long as $\sigma_i \le \sigma^w$ the cross-section remains in the active state (taut), its moment of inertia is $J_x = \pi\ r^3\, t$ and the following linear constitutive law holds between the local elastic curvature k_x^E and the bending moment *M*

$$k_x^E = \frac{1}{R_x} = \frac{\varepsilon_e - \varepsilon_i}{2\, r} = \frac{\sigma_e - \sigma_i}{2\, E\, r} = -\frac{M}{E\, J_x}\,, \tag{8}$$

where *E* is the Young's modulus of the material.

The Jourawski formula gives the expressions for the shear stress $\tau_z(\theta)$ and the corresponding shearing strain $\gamma_z(\theta)$ at point $P(r,\theta)$

$$\tau_z(\theta) = \frac{T_y \sin\theta}{\pi\, r\, t}\,, \qquad \gamma_z(\theta) = \frac{\tau_z(\theta)}{G} = \frac{T_y \sin\theta}{G\, \pi\, r\, t}\,, \tag{9}$$

where *G* is the shear modulus of the material. Finally, by means of the Clapeyron theorem, we derive the sought expression for the characteristic shearing strain of the cross-section

$$\gamma_y = \frac{T_y}{G\, \pi\, r\, t} = \frac{2\, T_y}{G\, A}\,, \tag{10}$$

where $A = 2\, \pi\, r\, t$ is the area of the complete section and $\chi_y = 2$ is the shear factor.

4 THE WRINKLED PHASE

When $M > M^w$, the neutral axis $n - n$ intersects the cross-section, so the mechanical behavior of the cylindrical beam changes considerably; in fact, since the membrane is unable to sustain compressive stresses, from a structural point of view this is equivalent to admit a loss of resisting material. Thus, for increasing M, the active zone of the cross section reduces progressively to that represented by a thicker line in Figure 2. We denote with $2\,\theta_0$ the angular amplitude of the wrinkled zone.

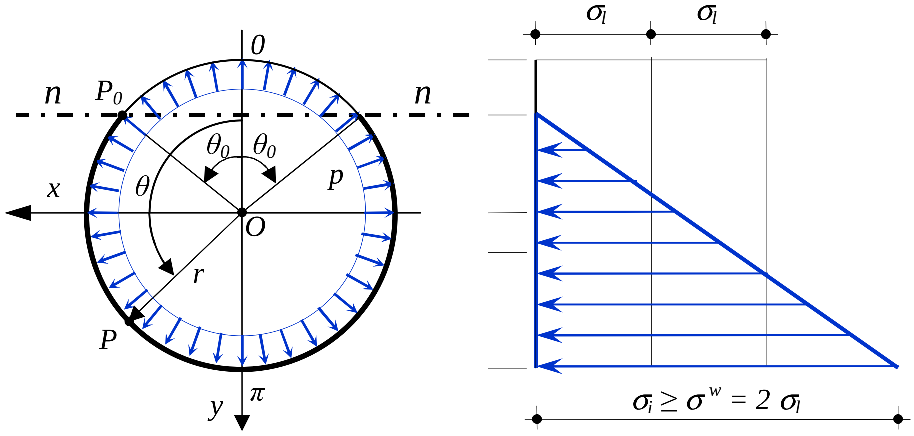

Figure 2: stress distributions along the wrinkled phase.

Since $\sigma_e = 0$ and $\sigma_i > \sigma^w$, the normal stress at point *P(r,θ)* now becomes

$$\sigma_z(\theta,\theta_0) = \frac{\cos\theta_0 - \cos\theta}{1+\cos\theta_0}\,\sigma_i\,, \tag{11}$$

and the equilibrium along *z*-axis gives the first relation of static equivalence

$$\sigma_i(\theta_0) = \frac{p\,\pi\,r}{2\,t}\,\frac{1+\cos\theta_0}{\sin\theta_0 + (\pi-\theta_0)\cos\theta_0}\,. \tag{12}$$

Putting together (11) and (12) we have

$$\sigma_z(\theta,\theta_0) = \frac{p\,\pi\,r}{2\,t}\,\frac{\cos\theta_0 - \cos\theta}{\sin\theta_0 + (\pi-\theta_0)\cos\theta_0}\,. \tag{13}$$

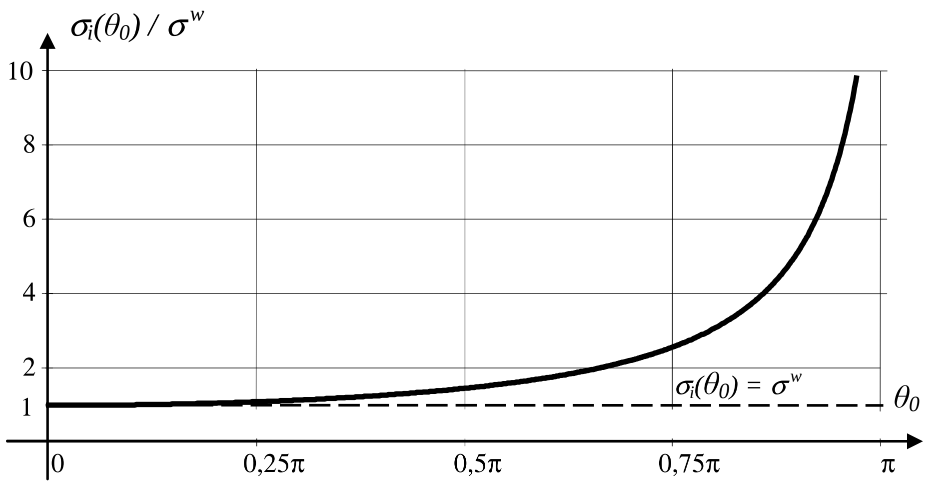

Figure 3: Law of variation of $\sigma_i(\theta_0)$.

It is interesting to consider the law of variation of $\sigma_i(\theta_0)/\sigma^w$ illustrated in the Figure 3. We observe how, just after the onset of wrinkling, *i.e.*, when the values of θ_0 are still small or moderate, $0 \le \theta_0 < \pi/4$, $\sigma_i(\theta_0)$ remains almost constant, $\sigma_i(\theta_0) \cong \sigma^w = p\,r/t$; conversely, when θ_0 approaches π, $\sigma_i(\theta_0)$ is not limited. In other words, the global equilibrium imposes locally the presence of a concentrated force.

The rotational equilibrium about the *x*-axis

$$2\int_{\theta_0}^{\pi} \sigma_z(\theta,\theta_0)\, r^2 \cos(\pi-\theta)\, t\, d\theta = M\,, \tag{14}$$

leads to the second static equivalence relation

$$\sigma_i(\theta_0) = \frac{M}{r^2 t}\frac{1+\cos\theta_0}{\pi-\theta_0+\sin\theta_0\cos\theta_0}\,, \tag{15}$$

which, by means of (12), furnishes

$$M = \frac{p\,\pi\,r^3}{2}\frac{\pi-\theta_0+\sin\theta_0\cos\theta_0}{\sin\theta_0+(\pi-\theta_0)\cos\theta_0}\,. \tag{16}$$

Once *M* is assigned, this equation gives the sought value of $\theta_0(M)$.

If we take the limit of the above expression for $\theta_0 \to \pi$, we obtain

$$M^u = \lim_{\theta_0\to\pi} M(\theta_0) = p\,\pi\,r^3 = 2M^w\,, \tag{17}$$

the *ultimate bending moment* that an inflated cylindrical beam of radius *r* subject to internal pressure *p* may sustain if the material of the membrane were able to resist at the intrados to the concentrated axial force $T_{\max} = p\,\pi\,r^2$.

Since $p\,\pi\,r^3/2 = M^w$, from (16) we obtain

$$\frac{M}{M^w} = \frac{\pi-\theta_0+\sin\theta_0\cos\theta_0}{\sin\theta_0+(\pi-\theta_0)\cos\theta_0} = m(\theta_0)\,, \tag{18}$$

an expression that permits to recover $M(\theta_0)$ once the angular amplitude $2\theta_0$ of the wrinkle zone is known. A graph of $m(\theta_0) = M/M^w$ is given in the next Figure 4.

We notice how a good approximation for $m(\theta_0)$ within the interval $(0, \pi)$ is given by the simpler function

$$\frac{M}{M^w} \cong \mu(\theta_0) = \frac{3-\cos\theta_0}{2}\,, \tag{19}$$

from which we obtain the value

$$\theta_0 = \arccos(3 - 2\,M/M^w)\,, \tag{20}$$

of the angle which individualizes the position of neutral axis $n - n$.

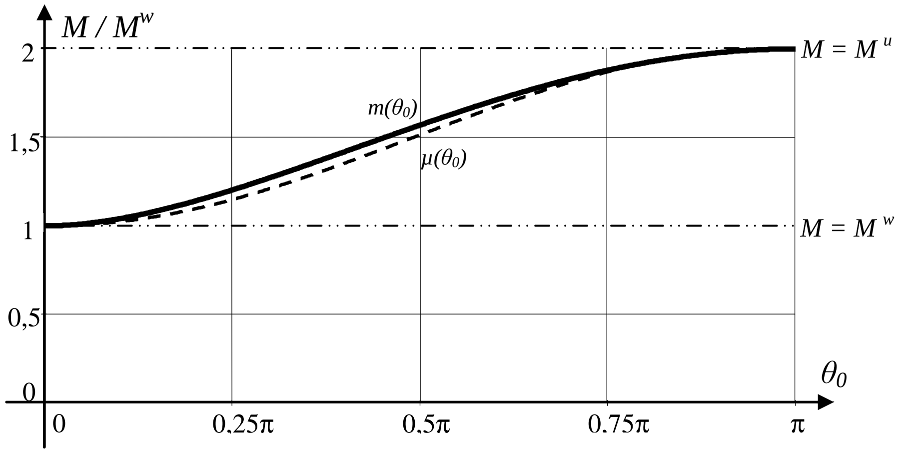

Figure 4: Law of variation of $M(\theta_0)/M^w$.

By analogy with (8), the local curvature of a wrinkled section is

$$k_x^w = \frac{1}{R^w} = -\frac{\varepsilon_i}{r\,(1+\cos\theta_0)} = -\frac{\sigma_i}{E\,r\,(1+\cos\theta_0)} = -\frac{M}{E\,J_x}\,\frac{\pi}{\pi-\theta_0+\sin\theta_0\,\cos\theta_0}, \tag{21}$$

from which we obtain the expression of the fictitious moment of inertia of a wrinkled section

$$J^w(\theta_0) = J_x\,\frac{\pi-\theta_0+\sin\theta_0\,\cos\theta_0}{\pi}, \tag{22}$$

able to simultaneously account for the two different static schemes that a wrinkled section uses to resist to an assigned bending moment: the first one is that of a couple deriving from two eccentric axial forces, and the second, of minor importance on a quantitative basis, of pure bending. In effect, the real moment of inertia of a wrinkled section is

$$J_{x'}(\theta_0) = J_x\left(\frac{\pi-\theta_0-\sin\theta_0\,\cos\theta_0}{\pi} - \frac{2\sin^2\theta_0}{\pi\,(\pi-\theta_0)}\right), \tag{23}$$

where x' is the axis parallel to the x-axis but containing the centroid of the wrinkled section. A plot of both the ratios $J^w(\theta_0)/J_x$ and $J_{x'}(\theta_0)/J_x$ is given in the following Figure 5. From this it is evident how the real moment of inertia decreases quite rapidly even for small values of θ_0, so that the capacity of a wrinkled cross-section to resist trough pure bending is soon frustrated. This feature is important also with regard to the shear deformations since the shearing force is resisted only through the unwrinkled zone.

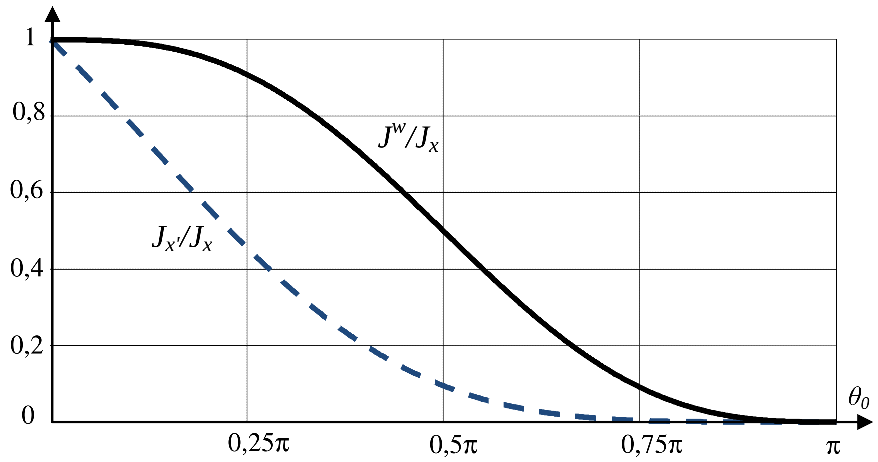

Figure 5: Law of variation of the moment of inertia of a wrinkled cross-section: (J^w fictitious, $J_{x'}$ effective).

The assessment of the shear deformations within a wrinkled zone is more involved with respect to that of a taut zone since now M and $\theta_0(M)$ both depend on z. As a consequence, the expressions of the shear stress $\tau_z(\theta,\theta_0)$ and the shearing strain $\gamma_z(\theta,\theta_0)$ at point $P(r,\theta)$ as well as for the characteristic shear strain $\gamma_y(\theta_0)$ of the overall cross-section are much more complicated and will be given in an incoming paper. Here, for the sake of simplicity, we consider the frequent case of distributed loads of small magnitude. Within this assumption, since the shear force T_y is small, the bending moment M changes slowly along the z-axis, so we may disregard the variation of the angular amplitude $\theta_0(M)$.

Thus, we may make use once again of the Jourawski formula which gives the following expressions for the shear stress $\tau_z^w(\theta)$ and the corresponding shearing strain $\gamma_z^w(\theta)$ at the point $P(r,\theta)$

$$\tau_z^w(\theta,\theta_0) = \frac{T_y}{r\,t}\,\frac{(\pi-\theta_0)\sin\theta-(\pi-\theta)\sin\theta_0}{(\pi-\theta_0)(\pi-\theta_0-\sin\theta_0\cos\theta_0)-2\sin^2\theta_0}, \quad \theta_0 \le \theta \le \pi, \qquad (24)$$

$$\gamma_z^w(\theta,\theta_0) = \frac{T_y}{G\,r\,t}\,\frac{(\pi-\theta_0)\sin\theta-(\pi-\theta)\sin\theta_0}{(\pi-\theta_0)(\pi-\theta_0-\sin\theta_0\cos\theta_0)-2\sin^2\theta_0}.$$

Finally, by means of the Clapeyron theorem, we derive the sought expression for the characteristic shearing strain of a partly wrinkled cross-section

$$\gamma_y^w(\theta_0) = \frac{\chi_y(\theta_0)\,T_y}{G\,A}, \qquad (25)$$

where $A = 2\,\pi\,r\,t$ is still the area of the complete section and $\chi_y(\theta_0)$ is the shear factor of the wrinkled cross-section whose expression is

$$\chi_y(\theta_0) = \frac{2\pi((2(\pi-\theta_0)^3 - 12(\pi-\theta)\sin^2\theta_0 + 3(\pi-\theta)^2\,(\pi-\theta-3\sin\theta_0\cos\theta_0))}{3(\pi-\theta_0)^2\,(\pi-\theta_0-\sin\theta_0\,\cos\theta_0 - 2\,\dfrac{\sin^2\theta_0}{\pi-\theta})}, \qquad (26)$$

whose graph is represented in the Figure 6.

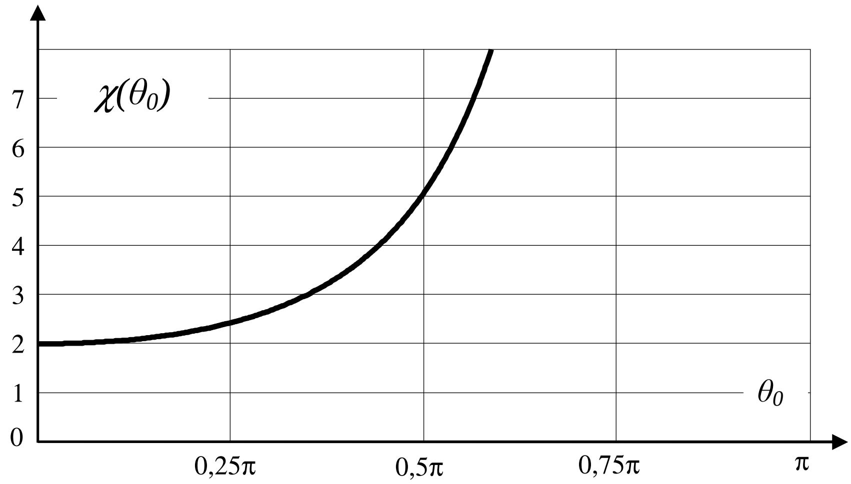

Figure 6: Law of variation of the shear factor of a wrinkled cross-section.

The graph gives a clear indication of the decreasing rate of the shear stiffness of a pressurized beam for increasing wrinkling.

CONCLUSIONS

In this paper, we analyzed the effects of the shear deformations on the load-displacement response of inflated cylindrical beams in bending. A particular attention was posed on the equilibrium state (stress-strain) within the partly wrinkled zones of the beam.

The analysis was performed by means of some classical structural models which account for the inability of the material of the cylindrical wall of sustaining compressive stresses.

The obtained system of nonlinear equations governing the equilibrium of the inflated beams appears suitable to be numerically solved by standard incremental-iterative algorithms, so the mechanical response of these beams after the onset of wrinkling can be accurately monitored.

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