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The $p$-restricted edge-connectivity of Kneser graphs

C. Balbuena, X. Marcote

Abstract

Given a connected graph $G$ and an integer $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$, a $p$-restricted edge-cut of $G$ is any set of edges $S \subseteq E(G)$, if any, such that $G - S$ is not connected and each component of $G - S$ has at least $p$ vertices; and the $p$-restricted edge-connectivity of $G$, denoted $\lambda_p(G)$, is the minimum cardinality of such a $p$-restricted edge-cut. When $p$-restricted edge-cuts exist, $G$ is said to be super-$\lambda_p$. If the deletion from $G$ of any $p$-restricted edge-cut $S$ of cardinality $\lambda_p(G)$ yields a graph $G - S$ that has at least one component with exactly $p$ vertices. In this work, we prove that Kneser graphs $K(n,k)$ are super-$\lambda_p$-connected for a wide range of values of $p$. Moreover, we obtain the values of $\lambda_p(G)$ for all possible $p$ and all $n \geq 5$ when $G = K(n,2)$. Also, we discuss in which cases $\lambda_p(G)$ attains its maximum possible value, and determine for which values of $p$ graph $G = K(n,2)$ is super-$\lambda_p$.

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1. Introduction

For other terminology and notation not defined here, we refer the reader to the book by Chartrand and Lesniak [9].

All graphs are considered hereafter as finite and simple, that is, with a finite number of vertices and without loops or multiple edges. If $G$ is such a graph, its sets of vertices and edges are denoted as $V(G), E(G)$, respectively. For a nonempty subset of vertices $X \subseteq V(G)$, $G[X]$ stands for the subgraph of $G$ induced by $X$. The clique number of $G$ is the maximum cardinality of $X \subseteq V(G)$ such that $G[X]$ is a complete graph. The connectivity (or vertex-connectivity) of $G$ is written $\kappa(G)$, and the edge-connectivity of $G$ is denoted as $\lambda(G)$. For nonempty disjoint sets $X, Y \subseteq V(G)$ let $X \cup Y$ be the set of edges with one end in $X$ and the other end in $Y$. Clearly, $[X \setminus V(G)]X$ is an edge-cut of $G$. Denote $\omega_{\Delta}(X) = |X \setminus V(G) \setminus X|$. The degree of a vertex $x \in V(G)$ is $\deg_G(x) = |\omega_{\Delta}(x)|$, and $\delta(G)$ stands for the minimum degree of $G$.

In [12,13] Fàbrega and Fiol proposed the concept of $p$-restricted edge-connectivity. Given a connected graph $G$ and an integer $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$, a $p$-restricted edge-cut of $G$ is any set of edges $S \subseteq E(G)$, if any, such that $G - S$ is not connected and all components of $G - S$ have at least $p$ vertices. If $p$-restricted edge-cuts of $G$ exist, then $G$ is said to be super-$\lambda_p$-connected. When $G$ is $\lambda_p$-connected, the $p$-restricted edge-connectivity of $G$, $\lambda_p(G)$, is defined as follows:

$$\lambda_p(G) = \min_{S \subseteq E(G)} \{ |S| \mid S \text{ is a } p\text{-restricted edge-cut of } G \}.$$ 

If $G$ is $\lambda_q$-connected for some $q > p$, note that $G$ is $\lambda_p$-connected and $\lambda_p(G) \leq \lambda_q(G)$ holds. When $p = 1$, $\lambda_p(G) = \lambda_1(G)$ is the standard edge-connectivity $\lambda(G)$; and for the case $p = 2$, $\lambda_2(G)$ is usually known as the edge-superconnectivity of $G$ (also denoted $\lambda^*(G)$). A $p$-restricted edge-cut of cardinality $\lambda_p(G)$ is called a $\lambda_p$-cut. When $p$-restricted edge-cuts of $G$ exist, $G$ is

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said to be super-$\lambda_p$ if the deletion from $G$ of any $\lambda_q$-cut $S$ yields a graph $G - S$ that has at least one component with exactly $p$ vertices. If $G$ is super-$\lambda_p$ and also $\lambda_q$-connected for some $q > p$, observe that $\lambda_p(G) < \lambda_q(G)$ necessarily. For the case $p = 1$, saying that $G$ is super-$\lambda_1$ and that $G$ is edge-supercritical are synonyms.

The optimization of $\lambda_p(G)$ requires an upper bound. Let

$$\xi_p(G) = \min_{X \subseteq V(G)} \left| \{V(G) \setminus X \} : |X| = p, \, G[X] \text{ is connected} \right|.$$ 

It has been shown that $\lambda_p(G) \leq \xi_p(G)$ for many graphs \cite{4,6,16,21,28,30} and sufficient conditions to establish that $\lambda_p(G) = \xi_p(G)$ have been given in \cite{4,18,26} among others.

It is worth noting that attaining super-$\lambda_p$ property implies minimizing the number of minimum $p$-restricted edge-cuts (see \cite{23} for the case $p = 1$). In general, to determine whether a graph is super-$\lambda_p$ is a hard problem, and only some special graphs have been shown to possess the super-$\lambda_p$ property.

Fábrega and Fiol also proposed the concept of $p$-restricted (vertex-)connectivity $\kappa_p$ and some results for this kind of connectivity have been obtained in \cite{2,3,27,29}. Other kind of connectivity measures involving both vertices and edges are studied in \cite{11,19}, for instance. Hellwig and Volkmann \cite{17} provide a comprehensive survey of sufficient conditions for a graph to achieve lower bounds on other index of connectivities.

In this paper, we are interested in studying the $p$-restricted edge-connectivity of Kneser graphs, which are a class of graphs introduced by Lovász \cite{20} to prove Kneser’s conjecture. Given integers $n \geq k \geq 1$, the Kneser graph $K(n,k)$ is the graph whose vertices are the $k$-subsets of the set $\{1, \ldots, n\}$, two vertices being adjacent if and only if they correspond to disjoint subsets. Therefore, $K(n,k)$ has $\binom{n}{k}$ vertices, and has no edges in case that $n < 2k$. When $n \geq 2k$, $K(n,k)$ is $\left(\binom{n}{k}\right)$-regular, then it has $\left(\binom{n}{k}\right)/2$ edges; hence for the case $n = 2k$, $K(n,k)$ consists of a set of $\left(\binom{n}{k}\right)/2$ independent edges. Note that $K(n,1)$ is the complete graph on $n$ vertices and also that $K(5,2)$ is the Petersen graph.

A number of structural properties are known for $K(n,k)$. Chen and Li [10] showed that Kneser graphs are vertex- and edge-transitive. Valencia-Pavon and Vera \cite{25} showed that the diameter of $K(n,k)$ is equal to $\left\lceil \left(\binom{n-1}{k-1}\right)/(n-k+1) \right\rceil$. When $n > 2k$, Lovász \cite{20} proved that the chromatic number of $K(n,k)$ is $n - 2k + 2$. Many of these results can be checked in the book by Aigner and Ziegler \cite{1}; for instance, the clique number of $K(n,k)$ is $\lceil n/k \rceil$, and its independence number is $\lceil (n-k)/2 \rceil$. It has long been conjectured that $K(n,k)$ is Hamiltonian (with the exception of $K(5,2)$), and this was verified by Shields and Savage \cite{22} for $n \leq 27$. It is also worth noting that the Kneser graph $K(n,2)$ is distance-regular with intersection array $(n-2(n-2))/2, 2(n-8), 1, (n-3)(n-4)/2$ (see \cite{24}, p. 86). Brouwer and Haemers proved in \cite{8} that distance-regular graphs are edge-supercritical, then $K(5,2)$ is edge-supercritical.

Concerning the connectedness of Kneser graphs the following results were obtained in \cite{7}. Note that $K(n,k)$ is connected whenever $n > 2k+1$, since it has a finite diameter (see again \cite{25}).

**Theorem 1.1** \cite{7} Let $n$, $k$ be two integers, $n \geq 2k+1 \geq 5$. The following statements hold:

(i) the graph $K(n,k)$ is maximally connected; that is, its (vertex-)connectivity is equal to $\binom{n}{k}$;

(ii) the graph $K(n,2)$ is (vertex-)superconnected;

(iii) the (vertex-superconnectivity of $K(n,2)$ is equal to $\binom{n}{2} - 6$.

The paper (Section 2) is organized into two subsections as follows. Section 2.1 is devoted to prove $G = K(n,k)$ that there exists some $n_0 \geq 2k+1$ such that $G$ is $\lambda_p$-connected and satisfies $\lambda_p \leq \xi_p$ for all $n \geq n_0$ and all $1 \leq p \leq |V(G)|/2$; moreover, we prove that $n_0 = 5$ when $k = 2$. In Section 2.2 we focus on $G = K(n,2)$, approaching the problem of finding for which values of $1 \leq p \leq |V(G)|/2$ the optimal result $\lambda_p = \xi_p$ holds, and we study if $G$ is super-$\lambda_p$ in the affirmative case. This is done by computing the exact values of $\xi_p$ for all $1 \leq p \leq |V(G)|/2$, from where all the values of $\lambda_p$ will follow.

For the sake of simplicity, most of quantities defined for a graph $G$ will be written from now on without any explicit reference to $G$, unless it is necessary; for instance, $\kappa$, $\lambda$, $\omega(X)$ will be written instead of $\kappa(G)$, $\lambda(G)$, $\omega_G(X)$, respectively.

2. Results

2.1. $\lambda_p \leq \xi_p$ for $K(n,k)$

Let $G_1, \ldots, G_s$ be $s$ copies of a complete graph $K_t$. The graph denoted as $G^t_s$ is obtained by adding a new vertex $u$ and joining $u$ to every vertex in $V(G_i)$, $i = 1,\ldots, s$. In \cite{30} it is proved the following result.

**Theorem 2.1** \cite{30}. Let $G$ be a connected graph with order at least $2(\delta(G) + 1)$ which is not isomorphic to any $G^t_s$ with $t = \delta(G)$. Then for any $p \leq \delta(G) + 1$, $G$ has $p$-restricted edge-cuts and $\lambda_p \leq \xi_p$.

In the following statement we prove a similar result for graphs of order less than $2(\delta(G) + 1)$.

**Lemma 2.1.** Let $G$ be a connected graph with vertex connectivity $\kappa$ and order $\nu \leq 2\kappa - 1$. Then $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ for all integer $p$ such that $1 \leq p \leq |V(G)|/2$.

**Proof.** Let $X \subseteq V(G)$ satisfying $|X| = p \leq |V(G)|/2$. Then $G - X$ is connected because $|X| = p \leq |V(G)|/2 \leq (\nu - 1)/2 = \kappa - 1$. Moreover, $|V(G) \setminus X| = \nu - p \geq \nu - |V(G)|/2 \geq |V(G)|/2$ holds. Hence, $\omega(X) = |V(G) \setminus X|$ is a $p$-restricted edge-cut yielding that $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$. $\square$

We now apply the above results to Kneser graphs $K(n, k)$.

**Theorem 2.2.** Let $n, k$ be two integers, $n \geq 2k + 1 \geq 5$, $G = K(n, k)$, and $p$ be an integer. Then $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ if

(i) $\binom{n-k}{k} \geq 2\left(\frac{n-k}{k}\right) + 2$ for $1 \leq p \leq \left(\frac{n-k}{k}\right) + 1$.

(ii) $\binom{n-k}{k} \leq 2\left(\frac{n-k}{k}\right) + 1$ for $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$.

**Proof.** Since $n \geq 2k + 1$, $G = K(n, k)$ is connected. Let $\nu = \binom{n-k}{k}$ and $\delta = \binom{n-k}{k}$ be the order and degree of $G$, respectively. If $\nu \geq 2d + 2$, then $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ for $p \leq d + 1$ by Theorem 2.1 because clearly $G$ is not isomorphic to $G_3^*$.

Hence item (i) holds. If $\nu \leq 2d - 1$, then $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ by Lemma 2.1, as $k = d$ by Theorem 1.1. Therefore it remains to study for item (ii) the case when either $\nu = 2d$ or $\nu = 2d + 1$. The former case (ii) is not possible because $\binom{n-k}{k} = \sum_{i=1}^{n-k} \binom{k}{i-1} + \binom{n-k}{k}$ and $\sum_{i=1}^{n-k} \binom{k}{i-1} \neq \binom{n-k}{k}$. The latter case (ii) is $\binom{n-k}{k} + 1$ only holds when $n = 7$ and $k = 2$.

For the rest of values of $n, k$ we also have $\sum_{i=1}^{n-k} \binom{k}{i-1} \neq \binom{n-k}{k} + 1$. When $n = 7$ and $k = 2$ let us take the following set of vertices:

$$X = \{x_1 = \{1, 2\}, x_2 = \{3, 4\}, x_3 = \{5, 6\}, x_4 = \{1, 7\}, x_5 = \{2, 4\}, x_6 = \{3, 5\}, x_7 = \{6, 7\}, x_8 = \{2, 7\}, x_9 = \{1, 6\}, x_{10} = \{4, 5\}\}.$$

It is not difficult to check that for all $p = 1, \ldots, \lfloor |V(G)|/2 \rfloor = 10$, both $X_p = \{x_1, \ldots, x_p\} \subseteq X$ and $G - X_p$ induce connected subgraphs of $G$, with $\omega(X_p) = \xi_p$. Hence, item(ii) holds, and the proof is complete. □

Observe from the above theorem that for all $k \geq 2$ there exists an integer $n_0 \geq 2k + 1$ such that for all $n \geq n_0$, $G = K(n, k)$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ for all $p$ with $1 \leq p \leq |V(G)|/2$. In the following corollary we prove that $n_0 = 5$ when $k = 2$.

**Corollary 2.1.** Let $n \geq 5$ be an integer, $G = K(n, 2)$, and $p$ be an integer such that $1 \leq p \leq |V(G)|/2$. Then $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$.

**Proof.** The result follows from Theorem 2.2 (ii) when $n \geq 7$. When $n = 5, 6$, from Theorem 2.2 (i) we have that $G$ is $\lambda_p$-connected and $\lambda_p \leq \xi_p$ for $p \leq \binom{n-k}{k} + 1$. This implies that the result is valid for $1 \leq p \leq |V(G)|/2$ when $n = 6$; and when $n = 5$ the result holds for $1 \leq p \leq 4$. Thus, the only remaining case is $n = p = 5 = |V(G)|/2$. The graph $G = K(5, 2)$ is isomorphic to Petersen graph and it can be described as two disjoint cycles of length 5 joined by a matching. Hence $G$ is $\lambda_5$-connected and $\lambda_5 \leq \xi_5 = 5$, ending the proof. □

2.2. $\lambda_p$-optimality and super-$\lambda_p$ in $K(n, 2)$

Let $G$ be a $\lambda_p$-connected graph and let $X \subseteq V(G)$ with $|X| \geq p$ such that $\omega_G(X)$ is a $\lambda_p$-cut. Then, $X$ is called a $\lambda_p$-fragment of $G$. Define

$$r_p(G) = \min \{|X| : X \text{ is a } \lambda_p\text{-fragment of } G\}.$$

Clearly, $p \leq r_p(G) \leq |V(G)|/2$. A $\lambda_p$-fragment $X$ is called a $\lambda_p$-atom of $G$ when $|X| = r_p(G)$. Next, we recall a result obtained by Wang et al. [28], where $\lambda_p$-connected $(q + 1)$-clique-free graphs were nicely addressed. Then a first result for the equality of $\lambda_p(K(n, 2))$ and $\xi_p(K(n, 2))$ will follow quite straightforwardly for some values of $p$.

**Theorem 2.3.** ([28]) Let $G$ be a $\lambda_p$-connected and $(q + 1)$-clique-free graph. If $\lambda_p(G) < \xi_p(G)$, then $r_p(G) \geq \max\{p + 1, \frac{q}{q-1}\delta(G) - p - 1\}$.

**Proposition 2.1.** Let $n \geq 7$ be an integer and $G = K(n, 2)$. Then $\lambda_p = \xi_p$ if

$$ p \leq \begin{cases} \frac{n(n-5)}{4} - 2, & \text{if } n \text{ is even} \\ \left\lfloor \frac{(n-1)(n-4)}{4} \right\rfloor - 2, & \text{if } n \text{ is odd} \end{cases} $$

**Proof.** We know that $G$ is a $(q + 1)$-clique-free graph, where $q = \lfloor n/2 \rfloor$. First, suppose that $n$ is even. Suppose $p \leq \frac{n(n-5)}{4} - 2$ and $\lambda_p < \xi_p$. From Theorem 2.3 it follows that $r_p(G) \geq \max\{p + 1, \frac{q}{q-1}\frac{n(n-2)}{2} - p - 1\}$, yielding that $r_p(G) \geq \frac{q}{q-1}\frac{n(n-2)}{2} - p - 1 \geq \sum_{\frac{n}{2}}^{\frac{n}{2}+1} \binom{n}{k} - p - 1 \geq 1 + \sum_{\frac{n}{2}}^{\frac{n}{2}+1} \binom{n}{k} + 1$, a absurdity. Similarly, when $n$ is odd and $p \leq \frac{(n-1)(n-4)}{4} - 2$ we have $r_p(G) \geq \frac{q}{q-1}\frac{(n-2)}{2} - p - 1 \geq \frac{n}{2} - \frac{(n-2)}{2} - p - 1 \geq \sum_{\frac{n}{2}-1}^{\frac{n}{2}+1} \binom{n}{k} + 1$ which is again a contradiction. Hence, $\lambda_p \geq \xi_p$, and by Corollary 2.1 we can conclude that $\lambda_p = \xi_p$. □

For $K(n, 2)$, our objectives now are to compute $\lambda_p$ for all $1 \leq p \leq \lfloor |V(K(n, 2))/2 \rfloor$ (extending the result in Proposition 2.1), and to study when $K(n, 2)$ is super-$\lambda_p$. As we show in the following lemma for a general graph $G$, these objectives can be reached provided that the values of $\xi_p(G)$ are known for all $1 \leq p \leq |V(G)|/2$. In the rest of the paper, by $\binom{V(G)}{p}$ we denote the set of those subsets of $V(G)$ having cardinality $p$.

Lemma 2.2. Let $G$ be a $\lambda_p$-connected graph with $\lambda_p \leq \xi_p$ for all $1 \leq p \leq |V(G)|/2$. The following statements hold:

(i) $\lambda_p = \min\{\xi_q : p \leq q \leq |V(G)|/2\}$.

(ii) For $p = |V(G)|/2$ it follows that $\lambda_p = \xi_p$ and $G$ is super-$\lambda_p$.

(iii) For $p \leq |V(G)|/2 - 1$ it follows that:

1) $\lambda_p = \xi_p$ if and only if $\xi_p \leq \xi_q$ for all $q$ such that $p < q \leq |V(G)|/2$.

2) $\lambda_p = \xi_p$ and $G$ is super-$\lambda_p$ if and only if $\xi_p < \xi_q$ for all $q$ such that $p < q \leq |V(G)|/2$.

Proof. (i) Let $t = t_p(G)$ be the cardinality of a $\lambda_p$-atom of $G$. Clearly $p \leq t \leq |V(G)|/2$. Let $X \in \binom{V(G)}{p}$ be such that $\omega(X)$ is a $\lambda_p$-cut (note that $|V(G) \setminus X| = |V(G)| - t \geq |V(G)| - |V(G)|/2 \geq |V(G)|/2 \geq p$), then $\lambda_p = |\omega(X)| = \xi_t$. But $\lambda_p \leq \lambda_t \leq \xi_t$, hence $\xi_t = \xi_p$. Suppose next that there exists some integer $q$ such that $p \leq q \leq |V(G)|/2$ and $\xi_q < \xi_t$. Then $\xi_q < \xi_t = \xi_p \leq \xi_q$, that is, $\xi_q < \xi_q$, an absurdity. As a consequence, $\xi_t \leq \xi_q$ for all $p \leq q \leq |V(G)|/2$ and therefore $\lambda_p = \xi_t = \min\{\xi_q(G) : p \leq q \leq |V(G)|/2\}$.

as claimed in (i).

When $p = |V(G)|/2$ we have $\lambda_p = \xi_p$ by (i), and note that every $p$-restricted edge-cut $\omega(Y)$ is such that $|Y| = p$ or $|V(G) \setminus Y| = p$. As a consequence, $G$ is super-$\lambda_p$. This proves item (ii).

Item (iii.1) follows directly from (i). For (iii.2), if $\lambda_p = \xi_p$ and $G$ is super-$\lambda_p$ then $\xi_p = \lambda_p < \lambda_q \leq \xi_q$ for all $q > p$, hence $\xi_p < \xi_q$. Conversely, suppose that $\xi_p < \xi_q$ for all $q > p$. Then $\lambda_p = \xi_p$ follows from (i). Moreover, if $G$ is not super-$\lambda_p$, we can consider some $Y \subseteq V(G)$ such that $|Y| \geq p + 1$. $|V(G) \setminus Y| \geq p + 1$. $G|Y$ and $G - Y$ are both connected and $|\omega(Y)| = \lambda_p$. Setting $m = \min\{|Y|, |V(G) \setminus Y|\}$ it follows that $\xi_p = \lambda_p \geq \xi_m > \xi_p$.

again an absurdity. Then $G$ must be super-$\lambda_p$, ending the proof of (iii.2). $\square$

As $G = K(n, 2)$ is a regular graph, minimizing the cardinality of $\omega(X)$ among all sets $X \subseteq V(G)$ on $p$ vertices that induce a connected subgraph is equivalent to finding such a set $X$ which maximizes $|E(G[X])|$. In the following result we present a set $X_p^*$ of $p$ vertices (for each $1 \leq p \leq |V(G)|/2$) with large $|E(G[X_p^*])|$, for which we will finally prove that $\xi_p(G) = |\omega(X_p^*)|$.

Proposition 2.2. Let $n \geq 5$ be an integer, and let $G = K(n, 2)$. For all integers $1 \leq p \leq |V(G)|/2$ there exists a set $X_p^* \in \binom{V(G)}{p}$ such that $G[X_p^*]$ is connected and

$$|E(G[X_p^*])| = \frac{1}{2} \left( p^2 + \left( 2p/|n| \right) \left( 1 + \left( 2p/|n| \right) \right) - n - \left( 1 + \left( 2p/|n| \right) \right) \right).$$

Proof. Suppose first that $n \geq 6$ is even. The following partition of $V(K(n, 2))$ is direct from some related known results, see for instance Baranyai’s Theorem ([5]):

$$V(K(n, 2)) = E_1 \cup \cdots \cup E_{n-1},$$

where the following statements hold for all $i = 1, \ldots, n - 1$:

- $E_i \cap E_j = \emptyset$, for all $j \neq i$;
- $|E_i| = n/2$;
- $e_k \in E_i$, $e_k \in E_j$, for all distinct $e_k, e_j \in E_i$;
- the union of all elements of $E_i$ is equal to $\{1, \ldots, n\}$.

Let $p$ be an integer, $1 \leq p \leq |V(G)|/2$, and set $c = [p/(n/2)] = [2p/n]$, for which $0 \leq c \leq n/2 - 1 < n - 1$. Hence we write $p = c + r$, where $0 \leq r < n/2 - 1$. Suppose $c \geq 1$ and consider the set $X_p^*$ of $p$ vertices defined as

$$X_p^* = E_1 \cup \cdots \cup E_c \cup R,$$

where $R \subseteq E_{n-1}$ is any subset of cardinality $r$. Observe that each $E_i$ induces a clique in $G$ of cardinality $\frac{n}{2}$, and $R$ (if nonempty) induces a complete graph on $r$ vertices. Hence

$$|E(G[E_i])| = \frac{1}{2} \left( \frac{n}{2} \right) \left( \frac{n}{2} - 1 \right), |E(G[R])| = \frac{1}{2} r(r - 1).$$

As the union of all elements of $E_i$ is equal to $\{1, \ldots, n\}$, note that each vertex in $E_i$ is adjacent to exactly $\frac{n}{2} - 2$ vertices in $E_i$, for $i \neq j$, and analogously, each vertex in $R$ is adjacent to exactly $\frac{n}{2} - 2$ vertices in $E_i$. As a consequence we have:

$$|E(G[X_p^*])| = \sum_{i=1}^{c} |E(G[E_i])| + |E(G[R])| + \sum_{i=1}^{c} |E_i|$$

$$+ \sum_{1 \leq i < j \leq c} |E_i \cap E_j|.$$
last expression obtained after replacing $c$ by $[2p/n]$ and $r$ by $p - [2p/n]$. Note that this expression for $|E(G[X_p^n])|$ still holds when $c = 0$, where $X_p^n = R$. Observe that $G[X_p^n]$ is connected by construction. Hence the proof is complete when $n$ is even.

Next we consider the case when $n \geq 5$ is odd. Note that (1) can be applied to $V(K(n,1,2))$ yielding $V(K(n,1,2)) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n$. Observe that after a suitable relabeling of the elements of $\{1, \ldots, n+1\}$ and, if necessary, a reordering of sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$, we can assume that

$$\{i, n+1\} \in \mathcal{E}_i \text{ for all } i = 1, \ldots, n; \text{ and } \mathcal{E}_n = \{t_j = \{2j-1, 2j\} : j = 1, \ldots, (n+1)/2\}.$$  

By defining $\mathcal{O}_i = \mathcal{E}_i \setminus \{i, n+1\}$ for all $i = 1, \ldots, n$, from (1) we can write

$$V(K(n,2)) = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n,$$

where $\mathcal{O}_i = \{t_j = \{2j-1, 2j\} : j = 1, \ldots, (n-1)/2\}$ and where the following statements hold for all $i = 1, \ldots, n$:

- $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$, for all $j \neq i$;
- $|\mathcal{O}_i| = (n-1)/2$;
- $e_k \cap e_j = \emptyset$, for all distinct $e_k, e_j \in \mathcal{O}_i$;
- the union of all elements of $\mathcal{O}_i$ is equal to $\{1, \ldots, n\} \setminus \{i\}$.

Let $p$ be an integer, $1 \leq p \leq |V(G)|$, and $c = \lfloor p/(n-1) \rfloor$. Suppose $c \geq 1$ and consider the set of $p$ vertices $X_p^n$ defined as

$$X_p^n = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_c \cup R,$$

where $R = \{t_j = \{2j-1, 2j\} : j = 1, \ldots, r \} \subset \mathcal{O}_n$. Again

$$|E(G[\mathcal{O}_i])| = \frac{1}{2} \frac{n-1}{2} (n-1/2 - 1), \quad |E(G[R])| = \frac{1}{2} \frac{r(r-1)}{2},$$

because the respective induced subgraphs are complete. Furthermore, all but one vertices in $\mathcal{O}_i$ are adjacent to exactly $
-1/2 - 1$ vertices in $\mathcal{O}_i = \mathcal{E}_i \setminus \{i, n+1\}$, for $i \neq j$, and one only vertex in $\mathcal{O}_j$ is adjacent to $n-1/2 - 1$ vertices in $\mathcal{O}_i$, precisely a vertex of the kind $(i, \alpha)$, with $\alpha \in \{1, \ldots, n\} \setminus \{i\}$. Then, for all $1 \leq j < i \leq c$ we have

$$|\mathcal{O}_i, \mathcal{O}_j| = 1 + \frac{n-1}{2} (n-1/2 - 2).$$

Notice now that vertex $t_j = \{2j-1, 2j\} \in R$ is adjacent to exactly $n-1/2 - 1$ vertices in $\mathcal{O}_i$ for all $i \in \{1, \ldots, c\} \setminus \{2j-1, 2j\}$ and $t_j = \{2j-1, 2j\} \in R$ is adjacent to $n-1/2 - 1$ vertices in $\mathcal{O}_{2j-1}$ and to $n-1/2 - 1$ vertices in $\mathcal{O}_{2j}$ whenever $2j-1 \leq c$ or $2j \leq c$ respectively. Therefore.

$$\sum_{i=1}^c |R, \mathcal{O}_i| = \begin{cases} \frac{r(c-2)}{2} - 2r & \text{if } c > 2r \\ \frac{r(c-2)}{2} + c & \text{if } c \leq 2r \end{cases} = \frac{r (n-1/2 - 2)}{2} + \min(2r, c).$$

Then we have:

$$|E(G[X_p^n])| = \sum_{i=1}^c |E(G[\mathcal{O}_i])| + |E(G[R])| + \sum_{i=1}^c |R, \mathcal{O}_i|$$

$$= c \frac{1}{2} \frac{n-1}{2} (n-1/2 - 1) + \frac{1}{2} \frac{r(r-1)}{2} + \frac{r(n-1/2 - 2)}{2} + \min(2r, c)$$

$$= c \frac{1}{2} (p^2 + [2p/(n-1)](1 + [2p/(n-1)])n - p(1 + 4[2p/(n-1)]))$$

$$- c + \min(2r, c).$$

Observe again that this expression for $|E(G[X_p^n])|$ still holds when $c = 0$, where $X_p^n = R$. Note that $G[X_p^n]$ is connected by construction.
We continue by discussing on the sign of $c - 2r$. When $c - 2r \leq 0$ we have $\left\lfloor \frac{2p}{n} \right\rfloor n - 2p \leq 0$, that is, $\left\lfloor \frac{2p}{n} \right\rfloor \leq \frac{2p}{n}$. As $\frac{2p}{n} < \frac{2p}{n+1}$ it follows that $\left\lfloor \frac{2p}{n+1} \right\rfloor = \frac{2p}{n+1}$. Since $\min(2r, c) = c$, by replacing $\left\lfloor \frac{2p}{n} \right\rfloor$ with $\left\lfloor \frac{2p}{n+1} \right\rfloor$ in (3) we get

$$|E(G[X_p])| = \frac{1}{2} \left( p^2 + 2p \left\lfloor \frac{2p}{n+1} \right\rfloor \right),$$

and the result follows in this case.

When $c - 2r > 0$ we have $\left\lfloor \frac{2p}{n} \right\rfloor n - 2p > 0$, hence $\left\lfloor \frac{2p}{n+1} \right\rfloor > \frac{2p}{n+1}$. Since we have $0 < \frac{2p}{n+1} < \frac{2p}{n} = \frac{2p}{n(n-1)/2} \leq 1$ it follows that $\left\lfloor \frac{2p}{n+1} \right\rfloor = 1 + \left\lfloor \frac{2p}{n+1} \right\rfloor$. As $-c + \min(2r, c) = 2r - c < 2p - \left\lfloor \frac{2p}{n+1} \right\rfloor n$, replacing $\left\lfloor \frac{2p}{n+1} \right\rfloor$ with $1 + \left\lfloor \frac{2p}{n+1} \right\rfloor$ in (3) we obtain

$$|E(G[X_p])| = \frac{1}{2} \left( p^2 + (1 + \left\lfloor \frac{2p}{n+1} \right\rfloor) (2 + \left\lfloor \frac{2p}{n+1} \right\rfloor) n - p(1 + 4(1 + \left\lfloor \frac{2p}{n+1} \right\rfloor)) + 2p - (1 + \left\lfloor \frac{2p}{n+1} \right\rfloor)n \right),$$

proving the result also in this case. The proof is so complete. □

The following theorem makes use of the adjacency matrix of $K(n, 2)$, and its proof follows similar lines of reasoning as those used for this topic in the literature (see, for instance, [14,15]). With this theorem we deduce the exact value of $\xi_p(K(n, 2))$ for all possible $p$.

**Theorem 2.4.** Let $n \geq 5$ be an integer, $G = K(n, 2)$, and let $p$ be an integer such that $1 \leq p \leq \lfloor |V(G)|/2 \rfloor$. Then it follows that

$$\max \left\{ |E(G[X])| : X \in \binom{V(G)}{p} \right\} = |E(G[X_p])|,$$

where $X_p \in \binom{V(G)}{p}$ is the set of vertices given in Proposition 2.2. As a consequence,

$$\xi_p = p \left( \frac{n - 2}{2} \right) - p^2 - \left\lfloor \frac{2p}{n+1} \right\rfloor (1 + \left\lfloor \frac{2p}{n+1} \right\rfloor) n + p(1 + 4 \left\lfloor \frac{2p}{n+1} \right\rfloor) .$$

**Proof.** Note that $G$ is connected because $n \geq 5$. Set $V(G) = \{v_1, \ldots, v_N\}$, with $N = |V(G)| = n(n - 1)/2$, and consider some $X \in \binom{V(G)}{p}$. Let us represent set $X$ of cardinality $p$ as $Z_X = [t_1, t_2, \ldots, t_N]^T$, with $t_j = 1$, if $v_j \in X$; 0, if $v_j \not\in X$ for all $j = 1, \ldots, N$.

If $A$ is the adjacency matrix of $G$, it is known (see [15] for a proof) that its eigenvalues are

$$\lambda_1 = \left( n - 2 \right) > \lambda_2 = \cdots = \lambda_{m+1} = 1 > \lambda_{m+2} = \cdots = \lambda_N = -(n - 3),$$

where $m = \lfloor n(n - 3)/2 \rfloor$. Then, we can write

$$Z_X = Z_1 + Z_2 + Z_3, \quad \text{with} \quad \begin{cases} \ Z_1^T Z_j = 0, \quad \text{for all} \ i \neq j \\ \ Z_1 = \frac{p}{n} 1 \\ \ AZ_1 = (\binom{n-2}{2})Z_1, \quad AZ_2 = Z_2, \quad AZ_3 = -(n - 3)Z_3, \end{cases}$$

(4)

where $1$ is a column matrix full of ones, with $N$ rows. Notice that

$$p = Z_1^T Z_X = Z_1^T Z_1 + Z_1^T Z_2 + Z_1^T Z_3, \quad \text{hence} \quad Z_1^T Z_2 = p - Z_1^T Z_1 - Z_1^T Z_3.$$

Since $AZ_X = (\binom{n-3}{2} Z_1 + Z_2 - (n - 3)Z_3)$, it turns out that

$$2|E(G[X])| = Z_X^T AZ_X = \binom{n-3}{2} Z_1^T Z_1 + Z_1^T Z_2 - (n - 3)Z_1^T Z_3$$

$$= (\binom{n-3}{2} Z_1^T Z_1 + (p - Z_1^T Z_1 - Z_1^T Z_2) - (n - 3)Z_1^T Z_3$$

$$= p + (\binom{n-2}{2} - 1)Z_1^T Z_1 - (n - 2)Z_1^T Z_3,$$

once replaced $Z_1^T Z_2$ with $p - Z_1^T Z_1 - Z_1^T Z_3$. As $Z_1^T Z_1 = \frac{p}{n} 1^T \cdot \frac{p}{n} 1 = \frac{p^2}{n}$, we get

$$2|E(G[X])| = |p + (\binom{n-2}{2} - 1) \frac{p^2}{n} - (n - 2)Z_1^T Z_3$$

$$= |p + (1 - 4/n)p^2 - (n - 2)Z_1^T Z_3|,$$

(5)

Let us next compute $Z_1^T Z_3$ in a more useful manner. To this end, for all $j \in \{1, \ldots, n\}$, let $Y_j$ be a column matrix on $N$ rows, with $i$-row entry equal to one if $j \in v_i$ (that is, when $v_i = \{j, \ell\}$ for some $\ell \neq j$), and zero otherwise (note that $Y_j$ has exactly $n - 1$ ones). Since for all $j \in \{2, \ldots, n\}$ we have

$$(Y_j - Y_1)^T \cdot 1 = Y_j^T \cdot 1 - Y_1^T \cdot 1 = (n - 1) - (n - 1) = 0,$$
following [14] (p. 34) we conclude that
\[ \{Y_j - Y_1 : j = 2, \ldots, n\} \] is a basis of the eigenspace associated to eigenvalue \(-n - 3\).

Therefore, there must exist some \(\mu_2, \ldots, \mu_n \in \mathbb{R}\) such that
\[ Z_3 = \sum_{j=2}^{n} \mu_j (Y_j - Y_1). \]

Since \((Y_i - Y_1)^T (Y_j - Y_1) = \begin{cases} 2(n-2), & \text{if } i = j \\ n-2, & \text{if } i \neq j \end{cases}\), we can write
\[ Z_3^T Z_3 = \sum_{i=2}^{n} \sum_{j=2}^{n} \mu_i \mu_j (Y_i - Y_1)^T (Y_j - Y_1) = (n-2) [\mu_2 \ldots \mu_n] (I + J) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}, \]

where \(I\) is the identity matrix of order \(n-1\), and \(J\) is a square matrix of order \(n-1\) full of ones. In order to obtain the values of \(\mu_2, \ldots, \mu_n\), we compute next \((Y_i - Y_1)^T Z_3\) in two different ways, for all \(i = 2, \ldots, n\). First,
\[ (Y_i - Y_1)^T Z_3 = \sum_{j=2}^{n} \mu_j (Y_i - Y_1)^T (Y_j - Y_1) = (n-2) \left( 2\mu_i + \sum_{j=2, j \neq i}^{n} \mu_j \right). \]

Secondly, taking into account that \((Y_i - Y_1)^T Z_1 = (Y_i - Y_1)^T Z_2 = 0\) by (4):
\[ (Y_i - Y_1)^T Z_3 = (Y_i - Y_1)^T Z_X = \sigma_i - \sigma_1, \]

where \(\sigma_j\) (for all \(j \in \{1, \ldots, n\}\)) is the number of elements in \(X \in \binom{V(G)}{p}\) of the kind \((j, h)\), with \(h \neq j\). Note then that \(\sum_{j=1}^{n} \sigma_j = 2p\). By combining these two expressions of \((Y_i - Y_1)^T Z_3\) we get
\[ 2\mu_i + \sum_{j=2, j \neq i}^{n} \mu_j = \frac{\sigma_i - \sigma_1}{n-2} \quad \text{for all } i = 2, \ldots, n; \]

or, in matrix form,
\[ (I + J) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \frac{1}{n-2} \begin{bmatrix} \sigma_2 - \sigma_1 \\ \vdots \\ \sigma_n - \sigma_1 \end{bmatrix}. \]

As \((I + J)^T = I + J\) and \((I + J)^{-1} = I - \frac{1}{n}J\) it follows for \(Z_3^T Z_3\) that:
\[ Z_3^T Z_3 = (n-2) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} (I + J) (I + J) \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \frac{1}{n-2} \left( \sum_{j=2}^{n} (\sigma_j - \sigma_1)^2 - \frac{1}{n} \left( \sum_{j=2}^{n} (\sigma_j - \sigma_1) \right)^2 \right). \]

which, after some algebra, can be written as
\[ Z_3^T Z_3 = \frac{1}{n(n-2)} \sum_{1 \leq i < j \leq n} (\sigma_i - \sigma_j)^2. \]

The minimum possible value of \(\sum_{1 \leq i < j \leq n} (\sigma_i - \sigma_j)^2\) occurs for the most possible balanced distribution of \(\sigma_j\)'s: when
\[ ([2p/n] + 1)n - 2p \] elements in \(\{\sigma_1, \ldots, \sigma_n\}\) are equal to \([2p/n]\), the remaining \(2p - [2p/n]n\) elements in \(\{\sigma_1, \ldots, \sigma_n\}\) being equal to \([2p/n] + 1\). That is,
\[ \sum_{1 \leq i < j \leq n} (\sigma_i - \sigma_j)^2 \geq ([2p/n] + 1)n - 2p \cdot (2p - [2p/n]n). \]

Hence, coming back to expression (5):
\[ 2|E(G[X])| \leq p + (1 - \frac{4}{n})p^2 - \frac{1}{n}([2p/n] + 1)n - 2p \cdot (2p - [2p/n]n). \]
It takes a few calculations to see that the right hand side of this inequality is precisely equal to \(2|E(G[X_p])|\). As a consequence,
\[
\max \left\{ 2|E(G[X])| : X \in \binom{V(G)}{p} \right\} = 2|E(G[X_p])|.
\]

Since \(G[X_p] \) is connected and \( G \) is \( (\frac{n-2}{2}) \)-regular we finally obtain
\[
\xi_p = |\omega(X_p)| = p\left( \frac{n-2}{2} \right) - 2|E(G[X_p])|.
\]

and the proof ends by replacing \( |E(G[X_p])| \) with the value given by Proposition 2.2. \( \square \)

From both Lemma 2.2 and Theorem 2.4 we get the following theorem, which constitutes the main result of this work.

**Theorem 2.5.** Let \( n \geq 5 \) be an integer, \( G = K(n, 2) \), and \( p \) be any integer such that \( 1 \leq p \leq \frac{|V(G)|}{2} \). Then, the following statements hold:

(i) \( \lambda_p = \xi_p+1 = \xi_p-1 < \xi_p \) when \( n \equiv 1 \mod 4 \) and \( p = \lfloor |V(G)|/2 \rfloor - 1 \).

(ii) \( \lambda_p = \xi_p \) but \( G \) is not super-\( \lambda_p \) in the following cases: \( n = 6 \) and \( p = 5 \); \( n \equiv 1 \mod 4 \) and \( p = \lfloor |V(G)|/2 \rfloor - 2 \); \( n \equiv 3 \mod 4 \) and \( p = \lfloor |V(G)|/2 \rfloor - 1 \).

(iii) \( \lambda_p = \xi_p \) and \( G \) is super-\( \lambda_p \) for all values of \( n \), \( p \) not considered in (i), (ii).

**Proof.** By Lemma 2.2 (ii), when \( p = \lfloor |V(G)|/2 \rfloor \) it turns out that \( \lambda_p = \xi_p \) and \( G \) is super-\( \lambda_p \), so the statement holds for this value of \( p \). Suppose then \( 1 \leq p \leq \lfloor |V(G)|/2 \rfloor - 1 \) from now on. By Corollary 2.1, \( G \) is \( \lambda_p \)-connected and \( \lambda_p \leq \xi_p \).

Let us consider \( n = 5, 6, 7 \), for which we get all possible values of \( \xi_p \) from Theorem 2.4. When \( n = 5 \equiv 1 \mod 4 \) and \( 1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 5 \):

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<th>3</th>
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<tbody>
<tr>
<td>( \xi_p )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
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From Lemma 2.2 (i) we get \( \lambda_1 = \xi_1, \lambda_2 = \xi_2, \lambda_3 = \xi_3 \text{ and } \lambda_4 = \xi_4 = \xi_4 - 1 < \xi_4 \); and by Lemma 2.2 (iii, ii), \( G \) is super-\( \lambda_p \) only when \( p = 1, 2 \). Hence the result holds. For \( n = 6 \) and \( 1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 7 \):

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</tr>
</thead>
<tbody>
<tr>
<td>( \xi_p )</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>16</td>
<td>18</td>
<td>18</td>
<td>20</td>
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Therefore, again from Lemma 2.2 (iii) it turns out that \( \lambda_p = \xi_p \) for all \( 1 \leq p \leq 6 \), and \( G \) is super-\( \lambda_p \) for all those values of \( p \) except for \( p = 5 \). And when \( n = 7 \equiv 3 \mod 4 \) and \( 1 \leq p \leq \lfloor |V(G)|/2 \rfloor = 10 \) we obtain

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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_p )</td>
<td>10</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>40</td>
<td>42</td>
<td>46</td>
<td>48</td>
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</tr>
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</table>

Then, \( \lambda_p = \xi_p \) for all \( 1 \leq p \leq 9 \), and \( G \) is super-\( \lambda_p \) for all \( 1 \leq p \leq 8 \).

So the statement holds for \( n = 5, 6, 7 \). Take \( n \geq 8 \) from now on, and let us next study the sign of \( \xi_{p+1} - \xi_p \) for all \( 1 \leq p \leq \lfloor |V(G)|/2 \rfloor - 1 \).

Let us write \( p = c_2^2 + r \), where \( c = \left\lfloor \frac{2p}{n} \right\rfloor \) and \( \left\{ \begin{array}{ll} 0 \leq r < \frac{n}{2} - 1, & \text{if } 0 \leq c < \frac{n-4}{2}; \\ 0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1, & \text{if } c = \frac{n-4}{2}. \end{array} \right. \)

Suppose first that \( \left\lfloor \frac{2(p+1)}{n} \right\rfloor = \left\lfloor \frac{2p}{n} \right\rfloor = c. \) Hence from Theorem 2.4 we obtain:
\[
\xi_{p+1} - \xi_p = \binom{n-2}{2} - c(n-4) - 2r. \tag{6}
\]

Observe that \( \left\lfloor \frac{2(p+1)}{n} \right\rfloor = \left\lfloor \frac{2p}{n} \right\rfloor \) implies \( r \leq \frac{n}{2} - 2 \) when \( c < \frac{n-4}{2} \). Then, for all \( c \leq \frac{n-4}{2} \) it follows easily from (6) that \( \xi_{p+1} - \xi_p > 0 \).

Suppose next that \( \left\lfloor \frac{2(p+1)}{n} \right\rfloor = c + 1 > c = \left\lfloor \frac{2p}{n} \right\rfloor \), then \( c \leq \frac{n-4}{2} \) and \( r = \frac{n}{2} - 1 \). Theorem 2.4 yields in this case:
\[
\xi_{p+1} - \xi_p = \binom{n-2}{2} - (n-4)c - (n-2) \geq \frac{n-6}{2} > 0.
\]

Having obtained \( \xi_{p+1} - \xi_p > 0 \) for all \( p \) when \( n \geq 8 \) is even, we get
\[
\xi_1 < \cdots < \xi_{\lfloor |V(G)|/2 \rfloor - 1} < \xi_{\lfloor |V(G)|/2 \rfloor}.
\]

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Then Lemma 2.2 (iii.2) allows us to assure that $\lambda_p = \xi_p$ and $G$ is super-$\lambda_p$ for all $p$, and we are done for the case that $n$ is even.

$n$ odd:

We write $p = c\frac{n-1}{2} + r$, with $c = \left\lfloor \frac{2p}{n-1} \right\rfloor$, with $0 \leq r \leq \frac{n-1}{2} - 1$, $0 \leq c \leq \frac{n-3}{4}$; $0 \leq c \leq \frac{n-3}{4}$.

In this case, it is more convenient to use expression (3) for obtaining $\xi_p$, instead of applying Theorem 2.4 directly. That is, from $\xi_p = p\left(\frac{n^2}{2} - 2|E(G[X_p])|\right)$ and expression (3) we write:

$$
\xi_p = p\left(\frac{n - 2}{2}\right) - p^2 - c(n + 1)n + p(1 + 4c) + 2c - 2\min\{2r, c\}.
$$

Suppose first that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = \left\lfloor \frac{2p}{n-1} \right\rfloor - c$. Hence from (7) it follows that:

$$
\xi_{p+1} - \xi_p = \left(\frac{n - 2}{2}\right) - c(n - 5) - 2r + 2\min\{2r, c\} - 2\min\{2r + 2, c\}.
$$

Observe that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = \left\lfloor \frac{2p}{n-1} \right\rfloor$ implies $r \leq \frac{n-1}{2} - 2$ when $c \leq \frac{n-3}{4}$. Then, for $c < \frac{n-1}{2}$ it follows easily from (8) that $\xi_{p+1} - \xi_p > 0$. Except for the following cases (for which $2\min\{2r, c\} - 2\min\{2r + 2, c\} = -4$):

\[\xi_{p+1} - \xi_p = -1, \quad \text{when } n \equiv 1 \pmod{4}, \ c = \frac{n-1}{2}, \ \text{and } r = \frac{n-1}{2}; \]

\[\xi_{p+1} - \xi_p = 0, \quad \text{when } n \equiv 3 \pmod{4}, \ c = \frac{n-1}{2}, \ \text{and } r = \frac{n-3}{4}.\]

Indeed, for the former case we have:

$$
\xi_{p+1} - \xi_p = \left(\frac{n - 2}{2}\right) - \frac{(n - 1)(n - 5)}{2} - \frac{(n - 1)}{2} - 2 \leq -1 < 0;
$$

and for the latter,

$$
\xi_{p+1} - \xi_p = \left(\frac{n - 2}{2}\right) - \frac{(n - 1)(n - 5)}{2} - \frac{(n - 3)}{2} - 2 = 0.
$$

Suppose next that $\left\lfloor \frac{2(p+1)}{n-1} \right\rfloor = c + 1 > c = \left\lfloor \frac{2p}{n-1} \right\rfloor$, then $c = \frac{n-3}{4}$ and $r = \frac{n+1}{2} - 1$. In this case expression (7) yields:

$$
\xi_{p+1} - \xi_p = \left(\frac{n - 2}{2}\right) - (c + 1)(n - 3) - 2 \geq \frac{n - 7}{2} > 0
$$

because $n \geq 9$ in the odd case.

Let us gather together all these deductions for $n \geq 9$ odd. Firstly, when $n \equiv 1 \pmod{4}$ we have obtained $\xi_{p+1} - \xi_p > 0$ for all $p$ except for the case $p = \left\lfloor |V(G)|/2 \right\rfloor - 1$, where $\xi_{p+1} - \xi_p = \xi_{\left\lfloor |V(G)|/2 \right\rfloor} - \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1} = -1$. As it is easy to compute from (3), $\xi_{\left\lfloor |V(G)|/2 \right\rfloor} - \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1} = 0$. That is,

$$
\xi_1 < \cdots < \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 2} < \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1} = \xi_{\left\lfloor |V(G)|/2 \right\rfloor} = \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 2}.
$$

Then from Lemma 2.2 (i) we have that $\lambda_p = \xi_p$ for all $p \neq \left\lfloor |V(G)|/2 \right\rfloor - 1$, and among these values of $p$ graph $G$ is super-$\lambda_p$ for all $p \neq \left\lfloor |V(G)|/2 \right\rfloor - 2$, so the statement holds. Finally, when $n \equiv 3 \pmod{4}$ we have obtained $\xi_{p+1} - \xi_p > 0$ for all $p$ except for the case $p = \left\lfloor |V(G)|/2 \right\rfloor - 1$, where $\xi_{p+1} - \xi_p = \xi_{\left\lfloor |V(G)|/2 \right\rfloor} - \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1} = 0$. Therefore,

$$
\xi_1 < \cdots < \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1} < \xi_{\left\lfloor |V(G)|/2 \right\rfloor} = \xi_{\left\lfloor |V(G)|/2 \right\rfloor - 1}.
$$

and Lemma 2.2 states that $\lambda_p = \xi_p$ holds for all $p$, $G$ being super-$\lambda_p$ for all those values of $p$ except for $p = \left\lfloor |V(G)|/2 \right\rfloor - 1$.

The proof is so complete. □

References


