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E. Oñate, D.R.J. Owen, D. Peric and B. Suárez (Eds)

RHEOLOGICAL METHOD FOR CONSTRUCTING CONSTITUTIVE EQUATIONS OF ONE-PHASE GRANULAR AND POROUS MATERIALS

VLADIMIR M. SADOVSKIY

Institute of Computational Modeling SB RAS
Akademgorodok 50/44, 660036 Krasnoyarsk, Russia
e-mail: sadov@icm.krasn.ru, http://icm.krasn.ru

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Abstract. To make possible the description of deformation of materials with different resistance to tension and compression, the rheological method is supplemented by a new element, a rigid contact, which serves for imitation of a perfectly granular material with rigid particles. By using a rigid contact in combination with conventional rheological elements the constitutive equations of granular materials and soils with elastic-plastic particles and of porous materials, like metal foams, are constructed.

1 INTRODUCTION

The theory of granular materials is among the most intensively developing fields of mechanics because the area of its application is very wide. In spite of the fact that the foundations of this theory have been laid even at the dawn of the development of continuum mechanics in the classical works by Coulomb and Reynolds, by now the theory is still far from completion. The main difficulties are caused by drastic difference in behaviour of granular materials in tension and compression experiments. Essentially all of known natural and artificial materials possess this property of heteroresistance (heterostrength) to some extent. For some of them, differences in modulus of elasticity, yield point, or creep diagram obtained with tension and compression are small to an extent that they should be neglected. However, in the studies of alternating-sign strains in granular materials, these differences may not be neglected. In addition, mechanical properties of granular materials, as a rule, depend on a number of side factors such as inhomogeneity in size of particles and in composition, anisotropy, fissuring, moisture etc. This results in low accuracy of experimental measurements of phenomenological parameters of models.

At the present time, two classes of mathematical models corresponding to two different conditions of deformation of a granular material (quasistatic conditions and fast motion ones) have been formed [1]. The first class describes behaviour of a closely packed medium at compression load on the basis of the theory of plastic flow. In the space of stress tensors conical domains of admissible stresses rather than cylindrical ones, as in the perfect plasticity theory, satisfy these conditions. In the second class, a loosened medium modeled as an ensemble of a large number of particles in the context of the kinetic gas theory is considered.

To study quasistatic conditions of deformation, the stress theory in statically determinate problems which is applied in soil mechanics is developed. The case of plane strain is best studied by Sokolovskii [2], and the axially symmetric case – by Ishlinskii [3]. Velocity fields in these problems are defined according to the associated flow rule considered by Drucker and Prager [4]. Mróz and Szymanski [5] showed that the special nonassociated rule provides more accurate results in the problem on penetration of a rigid stamp into sand. A common disadvantage of these approaches lies in the fact that, when unloading, in the kinematic laws of the plastic flow theory a strain rate tensor is assumed to be zero, hence, deformation of a material is possible only as stresses achieve a limiting surface. From this it follows, for example, that a loosened granular material whose stressed state corresponds to a vertex of admissible cone can not be compressed by hydrostatic pressure since to any state of hydrostatic compression there corresponds an interior point on the axis of the cone. This is in contradiction with a qualitative pattern.

Kinematic laws turn out to be applicable in practice in the case of monotone loading only. Constitutive equations of the hypoplasticity in application to soil mechanics have a similar disadvantage [6, 7], because tension and compression states in them differ from one another in sign of instantaneous strain rate rather than in sign of total strain.

The equations of uniaxial dynamic deformation of a granular material, correct from the mechanical point of view, being a limiting case of the equations of heteromodular elastic medium [8], were studied by Maslov and Mosolov [9]. Phenomenological models of a spatial stressed-strained state of a cohesive soil for finite strains were proposed by Grigoryan [10] and Nikolaevskii [11]. The works [12, 13] are devoted to generalization of fundamentals of the plasticity theory for description of dynamics and statics of granular materials.

A spatial model of fast motions was proposed by Savage [14], who compared the solution of the problem on channel flow with experimental results, in particular, with those of Bagnold. Goodman and Cowin [15] developed a model for the analysis of gravity flow of a granular material. Nedderman and Tüzün [16] constructed a simple kinematic model which allows one to simulate an experimental pattern of steady-state outflow from funnel-shaped bunkers.

Nevertheless there is no a simple mathematical model which can be applicable both in the case of quasistatics and in the case of fast motions to describe the stagnant zones in a granular flow. The efforts to construct such model give only some limited applications for one-dimensional shear motions.

Porous metals are new artificial materials that can find wide application in engineering, thanks to the low density and good damping properties. The ability of porous metals effectively absorb energy during plastic deformation opens up the prospects of their use for production of the car bumpers and elements of the car body, so called "crushed" zones. They can be also used in reducers and drives as destructible fuses which dissipate the energy of dynamic impact, preventing the destruction of all mechanical system.

Similar to granular materials, their deformation properties significantly differ in tension and compression, which is typical virtually for all porous materials. Under tension, the stages of elastic deformation of the skeleton and plastic flow up to fracture are distinguished. Under compression, the stages of elastic and plastic deformation of the skeleton up to the collapse of pores, and the subsequent stage of elastic or elastic-plastic deformation of a solid, non-porous material are distinguished. In the case of small pore sizes, the collapse may occur in the elastic stage with the appearance of plasticity only at sufficiently high levels of loading at the last stage.

Currently the technology of production of metal foams on the basis of aluminum, copper, nickel, tin, zinc and other metals is worked out. Extensive experimental researches of mechanical properties of such materials are carried out. The diagrams of uniaxial tension and uniaxial compression on an example of aluminum foam and porous copper were obtained in [17, 18]. The paper [19] deals with problems of the wear resistance and the cyclic fatigue of porous metals.

Theoretical results related to the construction of constitutive equations and to the analysis on this basis of a spatial stressed-strained state of structural elements of a metal foam, according to available publications, are practically not studied. Still more difficult to construct a universal model for the description of a spatial stressed-strained state. Performing of adequate computations based on discrete models of a metal foam as a structurally inhomogeneous material is possible only with using the multiprocessor systems which have high speed and large amounts of RAM.

In this paper a simple method for constructing constitutive equations of granular and porous materials based on the rheological approach is suggested.

2 GRANULAR MATERIALS

Rheology is the basis of the phenomenological approach to the description of a stressed-strained state of materials with complex mechanical properties. As a rule, for the models obtained with the help of rheological method, solvability of main boundary-value problems can be analyzed and efficient algorithms for numerical implementation can be easily constructed. At the same time, with the use of conventional rheological elements (a spring simulating elastic properties of a material, a viscous damper, and a plastic hinge) only, it is impossible to construct a rheological scheme for a medium with different resistance to tension and compression or for a medium with different ultimate strengths under tension and compression. To make it possible, the rheological method is supplemented by a new element, a rigid contact (see Fig. 1a), which serves for imitation of a perfectly granular

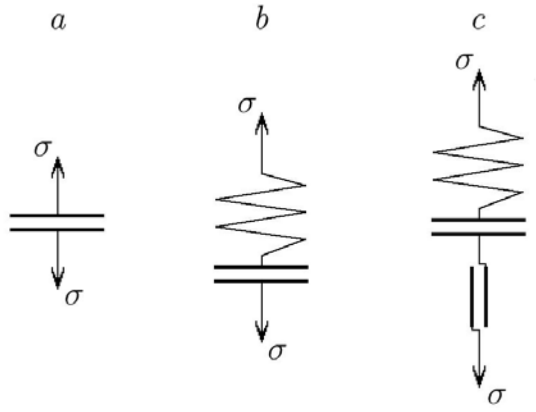

Figure 1: Rheological models of a granular material: rigid (a), elastic (b) and elastic-plastic (c)

material with rigid particles [20]. Under compressive stresses this element doesn't deformed. If stress is equal to zero then strain may be arbitrary positive value. Tensile stresses aren't admissible. Rheological models of perfectly elastic and elastic-plastic granular media are represented in Figs. 1b and 1c. In the case of compression such media are either in elastic state or in plastic one, but in the case of tension the stresses are equal to zero. By different combining these elements with viscous element, one can construct rheological models of more complex media.

Mathematical model of a rigid contact (perfectly granular medium with rigid particles) is reduced to the system of relationships

$$\sigma \le 0, \qquad \varepsilon \ge 0, \qquad \sigma\,\varepsilon = 0.$$

It is possible to represent it in the form of variational inequalities

$$(\tilde\varepsilon - \varepsilon)\,\sigma \le 0, \quad \varepsilon,\ \tilde\varepsilon \ge 0, \qquad (\tilde\sigma - \sigma)\,\varepsilon \le 0, \quad \sigma,\ \tilde\sigma \le 0,$$

each of which assumes the potential representation

$$\sigma \in \partial\varphi(\varepsilon), \qquad \varepsilon \in \partial\psi(\sigma). \tag{1}$$

Here φ and ψ – the potentials of stresses and strains – are the indicator functions, equal to zero on cones $C = \{\varepsilon \ge 0\}$ and $K = \{\sigma \le 0\}$ respectively, and equal to infinity outside of these cones. These functions are denoted as $\delta_C(\varepsilon)$ and $\delta_K(\sigma)$. The symbol ∂ serves for designation of a subdifferential, the arbitrary variable values are denoted by a wave.

Generalization of the model, schematically represented in Fig. 1a, on the case of a spatial stressed-strained state is easily constructed on the basis of inclusions (1). For that it is necessary to set the convex cone C in the space of strain tensors or the cone K in the space of stress tensors. If one of these cones is known then another one is found as conjugate:

$$K = \Big\{\sigma \,\big|\ \ \sigma : \varepsilon \le 0 \quad \forall\, \varepsilon \in C\Big\}, \qquad C = \Big\{\varepsilon \,\big|\ \ \sigma : \varepsilon \le 0 \quad \forall\, \sigma \in K\Big\}$$

(the colon denotes the convolution of tensors). Corresponding potentials – the indicator functions of cones C and K – are dual, i.e. they are determined one by another with the help of the Young transformation

$$\varphi(\varepsilon) = \sup_{\sigma}\Big\{\sigma : \varepsilon - \psi(\sigma)\Big\}, \qquad \psi(\sigma) = \sup_{\varepsilon}\Big\{\sigma : \varepsilon - \varphi(\varepsilon)\Big\}.$$

Known experimental results on the deformation properties of compact sands confirm the hypothesis about elastic state of a medium under stresses, close to hydrostatic compression. Such stresses are interior points of the cone K. For an elastic granular medium (Fig. 1b) $\psi = \sigma : a : \sigma/2 + \delta_K(\sigma)$, where a is the tensor of moduli of elastic compliance of fourth rank, corresponding to the model of an elastic element. The constitutive relationships (1) are reduced to the Haar–Karman inequality [20]

$$(\tilde{\sigma} - \sigma) : (a : \sigma - \varepsilon) \ge 0, \qquad \sigma,\ \tilde{\sigma} \in K. \tag{2}$$

Taking into account the symmetry and the positive definiteness of the tensor a, it is possible to show that the solution of inequality (2) is the tensor of stresses $\sigma = s^{\pi}$, equals to the projection of the conditional stress tensor s, determined from the linear Hooke law $a : s = \varepsilon$, onto K with respect to the norm $|\sigma|_a = \sqrt{\sigma : a : \sigma}$.

For a medium possessing plastic properties, rheological scheme of which is represented in Fig. 1c, the strain tensor is decomposed into the sum of elastic and plastic components: $\varepsilon = \varepsilon^e + \varepsilon^p$. The tensor of elastic strain satisfies the inequality (2), taking into account the property of granularity of a medium. For the plastic strain rate tensor the constitutive relationships of the flow theory

$$\sigma \in \partial\eta(\dot{\varepsilon}^p) \tag{3}$$

are correct. Here η is the dissipative potential of stresses being a convex positive homogeneous function of the strain rates, the dot over a symbol serves to indicate the time derivative. Homogeneity of this potential is the consequence of independence of the process of plastic deformation on time scale. By virtue of this property, the dual potential $\chi(\sigma)$ – the Young transformation of the function $\eta(\dot{\varepsilon})$ – is equal to the indicator function of the convex closed set

$$F = \Big\{\sigma \,\big|\ \ \sigma : \dot{\varepsilon} \le \eta(\dot{\varepsilon}) \quad \forall\, \dot{\varepsilon}\Big\}.$$

The boundary of F in the stress space defines the yield surface of a material. If the set F is a cylinder with the axis of hydrostatic stresses then the volume strain of a medium obeys the linearly elastic law. In the opposite case the model, being under consideration, describes irreversible volumetric contraction.

The inclusion (3) in the equivalent form $\dot{\varepsilon}^p \in \partial\chi(\sigma)$ is reduced to the Mises inequality

$$(\tilde{\sigma} - \sigma) : \dot{\varepsilon}^p \leq 0, \qquad \sigma,\ \tilde{\sigma} \in F. \tag{4}$$

The variational inequality (2) for elastic part of the strain tensor and the inequality (4) for its plastic part together with the equations of motion and the kinematic equations

$$\rho\,\dot{v} = \nabla \cdot \sigma, \qquad 2\,(\dot{\varepsilon}^e + \dot{\varepsilon}^p) = \nabla v + (\nabla v)^* \tag{5}$$

form a closed model describing the dynamics of a granular medium. Here ρ is the density, v is the velocity vector, ∇ is the gradient, an asterisk denotes the operation of transposition.

3 POROUS METALS

The porosity of a metal foam is determined as the ratio of the pore volume to the volume of a porous material: $\varepsilon_0 = V_0/V$. If ρ is the density of initial (solid) metal, then, ignoring the presence of gas in the pores, the density of a porous metal can be calculated by formula: $\rho_0 = \rho\,(V - V_0)/V$. Consequently, $\varepsilon_0 = (\rho - \rho_0)/\rho$. For highly porous materials the volume strain caused by the collapse of pores is much higher than the concomitant strain of volume compression of the skeleton, therefore the pores disappear when the volume strain is approximately equal to $\theta_0 \approx ((V - V_0) - V)/V = -\varepsilon_0$.

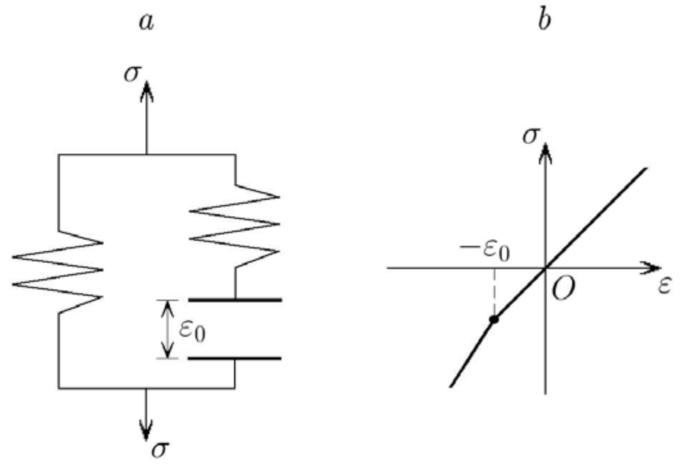

Figure 2: Rheological scheme (a) and diagram of uniaxial elastic deformation of a porous metal (b)

The simplest rheological scheme taking into account the main qualitative features of deformation of porous metals is represented in Fig. 2a. In this scheme the behaviour of a material under tension and under compression up to the moment of pores collapse is simulated by an elastic spring with the compliance modulus a, and the increasing of rigidity as the collapse of pores is simulated by a spring with the compliance modulus b. Segments of the diagram of uniaxial deformation with a break at the point $\varepsilon = -\varepsilon_0$ (see Fig. 2b) are defined by the equations: $\sigma = \varepsilon/a$ and $\sigma = \varepsilon/a + (\varepsilon + \varepsilon_0)/b$. This scheme describes an elastic process that occurs without the dissipation of mechanical energy.

Fig. 3a shows a more general rheological scheme with a plastic hinge. It is assumed that under tensile stress σ_s^+ the skeleton goes into the yield state, and under compressive stress $-\sigma_s^-$ the plastic loss of stability takes place. The corresponding diagram of uniaxial deformation is a four-segment broken line (see Fig. 3b). Elastic stage is described by the equation $\sigma = \varepsilon/a$, and the stage of elastic-plastic deformation of a solid material after collapse of the pores is described by the equation $\sigma = (\varepsilon + \varepsilon_0)/b - \sigma_s^-$. Transitions of a material in the unloading state are shown by arrows. The unloading of a porous material occurs by the law $d\sigma = d\varepsilon/a$, and the unloading of a solid material occurs by the law $d\sigma = d\varepsilon\,(1/a + 1/b)$. Specific dissipative energy, which is released during the collapse of the pores, is estimated by the product $\sigma_s^-\varepsilon_0$ in this model. The plastic flow, which occurs in a solid material at higher level of compressive stresses, is not considered.

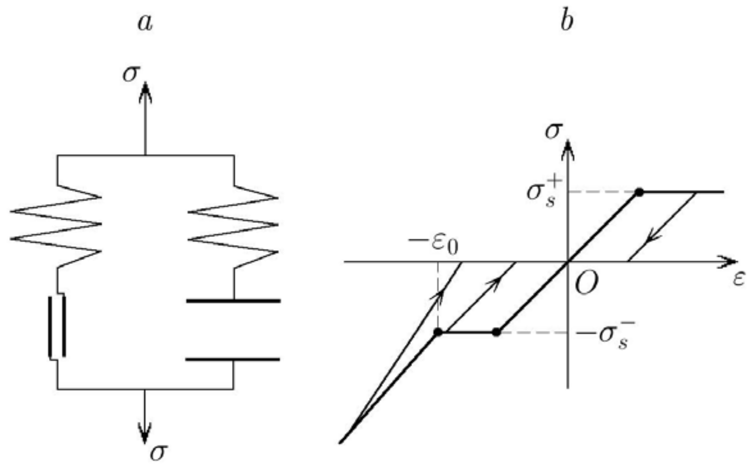

Figure 3: Rheological scheme (a) and diagram of elastic-plastic deformation of a skeleton (b)

In the general case of a spatial stressed-strained state, in accordance with the rheological scheme in Fig. 3b, the stress tensor σ is equal to the sum of the tensors σ^p of plastic stresses and σ^c of additional stresses acting after collapse of the pores. It is assumed that these tensors are symmetric. Elastic compliance of a material at small strains is characterized by the fourth-rank tensors a and b, satisfying the usual conditions of symmetry and positive definiteness. The series connection of an elastic spring and a plastic hinge in the scheme corresponds to the theory of elastic-plastic flow of Prandtl–Reuss. Within the framework of this flow theory the constitutive relationships are postulated in the form of principle of maximum of the energy dissipation rate:

$$(\tilde{\sigma} - \sigma^p) : (a : \dot{\sigma}^p - \dot{\varepsilon}) \geq 0, \qquad \tilde{\sigma},\ \sigma^p \in F. \tag{6}$$

Here ε is the actual strain tensor, F is the convex set in the stress space, bounded by the yield surface of a material. Assuming that the deformation of the jumpers of porous skeleton, distributed randomly on macrovolume of a material, can be described with satisfactory accuracy as a bar model, let us define concretely the set of admissible stresses:

$$F = \Big\{ \tilde{\sigma} \,\Big|\; -\sigma_s^- \leq \tilde{\sigma}_k \leq \sigma_s^+, \quad k = 1, 2, 3 \Big\},$$

where $\tilde{\sigma}_k$ are the principal values of σ.

Constitutive relationships of a rigid contact are formulated as the variational inequality

$$(\tilde{\sigma} - \sigma^c) : (\varepsilon^c + \varepsilon^0) \leq 0, \qquad \tilde{\sigma},\ \sigma^c \in K. \tag{7}$$

Here $\varepsilon^c = \varepsilon - b : \sigma^c$ is the strain tensor of porous skeleton, $\varepsilon^0 = \varepsilon_0\, \delta/3$ is the spherical tensor of initial porosity of a material, δ is the Kronecker delta. The transition of a material from porous state to continuous one is modeled by the convex cone K. As a simple variant of K one can use the Mises–Schleicher circular cone:

$$K = \Big\{ \tilde{\sigma} \,\Big|\; \tau(\tilde{\sigma}) \leq æ\, p(\tilde{\sigma}) \Big\},$$

where æ is the phenomenological parameter of a dilatancy, $p(\sigma) = -\sigma : \delta/3$ is the hydrostatic pressure, $\tau(\sigma)$ is the intensity of tangential stresses determined via the deviator of the stress tensor $\sigma' = \sigma + p(\sigma)\, \delta$ by means of the formula: $\tau^2(\sigma) = \sigma' : \sigma'/2$.

Taking into account these notations, the inequality (7) is converted to the form

$$(\tilde{\sigma} - \sigma^c) : b : (\sigma^c - s) \geq 0, \qquad \tilde{\sigma},\ \sigma^c \in K. \tag{8}$$

Here s is a tensor of conditional stresses, which is calculated by the law of linear elasticity with initial strains: $b : s = \varepsilon + \varepsilon^0$. If this tensor is admissible, i.e. if the inclusion $s \in K$ is fulfilled, then by (8) $\sigma^c = s$. If $s \notin K$ and for any $\tilde{\sigma} \in K$ the inequality $\tilde{\sigma} : b : s \leq 0$ is valid, which means precisely that the sum of tensors $\varepsilon + \varepsilon^0$ belongs to the cone $C = \Big\{ \tilde{\varepsilon} \,\Big|\; \tilde{\sigma} : \tilde{\varepsilon} \leq 0, \quad \tilde{\sigma} \in K \Big\}$ of admissible strains, dual to the cone K, then as follows from (8) $\sigma^c = 0$. In the general case, the variational inequality (8) allows to determine the tensor $\sigma^c = s^\pi$ as a projection of the tensor s onto K with respect to the norm $|s| = \sqrt{s : b : s}$, and the above two variants for setting s are special cases when the projection coincides with the original tensor and the projection is a vertex of cone. If the projection belongs to a conical surface, then the formulas for calculating the projection take the next form [20]:

$$p(\sigma) = \frac{\mu\, p(s) + æ\, k\, \tau(s)}{\mu + æ^2 k}, \qquad \sigma' = æ\, p(s)\, \frac{s'}{\tau(s)} \tag{9}$$

(for an isotropic medium the tensor b of elastic compliance is characterized by two independent parameters – the volume compression modulus k and the shear modulus μ). This variant is realized when both of the conditions $s \notin K$ and $\varepsilon + \varepsilon^0 \notin C$ are fulfilled. The cone C, dual to the Mises–Schleicher cone, is defined as

$$C = \Big\{ \tilde{\varepsilon} \;\Big|\; æ\, \gamma(\tilde{\varepsilon}) \le \theta(\tilde{\varepsilon}) \Big\},$$

where $\gamma(\tilde{\varepsilon}) = \sqrt{2\, \tilde{\varepsilon}' : \tilde{\varepsilon}'}$ is the shear intensity, and $\theta(\tilde{\varepsilon}) = \tilde{\varepsilon} : \delta$ is the volume strain.

The inclusion $\varepsilon + \varepsilon^0 \in C$ means that the rigid contact in rheological scheme is opened, i.e. the pores are in the open state. When the collapse of pores take place, the limit condition $æ\, \gamma(\varepsilon) = \varepsilon_0 + \theta(\varepsilon)$ is satisfied, which describes the dilatational volume increasing of a material due to the shear strain.

Note that in the simulation of real porous metals it is necessary to take into account a random character of distribution of a pore size, therefore the value ε_0 can vary randomly at each elementary portion of a sample (at each mesh of the grid domain). In principle, the law of distribution of pores by size is completely determined by technology of the production of metal foams, however in numerical computations (in order to describe qualitatively the effect of random distribution of pores on the stressed-strained state of a material) can be used, for example, the formula

$$\varepsilon_0 = \varepsilon_0^- + (\varepsilon_0^+ - \varepsilon_0^-)\, rand,$$

where $\varepsilon_0^\pm$ are the boundaries of porosity, $rand$ is a built-in function of the uniform distribution on the segment $[0, 1]$.

4 UNIVERSAL FORM OF MODELS

Mathematical model describing the dynamic deformation of porous metal under small strains and rotations of elements can be written in the next form:

$$\begin{aligned} &\rho_0\, \dot{v} = \nabla \cdot \sigma, \\ &(\tilde{\sigma} - \sigma^p) : (a : \dot{\sigma}^p - \nabla v) \ge 0, \qquad \tilde{\sigma},\ \sigma^p \in F, \\ &b : \dot{s} = (\nabla v + \nabla v^*)/2, \qquad \sigma = \sigma^p + \pi_K(s). \end{aligned} \tag{10}$$

Unknown functions are the velocity vector v and the tensors of plastic stresses σ^p and of conditional stresses s. The initial conditions, describing the natural (stress-free) state of a material, are formulated for the system (10) as

$$v\big|_{t=0} = 0, \qquad \sigma^p\big|_{t=0} = 0, \qquad s\big|_{t=0} = b^{-1} : \varepsilon^0.$$

The boundary conditions can be given in the terms of velocities: $v\big|_\Gamma = v^0(x)$, as well as in stresses: $\sigma\big|_\Gamma \cdot \nu(x) = q(x)$, where ν is the outward normal vector, v^0 and q are given functions.

It turns out that the relationships (10) and the relationships (2), (4), (5) of mathematical model of an elastic-plastic granular material can be represented in the universal matrix form

$$(\tilde{U} - U)\Big(A\,\dot{U} - \sum_{i=1}^{n} B^i\, U^{\pi}_{,i}\Big) \geq 0, \qquad \tilde{U},\ U \in F. \tag{11}$$

Here U is the unknown m-dimensional vector–function, A and B^i are the given matrices whose coefficients are the density and the mechanical coefficients of a material, subscripts after a comma denote partial derivatives with respect to spatial variables, superscript π denotes the projection of vector U onto the cone K of admissible variations with respect to the energy norm $|U| = \sqrt{UAU}$, $n = 1, 2,$ or 3 is the spatial dimension of the model.

The difference is that the vector–function U in the model of an elastic-plastic granular material consists of the projections of the velocity vector v and the components of the conditional stress tensor s. In the model of a porous metal it consists of the velocities and the components of two stress tensors – the plastic stress tensor σ^p, which is constrained by the plasticity condition, and the conditional stress tensor s.

The inequality (11) is very useful in constructing the numerical algorithms for the solution of initial-boundary problems. A variant of such algorithm is considered in our monograph [20]. In this monograph one can find the examples of numerical modeling of the processes of an elastic-plastic waves propagation in a loosened granular medium.

5 CONCLUSIONS

- Rheological method is supplemented by a new element, a rigid contact, which make it possible to describe mechanical properties of materials having different resistance to compression and tension.
- By means of this method constitutive relationships of granular materials with rigid, elastic and elastic-plastic particles are considered.
- Constitutive equations of metal foams of low porosity are obtained describing the phases of elastic deformation and plastic loss of stability of a skeleton and the phase of elastic deformation of a compact material after the pores collapse.

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