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Q1 # New small regular graphs of girth 5

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ABSTRACT

A (k, g)-graph is a k-regular graph with girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices. The *cage problem* consists of constructing (k, g)-graphs of minimum order $n(k, g)$. We focus on girth $g = 5$, where cages are known only for degrees $k \leq 7$. We construct $(k, 5)$-graphs using techniques exposed by Funk (2009) and Abreu et al. (2012) to obtain the best upper bounds on $n(k, 5)$ known hitherto. The tables given in the introduction show the improvements obtained with our results.

1. Introduction

All the graphs considered are finite and simple. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the size $g = g(G)$ of a shortest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called *k-regular* if all its vertices have the same degree k, and *bi-regular or* (k_1, k_2)*-regular* if all its vertices have either degree k_1 or k_2. A (k, g)-graph is a k-regular graph of girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices; the order of a (k, g)-cage is denoted by $n(k, g)$. Cages were introduced by Tutte [29] in 1947 and their existence was proved by Erdős and Sachs [14] in 1963 for any values of regularity and girth. The lower bound on the number of vertices of a (k, g)-graph is denoted by $n_0(k, g)$, and it is calculated using the distance partition either to a vertex (for odd g), or to an edge (for even g):

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

A graph that attains this lower bound is called a Moore (k, g)-cage. Biggs [11] calls *excess* of a (k, g)–graph G the difference $|V(G)| - n_0(k, g)$. There has been intense work related with *the cage problem*, focused on constructing the smallest (k, g)-graphs (for a complete survey of this topic see [16]).

In this paper we are interested in *the cage problem* for $g = 5$, in this case $n_0(k, 5) = 1 + k^2$. It is well known that this bound is attained for $k = 2, 3, 7$ and perhaps for $k = 57$ (see [11]) and that for $k = 4, 5, 6$, the known graphs of minimum order are cages (see [22–25,30–33]).

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Table 1
Current and our new values of $rec(k, 5)$ for $8 \leq k \leq 22$.

k	$rec(k, 5)$	Due to	Reference	New $rec(k, 5)$
8	80	Royle, Jørgensen	[20,26]	
9	96	Jørgensen	[20]	
10	124	Exoo	[15]	
11	154	Exoo	[15]	
12	203	Exoo	[15]	
13	230	Exoo	[15]	
14	284	Abreu et al.	[1]	
15	310	Abreu et al.	[1]	
16	336	Jørgensen	[20]	
17	448	Schwenk	[27]	*436*
18	480	Schwenk	[27]	*468*
19	512	Schwenk	[27]	*500*
20	572	Abreu et al.	[1]	*564*
21	682	Abreu et al.	[1]	*666*
22	720	Jørgensen	[20]	*704*

Table 2
Current and our new values of $rec(k, 5)$ for $32 \leq k \leq 52$.

k	$rec(k, 5)$	Due to	Reference	New $rec(k, 5)$
32	1680	Jørgensen	[20]	*1624*
33	1856	Funk	[17]	*1680*
34	1920	Jørgensen	[20]	*1800*
35	1984	Funk	[17]	*1860*
36	2048	Funk	[17]	*1920*
37	2514	Abreu et al.	[1]	*2048*
38	2588	Abreu et al.	[1]	*2448*
39	2662	Abreu et al.	[1]	*2520*
40	2736	Jørgensen	[20]	*2592*
41	3114	Abreu et al.	[1]	*2664*
42	3196	Abreu et al.	[1]	*2736*
43	3278	Abreu et al.	[1]	*3040*
44	3360	Jørgensen	[20]	*3120*
45	3610	Abreu et al.	[1]	*3200*
46	3696	Jørgensen	[20]	*3280*
47	4134	Abreu et al.	[1]	*3360*
48	4228	Abreu et al.	[1]	*3696*
49	4322	Abreu et al.	[1]	*4140*
50	4416	Jørgensen	[20]	*4232*
51	4704	Jørgensen	[20]	*4324*
52	4800	Jørgensen	[20]	*4416*

Jørgensen [20] established that $n(k, 5) \leq 2(q-1)(k-2)$ for every odd prime power $q \geq 13$ and $k \leq q+3$. Abreu et al. [1] proved that $n(k, 5) \leq 2(qk - 3q - 1)$ for any prime $q \geq 13$ and $k \leq q + 3$, which improved Jørgensen's bound except for $k = q + 3$ where both coincide.

In [17] Funk uses a technique that consists in finding regular graphs of girth greater or equal than five and performing some operations of amalgams and reductions of the (bipartite) incidence graph, also called Levi Graph of elliptic semiplanes of type C and L (see [5,13,17]). In this paper we improve some results of Funk finding the best possible regular graphs to amalgamate which allow us to obtain new better upper bounds. To do that, we also use the techniques given in [1,2] where the authors amalgamate not only regular graphs, but also bi-regular graphs. In this paper new small $(k, 5)$-graphs are constructed for $17 \leq k \leq 22$ using the incidence graphs of elliptic semiplanes of type C. The new upper bounds appear in the last column of Table 1, which also shows the current values for $8 \leq k \leq 22$. To evaluate our achievements, we follow the notation in [16,17], and let $rec(k, 5)$ denote the smallest currently known order of a k-regular graph of girth 5. Hence $n(k, 5) \leq rec(k, 5)$.

For $23 \leq k \leq 31$, the value $rec(k, 5)$ obtained by Funk in [17] remain untouched. However, for $32 \leq k \leq 52$, we construct a k-regular graph of girth five and provide in Table 2 a new value of $rec(k, 5)$.

Finally, when $q \geq 49$ is a prime power, the search for 6-regular suitable pairs of graphs allows us to establish the two following general results. Note that the bounds are different depending on the parity of q.

Theorem 1.1. *Given an integer $k \geq 53$, let q be the lowest odd prime power such that $k \leq q+6$. Then $n(k, 5) \leq 2(q-1)(k-5)$.*

Theorem 1.2. *Given an integer $k \geq 68$, let $q = 2^m$ be the lowest even prime power such that $k \leq q+6$. Then $n(k, 5) \leq 2q(k-6)$.*

Since the bounds of Theorems 1.1 and 1.2, associated to primes $q = 49$ and $q = 64$, represent a considerable improvement to the current known ones, we give them explicitly in Table 3.

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Table 3
Current and our new values of $rec(k, 5)$ for $k = 55, 70$.

k	$rec(k, 5)$	Due to	Reference	New $rec(k, 5)$
55	5510	Abreu et al.	[1]	*4800*
70	8976	Jørgensen	[20]	*8192*

To finalize the introduction we want to emphasize that Funk, in [17], gives a pair of 4-regular graphs of girth 5 suitable for amalgamation into some specific incidence graphs of elliptic semiplanes and he posed the question about the existence of a suitable pair of 5-regular graphs for the same objective. In this paper we exhibit these graphs which solve this open question. Furthermore, let us notice that all the bounds on $n(k, 5)$ contained in this paper are obtained constructively, that is, for each k, we construct a $(k, 5)$-graph with improved order $rec(k, 5)$.

2. Preliminaries

A useful tool to construct k-regular graphs of girth 5 is the operation of *amalgamation* on the incidence graph of an elliptic semiplane (Jørgensen [20] and Funk [17]).

Let q be a prime power and consider the Levi graphs or incidence graphs C_q and $\mathcal{L}_q$ of elliptic semiplanes of type C and L, respectively. Recall that a semiplane of type C is obtained from the projective plane of order q by choosing an incident point line pair (p, ℓ) and deleting all the lines incident with p and all the points belonging to ℓ. Thus, the Levi (or incidence) graph C_q is bipartite, q-regular and has $2q^2$ vertices, which correspond in the elliptic semiplane of type C to q^2 points and q^2 lines both partitioned into q parallel classes or blocks of q elements each. A semiplane of type L is obtained from the projective plane of order q by choosing a non-incident point line pair (p, ℓ) and deleting all the lines incident with p and all the points belonging to ℓ. Hence, the Levi graph $\mathcal{L}_q$ is also bipartite, q-regular and has $2(q^2 - 1)$ vertices, which correspond in the elliptic semiplane of type L to $q^2 - 1$ points and $q^2 - 1$ lines both partitioned into $q + 1$ parallel classes of $q - 1$ elements each.

To construct our new graphs we find regular and bi-regular graphs of girth greater or equal than five and we use them to perform some operations of amalgams and reductions in C_q or $\mathcal{L}_q$. In [20], Jørgensen exploits these ideas and proves that two graphs are suitable for amalgamation (one of them in each block of points and the other in each block of lines) if they have disjoint sets of Cayley colors. In [1] these ideas are also used to construct graphs using the elliptic semiplane of type C, and the main theorem of [1] was refined in [2] to construct bi-regular cages of girth 5. In fact, the suitable graphs to be amalgamated can have some specific edges with a common Cayley color.

The paper is organized as follows. In the next section we work with elliptic semiplanes of type C and with techniques used in [1,2]. In Section 4 we work with elliptic semiplanes of type L and with techniques given by Funk in [17]. Finally, in Section 5 we return to the elliptic semiplanes of type C for even prime powers because new descriptions are required.

3. Amalgamating into elliptic semiplanes of type C

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q; we recall the definition and properties of the incidence bipartite graph C_q of an elliptic semiplane of type C exactly as they appear in [1,2]. Notice that in these papers the authors call this graph B_q and here, as it is related to the elliptic semiplane of type C, we prefer to call it C_q.

Definition 3.1. Let C_q be a bipartite graph with vertex set (V_0, V_1) where $V_r = \mathbb{F}_q \times \mathbb{F}_q$, $r = 0, 1$; and edge set defined as follows:

$$(x, y)_0 \in V_0 \text{ adjacent to } (m, b)_1 \in V_1 \text{ if and only if } y = mx + b. \tag{1}$$

The graph C_q is also known as the incidence graph of the biaffine plane [18] and it has been used to find extremal graphs without short cycles (see [1–10,21]). The graph C_q is q-regular of order $2q^2$, has girth $g = 6$ for $q \geq 3$ and it is vertex transitive. The set of vertices can be described as the disjoint union of the sets $P_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $L_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$ for all $x, m \in \mathbb{F}_q$. Other well known properties of the graph C_q can be seen in [1,2,18,21].

Let Γ be a subgraph of G, and Γ' a graph of the same order of Γ and with the same labels on their vertices; an *amalgam of Γ' into Γ* is a graph obtained from G by adding all the edges of Γ into Γ'. In [1] the authors described a technique of amalgamation of two r-regular graphs H_0, H_1 and two $(r, r+1)$-regular graphs G_0, G_1 (all of them of girth at least 5 and with some specific properties) into a subgraph of C_q such that the resulting amalgam graph, denoted by $\mathcal{C}_q(H_0, H_1, G_0, G_1)$, is $(q + r)$-regular and has girth at least five.

Recall that if G is a graph with $V(G)$ labeled with the elements of $\mathbb{F}_q$ and $\alpha\beta$ is an edge of G, then the *Cayley Color* or *weight* of the edge $\alpha\beta$ is $\pm(\alpha - \beta) \in \mathbb{F}_q - \{0\}$. Theorem 3.1 is a reformulation of Theorem 5 in [1] (with a new strong hypothesis that also appears in Theorem 4.9 in [2]).

Theorem 3.1. *Let $q \geq 3$ be a prime power and $r \geq 2$ an integer. Consider graphs H_0, H_1, G_0 and G_1 with the following properties:*

(i) *$V(G_i) = \mathbb{F}_q$ and G_i is an $(r, r+1)$-regular graph of girth $g(G_i) \geq 5$ for $i = 0, 1$;*

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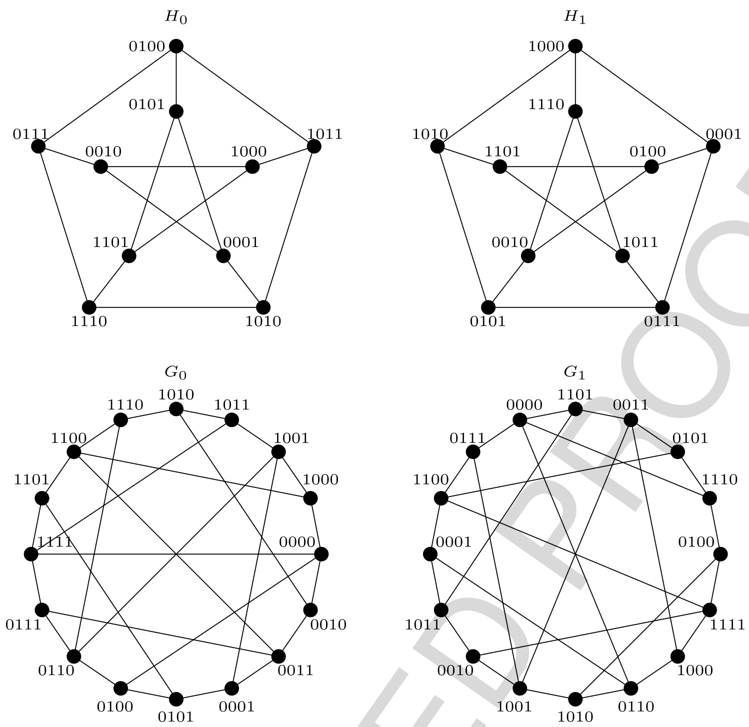

Fig. 1. The graphs H_i and G_i for $i = 0, 1$ where $q = 16$.

(ii) *H_i is an r-regular graph of girth $g(H_i) \geq 5$ and $V(H_i) = \{v \in \mathbb{F}_q : d_{G_j}(v) = r \text{ with } i \neq j\}$, for $i, j \in \{0, 1\}$.*

(iii) *$E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$, $E(H_1) \cap E(G_0) = \emptyset$ and G_0 and G_1 have disjoint Cayley colors.*

Let C_q be the graph given in Definition 3.1, and $A = V(C_q) \setminus (\{(0, y)_0 : y \notin V(H_0)\} \cup \{(0, b)_1 : b \notin V(H_1)\})$. Let $C_q[A]$ be the induced subgraph of C_q by A.

Let the sets of edges $E_0(0) = \{(0, y)_0(0, y')_0 : yy' \in E(H_0)\}$, $E_1(0) = \{(0, b)_1(0, b')_1 : bb' \in E(H_1)\}$, $E_0(x) = \{(x, y)_0(x, y')_0 : yy' \in E(G_0)\}$, $E_1(m) = \{(m, b)_1(m, b')_1 : bb' \in E(G_1)\}$ for all $m, x \in \mathbb{F}_q - \{0\}$.

The graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ with vertex set A and edge set $E(C_q[A]) \cup (\bigcup_{x \in \mathbb{F}_q} E_0(x)) \cup (\bigcup_{m \in \mathbb{F}_q} E_1(m))$ is $(q + r)$-regular and has girth at least five.

The proof is the same as the one of Theorem 4.9 in [2]. Notice that Theorem 3.1 can also be applied when G_0 and G_1 are regular graphs (then $H_0 = G_0$ and $H_1 = G_1$). In this case we denote the obtained graph by $\mathcal{C}_q(G_0, G_1)$.

Next, for $q \in \{16, 17, 19\}$, we construct graphs H_0, H_1, G_0, G_1, satisfying the conditions of Theorem 3.1.

Construction 1:

• For $q = 16$:

Let $(\mathbb{F}_{16}, +) \cong ((\mathbb{Z}_2)^4, +)$ be a finite field of order 16 with set of elements $\{(d, e, f, g) : d, e, f, g \in \mathbb{Z}_2\}$, we write $defg$ instead of (d, e, f, g). Consider the graphs H_0, H_1, G_0 and G_1 displayed in Fig. 1. The graphs G_0 and G_1 are not isomorphic, although both have girth 5 and order 16, with 6 vertices of degree 4 and 10 vertices of degree 3. We label the vertices of G_0 and G_1 such that the vertices of the set $S = \{0000, 1100, 0110, 1001, 0011, 1111\}$ have degree four and the other ones have degree three. The weights or Cayley colors of G_0 (and G_1) are $\{0001, 0010, 0100, 1000, 1111\}$ (and $\{0011, 0110, 0111, 1001, 1010, 1011, 1100, 1101, 1110\}$, respectively). Hence, G_0 and G_1 have disjoint sets of Cayley colors. Moreover, the graphs H_0 and H_1 are isomorphic to the Petersen graph and they are labeled with the elements of $(\mathbb{Z}_2)^4 - S$ in such a way that $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

• For $q = 17$:

Let H_0, H_1, G_0 and G_1 be the graphs of girth 5 displayed in Fig. 2. Graphs G_0 and G_1 are isomorphic with $V(G_0) = V(G_1) = \mathbb{Z}_{17}$, and both have the same set of vertices $S = \{0, 2, 5, 8, 10, 13, 15\}$ of degree 4. The Cayley colors of G_0 (and G_1) in $\mathbb{Z}_{17}$ are $\pm\{1, 5, 8\}$ (and $\pm\{2, 3, 4, 6, 7\}$, respectively). Regarding H_0 and H_1, it can be checked that $V(H_0) = V(H_1) = \mathbb{Z}_{17} - S$, $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

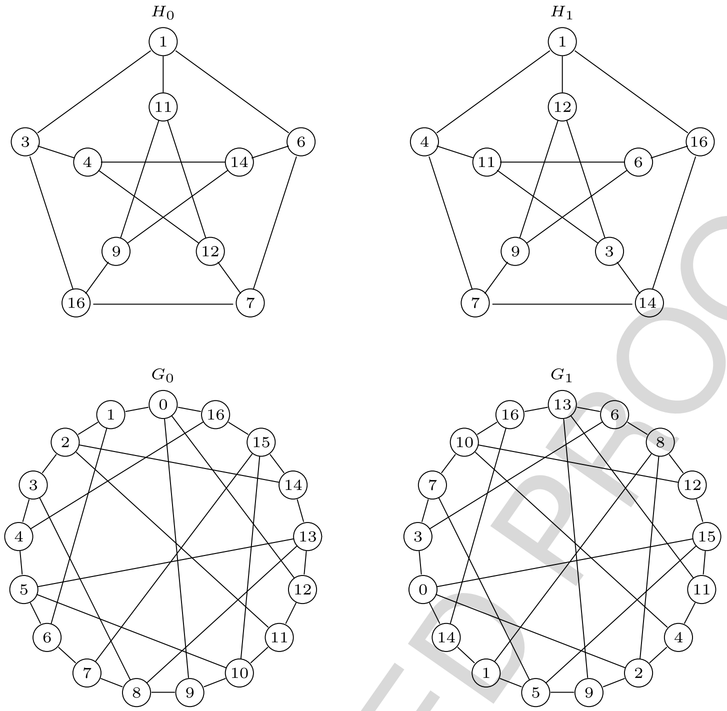

Fig. 2. The graphs H_i and G_i for $i = 0, 1$ where $q = 17$.

• For $q = 19$: Let H_0, H_1, G_0 and G_1 be the graphs of girth 5 shown in Fig. 3. Graphs G_0 and G_1 are isomorphic, $V(G_0) = V(G_1) = \mathbb{Z}_{19}$ and both have the same set $S = \{0, 2, 3, 5, 6, 12, 13, 16, 17\}$ of vertices degree 4. The weights or Cayley colors modulo 19 of G_0 (and G_1) are $\pm\{1, 4, 7, 8\}$ (and $\pm\{2, 3, 5, 6, 9\}$, respectively). The two graphs H_0 and H_1 have vertex set $\mathbb{Z}_{19} - S$ and verify $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

Next, we apply Theorem 3.1 to $q \in \{16, 17, 19\}$. The graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ is a $(q+3, 5)$-regular graph with less vertices than any other $(q + 3)$-regular graph of girth 5 so far known, and therefore the upper bound $rec(k, 5)$ for $k \in \{19, 20, 22\}$ is improved. As it is explained in the Reduction 2 in [1], referred as "Deletion" in [17], by removing pairs of blocks P_x and L_m from $\mathcal{C}_q(H_0, H_1, G_0, G_1)$, we also generate new graphs which improve $rec(k, 5)$ for $k \in \{17, 18, 21\}$.

Theorem 3.2. *The following upper bound* $rec(k, 5)$ *on the order* $n(k, 5)$ *of a k-regular cage of girth* 5 *holds*

k	17	18	19	20	21	22
$rec(k, 5)$	436	468	500	564	666	704

Proof. Using the graphs given in Construction 1, we obtain for $q \in \{16, 17, 19\}$ the graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ as in Theorem 3.1, which has girth 5. Moreover we have the following considerations:

For $q = 16$, $\mathcal{C}_{16}(H_0, H_1, G_0, G_1)$ is a $(19, 5)$-graph of order $2 \cdot 16^2 - 12 = 500$ implying that any $(19, 5)$-cage has at most 500 vertices. Removing of $\mathcal{C}_{16}(H_0, H_1, G_0, G_1)$ (using the operation called "Reduction 2" in [1]) a block of lines L_m and a block of points P_x, for $x, m \in (\mathbb{Z}_2)^4 - \{0000\}$, we construct a 18-regular graph with $500 - 2 \cdot 16 = 468$ vertices. Similarly, deleting from this last graph another pair of blocks we obtain a 17-regular graph of girth 5 with 436 vertices. Each of these k-regular graphs ($k = 17, 18, 19$) has 12 vertices less than the ones constructed by Schwenk in [27].

For $q = 17$, $\mathcal{C}_{17}(H_0, H_1, G_0, G_1)$ is a $(20, 5)$-graph of order $2 \cdot 17^2 - 14 = 564$, which implies that a $(20, 5)$-cage has at most 564 vertices.

For $q = 19$, $\mathcal{C}_{19}(H_0, H_1, G_0, G_1)$ is a $(22, 5)$-graph of order $2 \cdot 19^2 - 18 = 704$, which also implies that any $(22, 5)$-cage has at most 704 vertices. Newly, deleting any block of points and any block of lines (except P_0 and L_0 blocks), it is straightforward to check out that $n(21, 5) \leq 666$. ■

Remark 3.1. Note that the construction of a $(q+3)$-regular graph of girth at least 5 using bi-regular amalgams into a subgraph of C_q involves the existence of two 3-regular graphs H_0 and H_1 and two $(3, 4)$-regular graphs G_0 and G_1 all of them with girth at least 5. The graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ has order $2(q^2 - (q - n(H_0))) \geq 2(q^2 - q + 10)$. It means that our construction is the

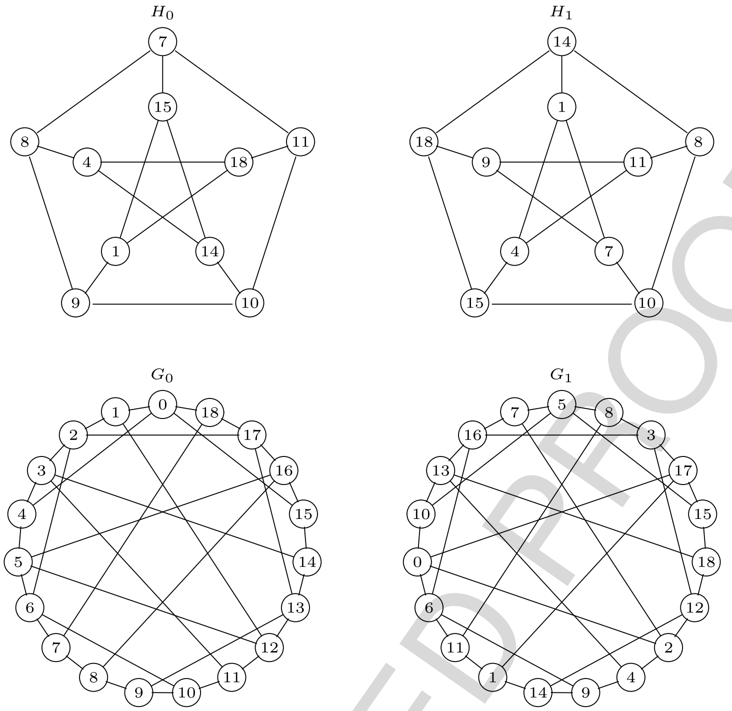

Fig. 3. The graphs H_i and G_i for $i = 0, 1$ where $q = 19$.

best possible one for $q = 16$ and $q = 17$, because a 4-regular amalgam could only be possible for $q \geq n(4, 5) = 19$ (the $(4, 5)$-cage is the Robertson Graph that has order 19).

4. Elliptic semiplanes of type *L*

In this section we use the technique given by Funk in [17] to amalgamate a pair of suitable regular graphs into the Levi graph $\mathcal{L}_q$ of an elliptic semiplane of type L. Recall that the semiplane of type L is obtained by deleting from the projective plane of order q a non-incident pair (p, ℓ), all the lines incident with the point p and all the points incident with the line ℓ. Moreover, the Levi graph $\mathcal{L}_q$ is bipartite, q-regular and has $2(q^2 - 1)$ vertices of which $q^2 - 1$ are points and $q^2 - 1$ are lines in the elliptic semiplane, both partitioned into $q + 1$ parallel classes of $q - 1$ elements each.

We divide the section into two parts. First we construct suitable regular graphs G_0, G_1 and then we describe the graph $\mathcal{L}_q(G_0, G_1)$ obtained after amalgamation.

4.1. Constructions of regular graphs of girth five

To apply Funk's technique we need to construct two regular graphs with the same order, girth at least five and having disjoint Cayley colors, one of them to be amalgamated into the point blocks and the other into the line blocks of $\mathcal{L}_q$.

Let $\mathbb{Z}_n$ be the set of integers modulo n, and $J = \{k_1, \ldots, k_w\} \subset \mathbb{Z}_n - \{0\}$. Recall that a *circulant graph* $Z_n(J)$ is a graph with vertex set $\mathbb{Z}_n$ and edges $\alpha\beta$ where $\beta - \alpha \in J$. Let $n = 2t$ and suppose that every element of J is odd. We denote by $S_{2t}(k_1, \ldots, k_w)$ the subgraph of the circulant graph $\mathbb{Z}_{2t}(k_1, \ldots, k_w)$ with vertex set $\mathbb{Z}_{2t}$ and edge set $\{\{2v, 2v + k_j\} : 0 \leq v \leq t - 1,\ 1 \leq j \leq w\}$ where the sum is taken modulo $2t$. Moreover, we denote by $S_\infty(k_1, \ldots, k_w)$ the (infinite) graph defined in a similar way over $\mathbb{Z}$. Next, we describe some relevant properties of this graph.

Lemma 4.1. *Given an integer $t \geq 5$, and a sequence $k_1, \ldots, k_w$ of different odd elements from $\mathbb{Z}_{2t}$, the graph $S_{2t}(k_1, \ldots, k_w)$ is w-regular, bipartite and has at most w Cayley colors in $\mathbb{Z}_{2t}$. Moreover, the girth of $S_{2t}(k_1, \ldots, k_w)$ is at least 6 iff all the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq w$. These properties hold even for $2t = \infty$.*

Proof. Given an odd element $k_j \in \mathbb{Z}_{2t}$, the set of edges $\{\{2v, 2v + k_j\} : 0 \leq v \leq t - 1\}$ defines a matching between the vertices with even label and the ones with odd label in $\mathbb{Z}_{2t}$. Therefore, for a given sequence $k_1, \ldots, k_w$ of different odd elements of $\mathbb{Z}_{2t}$, the graph $G = S_{2t}(k_1, \ldots, k_w)$ is w-regular, bipartite and has even girth $g \geq 4$.

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By hypothesis, the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq w$. We prove that the girth of G is greater or equal than 6. Suppose that there exists a 4-cycle $v_0v_1v_2v_3v_0$. By reordering, we may take v_0, v_2 even and v_1, v_3 odd. So, $v_1 = v_0 + k_i$, $v_2 = v_1 - k_j$, $v_3 = v_2 + k_p$, $v_0 = v_3 - k_q$ with $i \neq j$, $p \neq q$, $p \neq j$, $q \neq i$. Then, $k_i - k_j + k_p - k_q = 0$ and $k_i - k_j = k_q - k_p$ in $\mathbb{Z}_{2t}$ which is a contradiction, since by hypothesis all these numbers are different. Hence the girth of G must be at least 6 because it is bipartite. The proof is the same when $\mathbb{Z}_{2t} = \mathbb{Z}$, taking into account that in this case the equalities are considered in $\mathbb{Z}$. ■

The $(q + 1, 6)$-cages with q a prime power, are known examples of this type of graphs. For instance, the Heawood graph or the Moore $(3, 6)$-cage can be constructed as $S_{14}(1, -1, 5)$. It is also very known that Moore $(q + 1, 6)$-cages can also be represented by using *perfect difference sets* (see [20,28]) and as the Levi graphs of the projective plane over the field $\mathbb{F}_q$.

Definition 4.1. Given an integer $t \geq 5$, a sequence of different odd elements $k_1, \ldots, k_w$ and two different even elements $0 < P, Q < t$ from $\mathbb{Z}_{2t}$, we denote by $S_{2t}(P, Q; k_1, \ldots, k_w)$ the graph obtained by adding to $S_{2t}(k_1, \ldots, k_w)$ the new edges $\{2v, 2v + P\}$ and $\{2v + 1, 2v + 1 + Q\}$, where the sum is taken modulo $2t$. The graph $S_\infty(P, Q; k_1, \ldots, k_w)$ is defined in a similar way over $\mathbb{Z}$.

Notice that if P divides $2t$, the subgraph of $S_{2t}(P, Q; k_1, \ldots, k_w)$ induced by the even numbers is formed by $P/2$ cycles, each of them with size $2t/P$. Similar result holds when Q divides $2t$ and the subgraph of $S_{2t}(P, Q; k_1, \ldots, k_w)$ induced by the odd numbers. The standard Generalized Petersen Graphs with $2t$ vertices introduced by Coxeter in [12] are obtained as $S_{2t}(2, Q; 1)$ and the I-graph $I(t, j, k)$ in [34] as $S_{2t}(2j, 2k; 1)$. Funk uses in [17] a 4-regular generalization $P(k, \eta, \nu)$ of the Petersen graph which corresponds to $S_{2k}(2, 2\eta; 1, 2\nu + 1)$. Another example is one of the four $(5, 5)$-cages on 30 vertices, which can be described as $S_{30}(6, 12; 1, -1, 9)$. It can be checked that this graph is isomorphic to the $(5, 5)$-cage obtained from Hoffman–Singleton graph by deleting two Petersen graphs.

Next, we prove some useful properties of these graphs.

Lemma 4.2. *The graph $S_{2t}(P, Q; k_1, \ldots, k_w)$ defined over $\mathbb{Z}_{2t}$ is $(w + 2)$-regular and has at most $w + 2$ Cayley colors. Moreover, the girth of $S_{2t}(P, Q; k_1, \ldots, k_w)$ is at least 5 if and only if the following conditions hold:*

(i) *The numbers $3P$, $4P$, $3Q$, $4Q$ are different from 0 in $\mathbb{Z}_{2t}$.*
(ii) *All the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq w$.*
(iii) *No relation $k_i - k_j = a - a'$ holds for any pair $a, a' \in \Omega = \{0, \pm P, \pm Q\}$.*

The result also holds when $Z_{2t} = Z$.

Proof. According to Lemma 4.1, the graph $B = S_{2t}(k_1, \ldots, k_w)$ is an w-regular bipartite graph with girth at least 6 iff item (ii) is satisfied. The partite sets of B are the set of even vertices, denoted by Ev, and the set of odd vertices, denoted by Od, of $\mathbb{Z}_{2t}$. Consider T_0 and T_1 the circulant graphs whose vertices are Ev and Od respectively, and whose edges are $\{2v, 2v + P\}$ and $\{2v + 1, 2v + 1 + Q\}$, respectively. Clearly, T_0 and T_1 are 2-regular and condition (i) that $3P$, $4P$, $3Q$, $4Q \neq 0$ means that T_0 and T_1 have girth at least five. Denote $G = S_{2t}(P, Q; k_1, \ldots, k_w)$ and observe that the graph G is an amalgamation $B(T_0, T_1)$ obtained by adding to Ev all the edges of T_0 and by adding to Od all the edges of T_1. Hence G is $(w + 2)$-regular. Let us see that G has girth five.

Suppose that C is a cycle in G of size 3 or 4 which must contain even and odd vertices. If C has only one even vertex, then $k_i - k_j + a = 0$ for $a \in \{\pm Q, \pm 2Q\}$, depending on the size of C, contradicting (iii). If C contains two even and two odd vertices, we have $k_i + a - k_j - a' = 0$, for $a, a' \in \{\pm P, \pm Q\}$, again contradicting (iii). Therefore G has girth at least 5 iff conditions (i), (ii), (iii) are satisfied. ■

Notice that it is useful to take $Q = 2P$ because in this case there are only four differences $\pm\{P, 2P, 3P, 4P\}$ to be avoided. Furthermore, if $S_{2t}(P, Q; k_1, \ldots, k_w)$ has girth $g \geq 5$, the (infinite) graph $S_\infty(P, Q; k_1, \ldots, k_w)$ also satisfies $g \geq 5$. We are interested in the converse result.

Definition 4.2. We call span D of a graph $S_\infty(P, Q; k_1, \ldots, k_w)$ the maximum element of the set $\{|k_i|, k_i - k_j, a - a'\}$, with $a, a' \in \{0, \pm P, \pm Q\}$.

Lemma 4.3. *Let $P \neq Q$ be two positive even integers and $k_1, \ldots, k_w$ different odd integers. Consider a graph $S_\infty(P, Q; k_1, \ldots, k_w)$ with girth $g \geq 5$ and span D. If $t \geq D + 1$, then $S_{2t}(P, Q; k_1, \ldots, k_w)$ is $(w + 2)$-regular and has girth at least 5.*

Proof. By definition, $0 < P, Q \leq D$ and $-D \leq k_i \leq D$. As $t \geq D + 1$, then $0 < P, Q, |k_1|, \ldots, |k_w| < t$ and clearly $S_{2t}(P, Q; k_1, \ldots, k_w)$ is $(w + 2)$-regular. Let us see that $S_{2t}(P, Q; k_1, \ldots, k_w)$ has girth $g \geq 5$. Since $S_\infty(P, Q; k_1, \ldots, k_w)$ has girth $g \geq 5$, it follows that $k_i - k_j \neq k_p - k_q$ in $\mathbb{Z}$. Also, from the definition of D, we have $-D \leq k_i - k_j, k_p - k_q \leq D$, yielding that $-2t < (k_i - k_j) - (k_p - k_q) < 2t$. Hence, $k_i - k_j \neq k_p - k_q$ in $\mathbb{Z}_{2t}$. The same argument shows $k_i - k_j \neq a - a'$ in $\mathbb{Z}_{2t}$ for $a, a' \in \{\pm P, \pm Q\}$. These are the requirements (ii), (iii) of Lemma 4.2. Notice that (i) of Lemma 4.2 has been explicitly stated, since $2t \geq 2(D + 1) \geq \max\{2(2P + 1), 2(2Q + 1)\}$, which implies that $3P$, $4P$, $3Q$, $4Q$ are different from 0 in $\mathbb{Z}_{2t}$. ■

As an example, let us mention that the graph $S_\infty(2, 4; 3, -7)$ has girth 5 and span $D = 10$. Therefore, the graph $S_{2t}(2, 4; 3, -7)$ is a 4-regular with girth 5 for orders $2t \geq 22$.

In the next subsection we construct a pair of regular graphs of girth 5 suitable for amalgamation into $\mathcal{L}_q$ for some prime powers q.

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4.2. Amalgamating into elliptic semiplanes of type L

Following the terminology of Funk in [17] we say that two r-regular graphs G_0 and G_1 with girth at least five are *suitable for amalgamation into the elliptic semiplane* $\mathcal{L}_q$ if they are labeled with the elements of the cyclic group $(\mathbb{Z}_{q-1}, +)$ with disjoint sets of Cayley colors in this group. When q is odd, the fact that $\mathbb{Z}_{q-1}$ has $q-1$ elements suggests the use of this semiplane, because r-regular graphs with odd degree have even order.

As in Section 4, the amalgamation of a pair of r-regular suitable graphs into the elliptic semiplane $\mathcal{L}_q$ gives a $(q+r, 5)$-graph $\mathcal{L}_q(G_0, G_1)$. It has $2(q^2-1)$ vertices and deleting pairs of blocks of vertices from $\mathcal{L}_q(G_0, G_1)$, for regularities $k \le q+r$, we have

$$n(k, 5) \le 2(q-1)(k-r+1). \qquad (2)$$

For $q = 19$ there is a unique 4-regular graph with girth 5, the $(4, 5)$-cage due to Robertson in [24]. The use of the highest value of $r \ge 4$ for a given $q > 19$ increases the accuracy of the inequality (2). Funk in [17] constructs the best possible regular amalgams for $q \in \{23, 25, 27\}$. Next, we give a construction of graphs which provide accurate amalgams for $q \in \{29, 31, 37, 41, 43, 47\}$.

Construction 2:

- For $q = 29$:

Consider the graphs $G_0 = S_{28}(4, 8; 1, -1)$ and $G_1 = S_{28}(2, 6; 3, -7)$. It is a suitable pair, that is, both graphs are 4-regular, have girth five and have disjoint sets of Cayley colors, concretely $\pm\{1, 4, 8\}$ and $\pm\{2, 3, 6, 7\}$, respectively. Hence, the 33-regular graph $\mathcal{L}_{29}(G_0, G_1)$ has girth 5, order 1680 and diameter 4. Deletion provides entries $k = 32, 33$ of Table 2.

- For $q = 31$:

There exist four $(5, 5)$-cages (see [22,25,30,32,33]) one of them being the graph $G_0 = S_{30}(6, 12; 1, -1, 9)$. Another graph G_1 has been found with the following relabeling of the vertices.

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 28$	$2 \leftrightarrow 1$	$3 \leftrightarrow 27$	$4 \leftrightarrow 2$	$5 \leftrightarrow 19$
$6 \leftrightarrow 4$	$7 \leftrightarrow 7$	$8 \leftrightarrow 5$	$9 \leftrightarrow 22$	$10 \leftrightarrow 6$	$11 \leftrightarrow 3$
$12 \leftrightarrow 8$	$13 \leftrightarrow 24$	$14 \leftrightarrow 9$	$15 \leftrightarrow 20$	$16 \leftrightarrow 10$	$17 \leftrightarrow 15$
$18 \leftrightarrow 12$	$19 \leftrightarrow 23$	$20 \leftrightarrow 13$	$21 \leftrightarrow 11$	$22 \leftrightarrow 14$	$23 \leftrightarrow 29$
$24 \leftrightarrow 16$	$25 \leftrightarrow 21$	$26 \leftrightarrow 17$	$27 \leftrightarrow 25$	$28 \leftrightarrow 18$	$29 \leftrightarrow 26$

Since the Cayley colors of G_1 are the elements of the set $\mathbb{Z}_{30} - \{0, \pm 1, \pm 6, \pm 9, \pm 12\}$, the graphs G_0 and G_1 have disjoint Cayley colors, and therefore, $\mathcal{L}_{31}(G_0, G_1)$ has girth 5, regularity 36 and order $2(31^2-1) = 1920$. Block deletion provides $n(35, 5) \le 1860$ and $n(34, 5) \le 1800$.

- For $q = 37$:

Consider the graphs $G_0 = S_{36}(8, 14; 1, -1, 11)$ and $G_1 = S_{36}(2, 4; 3, -7, 15)$ defined on the cyclic group $(\mathbb{Z}_{36}, +)$. Both graphs are 5-regular, have girth five and disjoint Cayley colors, concretely $\pm\{1, 8, 11, 14\}$ and $\pm\{2, 3, 4, 7, 15\}$, respectively. Hence, the 42-regular graph $\mathcal{L}_{37}(G_0, G_1)$ has girth 5 and order 2736. Deletion provides $n(41, 5) \le 2664$, $n(40, 5) \le 2592$, $n(39, 5) \le 2520$, $n(38, 5) \le 2448$.

- For $q = 41$:

The $(6, 5)$-cage is unique and it is well known (see [23]) that it can be constructed by removing the vertices of a Petersen graph from the Hoffman–Singleton cage. We present a construction of the $(6, 5)$-cage as the graph $S_{40}(8, 16; 1, -1, 5, -13)$. We denote it by G_0. Due to the uniqueness of this cage, we construct a suitable graph G_1 according to the following relabeling of the vertices.

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 12$	$2 \leftrightarrow 1$	$3 \leftrightarrow 20$	$4 \leftrightarrow 2$	$5 \leftrightarrow 33$	$6 \leftrightarrow 3$	$7 \leftrightarrow 37$
$8 \leftrightarrow 7$	$9 \leftrightarrow 38$	$10 \leftrightarrow 8$	$11 \leftrightarrow 18$	$12 \leftrightarrow 9$	$13 \leftrightarrow 32$	$14 \leftrightarrow 10$	$15 \leftrightarrow 25$
$16 \leftrightarrow 14$	$17 \leftrightarrow 5$	$18 \leftrightarrow 15$	$19 \leftrightarrow 13$	$20 \leftrightarrow 16$	$21 \leftrightarrow 36$	$22 \leftrightarrow 17$	$23 \leftrightarrow 27$
$24 \leftrightarrow 21$	$25 \leftrightarrow 19$	$26 \leftrightarrow 22$	$27 \leftrightarrow 11$	$28 \leftrightarrow 23$	$29 \leftrightarrow 35$	$30 \leftrightarrow 24$	$31 \leftrightarrow 39$
$32 \leftrightarrow 28$	$33 \leftrightarrow 26$	$34 \leftrightarrow 29$	$35 \leftrightarrow 4$	$36 \leftrightarrow 30$	$37 \leftrightarrow 34$	$38 \leftrightarrow 31$	$39 \leftrightarrow 6$

Since G_0 and G_1 have no Cayley color in common, the 47-regular graph $\mathcal{L}_{41}(G_0, G_1)$ has girth 5 and order $2(41^2-1) = 3360$. Deletion and inequality (2) provide entries $k = 43, 44, 45, 46$ of Table 2.

• For $q = 43, 47$:

Since we resort to 5-regularity, there exist several pairs of suitable graphs. For the sake of uniformity, we consider the graphs $S_{q-1}(6, 12; 1, -1, 9)$ and $S_{q-1}(2, 4; 3, -7, 15)$. It proves that $n(48, 5) \leq 2(43^2 - 1) = 3696$ and $n(52, 5) \leq 2(47^2 - 1) = 4416$. As a curiosity, let us mention that the graph $S_{42}(1, -1, -7, 11, 15)$ is the (5, 6)-cage and it forms a suitable pair with $S_{42}(2, 4; 5, -5, 17)$.

Based on the above constructions and recalling that it is possible to delete blocks of points and lines we can write the following theorem.

Theorem 4.1. *The following upper bound on the order of a k-regular graph of girth* 5 *holds*

k	$rec(k, 5)$
32, 33	$56(k-3)$
34, 35, 36	$60(k-4)$
$38, \ldots, 42$	$72(k-4)$
$43, \ldots, 47$	$80(k$-$5)$
48	3696
$49, \ldots, 52$	$92(k-4)$

To finalize this section we prove Theorem 1.1. In this case we generate a pair of 6-regular suitable graphs to be amalgamated into $\mathcal{L}_q$, for an odd prime power $q \geq 49$. We start with $q = 49$; notice that this case is sharp because the Hoffman–Singleton graph is the Moore cage that attains the lower bound $n_0(7, 5) = 50$ (see [19]).

Theorem 1.1. *Given an integer $k \geq 53$, let q be the lowest odd prime power, such that $k \leq q + 6$. Then $n(k, 5) \leq 2(q-1)(k-5)$.*

Proof. First consider $q = 49$. Add to the 4-regular bipartite graph $S_{48}(1, -1, 5, -13)$ the edges $\{2v, 2v + 8\}$ over the even vertices of $\mathbb{Z}_{48}$, and the four cycles $\{1 + i, 17 + i, 41 + i, 25 + i, 9 + i, 33 + i, 1 + i\}$, for $i = 0, 2, 4, 6$, over the odd vertices. We call this (6, 5)-graph G_0. To construct a suitable graph G_1, we resort to the following relabeling of the vertices

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 42$	$2 \leftrightarrow 1$	$3 \leftrightarrow 39$	$4 \leftrightarrow 2$	$5 \leftrightarrow 23$	$6 \leftrightarrow 3$	$7 \leftrightarrow 47$
$8 \leftrightarrow 6$	$9 \leftrightarrow 4$	$10 \leftrightarrow 7$	$11 \leftrightarrow 28$	$12 \leftrightarrow 8$	$13 \leftrightarrow 34$	$14 \leftrightarrow 9$	$15 \leftrightarrow 43$
$16 \leftrightarrow 12$	$17 \leftrightarrow 35$	$18 \leftrightarrow 13$	$19 \leftrightarrow 36$	$20 \leftrightarrow 14$	$21 \leftrightarrow 29$	$22 \leftrightarrow 15$	$23 \leftrightarrow 44$
$24 \leftrightarrow 18$	$25 \leftrightarrow 37$	$26 \leftrightarrow 19$	$27 \leftrightarrow 5$	$28 \leftrightarrow 20$	$29 \leftrightarrow 40$	$30 \leftrightarrow 21$	$31 \leftrightarrow 10$
$32 \leftrightarrow 24$	$33 \leftrightarrow 45$	$34 \leftrightarrow 25$	$35 \leftrightarrow 46$	$36 \leftrightarrow 26$	$37 \leftrightarrow 38$	$38 \leftrightarrow 27$	$39 \leftrightarrow 16$
$40 \leftrightarrow 30$	$41 \leftrightarrow 41$	$42 \leftrightarrow 31$	$43 \leftrightarrow 17$	$44 \leftrightarrow 32$	$45 \leftrightarrow 11$	$46 \leftrightarrow 33$	$47 \leftrightarrow 22$

The graphs G_0 and G_1 have disjoint Cayley colors, namely $w(G_0) = \pm\{1, 5, 8, 13, 16, 24\}$ and $w(G_1) = \mathbb{Z}_{48} - (w(G_0) \cup \{0\})$. Hence, G_0 and G_1 is a suitable pair of graphs to be amalgamated into $\mathcal{L}_{49}$. Using these graphs and also the fact that we can delete blocks of points and lines we prove the theorem for $53 \leq k \leq 55$.

When $q \in \{53, 67, 71, 79, \ldots\}$ is an odd prime power, we consider the 6-regular graphs $G_0 = S_{q-1}(8, 16; 1, -1, 5, -13)$ and $G_1 = S_{q-1}(2, 4; 3, -7, 15, -21)$. Direct checking shows their suitability over $\mathcal{L}_q$ for $q = 53, 67, 71$. When $q \geq 79$, the suitability of G_0 and G_1 is a consequence of Lemma 4.3, because the infinite graphs $S_\infty(8, 16; 1, -1, 5, -13)$ and $S_\infty(2, 4; 3, -7, 15, -21)$ have girth 5 and spans 32 and 37, respectively. When $q \in \{59, 61, 73\}$, the graph $G_0 = S_{q-1}(8, 16; 1, -1, 5, -13)$ and $G_1 = S_{q-1}(2, 4; 3, -7, 15, \alpha)$, where $\alpha = -23$ for $q = 59, 73$ and $\alpha = -25$ for $q = 61$, is a suitable pair of graphs over $\mathcal{L}_q$. Therefore, for $q \geq 49$, the $(q + 6)$-regular graph $\mathcal{L}_q(G_0, G_1)$ has girth at least 5 and order $2(q^2 - 1)$. Also, according to inequality (2), $n(k, 5) \leq 2(q-1)(k-5)$, for regularities $56 \leq k \leq q + 6$. ■

Remark 4.1. Notice that Theorem 1.1 improves Jørgensen's result $n(q + \lfloor \frac{\sqrt{q-1}}{4} \rfloor, 5) \leq 2(q^2 - 1)$ (see [20]) for $q \leq 571$ and ties with it for $578 \leq k \leq 779$.

5. General constructions for $q = 2^m$

In this last section we amalgamate into C_q for $q = 2^m$ when $m \geq 5$ applying Theorem 3.1 on regular graphs. The case $m = 4$ was considered in Section 3, where we amalgamated bi-regular graphs. First, we deal with $m = 5$ or $q = 32$. Since an r-regular graph with 32 vertices and girth 5 can reach at most 5-regularity, we have the following sharp result.

Theorem 5.1. *There exists a* 37*-regular graph with girth* 5 *and order* 2048.

Proof. As in the case $q = 16$, denote the elements of $(\mathbb{F}_{32}, +) \cong ((\mathbb{Z}_2)^5, +)$ by *defgh* instead of $\{d, e, f, g, h\}$. Let G_0 be the (5, 5)-graph with order 32 and with the following adjacency list:

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Vertex	Adjacent vertices	Vertex	Adjacent vertices
00000	10000, 11010, 11100, 00001, 11111	00001	00000, 10001, 11011, 11101, 11110
10000	00000, 01011, 01101, 01110, 11001	10001	00001, 01010, 01100, 01111, 11000
01000	01001, 10010, 10101, 10110, 11000	01001	01000, 10011, 10100, 10111, 11001
11000	00011, 00100, 00111, 01000, 10001	11001	00010, 00101, 00110, 01001, 10000
00100	00101, 10100, 11000, 11010, 11110	00101	00100, 10101, 11001, 11011, 11111
10100	00100, 01001, 01011, 01111, 11101	10101	00101, 01000, 01010, 01110, 11100
01100	01101, 10001, 10011, 10110, 11100	01101	01100, 10000, 10010, 10111, 11101
11100	00000, 00010, 00111, 01100, 10101	11101	00001, 00011, 00110, 01101, 10100
00010	00011, 10010, 11001, 11100, 11110	00011	00010, 10011, 11000, 11101, 11111
10010	00010, 01000, 01101, 01111, 11011	10011	00011, 01001, 01100, 01110, 11010
01010	01011, 10001, 10101, 10111, 11010	01011	01010, 10000, 10100, 10110, 11011
11010	00000, 00100, 00110, 01010, 10011	11011	00001, 00101, 00111, 01011, 10010
00110	00111, 10110, 11001, 11010, 11101	00111	00110, 10111, 11000, 11011, 11100
10110	00110, 01000, 01011, 01100, 11111	10111	00111, 01001, 01010, 01101, 11110
01110	01111, 10000, 10011, 10101, 11110	01111	01110, 10001, 10010, 10100, 11111
11110	00001, 00010, 00100, 01110, 10111	11111	00000, 00011, 00101, 01111, 10110

The set $w(G_0) = \{00001, 01001, 10000, 11010, 11011, 11100, 11101, 11110, 11111\}$ contains the Cayley colors of G_0. As graph G_1, consider the isomorphic graph of G_0 with the following relabeling of the vertices:

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
00000 ↔ 00000	00001 ↔ 00011	00010 ↔ 00010	00011 ↔ 00001	00100 ↔ 00100	00101 ↔ 00111
00110 ↔ 00110	00111 ↔ 01110	01000 ↔ 11001	01001 ↔ 11100	01010 ↔ 11111	01011 ↔ 11011
01100 ↔ 10011	01101↔ 11101	01110 ↔ 11010	01111 ↔ 11110	10000 ↔ 01111	10001 ↔ 10100
10010 ↔ 01100	10011↔ 10000	10100 ↔ 01000	10101 ↔ 10001	10110 ↔ 01010	10111 ↔ 11000
11000 ↔ 10110	11001↔ 01101	11010 ↔ 10101	11011 ↔ 01001	11100 ↔ 00101	11101 ↔ 01011
11110 ↔ 10111	11111↔ 10010				

Since the set of Cayley colors of G_1 is $w(G_1) = \mathbb{F}_{32} - (w(G_0) \cup \{0000, 00110\})$, the graphs G_0 and G_1 have disjoint Cayley colors, and therefore, the amalgam graph $\mathcal{C}_{32}(G_0, G_1)$ has girth 5, regularity 37 and order $2 \cdot 32^2 = 2048$. ■

To give a general result for $m \geq 6$ we need some equivalences and definitions. As usual we identify the elements of $\mathbb{F}_{2^m} \cong (\mathbb{Z}_2)^m$ with a number of $\mathbb{Z}_{2^m}$ in the following way:

$$(v_{m-1}, \dots, v_0) \longleftrightarrow \sum_{i=0}^{m-1} 2^i v_i$$

for every $i = 0, \dots, m-1$ and $v_i \in \mathbb{Z}_2$. This induces a bijection $\phi : \mathbb{Z}_{2^m} \to (\mathbb{Z}_2)^m$ such that the elements of $(\mathbb{Z}_2)^m$ can be represented either by a vector or by a number. This bijective relationship allows us to translate the graph $S_{2^m}(P, Q; k_1, \dots, k_w)$ with vertex set $\mathbb{Z}_{2^m}$ into a new graph with vertex set $(\mathbb{Z}_2)^m$ defined as follows:

Definition 5.1. Given an integer $m \geq 4$, a sequence $k_1, \dots, k_w$ of different odd elements from $\mathbb{Z}_{2^m}$ and two even elements $0 < P, Q < 2^{m-1}$, we denote by $\bar{S}_{2^m}(P, Q; k_1, \dots, k_w)$ the graph with vertex set $(\mathbb{Z}_2)^m$ obtained by translating the vertices and edges of $S_{2^m}(P, Q; k_1, \dots, k_w)$ by means of the bijection $\phi : \mathbb{Z}_{2^m} \to (\mathbb{Z}_2)^m$.

Clearly, graphs $S_{2^m}(P, Q; k_1, \dots, k_w)$ and $\bar{S}_{2^m}(P, Q; k_1, \dots, k_w)$ are isomorphic. Notice that the Cayley colors of the graph $\bar{S}_{2^m}(P, Q; k_1, \dots, k_w)$ are computed in the additive group $(\mathbb{Z}_2)^m$, which implies that edges of $\bar{S}_{2^m}(P, Q; k_1, \dots, k_w)$ associated to an element of $\{P, Q; k_1, \dots, k_w\}$ might have different Cayley colors in $(\mathbb{Z}_2)^m$.

At this point we are able to prove Theorem 1.2, mentioned in the Introduction.

Theorem 1.2. *Given an integer $k \geq 68$, let $q = 2^m$ be the lowest even prime power such that $k \leq q+6$. Then $n(k, 5) \leq 2q(k-6)$.*

Proof. Consider $q = 2^m$ for an integer $m \geq 6$. Due to the bijection ϕ described above, we represent the elements of $(\mathbb{Z}_2)^m$ by the numbers of $\mathbb{Z}_{2^m}$ and vice versa.

For $q = 64$ we consider the 6-regular graph $G_0 = \bar{S}_{64}(4, 8; 1, 3, 41, 47)$ of girth five and set of Cayley colors $w(G_0) = \{1, 3, 4, 7, 8, 12, 15, 19, 23, 24, 25, 28, 31, 41, 47, 51, 55, 56, 57, 60, 63\}$. To obtain the graph G_1 we consider the following relabeling of the vertices:

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
0 ↔ 0	1 ↔ 44	2 ↔ 2	3 ↔ 39	4 ↔ 5	5 ↔ 41	6 ↔ 7	7 ↔ 19
8 ↔ 12	9 ↔ 50	10 ↔ 14	11 ↔ 28	12 ↔ 1	13 ↔ 52	14 ↔ 3	15 ↔ 21
16 ↔ 4	17 ↔ 25	18 ↔ 6	19 ↔ 22	20 ↔ 57	21 ↔ 20	22 ↔ 59	23 ↔ 31
24 ↔ 24	25 ↔ 45	26 ↔ 26	27 ↔ 56	28 ↔ 61	29 ↔ 48	30 ↔ 63	31 ↔ 29
32 ↔ 32	33 ↔ 10	34 ↔ 34	35 ↔ 8	36 ↔ 49	37 ↔ 23	38 ↔ 51	39 ↔ 27
40 ↔ 36	41 ↔ 62	42 ↔ 38	43 ↔ 54	44 ↔ 9	45 ↔ 35	46 ↔ 11	47 ↔ 43
48 ↔ 40	49 ↔ 46	50 ↔ 42	51 ↔ 30	52 ↔ 53	53 ↔ 33	54 ↔ 55	55 ↔ 17
56 ↔ 16	57 ↔ 58	58 ↔ 18	59 ↔ 60	60 ↔ 13	61 ↔ 47	62 ↔ 15	63 ↔ 37

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The Cayley colors of G_1 are $w(G_1) = \{1, \ldots, 63\} - w(G_0) - \{50\}$ and hence the $(70, 5)$-graph $\mathcal{C}_{64}(G_0, G_1)$ has order $2 \cdot 64^2$.

In general for $q = 2^m$ and $m \geq 7$ we use the previous graphs G_0 and G_1 defined over $(\mathbb{Z}_2)^6$ to construct new graphs G_0^m and G_1^m with vertex set $(\mathbb{Z}_2)^m$ in the following way: The neighbors of a vertex $(u_{m-1}, \ldots, u_0)$ in G_0^m are the six vertices of the set $\{(u_{m-1}, \ldots, u_6, v_5, \ldots, v_0) : (u_5, \ldots, u_0)(v_5, \ldots, v_0) \in E(G_0)\}$. Similar definition holds for G_1^m. Graphs G_0^m and G_1^m are formed by 2^{m-6} disconnected copies of G_0 and G_1, respectively, and therefore, both graphs are 6-regular with girth 5. Also, the sets of Cayley colors $w(G_0^m) = \{(0, \ldots, 0, \alpha_5, \ldots, \alpha_0) \in (\mathbb{Z}_2)^m : (\alpha_5, \ldots, \alpha_0) \in w(G_0)\}$ and $w(G_1^m) = \{(0, \ldots, 0, \beta_5, \ldots, \beta_0) \in (\mathbb{Z}_2)^m : (\beta_5, \ldots, \beta_0) \in w(G_1)\}$ are disjoint because $w(G_0) \cap w(G_1) = \emptyset$. Clearly, the graphs G_0^m and G_1^m are suitable for amalgamation into $\mathcal{C}_q$ and the graph $\mathcal{C}_q(G_0^m, G_1^m)$ has regularity $q+6$, order $2q^2$ and girth at least five. For $k \leq q+6$ removing $q+6-k$ blocks of points and $q+6-k$ blocks of lines we obtain a graph of order $2q^2 - 2q(q+6-k)$ and consequently $n(k, 5) \leq 2q(k-6)$. ■

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