XI International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS XI
E. Oñate, D.R.J. Owen, D. Peric and B. Suárez (Eds)

A MICROMORPHIC CONTINUUM FORMULATION FOR FINITE STRAIN INELASTICITY

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Key words: Generalized continua, Micromorphic continuum, Finite strain inelasticity, Scale effects

Abstract. This work proposes a generalized theory of deformation which can capture scale effects also in a homogenously deforming body. Scale effects are relevant for small structures but also when it comes to high strain concentrations as in the case of localised shear bands or at crack tips, etc. In this context, so-called generalized continuum formulations have been proven to provide remedy as they allow for the incorporation of internal length-scale parameters which reflect the micro-structural influence on the macroscopic material response. Here, we want to adopt a generalized continuum framework which is based on the mathematical description of a combined macro- and micro-space [8]. The approach introduces additional degrees of freedom which constitute a so-called micromorphic deformation. First the treatment presented is general in nature but will be specified for the sake of an example and the number of extra degrees of freedom will be reduced to four. Based on the generalized deformation description new strain and stress measures are defined which lead to the formulation of a corresponding generalized variational principle. Of great advantage is the fact that the constitutive law is defined in the generalized space but can be classical otherwise. This limits the number of the extra material parameters necessary to those needed for the specification of the micro-space, in the example presented to only one.

1 Introduction

Decades ago, it has been recognized that for some materials the kinematics on meso- and micro-structural scales needs to be considered, if the external loading corresponds to material entities smaller than the *representative volume element* (RVE) and the statistical average of the macro-scopical material behaviour does not hold anymore. In this sense the fluctuation of deformation on micro-structural level as well as relative motion of micro-structural constituens, such as granule, crystalline or other heterogeneous aggregates, influence the material response on macro-structural level. Consequently, field equations based on the assumption of micro-scopically homogeneous material have to be supplemented and enriched to also include non-local and higher-order contributions.

In particular, generalized continua aim to describe material behaviour based on a deeper understanding of the kinematics at smaller scales rather than by pure phenomenological approximation of experimental data obtained at macro-scopical level. The meso- or micro-structural kinematics and its nonlocal nature is then treated either by incorporating higher-order gradients or by introducing extra degrees of freedom. For the latter, the small-scale kinematics at each material point can be thought to be equipped with a set of directors which specify the orientation and deformation of a surrounding a micro-space. This results in a *micromorphic continuum* theory [3], if the directors are allowed to experience rotation, stretch and change of angles to each other.

Geometrically nonlinear micromorphic formulations are sparsely found in literature, e.g. in [5] issues related to material forces of in the hyperelastic case were discussed, or in [7] micromorphic plasticity two-scale models have been proposed addressing micro-structural damage as well as granular material behaviour.

So far formulations of generalized continua are faced with two major problems. The first one relates to the fully non-linear and inelastic material behaviour. Classical inelastic formulations are based on decompositions of strain measures. Since generalized continua exhibit more than one strain or deformation measure the question arises as to how these can be decomposed into elastic and inelastic parts. Few suggestions were made in [11, 4, 2]. These formulations remained, however, less satisfactory since the decomposition of the two deformation measures were, strictly speaking, independent of each other, which raises many questions regarding the adequate formulation of evolution laws for the inelastic parts. The second problem relates to the observation of scale effects also in a homogenously deforming specimen. *Cosserat* and higher gradient theories cannot predict such scale effects, because the extra strain measures are identically zero for homogenous deformation. Furthermore, it is desirable to set out from a general and unified formulation of continua with meso- and micro-structure. We propose a framework based on the mathematical concept of fibre bundles embedded into a generalized continuum formulation. More specifically, we want to consider the Cartesian product of the macroscopic and further meso- or microscopic spaces and, accordingly, the generalized deformation is composed of a macro-, meso- and/or micro-components. In principle, every point of the macroscopic space would have an infinite number of degrees of freedom and dimensions. In practice, the number of degrees of freedom is finite corresponding to the chosen level of accuracy. In this sense the micromorphic continuum appears just as special approximation of the general case [8]. From the micromorphic deformation description nonlinear strain measures are derived and corresponding stress measures are defined which allow for the formulation of generalized variational principles and corresponding Dirichlet boundary conditions.

The paper is organized as follows. In Sec. 2 the theory of the generalized continuum is outlined. Subsequently, in Sec. 3 a generalized micromorphic principle of virtual work is proposed. The approach allows for the incorporation of any conventional constitutive law, this fact is exemplified using an inelastic material law. Details of the inelastic formulation are elaborated in Sec. 4.1 and Sec. 4.2. Finally, the excellent performance is demonstrated by an example of scale effects in homogenously deforming body as well as by that of a shear band formation in Sec 5.

2 Generalized deformation and strain

The basic idea is that a generalized continuum $\mathcal{G}$ can be assumed to inherit the mathematical structure of a fibre bundle. In the simplest case, this is the Cartesian product of a macro space $\mathcal{B} \subset \mathbb{E}(3)$ and a micro space $\mathcal{S}$ which we write as $\mathcal{G} := \mathcal{B} \times \mathcal{S}$. This definition assumes an additive structure of $\mathcal{G}$ which implies that the integration over the macro- and the micro-continuum can be performed separately. The macro-space $\mathcal{B}$ is parameterized by the curvilinear coordinates ϑ^i, $i = 1, 2, 3$ and the micro-space or micro-continuum $\mathcal{S}$ by the curvilinear coordinates ζ^α. Here, and in what follows, Greek indices take the values 1, ... or n. The dimension of $\mathcal{S}$ denoted by n is arbitrary, but finite. Furthermore, we want to exclude that the dimension and topology of the micro-space is dependent on ϑ^i. Each material point $\tilde{\mathbf{X}} \in \mathcal{G}$ is related to its spatial placement $\tilde{\mathbf{x}} \in \mathcal{G}_t$ at time $t \in \mathbb{R}$ by the mapping $\tilde{\boldsymbol{\varphi}}\,(t) : \mathcal{G} \longrightarrow \mathcal{G}_t$. For convenience but without loss of generality we identify $\mathcal{G}$ with the un-deformed reference configuration at a fixed time t_0 in what follows. The generalized space can be projected to the macro-space in its reference and its current configuration by

$$\pi_0(\tilde{\mathbf{X}}) = \mathbf{X} \qquad \text{and} \qquad \pi_t\,(\tilde{\mathbf{x}}) = \mathbf{x} \tag{1}$$

respectively, where π_0 as well as π_t represent projection maps, and $\mathbf{X} \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{B}_t$. The tangent space $\mathcal{TG}$ in the reference and current configuration, respectively, are defined by the pairs $(\tilde{\mathbf{G}}_i \times \mathbf{I}_\alpha)$ and $(\tilde{\mathbf{g}}_i \times \mathbf{i}_\alpha)$, respectively, given by

$$\tilde{\mathbf{G}}_i = \frac{\partial \tilde{\mathbf{X}}}{\partial \vartheta^i}\,, \quad \mathbf{I}_\alpha = \frac{\partial \tilde{\mathbf{X}}}{\partial \zeta^\alpha}\,, \quad \tilde{\mathbf{g}}_i = \frac{\partial \tilde{\mathbf{x}}}{\partial \vartheta^i} \quad \text{and} \quad \mathbf{i}_\alpha = \frac{\partial \tilde{\mathbf{x}}}{\partial \zeta^\alpha}\,, \tag{2}$$

where the corresponding dual contra-variant vectors are denoted by $\tilde{\mathbf{G}}^i$ and $\mathbf{I}^\alpha$, respectively. The generalized tangent space can also be projected to its corresponding macro-space by

$$\pi_0^*(\tilde{\mathbf{G}}_i) = \mathbf{G}_i \qquad \text{and} \qquad \pi_t^*\,(\tilde{\mathbf{g}}_i) = \mathbf{g}_i \tag{3}$$

respectively, where the tangent vectors $\mathbf{I}_\alpha$ are assumed to be constant throughout $\mathcal{S}$ for simplicity. Note that the definition of a projection map is not trivial. The tangent of the projection map defines the geometry of the extra space and so the metric which is to be used to evaluate the integral over the generalized space. The concept is rich in its structure.

Now, we assume that the placement vector $\tilde{\mathbf{x}}$ of a material point P ($\tilde{\mathbf{X}} \in \mathcal{G}$) is of an additive nature and is the sum of its position in the macro-continuum $\mathbf{x} \in \mathcal{B}_t$ and in the micro-continuum $\boldsymbol{\xi} \in \mathcal{S}_t$ as follows

$$\tilde{\mathbf{x}}\left(\vartheta^k, \zeta^\beta, t\right) = \mathbf{x}\left(\vartheta^k, t\right) + \boldsymbol{\xi}\left(\vartheta^k, \zeta^\beta, t\right) . \tag{4}$$

Thereby, the macro-placement vector $\mathbf{x}$ defines the origin of the micro co-ordinate system such that the micro-placement $\boldsymbol{\xi}$ is assumed to be relative to the macro-placement. The definition of the generalized continuum and so of the extra degrees of freedom depends directly on the choices to be made for the micro deformation $\boldsymbol{\xi}\left(\vartheta^k, \zeta^\beta, t\right)$. The theory is based on the fact that the dependency on the micro co-ordinates ζ^β must be determined apriori. Specific choices define specific continua. The following quadratic ansatz

$$\tilde{\mathbf{x}} = \mathbf{x}\left(\vartheta^k, t\right) + \zeta^\alpha \left(1 + \zeta^\beta \chi_\beta\left(\vartheta^k, t\right)\right) \mathbf{a}_\alpha\left(\vartheta^k, t\right) . \tag{5}$$

results adequate strain measures of full rank shown in [10]. The vector functions $\mathbf{a}_\alpha\left(\vartheta^k, t\right)$ and scalar functions χ_α, with their corresponding micro co-ordinates ζ^α, are independent degrees of freedom. The number α must be chosen according to the specific topology of the micro-space as well as depending on the physical properties of the material due to its intrinsic structure.

In computations we have to deal with four additional independent functions per micro co-ordinate. These are the three components of the vector $\mathbf{a}_\alpha$ as well as the independent displacement-like functions χ_α. Note, however, $\mathbf{a}_\alpha$ as well as χ_α are constant over $\mathcal{S}$. While the functions χ_α contribute to the definition of the strains, their special importance lies in the fact that they allow for the complete definition of linear distribution of strain in the extra dimensions. Also, it is important to realize that the dimension of the micro-space does not have to coincide with the dimension of the macro-space.

Now we proceed to define the strain measures. Taking the derivatives of $\tilde{\mathbf{x}}$ (Eq. 5) with respect to the macro-coordinates ϑ^i as well as with respect to the micro co-ordinates ζ^α, the generalized deformation gradient tensor can be expressed as follows

$$\begin{aligned}\tilde{\mathbf{F}} = & \Big[\mathbf{x}_{,i}\left(\vartheta^k, t\right) + \zeta^\alpha \zeta^\beta \chi_{\beta,i}\left(\vartheta^k, t\right) \mathbf{a}_\alpha\left(\vartheta^k, t\right) \\ & \quad + \zeta^\alpha \left(1 + \zeta^\beta \chi_\beta\left(\vartheta^k, t\right)\right) \mathbf{a}_{\alpha,i}\left(\vartheta^k, t\right)\Big] \otimes \tilde{\mathbf{G}}^i + \\ & \Big[\mathbf{a}_\alpha\left(\vartheta^k, t\right) + \zeta^\beta \left(\chi_\beta\left(\vartheta^k, t\right) \mathbf{a}_\alpha\left(\vartheta^k, t\right) + \chi_\alpha\left(\vartheta^k, t\right) \mathbf{a}_\beta\left(\vartheta^k, t\right)\right)\Big] \otimes \mathbf{I}^\alpha .\end{aligned} \tag{6}$$

Similar to its classical definition, a generalized right *Cauchy-Green* deformation tensor based on Eq. (6) is formulated as $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^T \tilde{\mathbf{F}}$ and neglecting higher order terms in ζ^α and extracting only the dominant parts of $\tilde{\mathbf{C}}$ (constant and linear in ζ^α) we arrive at

$$\begin{aligned}\tilde{\mathbf{C}} \;=\; & \Big(\mathbf{x}_{,k}\cdot\mathbf{x}_{,l} + \zeta^\alpha\,(\mathbf{a}_{\alpha,k}\cdot\mathbf{x}_{,l} + \mathbf{x}_{,k}\cdot\mathbf{a}_{\alpha,l})\Big)\;\tilde{\mathbf{G}}^k\otimes\tilde{\mathbf{G}}^l\\ & + \Big(\mathbf{x}_{,k}\cdot\mathbf{a}_\beta + \zeta^\alpha\,\mathbf{a}_{\alpha,k}\cdot\mathbf{a}_\beta + \zeta^\alpha\,\mathbf{x}_{,k}\cdot(\chi_\alpha\,\mathbf{a}_\beta + \chi_\beta\,\mathbf{a}_\alpha)\Big)\Big(\tilde{\mathbf{G}}^k\otimes\mathbf{I}^\beta + \mathbf{I}^\beta\otimes\tilde{\mathbf{G}}^k\Big)\\ & + \Big(\zeta^\alpha\,(\chi_\gamma\,\mathbf{a}_\alpha\cdot\mathbf{a}_\beta + \chi_\beta\,\mathbf{a}_\alpha\cdot\mathbf{a}_\gamma) + 2\,\zeta^\alpha\,\chi_\alpha\,\mathbf{a}_\beta\cdot\mathbf{a}_\gamma + \mathbf{a}_\beta\cdot\mathbf{a}_\gamma\Big)\;\mathbf{I}^\beta\otimes\mathbf{I}^\gamma = \mathbf{C} + \zeta^\alpha\,\mathbf{K}_\alpha\,. \quad (7)\end{aligned}$$

Note in order to obtain Eq. (7) the geometry of the micro-space must be specified. Specifically, one has to decide about the projection map $\pi_0^*(\tilde{\mathbf{G}}_i)$ which defines the transition from the tangent vectors $\tilde{\mathbf{G}}^i$, defined in the generalized space, to the tangent vectors $\mathbf{G}^i$, defined in $\mathcal{TB}$.

3 Generalized principle of virtual work

A micromorphic variational principle is established based on the generalized strain tensor $\tilde{\mathbf{C}}$ (Eq. 7). From a non-linear boundary value problem in the domain $\mathcal{B}\times\mathcal{S}$ considering the static case and considering only mechanical processes, the *first law of thermodynamics* provides the following variational statement

$$\delta\Psi - \mathcal{W}_{ext} = 0\,. \quad (8)$$

The external virtual work $\mathcal{W}_{ext}$ is defined in the Lagrangian form as follows

$$\mathcal{W}_{ext}\,(\mathbf{u}) = \int_{\mathcal{B}} \mathbf{b}\cdot\delta\mathbf{u}\;dV + \int_{\mathcal{B}} \mathbf{l}^\alpha\cdot\delta\mathbf{a}_\alpha\;dV + \int_{\partial\mathcal{B}_N} \mathbf{t}^{(\mathbf{n})}\cdot\delta\mathbf{u}\;dA + \int_{\partial\mathcal{B}_N} \mathbf{q}^{(\mathbf{n})\,\alpha}\cdot\delta\mathbf{a}_\alpha\;dA \quad (9)$$

where the external body force and moment $\mathbf{b}$ and $\mathbf{l}$, respectively, acting in $\mathcal{B}$ and the external traction and surface moment $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{q}^{(\mathbf{n})}$, respectively, acting on the Neumann boundary $\partial\mathcal{B}_N$ are obtained by integrating corresponding quantities over the micro-space $\mathcal{S}$. For more details refer to [10].

With Eq. (7) the internal virtual power in the Lagrangian form is given by

$$\delta\Psi = \int_{\mathcal{B}}\int_{\mathcal{S}} \tilde{\rho}_0\,\frac{\partial\psi(\tilde{\mathbf{C}})}{\partial\tilde{\mathbf{C}}}\,dS\,dV = \int_{\mathcal{B}} \frac{1}{2}\Big\{\mathbf{S}:\delta\mathbf{C} + \mathbf{M}^\alpha:\delta\mathbf{K}_\alpha\Big\}\;dV\,, \quad (10)$$

with the force stress and the higher-order size-scale relevant stress

$$\mathbf{S}\left(\vartheta^k\right) = \frac{1}{V_{\mathcal{S}}}\int_{\mathcal{S}} 2\,\tilde{\rho}_0\,\frac{\partial\psi(\tilde{\mathbf{C}})}{\partial\tilde{\mathbf{C}}}\;dS\,,\quad \mathbf{M}\left(\vartheta^k\right) = \frac{1}{V_{\mathcal{S}}}\int_{\mathcal{S}} 2\,\zeta^\alpha\,\tilde{\rho}_0\,\frac{\partial\psi(\tilde{\mathbf{C}})}{\partial\tilde{\mathbf{C}}}\,dS\,. \quad (11)$$

Then, substituting Eqs. (10) and (9) into Eq. (8) we end up with a micromorphic variational principle:

$$\int_{\mathcal{B}} \Big\{\mathbf{S}:\delta\mathbf{C} + \mathbf{M}^\alpha:\delta\mathbf{K}_\alpha\Big\}\;dV - \mathcal{W}_{ext} = 0\,. \quad (12)$$

The generalized principle of virtual work is supplemented by essential boundary conditions, the so-called Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{h}_u \quad \text{on } \partial\mathcal{B}_D\,, \qquad \mathbf{a}_\alpha = \mathbf{h}_{\gamma,\alpha} \quad \text{on } \partial\mathcal{B}_D\,, \tag{13}$$

where $\mathbf{h}_u$ and $\mathbf{h}_{\gamma,\alpha}$ are prescribed values at the boundary $\partial\mathcal{B}_D$.

4 The inelastic formulation

As discussed in the introduction, the inelastic constitutive law can be any classical one which is now to be defined at the level of the micro-continuum. In what follows we adopt and tailor to our purposes the formulation of finite strain inelasticity based on unified constitutive models as developed in ([9]). While the choice is convenient we stress that any alternative inelastic law could serve the purpose as well.

4.1 Generalized kinematics of the elastic-inelastic body

A point of departure for an inelastic formulation constitutes the multiplicative decomposition of the generalized deformation gradient Eq. (6) into an elastic and an inelastic part

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_e\tilde{\mathbf{F}}_p. \tag{14}$$

For metals, the above decomposition is accompanied with the assumption $\tilde{\mathbf{F}}_p \in SL^+(3,\mathbb{R})$ which reflects the incompressibility of the inelastic deformations, where $SL^+(3,\mathbb{R})$ denotes the special linear group with determinant equal one.

The following generalized right *Cauchy-Green*-type deformation tensors are defined

$$\tilde{\mathbf{C}} := \tilde{\mathbf{F}}^T\tilde{\mathbf{F}}\,, \qquad \tilde{\mathbf{C}}_e := \tilde{\mathbf{F}}_e^T\tilde{\mathbf{F}}_e\,, \qquad \tilde{\mathbf{C}}_p := \tilde{\mathbf{F}}_p^T\tilde{\mathbf{F}}_p\,. \tag{15}$$

Since the deformation gradient $\tilde{\mathbf{F}}$ is also an element of $GL^+(3,\mathbb{R})$ with positive determinant, we can attribute to its time derivative a left and right rate

$$\dot{\tilde{\mathbf{F}}} = \tilde{\mathbf{l}}\tilde{\mathbf{F}}\,, \qquad \dot{\tilde{\mathbf{F}}} = \tilde{\mathbf{F}}\tilde{\mathbf{L}}\,. \tag{16}$$

Both rates are mixed tensors (contravariant-covariant). They are related by means of the equation

$$\tilde{\mathbf{L}} = \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{l}}\tilde{\mathbf{F}}\,. \tag{17}$$

Since $\tilde{\mathbf{F}}_p \in SL^+(3,\mathbb{R})$ we can define a right rate according to

$$\dot{\tilde{\mathbf{F}}}_p = \tilde{\mathbf{F}}_p\tilde{\mathbf{L}}_p \tag{18}$$

which proves more appropriate for a numerical treatment in a purely material context.

4.2 The constitutive model

4.2.1 General considerations

Let $\tilde{\boldsymbol{\tau}}$ be the generalized *Kirchhoff* stress tensor. Consider the expression of the internal power in terms of spatial and material tensors, respectively

$$\mathcal{W} = \tilde{\boldsymbol{\tau}} : \tilde{\mathbf{l}}\,, \qquad \mathcal{W} = \tilde{\mathbf{\Gamma}} : \tilde{\mathbf{L}} \tag{19}$$

where $\tilde{\mathbf{l}}$ is defined in Eq. (16a). The comparison of Eq. (19a) with (19b) leads with the aid of Eq. (17) to the definition equation of the material stress tensor $\tilde{\mathbf{\Gamma}}$:

$$\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{F}}^{\mathrm{T}} \tilde{\boldsymbol{\tau}} \tilde{\mathbf{F}}^{-T}\,. \tag{20}$$

The tensor $\tilde{\mathbf{\Gamma}}$ is, accordingly, the mixed variant pull-back of the generalized *Kirchhoff* tensor. It coincides with Noll's intrinsic stress tensor and determines up to a spherical part the Eshelby stress tensor.

A common feature of inelastic constitutive models is the introduction of phenomenological internal variables. We denote a typical internal variable as $\boldsymbol{Z}$. Assuming the existence of a free energy function according to $\psi = \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})$, the localized form of the dissipation inequality for an isothermal process takes

$$\mathcal{D} \;=\; \tilde{\boldsymbol{\tau}} : \tilde{\mathbf{l}} - \tilde{\rho}_{\mathrm{ref}}\, \dot{\psi} = \tilde{\mathbf{\Gamma}} : \tilde{\mathbf{L}} - \tilde{\rho}_{\mathrm{ref}}\, \dot{\psi} \geq 0\,, \tag{21}$$

where ρ_{ref} is the density at the reference configuration. This inequality can be transferred to (see [10])

$$\begin{aligned}\mathcal{D} = &\left(\tilde{\mathbf{\Gamma}} - 2\tilde{\rho}_{\mathrm{ref}} \tilde{\mathbf{C}} \tilde{\mathbf{F}}_p^{-1} \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \tilde{\mathbf{C}}_e} \tilde{\mathbf{F}}_p^{-T} \right) : \tilde{\mathbf{L}} \\ &+ 2\tilde{\rho}_{\mathrm{ref}} \tilde{\mathbf{C}} \tilde{\mathbf{F}}_p^{-1} \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \tilde{\mathbf{C}}_e} \tilde{\mathbf{F}}_p^{-T} : \tilde{\mathbf{L}}_p - \tilde{\rho}_{\mathrm{ref}} \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \boldsymbol{Z}} \cdot \dot{\boldsymbol{Z}} \;\geq\; 0\,.\end{aligned}$$

By defining $\boldsymbol{Y}$ as the thermodynamical force conjugate to the internal variable $\boldsymbol{Z}$

$$\boldsymbol{Y} := -\tilde{\rho}_{\mathrm{ref}} \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \boldsymbol{Z}}, \tag{22}$$

and making use of standard thermodynamical arguments, from Eq. (22) follows the elastic constitutive equation

$$\tilde{\mathbf{\Gamma}} = 2\tilde{\rho}_{\mathrm{ref}} \tilde{\mathbf{C}} \tilde{\mathbf{F}}_p^{-1} \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \tilde{\mathbf{C}}_e} \tilde{\mathbf{F}}_p^{-T} = 2\tilde{\rho}_{\mathrm{ref}} \tilde{\mathbf{F}}_p^{T} \tilde{\mathbf{C}}_e \frac{\partial \psi(\tilde{\mathbf{C}}_e,\, \boldsymbol{Z})}{\partial \tilde{\mathbf{C}}_e} \tilde{\mathbf{F}}_p^{-T} \tag{23}$$

as well as the reduced local dissipation inequality

$$\mathcal{D}_p := \tilde{\mathbf{\Gamma}} : \tilde{\mathbf{L}}_p + \boldsymbol{Y} \cdot \dot{\boldsymbol{Z}} \geq 0\,, \tag{24}$$

where Eq. (22) has been considered. $\mathcal{D}_p$ is the plastic dissipation function. From Eq. (24) follows as an essential result that the stress tensor $\tilde{\mathbf{\Gamma}}$ and the plastic rate $\tilde{\mathbf{L}}_p$ are conjugate variables. Observe that the tensor $\tilde{\mathbf{L}}_p$ is defined in Eq. (18).

4.2.2 The elastic constitutive model

Further we assume that the elastic potential can be decomposed additively into one part depending only on the elastic generalized right *Cauchy-Green* deformation tensor $\tilde{\mathbf{C}}_e$ and the other one depending only on the internal variable $\boldsymbol{Z}$

$$\psi = \psi_{\mathrm{e}}(\tilde{\mathbf{C}}_e) + \psi_{\mathrm{Z}}(\boldsymbol{Z})\,. \tag{25}$$

Defining the logarithmic strain measure

$$\boldsymbol{\alpha} := \ln \tilde{\mathbf{C}}_{\mathrm{e}}\,, \qquad \tilde{\mathbf{C}}_e = \exp \boldsymbol{\alpha} \tag{26}$$

and assuming that the material is elastically isotropic, one can prove that the relation holds

$$\tilde{\mathbf{C}}_e \frac{\partial \psi_e(\tilde{\mathbf{C}}_e)}{\partial \tilde{\mathbf{C}}_e} = \frac{\partial \psi_{\mathrm{e}}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}\,, \tag{27}$$

where $\psi_{\mathrm{e}}(\boldsymbol{\alpha})$ is the potential expressed in the logarithmic strain measure $\boldsymbol{\alpha}$. Eq. (23) results then in

$$\tilde{\boldsymbol{\Gamma}} = 2\rho_{ref}\, \tilde{\mathbf{F}}_p^T \frac{\partial \psi_{\mathrm{e}}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \tilde{\mathbf{F}}_p^{-T}\,. \tag{28}$$

Note that ψ_{e} is an isotropic function of $\boldsymbol{\alpha}$. The last equation motivates the introduction of a modified logarithmic strain measure

$$\bar{\boldsymbol{\alpha}} := \tilde{\mathbf{F}}_p^{-1} \boldsymbol{\alpha} \tilde{\mathbf{F}}_p\,. \tag{29}$$

Since the following relation for the exponential map holds

$$\tilde{\mathbf{F}}_p^{-1}(\exp \boldsymbol{\alpha})\tilde{\mathbf{F}}_p = \exp \bar{\boldsymbol{\alpha}}, \tag{30}$$

Eq. (28) takes

$$\tilde{\boldsymbol{\Gamma}} = 2\rho_{ref}\, \frac{\partial \psi(\bar{\boldsymbol{\alpha}})}{\partial \bar{\boldsymbol{\alpha}}}\,. \tag{31}$$

It is interesting to note that Eq. (30) together with Eqs. (26), (15a), and (15c) lead to a direct definition of $\bar{\boldsymbol{\alpha}}$. The relation holds

$$\bar{\boldsymbol{\alpha}} = \ln(\tilde{\mathbf{C}}_p^{-1} \tilde{\mathbf{C}}). \tag{32}$$

For computational simplicity a linear relation is assumed and therefore the elastic constitutive model Eq. (31) takes its final form

$$\tilde{\boldsymbol{\Gamma}} = K\, \mathrm{tr}\, \bar{\boldsymbol{\alpha}}^T\, \mathbf{1} + \mu\, \mathrm{dev} \bar{\boldsymbol{\alpha}}^T \tag{33}$$

where

$$\bar{\boldsymbol{\alpha}}^T = \ln(\tilde{\mathbf{C}}\tilde{\mathbf{C}}_p^{-1})\,, \tag{34}$$

and K is the bulk modulus and μ the shear modulus.

It should be stressed that the reduction of the elastic constitutive law to that given by Eq. (31) results in a considerable simplification of the computations necessary for the formulation of the weak form of equilibrium and its corresponding linearisation. The only assumption we used was the very natural one of having an internal potential depending on $\tilde{\mathbf{C}}_e$. The following reduction is carried out systematically.

4.2.3 Inelastic constitutive model

The presented framework of generalized continua allows for the application of any set of classical constitutive laws. In what follows we confine ourselves to a unified constitutive law of the Bodner and Partom type as generalized in the first author's previous work (see e.g. [9]). We concluded from Eq. (24) that the tensors $\tilde{\boldsymbol{\Gamma}}$ and $\tilde{\mathbf{L}}_p$ are conjugate. Essentially we have to consider the stress tensor $\tilde{\boldsymbol{\Gamma}}$ as the driving stress quantity, while the plastic rate for which an evolution equation is to be formulated is taken to be $\tilde{\mathbf{L}}_p$. This leads to the following set of evolution equations

$$\begin{aligned}
\tilde{\mathbf{L}}_p &= \dot{\phi}\boldsymbol{\nu}^{\mathrm{T}}\,, \quad \dot{Z} = \frac{M}{Z_0}(Z_1 - Z)\dot{W}_p\,, \quad \dot{W}_p = \Pi_{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}}\dot{\phi}(\Pi_{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}}, Z)\,, \quad \boldsymbol{\nu} = \frac{3}{2}\frac{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}}{\Pi_{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}}} \\
\Pi_{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}} &= \sqrt{\frac{3}{2}\mathrm{dev}\tilde{\boldsymbol{\Gamma}} : \mathrm{dev}\tilde{\boldsymbol{\Gamma}}}\,, \quad \dot{\phi} = \frac{2}{\sqrt{3}}D_0 \exp\left[-\frac{1}{2}\frac{N+1}{N}\left(\frac{Z}{\Pi_{\mathrm{dev}\tilde{\boldsymbol{\Gamma}}}}\right)^{2N}\right]\,.
\end{aligned} \tag{35}$$

Here, Z_0, Z_1, D_0, N, M are material parameters. The choice of the transposed quantity in Eq. (35a) reflects the form given by associative viscoplasticity, when the classical flow functions are generalized and formulated in terms of nonsymmetric quantities.

5 Numerical examples

In this section two numerical examples are presented to demonstrate the applicability of the micromorphic theory. In this specific case the micro-deformation and so micro-continuum are assumed to be one-dimensional, i.e. we consider only $\alpha = 1$ in Eq. 5). There vector $\mathbf{a}$ in the generalized reference configuration is defined to be parallel to the $x-$axis. The material parameters, typical for metals, are chosen as follows: $K = 1.64206E02$ N/mm^2, $\mu = 1.6194E02$ N/mm^2, $D_0 = 10000$ 1/sec, $Z_0 = 1150$ N/mm^2, $Z_1 = 1400$ N/mm^2, $N = 1$ and $M = 100$. The inelastic parameters are reported in the literature for titanium; e.g. [1].

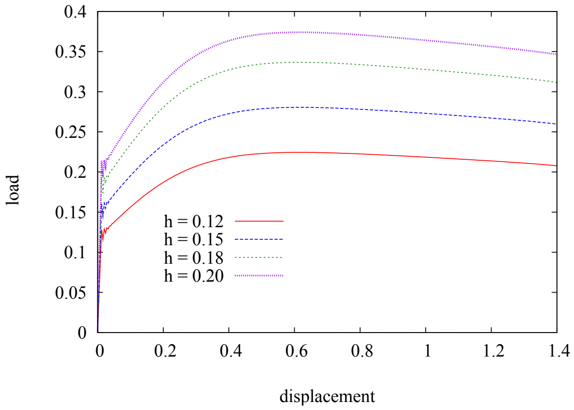

Figure 1: load displacement graph illustrating size-scale effects for different values of the internal length parameter h

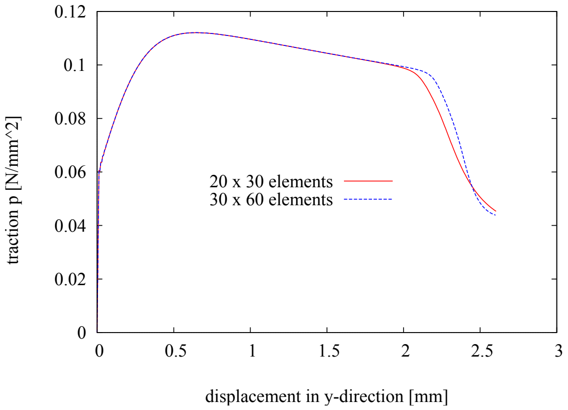

Figure 2: load displacement graph illustrating computations using 20×30 elements and 30×60 elements

5.1 Simple tension

The first example is a thin sheet of dimensions 26×10 subjected to simple tension. One quarter of the sheet is discretised using 5×5 enhanced 4-node finite elements of the type developed in [9], which, in this specific case, are equivalent to three-dimensional enhanced 8-node elements with thickness 1. The aim here is to illustrate size-scale effects in the viscoplastic regime at homogenous deformations. We consider four different internal length-scale parameters, denoted by h, which are nothing but the size/length of the microspace $\mathcal{S}$, and take them to be of the values 0.12, 0.15, 0.18 and 0.20 mm. The time step used is $0.1 sec$ for the displacement at the top increasing by a velocity of $0.02 N/sec$. While the specimen is under force loading with no prescribed displacements at the loading side, the computations are carried out displacement-controlled with the value of the external loading being scaled and determined to provide the prescribed displacement velocity.

The corresponding load-displacement graphs are depicted in Fig. 1. With increasing internal length-scale parameter it can be clearly seen that the onset of the plastic deformation takes place at larger loading values. During the plastic deformation the relative loading offset between the curves is maintained.

Now, this case of simple tension particularly illustrates the attractiveness of the proposed generalized theory as it predicts scale effects also in a homogenously deforming specimen. In fact alternative theories, such as micropolar (Cosserat) or strain gradient approaches, lack the means to predict this kind of scale effect. This is clear, because the former necessitates the rotation gradient and the latter the deformation gradient of higher order not to vanish. In this example, however, both of them do not arise and consequently, no scale effects would be observed.

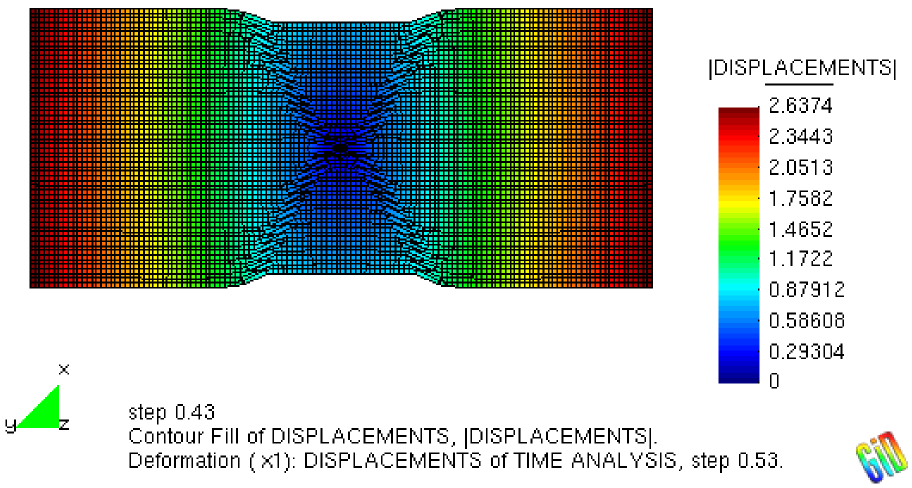

Figure 3: final deformed configuration displaying the shearband formation using 30×60 elements

5.2 Shearband formation

The second numerical example is the same as before in terms of geometry, loading, and time step - a thin sheet under tension. Shearbanding is initialized by decreasing the material parameter Z_0 by 10% within the first element (at the centre of the specimen). The internal length is considered to be $h = 0.1\,mm$. One quarter of the sheet is modeled using 20×30 and 30×60 4-node elements of the type described above. From the load-displacement curve in Fig. (2) it is clear that heavy softening related to the shearband formation takes place. This softening is independent of the mesh since both meshes give essentially the same results. The deformed configuration is pictured in Fig. (3).

Note that the constitutive law is of the viscoplastic type. However, the Bodner-Partom model covers in the limit the time-independent case as well. The present choice of material parameters together with the applied loading velocity renders the time-dependent effect rather very small. Also, from the previous example we can conclude that the scale effect due to the micromorphic formulation is dominant here.

6 Conclusion

A general framework for a micromorphic continuum has been developed which is especially attractive for non-linear material behaviour. This approach motivates research into experimental verification of the mentioned extra degrees of freedom which is still elusive at large. While it is clear that generalized degrees of freedom and the internal lengths as well as the scale effects associated with them are related to the internal structure of the material, the direct deformation mechanisms at the micro-scale giving rise to such degrees of freedom are widely subject to intensive research in many areas of mechanics and physics with many open questions. It is very likely that more than one mechanism could lead to a certain type of degrees of freedom. While these questions are beyond the scope of the present work we do acknowledge their importance. Multi-scale modelling and experimentation will be at the heart of any answer.

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