An alternate description of a \((q + 1, 8)\)-cage

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Abstract

Let \( q \geq 2 \) be a prime power. In this note we present an alternate description of the known \((q + 1, 8)\)-cages which has allowed us to construct small \((k, g)\)-graphs for \( k = q - 1 \), \( q \) and \( g = 7, 8 \) in other papers on this same topic.

Keywords: Cages, girth, Moore graphs, perfect dominating sets.

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [14] for terminology and notation.

Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The girth of $G$ is the number $g = g(G)$ of edges in a shortest cycle. For every $v \in V$, $N_G(v)$ denotes the neighbourhood of $v$, i.e. the set of all vertices adjacent to $v$, and $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighbourhood of $v$. The degree of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $S \subset V(G)$, then we denote by $N_G(S) = \cup_{s \in S} N_G(s) - S$ and by $N_G[S] = S \cup N_G(S)$.

A graph is called regular if all its vertices have the same degree. A $(k, g)$-graph is a $k$-regular graph with girth $g$. Erdős and Sachs [15] proved the existence of $(k, g)$-graphs for all values of $k$ and $g$ provided that $k \geq 2$. Since then most work carried out has focused on constructing a smallest $(k, g)$-graph (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 20, 21, 22]). A $(k, g)$-cage is a $k$-regular graph with girth $g$ having the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [25] in 1947. More details about constructions of cages can be found in the recent survey by Exoo and Jajcay [17].

In this note we are interested in $(k, 8)$-cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$-cage:

$$n_0(k, 8) = 2(1 + (k - 1) + (k - 1)^2 + (k - 1)^3).$$

A $(k, 8)$-cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$-graph (cf. [14]). These graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4, q)$ and $W(q)$ [12, 17, 24], which are known to exist for $q$ a prime power and $k = q + 1$ and no example is known when $k - 1$ is not a prime power (cf. [11, 13, 19, 27]). Since they are incidence graphs, these cages are bipartite and have diameter 4. Recall also that if $q$ is even, $Q(4, q)$ is isomorphic to the dual of $W(q)$ and viceversa. Hence, the corresponding $(q + 1, 8)$-cages are isomorphic.

In this note we present an alternate description of the known $(q + 1, 8)$-cages with $q \geq 2$ a prime power as follows:

**Definition 1.1.** Let $\mathbb{F}_q$ be a finite field with $q \geq 2$ a prime power and $\varrho$ be a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[W_0, W_1]$ denote a bipartite graph with vertex sets $W_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}, i = 0, 1$, and edge set defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{\Gamma_q}((a, b, c)_i) = \{(w, aw + b, a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\};$$

$$N_{\Gamma_q}((\varrho, b, c)_i) = \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\};$$

$$N_{\Gamma_q}((\varrho, \varrho, c)_i) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\};$$

$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_i) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}.$$
Or equivalently,

For all \( i, j, k \in \mathbb{F}_q \)

\[
N_{\Gamma_q}((i, j, k)_0) = \{(w, j - wi, w^2i - 2wj + k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\}
\]

\[
N_{\Gamma_q}((j, k)_1) = \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\}
\]

\[
N_{\Gamma_q}((\varrho, k)_0) = \{(w, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\};
\]

\[
N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\}.
\]

Note that \( \varrho \) is just a symbol not belonging to \( \mathbb{F}_q \) and no arithmetical operation will be performed with it.

**Theorem 1.2.** The graph \( \Gamma_q \) given in Definition 1.1 is a Moore \((q + 1, 8)\)-graph for each prime power \( q \geq 2 \).

The proof of the above theorem shows that the graph \( \Gamma_q \) described in Definition 1.1 is in fact a labelling for a \((q + 1, 8)\)-cage, for each prime power \( q \geq 2 \). We need to settle this alternate description because it is used in [2, 3, 4] to construct small \((k, g)\)-graphs for \( k = q - 1, q \) and \( g = 7, 8 \).

## 2 Proof of Theorem 1.2

### 2.1 Preliminaries: the graphs \( H_q \) and \( B_q \)

In order to prove Theorem 1.2 we will first define two \( q \)-regular bipartite graphs \( H_q \) and \( B_q \) (cf. Definitions 2.1 and 2.4). The graph \( H_q \) was also introduced by Lazebnik, Ustimenko and Woldar [20] with a different formulation.

**Definition 2.1.** Let \( \mathbb{F}_q \) be a finite field with \( q \geq 2 \). Let \( H_q = H_q[U_0, U_1] \) be a bipartite graph with vertex set \( U_r = \mathbb{F}_q^3, r = 0, 1 \), and edge set \( E(H_q) \) defined as follows:

For all \( a, b, c \in \mathbb{F}_q \)

\[
N_{H_q}((a, b, c)_1) = \{(w, aw + b, a^2w + c)_0 : w \in \mathbb{F}_q\}.
\]

Note that throughout the proofs equalities and operations are intended in \( \mathbb{F}_q \).

**Lemma 2.2.** Let \( H_q \) be the graph from Definition 2.1. For any given \( a \in \mathbb{F}_q \), the vertices in the set \( \{(a, b, c)_1 : b, c \in \mathbb{F}_q\} \) are mutually at distance at least four. And, for any given \( i \in \mathbb{F}_q \), the vertices in the set \( \{(i, j, k)_0 : j, k \in \mathbb{F}_q\} \) are mutually at distance at least four.

**Proof.** Suppose that there exists a path of length two between distinct vertices of the form \((a, b, c)_1 (w, j, k)_0 (a, b', c')_1\) in \( H_q \). By Definition 2.1, \( j = aw + b = aw + b' \) and \( k = a^2w + c = a^2w + c' \). Combining the equations we get \( b = b' \) and \( c = c' \) which implies that \((a, b, c)_1 = (a, b', c')_1\) contradicting the assumption that the path has length two. Similarly suppose that there exists a path of length two \((i, j, k)_0 (a, b, c)_1 (i, j', k')_0\). Reasoning as before, we obtain \( j = ai + b = j' \), and \( k = a^2i + c = k' \) yielding \((i, j, k)_0 = (i, j', k')_0\) which is a contradiction.

**Proposition 2.3.** The graph \( H_q \) from Definition 2.1 is \( q \)-regular, bipartite, of girth 8 and order \( 2q^3 \).
Definition 2.4. Let 
\[ N_{H_q}((x, y, z)_0) = \{(a, y - ax, z - a^2x) : a \in \mathbb{F}_q \}. \]  

(2.1)

Hence every vertex of \( U_0 \) has also degree \( q \) and \( H_q \) is \( q \)-regular. Next, let us prove that \( H_q \) has no cycles of length smaller than 8. Otherwise suppose that there exists in \( H_q \) a cycle 
\[ C_{2t+2} = (a_0, b_0, c_0) \cdots (x_t, y_t, z_t) (a_0, b_0, c_0) \]

of length \( 2t + 2 \) with \( t \in \{1, 2\} \). By Lemma 2.2, \( a_k \neq a_{k+1} \) and \( x_k \neq x_{k+1} \) (subscripts being taken modulo \( t + 1 \)). Then 
\[
y_k = a_kx_k + b_k = a_{k+1}x_k + b_{k+1}, \quad k = 0, \ldots, t, \\
z_k = a_k^2x_k + c_k = a_{k+1}^2x_k + c_{k+1}, \quad k = 0, \ldots, t,
\]

subscripts \( k \) being taken modulo \( t + 1 \). Summing all these equalities we get
\[
\sum_{k=0}^{t-1} (a_k - a_{k+1})x_k = (a_0 - a_t)x_t, \quad t = 1, 2; \\
\sum_{k=0}^{t-1} (a_k^2 - a_{k+1}^2)x_k = (a_0^2 - a_t^2)x_t, \quad t = 1, 2.
\]

(2.2)

If \( t = 1 \), then (2.2) leads to \((a_0 - a_1)(x_1 - x_0) = 0\). System (2.2) gives \( x_0 = x_1 = x_2 \) which is a contradiction to Lemma 2.2. This means that \( H_q \) has no squares so that we may assume that \( t = 2 \). The coefficient matrix of (2.2) has a Vandermonde determinant, i.e.
\[
\left| \begin{array}{cc}
a_1 - a_0 & a_0 - a_2 \\
a_1^2 - a_0^2 & a_0^2 - a_2^2 \\
\end{array} \right| = \left| \begin{array}{ccc} 1 & 1 & 1 \\
a_1 & a_0 & a_2 \\
a_1^2 & a_0^2 & a_2^2 \\
\end{array} \right| = \prod_{0 \leq k < j \leq 2} (a_j - a_k).
\]

This determinant is different from zero because by Lemma 2.2, \( a_{k+1} \neq a_k \) (the subscripts being taken modulo 3). Using Cramer’s rule to solve it we obtain \( x_1 = x_0 = x_2 \) which is a contradiction to Lemma 2.2.

Hence, \( H_q \) has girth at least 8. Furthermore, when \( q \geq 3 \) the minimum number of vertices of a \( q \)-regular bipartite graph of girth greater than 8 must be greater than \( 2q^3 \). Thus we conclude that the girth of \( H_q \) is exactly 8.

Next, we will make use of the following induced subgraph \( B_q \) of \( \Gamma_q \).

Definition 2.4. Let \( B_q = B_q[V_0, V_1] \) be a bipartite graph with vertex set \( V_i = \mathbb{F}_q^3, i = 0, 1 \), and edge set \( E(B_q) \) defined as follows:

For all \( a, b, c \in \mathbb{F}_q \)
\[
N_{B_q}(a, b, c) = \{(j, aj + b, a^2j + 2ab + c) : j \in \mathbb{F}_q \}.
\]

Lemma 2.5. The graph \( B_q \) is isomorphic to the graph \( H_q \).
Proof. Let $H_q$ be the bipartite graph from Definition 2.1. Since the map $\sigma : B_q \to H_q$ defined by $\sigma((a,b,c)_1) = (a,b,2ab+c)_1$ and $\sigma((x,y,z)_0) = (x,y,z)_0$ is an isomorphism, the result holds.

Hence, the graph $B_q$ is also $q$-regular, bipartite, of girth 8 and order $2q^3$.

In what follows, we will obtain the graph $\Gamma_q$ from the graph $B_q$ by adding some new vertices and edges. We need a preliminary lemma.

Lemma 2.6. Let $B_q$ be the graph from Definition 2.4. Then the following hold:

(i) The vertices in the set $\{(a,b,c)_1 : b,c \in \mathbb{F}_q\}$ are mutually at distance at least four for all $a \in \mathbb{F}_q$.

(ii) The vertices in the set $\{(i,j,k)_0 : j,k \in \mathbb{F}_q\}$ are mutually at distance at least four for all $i \in \mathbb{F}_q$.

(iii) The $q$ vertices of the set $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least six for all $x,y \in \mathbb{F}_q$.

Proof. The proof of items (i) and (ii) is almost identical to that of Lemma 2.2.

(iii): By (ii), the vertices in $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least four. Suppose by contradiction that $B_q$ contains the following path of length four:

$$(x,y,j)_0 (a,b,c)_1 (x',y',j')_0 (a',b',c')_1 (x,y,j'')_0, \text{ for some } j'' \neq j.$$ 

Then $y = ax+b = a'x+b'$ and $y' = ax'+b = a'x'+b'$. It follows that $(a-a')(x-x') = 0$, which is a contradiction since, by the previous statements, $a \neq a'$ and $x \neq x'$.

2.2 The conclusion

Figure 1 shows a spanning tree of $\Gamma_q$ with the vertices labelled according to Definition 1.1. Note that the lower level of such a tree corresponds to the set of vertices of $B_q$.

![Figure 1: Spanning tree of $\Gamma_q$.](image)
Proof of Theorem 1.2. Let $B'_q = B'_q[V_0, V'_1]$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding $q^2$ new vertices to $V_1$ labeled $(q, b, c)_1, b, c \in \mathbb{F}_q$ (i.e., $V'_1 = V_1 \cup \{(q, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B'_q}((q, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 1). Then $B'_q$ has $|V'_1| + |V_0| = 2q^3 + q^2$ vertices, every vertex of $V_0$ has degree $q + 1$, and every vertex of $V'_1$ has still degree $q$. Note that the girth of $B'_q$ is 8 by Lemma 2.6(iii). The statements from Lemma 2.6 still partially hold in $B'_q$, as stated in the following claim.

Claim 1. For any given $a \in \mathbb{F}_q \cup \{q\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in $B'_q$.

Proof. For $a = q$, it is clear from Lemma 2.6(i), since the new vertices do not change the distance among the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$. For $a \neq q$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four since each vertex of the form $(i, j, k)_0$ has exactly one neighbour in this set, so the result follows from the bipartition of $B'_q$. 

Claim 2. For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $c \in \mathbb{F}_q$, the $q$ vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in $B'_q$.

Proof. By Claim 1, for all $a \in \mathbb{F}_q \cup \{q\}$ the $q$ vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in $B'_q$. Suppose that there exists in $B'_q$ the following path of length four:

$$ (a, t, c)_1 (x, y, z)_0 (a', t', c')_1 (x', y', z')_0 (a, t'', c)_1, \text{ for some } t'' \neq t. $$

If $a = q$, then $x = x' = c, y = t, y' = t''$ and $a' \neq q$ by Claim 1. Then $y = a'x + t' = a'x' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq q$. If $a' = q$, then $x = x' = c'$ and $y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume $a' \neq q$ and $a \neq a'$ by Claim 1. In this case we have:

$$ y = ax + t = a'x + t'; 
\quad y' = ax' + t'' = a'x' + t'; 
\quad z = a^2x + 2at + c = a'^2x + 2a't' + c'; 
\quad z = a'^2x' + 2a't'' + c = a'^2x' + 2a't' + c'. $$

Thus,

$$ (a - a')(x - x') = t'' - t; \quad (2.3) $$
$$ (a^2 - a'^2)(x - x') = 2a(t'' - t). \quad (2.4) $$

If $q$ is even, (2.4) leads to $x = x'$ and (2.3) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume $q$ is odd. If $a + a' = 0$, then (2.4) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (2.3) by $a + a'$ and subtracting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 1. Therefore, Claim 2 holds. \qed
Let $B''_q = B''_q[V'_0, V'_1]$ be the graph obtained from $B'_q = B'_q[V_0, V'_1]$ by adding $q^2 + q$ new vertices to $V_0$ labeled $(\varrho, a, c)_0, a \in \mathbb{F}_q \cup \{\varrho\}, c \in \mathbb{F}_q$, and new edges $N_{B''_q}((\varrho, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 1). Then $B''_q$ has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree $q$. Moreover the girth of $B''_q$ is 8 by Claim 2.

**Claim 3.** For all $a \in \mathbb{F}_q \cup \{\varrho\}$, the $q$ vertices of the set $\{(\varrho, a, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in $B''_q$.

**Proof.** Clearly these $q$ vertices are mutually at distance at least 4 in $B''_q$. Suppose that there exists in $B''_q$ the following path of length four:

$$(\varrho, a, j_0) (a, b, j)_1 (x, y, z)_0 (a, b', j')_1 (\varrho, a, j')_0, \text{ for some } j' \neq j.$$ 

If $a = \varrho$ then $x = j = j'$ which is a contradiction. Therefore $a \neq \varrho$. In this case $y = ax + b = ax + b'$ which implies that $b = b'$. Hence $z = a^2 x + 2ab + j = a^2 x + 2ab + j'$ yielding that $j = j'$ which is again a contradiction. \hfill \Box

Let $B'''_q = B'''_q[V'_0, V'_1]$ be the graph obtained from $B''_q$ by adding $q + 1$ new vertices to $V'_1$ labeled $(\varrho, \varrho, a)_1, a \in \mathbb{F}_q \cup \{\varrho\}$, and new edges $N_{B'''_q}(\varrho, \varrho, a)_1 = \{(\varrho, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 1. Then $B'''_q$ has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree $q$. Moreover the girth of $B'''_q$ is 8 by Claim 3 and clearly these $q + 1$ new vertices are mutually at distance 6. Finally, the graph $\Gamma_q$ is obtained by adding to $B'''_q$ another new vertex labeled $(\varrho, \varrho, \varrho)_0$ and edges $N_{C'_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, i)_1 : i \in \mathbb{F}_q \cup \{\varrho\}\}$. The graph $\Gamma_q$ has $2(q^3 + q^2 + q + 1)$ vertices, it is $(q + 1)$-regular and has girth 8, so by the uniqueness of a $(q + 1, 8)$-cage (see e.g. [29]), $\Gamma_q$ is indeed a $(q + 1, 8)$ Moore graph. \hfill \Box

**Remark 2.7.** Coordinatizations of classical generalized quadrangles $Q(4, q)$ and $W(q)$ in four dimensions are discussed in [23, 26, 28]. The alternate description of a Moore $(q + 1, 8)$-graph given in Theorem 1.2 in three dimensions is equivalent to this coordinatization.

**References**


