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AN ELASTO-PLASTIC DAMAGE MODEL FOR CONCRETE

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Abstract. Constitutive modeling of concrete using continuum damage mechanics and plasticity theory is presented in this work. In order to derive the constitutive equations the strain equivalence hypothesis is adopted. Menetrey-William type yield function (in the effective stress space) with multiple hardening functions is used to define plastic loading of the material. Non-associated plastic flow rule is used to control inelastic dilatancy. Drucker-Prager type function is chosen as a plastic potential. Damage is assumed to be isotropic and two damage variables are used to represent tensile and compressive damage independently. Damage parameter is driven based on the plastic strain. Fully implicit integration scheme is employed and the consistent elastic-plastic-damage tangent operator is also derived. The overall performance of the proposed model is verified by comparing the model predictions to various numerical simulations, cyclic uniaxial tensile and compressive tests, monotonic biaxial compression test and reinforced concrete beam test.

1 INTRODUCTION

Concrete is widely used material due to its ability to be cast on site and to be formed in different shapes. Therefore its mechanical behaviour under different loading conditions must be better understood and it can be simulated by numerical methods. Several concrete constitutive equations have been developed based on nonlinear elasticity, plasticity theory, continuum damage mechanics (CDM), fracture mechanics and microplane model.

Concrete is a highly nonlinear material. Most prominent characteristic of concrete is its low tensile strength compared with its compressive strength. This causes micro-cracking of concrete even under very low loads which reduces the stiffness of concrete element. This leads to use of damage mechanics to model constitutive equations of concrete. On the other hand concrete exhibits some irreversible strain under compressive loads which can be simulated by using plasticity theory. Therefore accurate modeling of concrete behaviour needs to use plasticity theory and damage mechanics simultaneously.

Plasticity theory has been used successfully in modeling behavior of concrete by many researchers such as Grassl 2002, Papanikolaou and Kappos 2007, Kang and William, Imran and Pantazopoulou 2001, Etse and William 1994, Menetrey and William 1995. The main feature of these models is a pressure sensitive yield surface with parabolic meridians, non-associated flow and nonlinear hardening rule. However these models cannot take into account the degradation of material stiffness due to micro-cracking. On the other hand some researchers used continuum damage mechanics alone to simulate concrete behaviour Mazar and Cabot 1989, Simo and Ju 1987, Ortiz and Popov 1982, Tao and Phillips 2005.

Since both micro-cracking and irreversible deformations are two main distinct aspects of nonlinear response of concrete, several combined plasticity and CDM models have been developed in recent years. Combinations of plasticity and CDM are usually based on isotropic hardening plasticity with isotropic damage model. However some researchers use anisotropic damage model such as Çiçekli and Voyiadjis (2007), Carol et al (2001), Abu Al-Rub and Voyiadjis (2009). Most popular combination type is stress-based plasticity in effective space with damage because coupled plastic-damage models formulated in the effective space are more stable and attractive [2].

In this study concrete constitutive model is developed based on scalar damage with plasticity in effective stress space. Damage is modeled as the functions of plastic strain following Lee and Fenves (1998).

2 PLASTICITY FORMULATION

Three parameter Menetrey-William type yield function (in the effective stress space) with multiple hardening parameters is chosen to define plastic loading of the material. This criterion has been successfully used in simulating the concrete behaviour under uniaxial, biaxial and multiaxial loadings by many researchers [7,16]. It is smooth and convex, except the point where parabolic meridians intersect the hydrostatic axis. The yield function is formulated as follows:

$$f = 1.5\frac{\rho^2}{k f_c} + m\frac{\rho\, r(\theta)}{\sqrt{6}} + m\frac{\xi}{\sqrt{3}} - k f_c = 0 \tag{1}$$

in terms of Haigh-Westergaard coordinates in the effective stress space. Here ξ, ρ and θ is hydrostatic length, deviatoric length and Lode angle respectively and they are the functions of stress invariants according to following equations:

$$\xi = \frac{I_1}{\sqrt{3}} \tag{2}$$

$$\rho = \sqrt{2J_2}$$

$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right)$$

where I_1 is first invariant of stress tensor and J_2, J_3 are second and third invariant of deviatoric stress tensor respectively.

Given yield surface possess parabolic meridians and triangular sections at low confinement to almost circular sections at high confinement on deviatoric plane shown in Figure 1. Deviatoric sections shape is controlled by the function:

$$r(\theta,e) = \frac{4\left(1-e^2\right)\cos^2\theta + \left(2e-1\right)^2}{2\left(1-e\right)^2\cos\theta + \left(2e-1\right)\left[4\left(1-e^2\right)\cos^2\theta + 5e^2 - 4e\right]^{1/2}} \tag{3}$$

which is proposed by Willam and Warnke (1974). Here e is eccentricity parameter and it must be calibrated according to the uniaxial tensile and compressive strength and biaxial compressive strength.

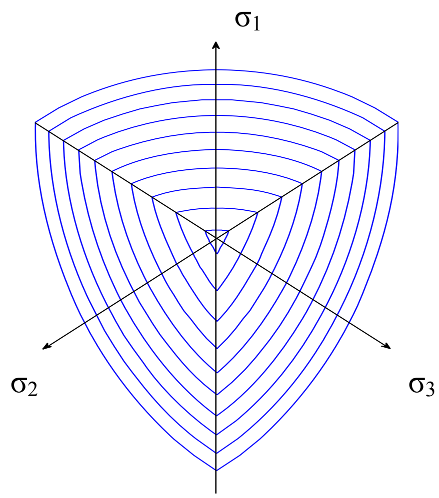

Figure 1. Deviatoric sections of yield functions

In yield surface equation f_c is the uniaxial compressive strength and m is friction parameter respectively. Friction parameter formulated in terms of compressive and tensile strength as following equations:

$$m = 3\frac{(k f_c)^2 - (c f_t)^2}{k c f_c f_t}\frac{e}{e+1} \tag{4}$$

where k and c is compressive and tensile hardening-softening parameter respectively.

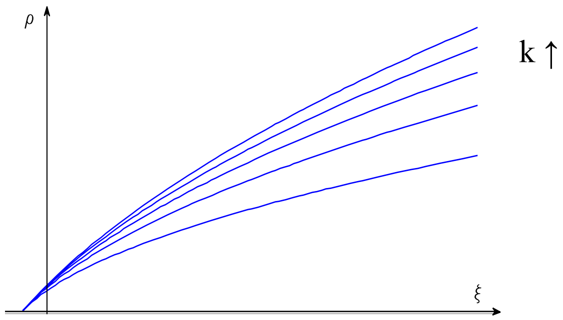

Figure 2. Evolution of Loading Surfaces

2.1 Flow Rule

In this study non-associated flow rule is adopted due to control excessive dilatancy. Non-associated means yield function and plastic potential are different each other and, therefore, plastic flow direction is not normal to the yield surface. This is important for realistic modelling of cohesive frictional material such as concrete and rock. The plastic strain rate obtained as:

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial g}{\partial \bar{\sigma}} \tag{5}$$

Where $\dot{\lambda}$ is plastic multiplier which can be obtained from plastic consistency condition and g is plastic potential. Drucker-Prager type potential function is chosen as follows

$$g = \alpha I_1 + \sqrt{3J_2} \tag{6}$$

such that;

$$\frac{\partial g}{\partial \bar{\sigma}} = \alpha\,\delta_{ij} + \frac{3}{2}\frac{s_{ij}}{\sqrt{3J_2}} \tag{7}$$

Here α is dilatation parameter and it controls inelastic volume expansion. Plastic consistency condition is obtained by taking the time derivative of yield function and satisfying Kuhn-Tucker conditions:

$$f < 0 \quad \rightarrow \quad \dot{\lambda} = 0 \quad (Elastic)$$

$$f = 0 \; and \; \dot{f} < 0 \quad \rightarrow \quad \dot{\lambda} = 0 \quad (unloading) \tag{8}$$

$$f = 0 \; and \; \dot{f} = 0 \quad \rightarrow \quad \dot{\lambda} > 0 \quad (plasticity)$$

$$f \leq 0, \dot{\lambda} \geq 0 \quad \rightarrow \quad \dot{\lambda} \; f = 0 \quad Kuhn \, Tuc\mathrm{ker}$$

2.2 Hardening and Softening Rule

The nonlinear behaviour of concrete in the pre-peak and post-peak region is described by isotropic hardening/softening rule. Hardening/softening and damage states are defined independently by two variables, κ_c and κ_t due to different behaviour under compressive and tensile loading. For uniaxial loading κ_c and κ_t is defined as axial plastic strain under compression and tension respectively [12].

$$\kappa = \begin{bmatrix} \kappa_t \\ \kappa_c \end{bmatrix} \tag{9}$$

$$\dot{\kappa}_t = \dot{\varepsilon}_1^p \qquad \kappa_t = \int_0^t \dot{\kappa}_t \, dt$$

$$\dot{\kappa}_c = -\dot{\varepsilon}_3^p \qquad \kappa_c = \int_0^t \dot{\kappa}_c \, dt$$

Under multiaxial loading the evolution of hardening variables is given as follows (Lee and Fenves 1998):

$$\Delta\kappa = h\left(\bar{\sigma}, \varepsilon^p\right) \Delta\hat{\varepsilon}^p \tag{10}$$

where $\hat{\varepsilon}^p$ represents eigenvalues of strain tensor

$$\Delta\hat{\varepsilon}^p = \begin{bmatrix} \Delta\hat{\varepsilon}_1^p & \Delta\hat{\varepsilon}_2^p & \Delta\hat{\varepsilon}_3^p \end{bmatrix}^T \tag{11}$$

and

$$h\left(\hat{\bar{\sigma}}, \tilde{\varepsilon}^p\right) = \begin{bmatrix} r\left(\hat{\bar{\sigma}}\right) & 0 & 0 \\ 0 & 0 & -\left(1 - r\left(\hat{\bar{\sigma}}\right)\right) \end{bmatrix} \tag{12}$$

The scalar $0 \leq r\left(\hat{\bar{\sigma}}\right) \leq 1$ is a weight factor and defined as

$$r\left(\hat{\bar{\sigma}}\right)=\frac{\sum_{i=1}^{3}\left\langle\hat{\bar{\sigma}}_i\right\rangle}{\sum_{i=1}^{3}\left|\hat{\bar{\sigma}}_i\right|} \tag{13}$$

Where $\langle x\rangle=\left(|x|+x\right)/2$ denotes the Macaulay bracket function and $\hat{\bar{\sigma}}$ is effective principal stress.

Under Tension

Concrete assumed linear elastic up to tensile strength. After that concrete exhibits strain softening. Descending part of tensile stress-strain curve is formulated by stress-crack opening relations given by Hordjik(1991).

$$\sigma_t=f_t\left[\left(1+\left(c_1\frac{w}{w_c}\right)^3\right)\exp\left(-c_2\frac{w}{w_c}\right)-\frac{w}{w_c}\left(1+c_1^3\right)\exp\left(-c_2\right)\right] \tag{14}$$

Where f_t tensile strength, w crack opening, w_c critical crack opening and c_1, c_2 are material constants. Hordjik gives material constants values as c_1=3, c_2=6.93. To prevent mesh dependent result Hordjik stress-crack opening equation formulated in terms of inelastic strain and stress by incorporating fracture energy and characteristic length as follows:

$$G_f=\int\sigma_t\cdot dw \tag{15}$$

$$w=l_c\cdot\varepsilon^{cr}$$

$$G_f=l_c\int_0^{\kappa_{tu}}\sigma_t\,d\varepsilon^{cr}$$

$$\kappa_{tu}=5.14\frac{G_f}{l_c\,f_t}$$

Where G_f, l_c, σ_t are crushing energy, characteristic length and stress in the direction of crack normal, respectively.

Under Compression:

Strength parameter k, which controls the evolution of the yield surface under compression, is defined in terms of hardening variable κ_c as follows [5,9]:

$$k = k_0 + \frac{\left(2\sqrt{\kappa_c\,\kappa_{c\max}} - \kappa_c\right)}{\kappa_{c\max}}\left(1-k_0\right) \qquad 0 \le \kappa_c \le \kappa_{c\max}$$
$$k = 1 - \left(\frac{\kappa_c - \kappa_{c\max}}{\kappa_{cu} - \kappa_{c\max}}\right)^2 \qquad \kappa_c > \kappa_{c\max} \tag{16}$$

Where $k_0 = f_{c0}/f_c$ and $\kappa_{c\max}$ and κ_{cu} is equivalent plastic strain at peak stress and ultimate equivalent plastic strain respectively. Second part gives compressive softening which also causes mesh dependent results. To prevent mesh dependency same procedure is followed as tension softening.

$$\kappa_{cu} = \kappa_{c\max} + 1.5\frac{G_c}{l_c\, f_c} \tag{17}$$

Where G_c is crushing energy.

3 DAMAGE MODEL

Isotropic damage is responsible for the degradation in elastic stiffness in this work. Two damage variables, one for tensile damage ω_t and one for compressive damage ω_c, are defined independently following Lee and Fenves (1998). It is assumed that damage variables are increasing functions of the equivalent plastic strains and they can take values ranging from zero, for the undamaged material, to one, for the fully damaged material.

It is assumed that the degradation takes the following exponential form [12,13]:

$$\omega_t = 1 - e^{-a_t\,\kappa}$$
$$\omega_c = 1 - e^{-a_c\,\kappa} \tag{18}$$

Where a_t and a_c are material constant for uniaxial tension and compression respectively and they must be calibrated from uniaxial tests. When uniaxial tensile and compressive damage variables are obtained then total damage variables calculated as the following form [1,12]:

$$1 - \omega = 1 - \left(1 - s_c\,\omega_t\right)\left(1 - s_t\,\omega_c\right) \tag{19}$$

Here s_t and s_c are used for to take into account closing and reopening of cracks.

4 NUMERICAL INTEGRATION

The implemented integration scheme is divided into two sequential steps, corresponding to the plastic and damage parts. In the plastic part, the plastic strain ε_p and the effective stress $\bar{\sigma}$ at the end of the step are determined by using the implicit backward-Euler return-mapping scheme. In the damage part, damage variable ω and nominal stress σ at the end of the step are determined.

Implementation of return-mapping algorithm requires integrating the rate form of constitutive relations in finite time step $\Delta t = t^{n+1} - t^{n}$ to obtain the stress changes $\Delta\sigma$ and the state variables corresponding to a total change of displacement $\Delta\varepsilon$ within the load increment.

$$\bar{\sigma}_{n+1} = \bar{\sigma}^{t} - D\,\Delta\varepsilon^{p}_{n+1} = \bar{\sigma}^{t} - \Delta\lambda_{n+1} D \frac{\partial g}{\partial \bar{\sigma}_{n+1}} \tag{20}$$

Where $\bar{\sigma}^{tr}$ is the effective trial stress which is evaluated from given strain increment assuming that plastic strain increment is zero. If trial stress is not outside the yield stress, $f \leq 0$, then step is elastic and plastic strain increment is zero. On the other hand if the trial stress is outside the yield surface then $\bar{\sigma}_{n+1}, \varepsilon^{p}_{n+1}, \kappa^{p}_{n+1}$ are determined according to calculated $\Delta\lambda$.

At the end of the loading step following four equations must be satisfied:

$$\bar{\sigma}_{n+1} = D\left(\varepsilon_{n+1} - \varepsilon^{p}_{n+1}\right) \tag{21}$$

$$\varepsilon^{p}_{n+1} = \varepsilon^{p}_{n} + \Delta\varepsilon^{p}_{n+1}$$

$$\kappa_{n+1} = \kappa_{n} + \Delta\kappa_{n+1}$$

$$f \leq 0$$

If one defines the residuals for the equations (4.17), (4.18) and (4.19) as follows:

$$R_{n+1} = \begin{Bmatrix} R_{\varepsilon,n+1} \\ R_{\kappa,n+1} \\ R_{f,n+1} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{p}_{n+1} - \varepsilon^{p}_{n} - \Delta\lambda_{n+1} \dfrac{\partial g}{\partial \bar{\sigma}_{n+1}} \\ \kappa_{n+1} - \kappa_{n} - \Delta\lambda_{n+1} H_{n+1} \\ f_{n+1} \end{Bmatrix} \tag{22}$$

and linearizates these according to Taylor expansion following equations are obtained:

$$\begin{aligned} & R_{\varepsilon,n+1} + \Delta\varepsilon^{p}_{n+1} - \Delta\lambda_{n+1}\,\Delta b_{n+1} - \delta\lambda_{n+1}\,b_{n+1} = 0 \\ & R_{\kappa,n+1} + \Delta\kappa_{n+1} - \Delta\lambda_{n+1}\,\Delta H_{n+1} - \delta\lambda_{n+1}\,H_{n+1} = 0 \\ & f + \frac{\partial f_{n+1}}{\partial \bar{\sigma}_{n+1}} \Delta\bar{\sigma}_{n+1} + \frac{\partial f_{n+1}}{\partial \kappa_{n+1}} \Delta\kappa_{n+1} = 0 \end{aligned} \tag{23}$$

After few manipulations, Δλ can be determined as follows:

$$\delta\lambda_{n+1} = \frac{f - \begin{bmatrix} \dfrac{\partial f_{n+1}}{\partial \bar{\sigma}_{n+1}} & \dfrac{\partial f_{n+1}}{\partial \kappa} \end{bmatrix}^T [A] \begin{Bmatrix} R_{\varepsilon,n+1} \\ R_{\kappa,n+1} \end{Bmatrix}}{\begin{bmatrix} \dfrac{\partial f_{n+1}}{\partial \bar{\sigma}_{n+1}} & \dfrac{\partial f_{n+1}}{\partial \kappa} \end{bmatrix}^T [A] \begin{Bmatrix} b_{n+1} \\ H_{n+1} \end{Bmatrix}} \tag{24}$$

Where

$$[A]^{-1} = \begin{bmatrix} I + D\,\Delta\lambda_{n+1} \dfrac{\partial b_{n+1}}{\partial \bar{\sigma}_{n+1}} & -\Delta\lambda_{n+1} \dfrac{\partial b_{n+1}}{\partial \kappa_{n+1}} \\ D\,\Delta\lambda_{n+1} \dfrac{\partial H_{n+1}}{\partial \bar{\sigma}_{n+1}} & I - \Delta\lambda_{n+1} \dfrac{\partial H_{n+1}}{\partial \kappa_{n+1}} \end{bmatrix} \tag{25}$$

and $b = \dfrac{\partial g}{\partial \bar{\sigma}}$ is the gradient of the plastic potential.

Once the effective stress $\bar{\sigma}_{n+1}$ is computed in the elastic predictor/plastic corrector steps, the damage parameter is then calculated from equation (3.10):

$$1 - \omega = 1 - (1 - s_c\,\omega_t)(1 - s_t\,\omega_c) \tag{26}$$

and the stress is updated as:

$$\sigma_{n+1} = (1 - \omega)\,\bar{\sigma}_{n+1} \tag{27}$$

5 NUMERICAL EXAMPLES

The present concrete model is implemented in Abaqus 6.8 by user element subroutine Umat. Its performance is denoted by comparing with uniaxial tensile and compressive, biaxial compressive and cyclic experimental test from literature.

In Figure 3 the cyclic uniaxial tensile test of Taylor (1992) and the cyclic compressive test of Karson and Jirsa (1969) are evaluated numerically to demonstrate the capability of the proposed model under cyclic load conditions. The following properties are adopted: for Taylor's simulation, E_c=3.1×10^4 MPa, f_t= 3.5 MPa, G_f=100 N/m; and for Karsan and Jirsa's one, E = 3.17×10^4 MPa, f_t= 3.0 MPa and f_{c0}=10.2 MPa. As shown in Figure 3, the experimentally observed strain softening, stiffness degrading, and irreversible strains, are agree well with the proposed results under both tension and compression.

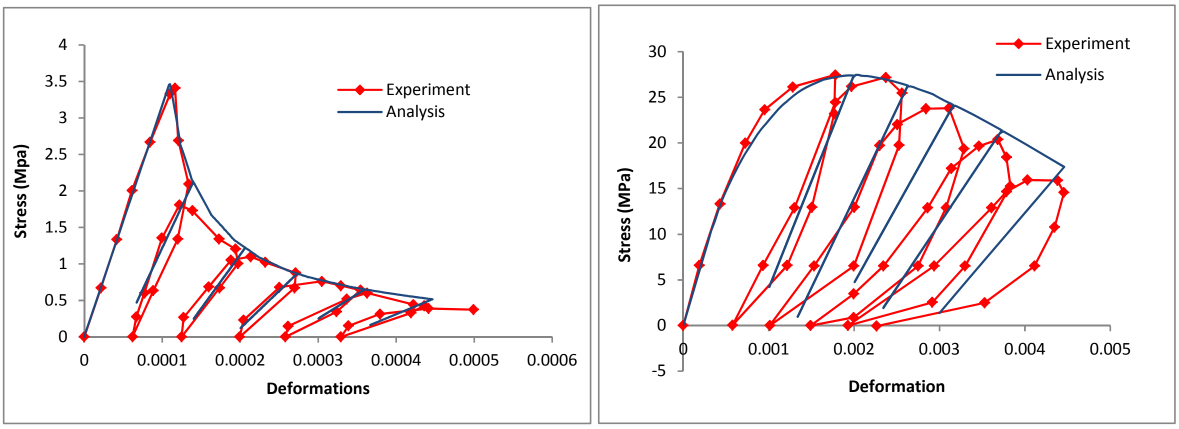

Figure 3 Uniaxial Cyclic Tensile (Taylor, 1992) and Compressive test (Karsan and Jirsa, 1969)

The proposed model is also validated with the results of biaxial compression test reported in Kupfer et al. (1969). The material properties adopted in the analysis are: E_c=3.1×10^4 MPa f_t=3.0 Mpa and G_f=75 N/m. For specimens under load conditions σ_2/σ_1=1/0, σ_2/σ_1=1/1 and σ_2/σ_1=1/0.5, the predicted stress–strain curves given in Figure 4a–c agree well with the experimental results.

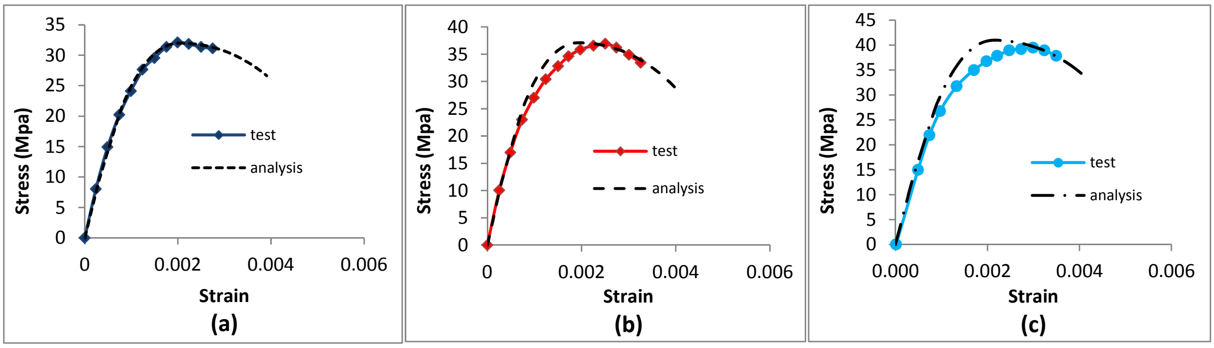

Figure 4 Monotonic Biaxial Compressive test (Kupfer et al, 1969)

Finally Bresler–Scordelis beam is used to validate the model performance for RC element. It is simply supported beam with 3.7m long span and subjected to concentrated load at midspan. The longitudinal reinforcement consists of four steel bars with total area of 2580 mm^2. The concrete has a compressive strength of 24.5 MPa and elastic modulus of 21300 MPa. The elastic modulus and yield stress of steel bars is 191.4 GPa and 444 Mpa respectively. In the finite element modeling, 4-noded rectangular plane stress element is used for concrete and truss elements for steel bars. Perfect bond between concrete and reinforcement is assumed. Load-displacement curve given in Figure 5 shows that analysis results is agree well with test results.

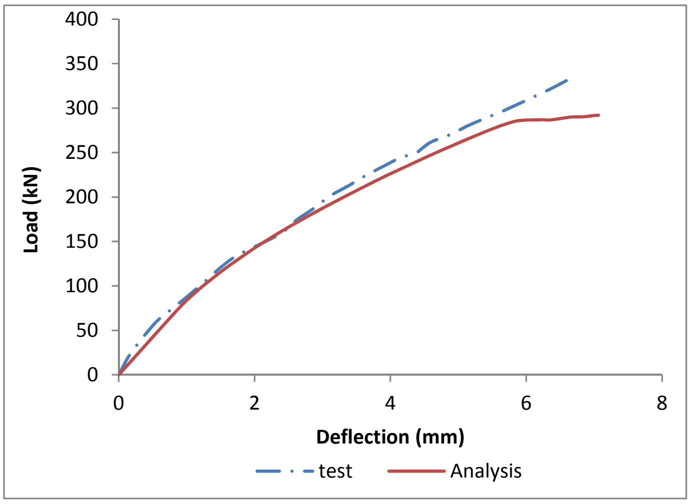

Figure 5 Reinforced Concrete Simply Supported Beam (Bresler-Scordelis,1963)

6 CONCLUSIONS

A constitutive model for concrete using continuum damage mechanics and plasticity theory is presented. The plastic part formulated in effective stress space and isotropic damage is formulated in terms of plastic strain. Multiple hardening and damage parameter are used due to different behaviour under tensile and compressive loading. The model predictions are found to be in good agreements with experimental results in uniaxial and biaxial loadings. Localization of deformations is considered by the fracture/crushing energy approach. This model may be enhanced by taking into account lateral confinement.

REFERENCES

[1] ABAQUS standard User's Manual. Hibbitt, Karlson and Sorenson, Inc. vol. 1,2 and 3, Version 6.4, USA, 2004.

[2] Abu Al-Rub, R.K. and Sun-Myung, K., Computational applications of a coupled plasticity-damage constitutive model for simulating plain concrete fracture, *Engineering Fracture Mechanics* (2010), **77**:1577-1603.

[3] Bresler, B. and Scordelis, A.C. Shear Strength of Reinforced Concrete Beams, *Journal of ACI* (1963), **60**:51-72.

[4] Etse, G. and William, K., Fracture energy formulation for inelastic behaviour of plain concrete, *Journal of Engineering Mechanics*, *ASCE* (1994), **120**:1983-2011.

[5] Feenstra, P.H. and de Borst, R. A composite plasticity model for concrete, *Int. J. Solids Struct.* (1996), 33:707-730.

[6] Gopalaratnam, V.S. and Shah, S.P. Softening response of plain concrete in direct tension, *ACI Journal* (1985), **85**:310-323.

[7] Grassl, P., Lundgren, K., Kent, G. Concrete in compression: a plasticity theory with a novel hardening law, *Int. J. Solids and Structures* (2002), **39**:5205-5223.

[8] Grassl, P. and Jirasek, M. Damage-plastic model for concrete failure, *Int. J. Solid and Structures* (2006), **43**:7166-7196.

[9] Imran, I. and Pantazopoulou, and S.J. Plasticity Model for Concrete under Triaxial Compression, *ASCE J. Engng. Mech* (2001), **127**: 281-290.

[10] Kang, H.D. *Triaxial constitutive model for plain and reinforced concrete behavior*, Phd Thesis, University of Colorado at Boulder, 1997.

[11] Kupfer, H.B., Hildorf, H.K. and Rusch, H. Behavior of concrete under biaxial stresses, ACI Journal (1969), **66**:656-666.

[12] Lee, J. and Fenves, G.L. Plastic-damage model for cycling loading of concrete structures, *Journal of Engineering Mechanics* (1998), **124**:892-900.

[13] Lubliner, J., Oliver, J., Oller, S. and Onate, E. A plastic-damage model for concrete, *Int. J. Solids Struct*. (1989), **25**:299-326.

[14] Mazars, J. and Cabot, G.P. Continuum damage theory-application to concrete, *Journal of Engineering Mechanics* (1989), **115**:345-365.

[15] Nguyen, G. D. and Houlsby, G.T. A coupled damage-plasticity model for concrete based on thermodynamic principles: Part I: model formulation and parameter identification, Int. J. Numer. Anal. Meth. Geomec. (2008), **32**:353-389.

[16] Papanikolaou, V.K. and Kappos, A.J. Confinement-sensitive plasticity constitutive model for concrete in triaxial compression, *Int. J. Solids and Structures* (2007), **44**:7021-7048.

[17] Salari, M.R., Saeb, S., Willam, K.J., Patchet, S.J. and Carrasco R.C. A coupled elastoplastic damage model for geomaterials, *Comput. Methods Appl. Mech. Engrg* (2004), **193**:2625-2643.

[18] Simo, J.C., and Ju, J.W. Strain and stress based continuum damage models, Part I: Formulation*, Int. J. Solids and Structures* (1987), 23: 821—840.

[19] Tao, X. and Phillips, D. V. A simplified isotropic damage model for concrete under bi-axial stress states, *Cement and Concrete Composites* (2005), **27**:716-726.

[20] Voyiadjis, G.Z. and Taqieddin, Z.N. Elastic Plastic and Damage Model for Concrete Materials: Part I – Theoretical Formulation, *Int. J. Struct. Chan. in Solids* (2009) **1**: 31-59.

[21] Wu, J.Y., Li, J., Faria, R. An energy release rate-based plastic-damage model for concrete, *Int. J. Solids and Structures* (2007), **43**:583-612.