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THE TIKHONOV REGULARIZATION METHOD IN ELASTOPLASTICITY

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Abstract. The numeric simulation of the mechanical behaviour of industrial materials is widely used in the companies for viability verification, improvement and optimization of designs. The eslastoplastic models have been used for forecast of the mechanical behaviour of materials of the most several natures (see [1]). The numerical analysis from this models come across ill-conditioning matrix problems, as for the case to finite or infinitesimal deformations. A complete investigation of the non linear behaviour of structures it follows from the equilibrium path of the body, in which come the singular (limit) points and/or bifurcation points. Several techniques to solve the numerical problems associated to these points have been disposed in the specialized literature, as for instance the call Load controlled Newton-Raphson method and displacement controlled techniques. Although most of these methods fail (due to problems convergence for ill-conditioning) in the neighbour of the limit points, mainly in the structures analysis that possess a snap-through or snap-back equilibrium path shape (see [2]). This work presents the main ideas formalities of Tikhonov Regularization Method (for example see [12]) applied to dynamic elastoplasticity problems (J2 model with damage and isotropic-kinetic hardening) for the treatment of these limit points, besides some mathematical rigour associated to the formulation (well-posed/existence and uniqueness) of the dynamic elastoplasticity problem. The numeric problems of this approach are discussed and some strategies are suggested to solve these misfortunes satisfactorily. The numerical technique for the physical problem is by classical Gelerkin method.

1 INTRODUCTION

Elastoplastic models have been widely used to forecast the behaviour of rate independent (in deformation sense) materials (see [1]). The numerical solution of these models involves handling of ill-conditioned matrices, for finite or infinitesimal deformations (see [**?**]). Such instabilities are due to the tangent operator being close to an identically null forth order tensor operator at the neibourhood of critical or limit points.

A complete investigation of non linear structural behaviour involves following the bodies equilibrium path through singular (limit) points and/or bifurcation points. In order to solve the numerical problems associated to these points several techniques have been considered in the specialized literature, for instance the so-called load controled Newton-Raphson method and displacement controled techniques. Due to ill-conditioning convergence problems, most of these methods fail, specially in the case of structures which present (λ-load factor,u-displacement) snap-through or snap-back equilibrium paths ([2]), as shown in figure (Fig.1).

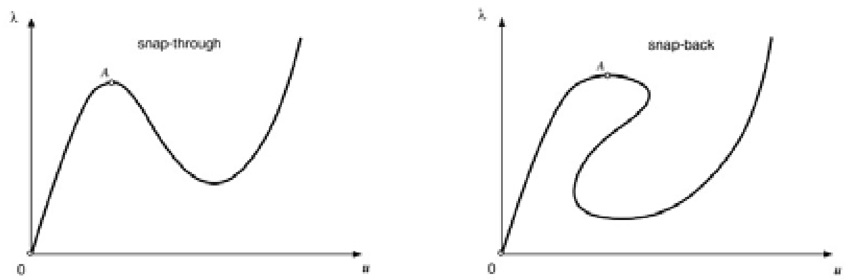

Figure 1: Snap-through and snap-back behaviour

Aiming at transposing these difficulties, this study proposes use of the L-curve Tikhonov regularization method ([14], [15], [6] and [12]). One of the objectives of the study is to investigate the potential of this approach in the solution of elastoplastic problems of infinitesimal strain. An overview of elastoplastic contitutive model is shown in section 2. Details about incremental approach are presented in section 3. In sections 4 and 5, it is presented the L-curve Thikhonov regularization method and main properties are shown. In section 6, a numerical problem case are presented to verify the efficacy of this proposed approch and concluding remarks are made in section 7.

2 YIELDING AND HARDENING LAWS (THE ELASTOPLASTIC CONSTITUTIVE MODEL)

A complete characterization of general elastoplastic model request definition of evolutionary laws of internal variables, i. e., variables associated to dissipative phenomena (ε^p and α_k - associated with the kinematic hardening mechanism). The first point in this analysis is determination of the plastic multiplier $\dot{\lambda}$ which is computed from consistence condition ($\mathcal{F} = 0$ and $\dot{\lambda} > 0$). Hence, from definition of α_k, we obtain

$$\dot{\lambda} = \frac{\frac{\partial \mathcal{F}}{\partial \sigma} : \mathbb{D}\dot{\varepsilon}}{\left\{ \frac{\partial \mathcal{F}}{\partial \sigma} : \mathbb{D}\mathbf{N} - \rho \frac{\partial \mathcal{F}}{\partial \alpha_k} . \left[\frac{\partial^2 \Psi^p}{\partial \beta_k^2} \right] \mathbf{H} \right\}}. \tag{1}$$

More details about constitutive Lemaitre's elastoplastic-damage simplified model with isotropic hardening can be found in [4] and [3]. In this sense, the elastoplastic constitutive model is described in following steps

Elastoplastic Constitutive Model

1. **Strain Tensor Additive Decomposition**

$$\varepsilon = \varepsilon^e + \varepsilon^p.$$

2. **Free Energy Potential Definition**

$$\Psi\left(\varepsilon^e, r, \alpha^D, D\right) = \Psi^e\left(\varepsilon^e, D\right) + \Psi^p\left(r, \alpha^D\right)$$

where α^D is the deviator part of α (backstrain tensor), r is the accumulated plastic strain, D is the isotropic damage variable.

3. **Constitutive equation for σ and thermodynamics forces β_k**

$$\sigma = \rho \frac{\partial \Psi^e}{\partial \varepsilon^e} \qquad \text{and} \qquad \beta_k = \rho \frac{\partial \Psi^p}{\partial \alpha_k}.$$

4. **Elastic-damage Coupling** $\sigma = (1 - D)\mathbb{D}\varepsilon^e$.

5. **Yield Function/Dissipation Potential(Associative Approach)**

$\mathcal{F}_p = \|\tilde{\sigma}^D - \chi^D\| - (R + \sigma_y)$ where $\tilde{\sigma}^D_{eq} = \left\{\frac{3}{2}\tilde{\sigma}^D : \tilde{\sigma}^D\right\}^{\frac{1}{2}}$;
$\tilde{\sigma}^D = \frac{1}{(1-D)}\left\{\sigma - \sigma_H I\right\}$ and $\sigma_H = \frac{1}{3}tr(\sigma)$.

6. **Hardening and Evolutionary Plastic Laws**

$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \mathcal{F}_p}{\partial \sigma}$, $\dot{r} = -\dot{\lambda}\frac{\partial \mathcal{F}_p}{\partial R}$ and $\dot{D} = \dot{\lambda}\frac{\partial \mathcal{F}_D}{\partial Y}$
where
$\mathcal{F} = \mathcal{F}_p + \mathcal{F}_D$ with $\mathcal{F}_p = \|\tilde{\sigma}^D - \chi^D\| - (R + \sigma_y)$ and $\mathcal{F}_D = \frac{Y^2}{2S(1-D)}H(p - p_d)$.

From these potentials it follows that
$\dot{\varepsilon}^p = \frac{3}{2}\frac{\dot{\lambda}}{(1-D)}\frac{\sigma^D}{\sigma^D_{eq}}$, $\dot{\chi} = \gamma(\chi_\infty \dot{\varepsilon}^p - \chi\dot{\lambda})$, $\dot{R} = b(R_\infty - R)\dot{\lambda}$ and $\dot{D} = \frac{Y}{S}\,\dot{p}\,H(p - p_d)$.
Then
$\dot{p} = \frac{\dot{\lambda}}{(1-D)}$ and $Y = \frac{\left(\tilde{\sigma}^D\right)^2}{2E}\left\{\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma^D_{eq}}\right)^2\right\}$.

7. **Consistence Condition under Plastic Yielding** $\left(\dot{\lambda} \neq 0\right)$

$$\mathcal{F}(\sigma, \alpha_k) \leq 0, \qquad \dot{\lambda} \geq 0, \qquad \mathcal{F}(\sigma, \alpha_k)\,\dot{\lambda} = 0$$

and $\dot{\lambda}\,\dot{\mathcal{F}}(\sigma, \alpha_k) = 0$.

3 INCREMENTAL FORMULATION

In this section we describe the incremental formulation of the problem between t_n and t_{n+1} instants. We consider that all state variables are known on Ω_n and equilibrium equations are imposed in Ω_{n+1}. In this way, on t_{n+1}, the weak formulation of the problem can be written as:

Problem 1. *Determine* $u_{n+1} \in Kin_o^u$ *such that*

$$F\left(u_{n+1};\hat{\mathbf{v}}\right) = 0, \qquad \forall \;\; \hat{v} \in Var_o^u, \tag{2}$$

where

$$F\left(u_{n+1};\hat{\mathbf{v}}\right) = \int_{\Omega_o} \mathbf{P}\left(u_{n+1}\right) : \nabla\hat{\mathbf{v}} d\Omega_o - \int_{\Omega_o} \rho_o \left(\bar{\mathbf{b}} - \ddot{u}_n\right) \cdot \hat{\mathbf{v}} d\Omega_o - \int_{\Gamma_o^t} \mathbf{t} \cdot \hat{\mathbf{v}} dA_o. \tag{3}$$

To solve above non linear problem in terms of u_{n+1} is used the Newton method. Hence, taking

$$u_{n+1}^0 = u_n, \qquad k = 0 \tag{4}$$

where k denotes the Newton method iteration step. Supposing the initial condition is given by last increment step converged solution u_n, then on k-th iteration we have

$$u_{n+1}^{k+1} = u_{n+1}^k + \Delta u_{n+1}^k. \tag{5}$$

To determine $\Delta\mathbf{u}_{n+1}^k$, one has

$$DF\left(u_{n+1}^k;\hat{\mathbf{v}}\right)\left[\Delta u_{n+1}^k\right] = -F\left(u_{n+1}^k;\hat{\mathbf{v}}\right), \tag{6}$$

with

$$DF\left(u_{n+1}^k;\hat{\mathbf{v}}\right)\left[\Delta u_{n+1}^k\right] = \int_{\Omega_o} \frac{d}{d\epsilon}\left[\mathbf{P}\left(u_{n+1}^k + \epsilon\Delta u_{n+1}^k\right)\right]_{\epsilon=0} : \nabla\hat{\mathbf{v}}\; d\Omega_o, \tag{7}$$

where Ω_o is fixed in space and it is supposing that $\mathbf{t}_{o_{n+1}}$ and $\bar{\mathbf{b}}_{n+1}$ are non depended of u. After some algebraic manipulations, we obtain

$$DF\left(u_{n+1}^k;\hat{\mathbf{v}}\right)\left[\Delta u_{n+1}^k\right] = \int_{\Omega_o} \left[\mathbb{A}\left(u_{n+1}^k\right)\right] \nabla\left(\Delta u_{n+1}^k\right) : \nabla\hat{\mathbf{v}}\; d\Omega_o, \tag{8}$$

where $\mathbb{A}$ (fourth order tensor) is the global tangent modulus, that is given by

$$\left[\mathbb{A}\left(\mathbf{u}_{n+1}^k\right)\right]_{ijkl} = \left.\frac{\partial P_{ij}}{\partial F_{kl}}\right|_{\mathbf{u}_{n+1}^k}. \tag{9}$$

On the other hand,observing the problem from an Eulerian approach, it is defined a couple of sets for each $t \in S$

$$Kin_u(\Omega) = \left\{u_i : \Omega \to \mathbb{R} \mid u_i \in H^1(\Omega), u\left(\mathbf{x},t\right) = \bar{u}\left(\mathbf{x},t\right), \forall\mathbf{x} \in \Gamma^u\right\}; \tag{10}$$

$$Var_u(\Omega) = \left\{\hat{v}_i : \Omega \to \mathbb{R} \mid \hat{v}_i \;\in H^1(\Omega_t), \hat{v}_i\left(\mathbf{x}\right) = 0, \forall\mathbf{x} \in \Gamma^u\right\}. \tag{11}$$

Hence, the weak formulation of the problem can be written as

Problem 2. *Determine* $u(\mathbf{x},t) \in Kin_u(\Omega)$*, for each* $t \in S$*, such that*

$$\int_\Omega \sigma \colon \nabla \hat{\mathbf{v}} d\Omega = \int_\Omega \rho\,(\mathbf{b} - \ddot{u}) \cdot \hat{\mathbf{v}} d\Omega + \int_{\Gamma^t} \mathbf{t} \cdot \hat{\mathbf{v}} dA, \qquad \forall\ \ \hat{\mathbf{v}} \in Var_u(\Omega), \tag{12}$$

and in this case the tangent operator (or the global tangent modulus) can be described as

$$\left[\mathbb{A}\left(u_{n+1}^k\right)\right]_{ijkl} = \left.\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}}\right|_{\mathbf{u}_{n+1}^k}. \tag{13}$$

It is important to comment that in bouth cases (Lagrangian or Eulerian approach) the global tangent modulus is defined by a rate of conjugated pairs.

4 THE TIKHONOV REGULARIZATION METHOD

After the Galerkin method discretization the problem described above belongs

$$\min_{\mathbf{f}\in\mathbb{R}^n} \|\mathbf{Af} - \mathbf{g}\|_2, \quad \mathbf{A} \in \mathbb{R}^{n\times n} \quad \mathbf{g} \in \mathbb{R}^n, \tag{14}$$

where $\mathbf{A}$ (matrix representation for discretized tangent operator $\left[\mathbb{A}\left(\mathbf{u}_{n+1}^k\right)\right]_{ijkl}$) has high condition number (ill-conditioned and singular values decreasing to zero without a gap on spectrum) on limit points neighbourhood ($\partial\sigma_{ij} / \partial\epsilon_{kl} \approx$ null fourth order tensor) due to the shape of the equilibrium path response. The $\mathbf{g}$ consists to discretized vectorial representation of $-\digamma\left(\mathbf{u}_{n+1}^k; \hat{\mathbf{v}}\right)$. Unfortunately for the standard least square (LS) the solution can be presented as $\mathbf{f}_{\text{ls}} = \mathbf{A}^\dagger \mathbf{g}$ (where $\mathbf{A}^\dagger$ denotes the pseudoinverse of $\mathbf{A}$) has serious numerical spurious error. In this sense, the Tikhonov regularization method is a natural way to computate a solution less susceptible to numerical errors. The classical Tikhonov method ([5] and [6]) consists in

$$\min_{\mathbf{f}\in\mathbf{R}^n} \daleth(\mathbf{f}) \tag{15}$$

where $\daleth(\mathbf{f}) = \|\mathbf{Af} - \mathbf{g}\|^2 + \tilde{\lambda}\|\mathbf{f}\|^2$ and $\tilde{\lambda} > 0$ is the regularization parameter. This problem (15) is equivalent to research solution of the regularized normal equation

$$(\mathbf{A}^T\mathbf{A} + \tilde{\lambda}\mathbf{I}_n)\mathbf{f} = \mathbf{A}^T\mathbf{g}, \tag{16}$$

whose solution is $\mathbf{f}_{\tilde{\lambda}} = (\mathbf{A}^T\mathbf{A} + \tilde{\lambda}\mathbf{I}_n)^{-1}\mathbf{A}^T\mathbf{g}$, and $\mathbf{I}_n$ is the identity matrix $n \times n$. Now the problem is how to determine $\tilde{\lambda}$ parameter such that $\mathbf{f}_{\tilde{\lambda}}$ be the nearest solution of the solution without numeric errors. A lot of techniques for the regularization parameter choice were developed and they are presented in the specialized literature. hese techniques can be organized in two classes: techniques that involves the pre-known (or estimative) of the norm error e behaviour, as discrepancy principle (DP) evidenced in Morozov [8], and techniques that do not explore this information. In this second class it can be cited the L-curved method (see [9]), generalized cross-validation (GCV) (see [10]), weighted-GCV

(W-GCV) (see [11]), and a fixed point method (FP-method) (see [12]). For an overview of parameter-choice techniques for Tikhonov regularization method see [6] and recently [12].

Considering SVD of $\mathbf{A}$, $\mathbf{A} = \hat{\mathbf{S}}_1\hat{\mathbf{S}}_2\hat{\mathbf{S}}_3^T$, where $\hat{\mathbf{S}}_2 \in \mathbb{R}^{n\times n}$ is a singular value diagonal matrix, and $\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_3 \in \mathbb{R}^{n\times n}$ are unitary matrixes, with $\hat{\mathbf{S}}_3$ non sigular matrix, the Thikhonov problem (15) can be written as

$$(\mathbf{A}^T\mathbf{A} + \tilde{\lambda}\mathbf{I}_n)\mathbf{f}_{\tilde{\lambda}} = \mathbf{A}^T\mathbf{g} \therefore \mathbf{f}_{\tilde{\lambda}} = \hat{\mathbf{S}}_3(\hat{\mathbf{S}}_2^2 + \tilde{\lambda}\mathbf{I}_n)^{-1}\hat{\mathbf{S}}_2\hat{\mathbf{S}}_1^T\mathbf{g}, \tag{17}$$

or $\mathbf{f}_{\tilde{\lambda}} = \sum_{i=1}^{n} \frac{\hat{S}_{2_i}^2}{\hat{S}_{2_i}^2+\tilde{\lambda}^2} \frac{\hat{\mathbf{S}}_{1_i}^T\mathbf{g}}{\hat{S}_{2_i}} \hat{\mathbf{S}}_{3_i}$ with $\hat{S}_{2_i}^2$ representing the i-th singular value, $\hat{\mathbf{S}}_{1_i}$ is the i-th colum vector of $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_{3_i}$ is the i-th colum vector of $\hat{\mathbf{S}}_3$.

Observing the problem (15), it is expected that the solution of this optimization problem converges to the solution of the equation $\mathbf{Af} = \mathbf{g}$ as $\tilde{\lambda}$ tends to zero. In this sense, some properties of Tikhonov regularization method are shown in following theorem

Theorem 1. *Let* $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ *be bounded. For every* $\tilde{\lambda} > 0$ *there exists a unique minimum* $\mathbf{f}_{\tilde{\lambda}}$ *of (15). Furthermore,* $\mathbf{f}_{\tilde{\lambda}}$ *satisfies the normal equation*

$$\tilde{\lambda}\left\langle \mathbf{f}_{\tilde{\lambda}}, \omega\right\rangle + \left\langle \mathbf{Af}_{\tilde{\lambda}} - \mathbf{g}, \mathbf{A}\omega\right\rangle = 0, \forall\omega \in \mathbb{R}^n, \tag{18}$$

or, using the adjoint $\mathbf{A}^* = \mathbf{A}^T : \mathbb{R}^n \to \mathbb{R}^n$ *of* $\mathbf{A}$,

$$(\mathbf{A}^T\mathbf{A} + \tilde{\lambda}\mathbf{I}_n)\mathbf{f}_{\tilde{\lambda}} = \mathbf{A}^T\mathbf{g}. \tag{19}$$

If, in addition, $\mathbf{A}$ *is one-to-one and* $\mathbf{f} \in \mathbb{R}^n$ *is the (unique) solution of the equation* $\mathbf{Af} = \mathbf{g}$ *then* $\mathbf{f}_{\tilde{\lambda}} \to \mathbf{f}$ *as* $\tilde{\lambda}$ *tends to zero. Finally, if* $\mathbf{f} \in \mathbf{A}^T(\mathbb{R}^n)$ *or* $\mathbf{f} \in \mathbf{A}^T\mathbf{A}(\mathbb{R}^n)$*, then* $\exists c > 0$ *with* $\|\mathbf{f}_{\tilde{\lambda}} - \mathbf{f}\| = c\sqrt{\tilde{\lambda}}$ *or* $\|\mathbf{f}_{\tilde{\lambda}} - \mathbf{f}\| = c\tilde{\lambda}$*, respectively.*

5 THE L-CURVE TECHNIQUE

In this section it is presented some ideas about the L-curve thechnique for choosing the regularization parameter. In this sense, let $\mathbf{f}_{\tilde{\lambda}}$ for be the family of solutions of the method of Tikhonov problem (15) and set

$$\vartheta_{1\tilde{\lambda}} := \|\mathbf{Af}_{\tilde{\lambda}} - \mathbf{g}\|^2 \text{ and } \vartheta_{2\tilde{\lambda}} := \|\mathbf{f}_{\tilde{\lambda}}\|^2. \tag{20}$$

It can be verified that $\mathbf{f}_{\tilde{\lambda}}$ is a solution of the method of residuals ($e_{1\tilde{\lambda}} := \sqrt{\vartheta_{1\tilde{\lambda}}}$) and quasisolutions ($e_{2\tilde{\lambda}} := \sqrt{\vartheta_{2\tilde{\lambda}}}$). Defining the bounded set

$$C := \left\{(c_1, c_2) \in \mathbb{R}^2 | \exists\mathbf{f} \in \mathbb{R}^n \text{ with } \|\mathbf{Af} - \mathbf{g}\| \le c_1 \text{ and } \|\mathbf{f}\| \le c_2\right\}, \tag{21}$$

it can be shown that the function $\tilde{\lambda} \mapsto e_{1\tilde{\lambda}}$ is increasing, $\tilde{\lambda} \mapsto e_{2\tilde{\lambda}}$ is decreasing and C is a convex set with boundary given by the curve $\tilde{\lambda} \mapsto (e_{1\tilde{\lambda}}, e_{2\tilde{\lambda}})$. Although if it cannot determine the rate $\frac{e_{1\tilde{\lambda}}}{e_{2\tilde{\lambda}}}$, it must be have to specify a method/technique to determine $\tilde{\lambda}$ in an optimal sense with using $\vartheta_{1\tilde{\lambda}}$ and $\vartheta_{2\tilde{\lambda}}$. In this way, the L$-$curve criterion consists in determine $\tilde{\lambda}$ which maximizes the curvature in the typical L-shaped curve $\ell : \tilde{\lambda} \in (0,\infty) \mapsto (\ln(e_1), \ln(e_2)) \in \mathbb{R}^2$. The main motivation comes from the fact that in almost vertical portion of $\ell-$graph for very small changes of $\tilde{\lambda}$ values corresponds to rapidly varying to regularized solutions norm with very little change in $\vartheta_{1\tilde{\lambda}}$, while on horizontal part of the graphic for larger values of $\tilde{\lambda}$ corresponds to regularized solutions norm where the plot is flat or slowly decreasing, for more detail see [9]. From these arguments, the L-curve corner is located in a natural transition point that links these two regions, for more details and substantial results see [6].

The evaluation of second derivatives shows that curve is convex and steeper as $\tilde{\lambda}$ approaches to the smallest singular value. The L$-$curve consists of a vertical part where e_2 is near of the maximum value and adjacent part with smaller slope and the more horizontal part corresponds to solutions dominated by regularization errors where the regularization parameter is too large. In this sense, the problem is to seek the L$-$curve point where the maximum curvature is reached.

Supposing L$-$curve is sufficiently smooth (in continuous sense) curvature $\kappa\left(\tilde{\lambda}\right)$ can be computed as

$$\kappa\left(\tilde{\lambda}\right) = \frac{e_1^{'}e_2^{''} - e_1^{''}e_2^{'}}{\left(\left(e_1^{'}\right)^2 + \left(e_2^{'}\right)^2\right)^{\frac{3}{2}}}, \tag{22}$$

where ($'$) denotes a derivative with respect to $\tilde{\lambda}$ regularization parameter and any one dimensional optimization method can be used to solve $\tilde{\lambda}$ for the maximum curvature problem. It must be to point out that numerical effort involved in minimization is smaller than SVD computation. Although, in many cases it is limited a finite set of points on L$-$curve, hence the curvature $\kappa\left(\tilde{\lambda}\right)$ cannot be computed as (22) . In a numerical sense the L$-$curve consists of a number of discrete points corresponding to different regularization parameter ($\tilde{\lambda}$) values at which we have evaluated e_1 and e_2. Thus, it is defined a sufficiently smooth curve associated to the set of discrete points in such way that the overall shape of L$-$curve is maintained. This procedure consists in determine an approximating smooth curve for L$-$curve. A reasonable approach for this is a cubic spline pair fitting for e_1 and e_2. Such a curve has some interesting properties as twice differentiable, numerically differentiable in stable way and local shape preserving features. It is important to comment that computational implementation of Tikhonov L$-$curve regularization technique is based on criteria described in [7] and [9]. In the next section we will point out the performance of proposed Tikhonov L$-$curve regularization method for dynamic elastoplasticity problem.

6 NUMERICAL EXAMPLE

The objective of presented numerical examples are to attest efficiency of the numerical regularization technique proposed for the time evolutionary analysis in elastoplasticity problems. Our implementation was made in MATLAB and results analysis are given by a comparative response between regularized (Tikhonov L−curve parameter choice) numerical solution and non-regularized numerical solution. The numerical examples presented here consists of 1-D low cycle fatigue applications and a monotonic load test. The body initial length is 100 mm, its elasticity modulus is $E = 2 \times 10^5\ MPa$, Poisson ratio is $\nu = 0.3$, yielding limit is $\sigma_y = 260\ MPa$, kinematic hardening constants is $\chi_\infty = 200\ MPa$ (kinematic hardening amplitude) and $\gamma = 2.0$ (controls the kinematic hardening increase rate), isotropic hardening constants is $R_\infty = 300\ MPa$ (isotropic hardening amplitude) and $b = 1$ (controls the isotropic hardening increase rate), and damage constants are $P_d = 0.0005$ and $D_c = 0.2$ (critical value of damage). This last value depends upon the material and the loading conditions. D_c represents the final decohesion of atoms is characterized by a critical value of effective stress acting on the resisting area. It is important to cite that D_c gives the critical value of the damage at a mesocrack initiation occurring for unidimensional stress, usually $D_c \in [0.2, 0.5]$. A sketch of the problem cases may be seen in figure below (see Fig.2).

Figure 2: Problem Case Domain Sketch

The load, in this example, is given by $\bar{u}(x,t) = 0.8 \sin^2(2\pi t)$ where t is in cycles. The regularized numerical solution (rns) and the non-regularized numerical solution (nrns) for analysis over $t \in [0, 4]$ are computed under 10^{-4} tolerance value. A fictitious exact solution (fes) was too construct for this application. A important fact that must be noted is bouth numerical solution didn't get to realize entire analysis over range $t \in [0, 4]$. The "nrns" was capable to continue the analysis until $t = 2.787$ cycles. The "rns-analysis", that use the Tikhonov regularization technique, can be cover range $t \in [0, 3.137]$ cycles with a excellent agreement with the "fes" as presented in figure (Fig. 3) below.

The nrns-analysis failed due to ill-condition problems. At point $t = 2.787$ cycles the condition number associated to the linearised system on Newton method iteration is 2.4×10^8. For this case the number of iteration extrapolated a lot allowed limit (500 iterations) with residual norm value oscillating in one belittles strip around 10^{-3}. If we grow up the allowed limit of iteration same pattern is the reached until 624710 iterations. In following figure (4) it can be seen a good agreement between "fes" and "rns". Note that the rns-response was capable to reproduce the beginning of softening behaviour.

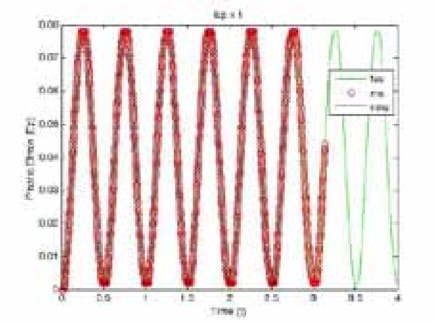

Figure 3: Plastic Strain vs. Time

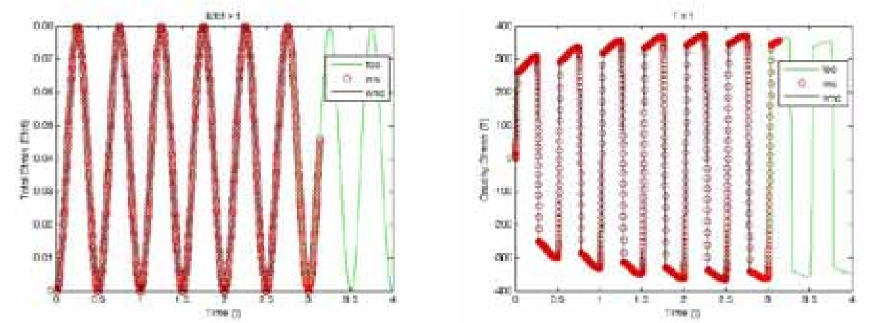

Figure 4: Total Strain vs. Time / Cauchy Stress vs. Time

In next figure (5) the hardening behaviour during analyzed time can be seen. Again, a good agreement among the numerical results (”fes” and ”rns”) can be noted.

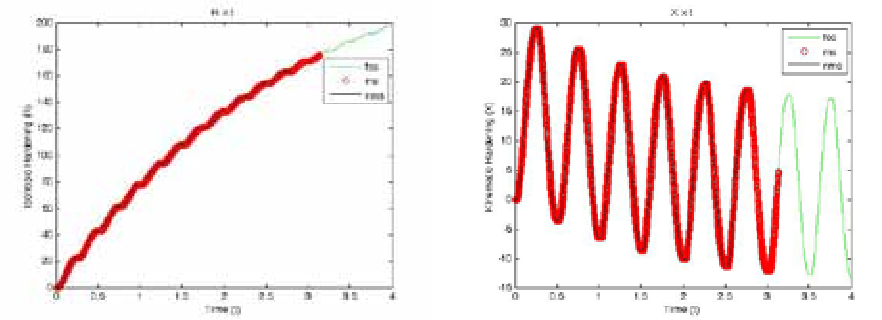

Figure 5: Isotropic Hardening vs. Time / Kinematic Hardening vs. Time

At this point, it is presented the responses about damage variable and storage plastic strain (see Fig. 6). A perfect ”fes-rns” agreement has been noted in storage plastic strain behaviour. The damage variable evolutionary profile shows a little bit discrepancy between "fes" and "rns" at $t = 2.75$ cycles (maximum difference) with 1.2% as relative error. It is important to stand that there is a tendency to both graphs ("fes" and "rns") coincides. The Tikhonov regularization process is setting to start when condition number is equal or greater than 2.4×10^8. Other settings are tested but the same unexpected pattern on rns-response was observed and non significant changes are noted.

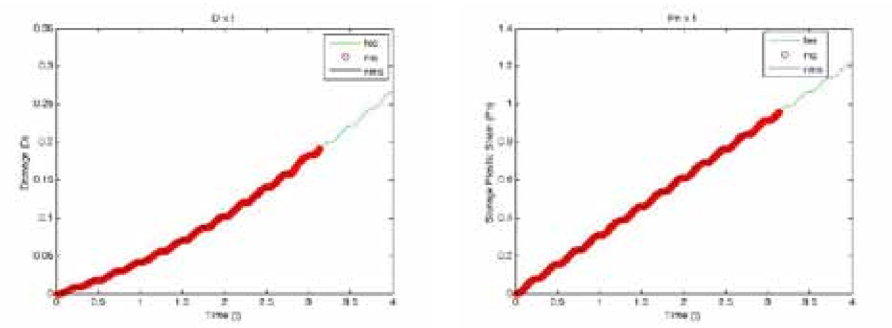

Figure 6: Damage vs. Time / Storage Plastic Strain vs. Time

The Tikhonov regularization method allowed that the numerical analysis continues until 3.06×10^8 as condition number. The regularization parameter computed for last Newton's iteration was $\tilde{\lambda} \approx 0.0525$ (see Fig. 7).

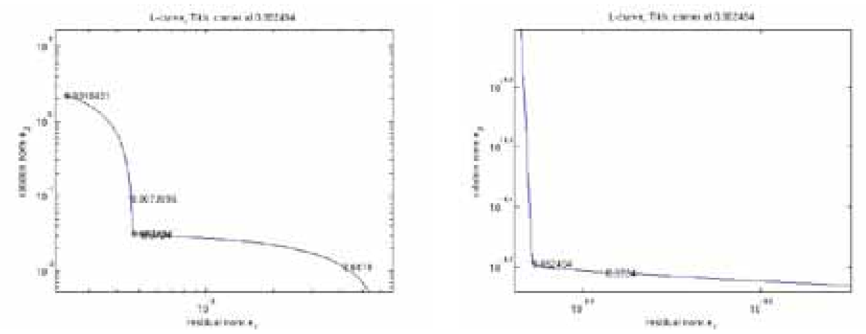

Figure 7: L−curve: e_2 vs. e_1/ L−curve (zoom): e_2 vs. e_1

7 CONCLUSION

In this work, it has discussed/analyzed the computational implementation of elasto-plsticity problem. As mentioned above to treat the critical points on equilibrium-path it was proposed a Thikhonov L-cruve regularization approach over Newton method. In this sense it has prsented some theoretical results from Thikonov regularization method and your application over numerical dynamic elastoplastic problem as an efficient form of transposing the numerical problems associated to ill-conditioning happened in neighbourhoods of critical points.

It is important comment that the Thikonov L-curve regularization method approach in elastoplastcity numerical analysis showed robustness, efficiency and potential as it can be seen in the comparative numerical example here presented. The used tolerance convergence criterion (10^{-4}) was obtained after tests with larger and smaller tolerance values, in that none differences in the pattern of the responses was noticed. In this numerical example it was verified the consistency, performance and computational accuracy of the approach proposed. In fact, there was an excellent agreement between the regularized numerical response and fictitious exact solution, adding numerical stability and possibiliting advances in the time of analysis over permanent deformation computational modelling. Although, it is clear that new numerical experiments in terms of applications to explore as problems involving time rate dependences (viscoplasticity) over permanent/plastic deformations.

Additionally it is important to point out that besides new applications, other choosing parameters techniques (see [13] and [12]) must be investigated in terms of computational efforts, accuracy and performance in relation to L-curve approach. In particular, some experience is needed with large problems from distinct application requiring the use of general-form Tikhonov regularization. These are the subject of a research that should be continued.

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