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# THERMODYNAMIC CONSISTENT GRADIENT-POROPLASTICITY THEORY FOR POROUS MEDIA

**GUILLERMO ETSE* AND JAVIER L. MROGINSKI†**

*CONICET and Center for Numerical and Computational Methods in Engineering (CEMNCI)
Faculty for Exact Sciences and Technology, Tucuman National University
Av. Roca 1800, (4000) Tucumán, Argentina
e-mail: getse@herrera.unt.edu.ar

†CONICET and Applied Mechanics Department
Faculty of Engineering, Northeaster National University
Av. Las Heras 727, Resistencia, Chaco, Argentina
e-mail: javierm@ing.unne.edu.ar

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**Abstract.** Complex degradation processes of partial saturated media like soils during post-peak regime are strongly dependent on humidity, stress state, boundary conditions and material parameters, particularly porosity. To realistically and objectively describe the dramatic change from diffuse to localized failure mode or from ductile to brittle ones, accurate constitutive theories and numerical approaches are required. In this paper, a non-local gradient poroplastic model is proposed for partial saturated media based on thermodynamic concepts. A restricted non-local gradient theory is considered, following (Mroginski, et al. Int. J. Plasticity, 27:620-634) whereby the state variables are the only ones of non-local character. The non-local softening formulation of the proposed constitutive theory incorporates the dependence of the gradient characteristic length on both the governing stress and hydraulic conditions to realistically predict the size of the maximum energy dissipation zone. The material model employed in this work to describe the plastic evolution of porous media is the Modified Cam Clay, which is widely used in saturated and partially saturated soil mechanics. To evaluate the dependence of the transition point between ductile and brittle failure regime in terms of the hydraulic and stress conditions, the localization indicator for discontinuous bifurcation is formulated for both drained and undrained conditions, based on wave propagation criterion.

## 1 INTRODUCTION

The mechanic of porous media constitutes a discipline of great relevance in several knowledge areas like the Geophysics, the Civil Engineering, the Biomechanics and the Materials Science. The main purpose of the mechanic of porous media is the deformation modelling and pore pressure prediction, when the body is been subjected to several external actions and physical phenomena. By the way, the complexity of the real engineering materials implies that for its appropriate modelling they should be included in the concept or theory of partially saturated porous media with cohesive-frictional properties diverse. Besides, the failure behaviour of engineering materials during monotonic loading processes demonstrate a strong dependence on both the stress state and the hydraulic conditions governed during the process. In spite of this fact most of the proposals for continuous formulations of engineering materials like concrete and soils are based on non-porous continua theory [9, 12]. In fact, the traditional formulations commonly accepted by the scientific community for the study of this kind of materials are experimental evidences founded, and a consistent elastoplastic thermodynamic framework is not fully considered [8, 14, 18]. Although it provides a general useful approach for a lot of engineering problems.

Even though noteworthy theoretical developments based on the theory of porous media were recently presented [4, 5, 10]. Nevertheless, it can be observed a necessity of new non-local formulations based on the theory of porous media in order to solve the critical problem of uniqueness loss of the numerical solution in post-peak regime or in pre-peak regime when the volumetric elastoplastic behavior is non-associate.

In this work the thermodynamically consistent formulation for gradient-based elastoplasticity by [22] that follows general thermodynamic approach by [19] for non-local damage formulation is extended for porous media. Main feature of present proposal is the definition of a gradient-based characteristic length in terms of both the governing stress and hydraulic conditions to capture the variation of the transition from brittle to ductile failure mode of cohesive-frictional porous materials with the level of confinement pressure and saturation. Relevant items in this work are, on the one hand, the particularization of the proposed thermodynamically consistent theory for non-local elastoplastic porous media to partially saturated soils. On the other hand, the formulation of the discontinuous bifurcations condition and related failure indicator as well as their evaluations for different hydraulic conditions.

## 2 POROUS MEDIA DESCRIPTION

Porous media are multiphase systems with interstitial voids in the grain matrix filled with water (liquid phase), water vapor and dry air (gas phase) at microscopic level [6, 9] (see Fig. 1a).

Key argument to reconcile continuum mechanics with the intrinsic microscopic discontinuities of porous like materials composed by several interacting phases, is to consider them as thermodynamically open continuum systems (see Fig. 1b). Thus, their kinematics and deformations are referred to those of the skeleton. Contrarily to mixture theories based upon an averaging process [13, 14], the representation of porous media is made by a superposition, in time and space, of two or more continuum phases. In case of non-saturated porous continua we recognize three phases, the skeleton, the liquid and the gaseous phases.

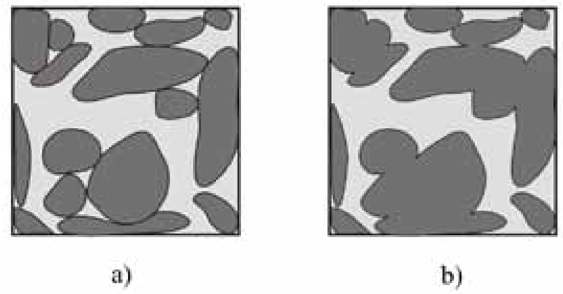


**Figure 1**: Porous media description. a) Microscopic level ; b) Macroscopic level

### 2.1 Stress tensors

The mechanical behavior of partially saturated porous media is usually described by the effective stress tensor $\boldsymbol{\sigma}'$, as follows

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\delta} p^w = \boldsymbol{\sigma}^n + \boldsymbol{s} \tag{1}$$

being $\boldsymbol{\sigma}$, $\boldsymbol{s} = \boldsymbol{\delta}\left(p^a - p^w\right)$ and $\boldsymbol{\sigma}^n = \boldsymbol{\sigma} - \boldsymbol{\delta} p^a$ the total, net, and suction stress tensors, respectively, while $\boldsymbol{\delta}$ is the Kronecker delta. Moreover, $p^a$ and $p^w$ are the gas and water pore pressures, respectively. In several geotechnical problems the gas pore pressure can be considered as a constant term that equals the atmospheric pressure [18]. In these cases the suction tensor is counterpart to the water pore pressure, $p$.

### 2.2 Flow theory of poroplasticity

Plasticity is a property exhibited by various materials to undergo permanent strains after a complete process of loading and unloading. Hence, poroplasticity is that property of porous media which defines their ability to undergo not only permanent skeleton strains, but also permanent variations in fluid mass content due to related porosity variations. To characterize current stages of thermodynamically consistent poroelastoplastic media and to describe their irreversible evolutions, internal variables such as the plastic porosity $\phi^p$ or the plastic fluid mass content $m^p$ must be considered in addition to the plastic strain $\boldsymbol{\varepsilon}^p$, and the irreversible entropy density $s^p$.

Assuming the additive decomposition of Prandtl-Reuss type to the thermodynamic variables into elastic and plastic parts

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \\ \dot{m} &= \dot{m}^e + \dot{m}^p \\ \dot{s} &= \dot{s}^e + \dot{s}^p \end{aligned} \tag{2}$$

Both, the rate of skeleton plastic strains $\dot{\boldsymbol{\varepsilon}}^p$ and the rate of plastic fluid mass content $\dot{m}^p$ are related to the irreversible evolution of the skeleton. Indeed, let $\dot{\phi}^p$ be the rate of plastic porosity

$$\dot{\phi}^p = \frac{\dot{m}^p}{\rho_0^{fl}} \tag{3}$$

with $\rho_0^{fl}$ the initial fluid mass density.

## 3 GRADIENT-POROPLASTICITY

In this section the fundamentals of the thermodynamically consistent gradient plasticity theory for porous media by Mrognski, Etse and Vrech (2011) [15] are shortly described.

### 3.1 Dissipative stress in non-local porous media

Based on previous studies developed by [6, 19], we assume that arbitrary thermodynamic states of the dissipative material during isothermal processes are completely determined by the elastic strain tensor $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$, and the internal variables $q_\alpha$ with $\alpha = s, p$ for solid or porous phase, respectively, which are considered here as scalar variables. When considering poroplastic materials the elastic variation of fluid mass content $m^e = m - m^p$ needs also to be included as a thermodynamic argument of the free energy, see [6]. Also, in order to capture non-local effects produced by monotonic external actions on each phase of the porous media we further assume that the gradient of the internal variables $\nabla q_\alpha$, are the only ones of non-local character [15, 19, 22]. The extension to more than two scalar internal variables is straightforward. Hence, both $q_\alpha$ and $\nabla q_\alpha$ will appear as arguments in the Helmholtzs free energy

$$\Psi = \Psi\left(\boldsymbol{\varepsilon}^e, m^e, q_\alpha, \nabla q_\alpha\right) \tag{4}$$

While the Clausius-Duhem inequality (CDI),

$$\int_\Omega \frac{1}{\theta}\left[ \left(\boldsymbol{\sigma} - \rho\partial_{\boldsymbol{\varepsilon}^e}\Psi\right) : \dot{\boldsymbol{\varepsilon}} + \left(p - \rho\partial_{m^e}\Psi\right)\dot{m} + \rho\partial_{\boldsymbol{\varepsilon}^e}\Psi : \dot{\boldsymbol{\varepsilon}}^p + \rho\partial_{m^e}\Psi\dot{m}^p + \right.$$
$$\left. - \sum_\alpha \rho\partial_{q_\alpha}\Psi\dot{q}_\alpha - \sum_\alpha \rho\partial_{\nabla q_\alpha}\Psi\nabla\dot{q_\alpha}\right] \mathrm{d}\Omega \geq 0 \quad (5)$$

where $\boldsymbol{\sigma}$ is the stress tensor, $p$ is the pore pressure and $\rho$ the mass density. Also, a following compact notation for partial derivative was adopted, $\partial_x(\bullet) = \frac{\partial(\bullet)}{\partial x}$. By, integrating the gradient term by parts and using de Divergence Theorem results

$$\int_\Omega \frac{1}{\theta}\left[(\boldsymbol{\sigma} - \rho\partial_{\boldsymbol{\varepsilon}^e}\Psi) : \dot{\boldsymbol{\varepsilon}} + (p - \rho\partial_{m^e}\Psi)\,\dot{m} + \rho\partial_{\boldsymbol{\varepsilon}^e}\Psi : \dot{\boldsymbol{\varepsilon}}^p + \rho\partial_{m^e}\Psi\dot{m}^p + \right. \\ \left. + \sum_\alpha Q_\alpha \dot{q}_\alpha\right] \mathrm{d}\Omega + \int_{\partial\Omega} \sum_\alpha Q_\alpha^{(b)}\dot{q}_\alpha \;\mathrm{d}\partial\Omega \geq 0 \quad (6)$$

Where the dissipative stresses $Q_\alpha$ and $Q_\alpha^{(b)}$ defined in the domain $\Omega$ and on the boundary $\partial\Omega$, respectively, as

$$Q_\alpha = -\rho\partial_{q_\alpha}\Psi - \nabla\cdot(\rho\partial_{\nabla q_\alpha}\Psi) \qquad \text{in } \Omega \qquad (7)$$

$$Q_\alpha^{(b)} = -\rho\partial_{\nabla q_\alpha}\Psi\mathbf{n} \qquad \text{on } \partial\Omega \qquad (8)$$

where introduced.

In the standard local theory it is postulated that the last inequality of Eq. (6) must hold for any choice of domain and for any independent thermodynamic process. As a result, Colemans equation are formally obtained like in local plasticity.

$$\boldsymbol{\sigma} = \rho\partial_{\boldsymbol{\varepsilon}^e}\Psi \qquad (9)$$

$$p = \rho\partial_{m^e}\Psi \qquad (10)$$

$$\mathfrak{D} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^e + p\dot{m}^p + \sum_\alpha Q_\alpha\dot{q}_\alpha \geq 0 \qquad \text{in } \Omega \qquad (11)$$

$$\mathfrak{D}^{(b)} = \sum_\alpha Q_\alpha^{(b)}\dot{q}_\alpha \geq 0 \qquad \text{on } \partial\Omega \qquad (12)$$

In case of $p = 0$ above equations takes similar form to those obteined by [19, 22] for non-porous media. Also, from Eqs. (11) and (12) it can be concluded that the dissipative stress $Q_\alpha$ can be decomposed into the local and non-local components

$$Q_\alpha = Q_\alpha^{loc} + Q_\alpha^{nloc} = -\rho\partial_{q_\alpha}\Psi - \rho\nabla\cdot(\partial_{\nabla q_\alpha}\Psi) \qquad (13)$$

### 3.2 Thermodynamically consistent gradient-based constitutive relationship

Based on previous works [19, 22], the following additive expression is adopted for the free energy density of non-local gradient poroplastic materials

$$\Psi(\boldsymbol{\varepsilon}^e, m^e, q_\alpha, \nabla q_\alpha) = \Psi^e(\boldsymbol{\varepsilon}^e, m^e) + \Psi^{p,loc}(q_\alpha) + \Psi^{p,nloc}(\nabla q_\alpha) \qquad (14)$$

with the elastic energy density,

$$\rho\Psi^e = \boldsymbol{\sigma}^0 : \boldsymbol{\varepsilon}^e + p^0 m^e + \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{C}^0 : \boldsymbol{\varepsilon}^e + \frac{1}{2}M\left(\mathbf{B} : \boldsymbol{\varepsilon}^e - m^e\right)^2 \tag{15}$$

$\Psi^{p,loc}$ and $\Psi^{p,nloc}$ are the local and non-local gradient contributions due to dissipative hardening/softening behaviors, which are expressed in terms of both the internal variables and their gradient, $q_\alpha$ and $\nabla q_\alpha$, respectively.

### 3.3 Non-local plastic flow rule

For general non-associative flow and hardening rule, we introduce the dissipative potential $\Phi^*$ such that non-associative flow and hardening rules are defined

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\partial_{\boldsymbol{\sigma}}\Phi^* = \dot{\lambda}\mathbf{m}_\sigma \quad ; \quad \dot{m}^p = \dot{\lambda}\partial_p\Phi^* = \dot{\lambda}\mathrm{m}_p \quad ; \quad \dot{q_\alpha} = \dot{\lambda}\partial_{Q_\alpha}\Phi^* = \dot{\lambda}\mathrm{m}_{Q_\alpha} \tag{16}$$

where $\mathbf{m}_\sigma = \partial_{\boldsymbol{\sigma}}\Phi^*$ , $\mathrm{m}_p = \partial_p\Phi^*$ and $\mathrm{m}_{Q_\alpha} = \partial_{Q_\alpha}\Phi^*$, being $\Phi^*$ the plastic dissipative potential. To complete problem formulation in $\Omega$, the Kuhn-Tucker complementary conditions are introduced as follow

$$\dot{\lambda} \geq 0 \qquad ; \qquad \Phi\left(\boldsymbol{\sigma}, p, Q_\alpha\right) \leq 0 \qquad ; \qquad \dot{\lambda}\Phi\left(\boldsymbol{\sigma}, p, Q_\alpha\right) = 0 \tag{17}$$

### 3.4 Rate constitutive equations

In the undrained condition and considering the additive decomposition of the free energy potential in Eq. (14) and the flow rule of Eq. (16), the following rate expressions of the stress tensor $\dot{\boldsymbol{\sigma}}$ and pore pressure $\dot{p}$ are obtained

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\mathbf{C} : \mathbf{m}_\sigma - M\mathbf{B}\dot{m} + \dot{\lambda}M\mathbf{B}\mathrm{m}_p \tag{18}$$

$$\dot{p} = -M\mathbf{B} : \dot{\boldsymbol{\varepsilon}} + \dot{\lambda}M\mathbf{B} : \mathbf{m}_\sigma + M\dot{m} - \dot{\lambda}M\mathrm{m}_p \tag{19}$$

being $M$ the Biot's module, $\mathbf{B} = b\mathbf{I}$ with $b$ the Biot coefficient and $\mathbf{I}$ the second-order unit tensor, and $\mathbf{C} = \mathbf{C}^0 + M\mathbf{B}\otimes\mathbf{B}$, whereby $\mathbf{C}^0$ is the fourth-order elastic tensor which linearly relates stress and strain.

After multiplying Eq. (19) by $\mathbf{B}$ and combining with Eq. (18), a more suitable expression of the rate of the stress tensor for drained condition is achieved

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^0 : \dot{\boldsymbol{\varepsilon}} - \mathbf{B}\dot{p} - \dot{\lambda}\mathbf{C}^0 : \mathbf{m}_\sigma \tag{20}$$

while the evolution laws of the local and non-local dissipative stress in Eq. (13) results

$$\dot{Q}_\alpha^{loc} = -\dot{\lambda}H_\alpha^{loc}\mathbf{m}_Q \tag{21}$$

$$\dot{Q}_\alpha^{nloc} = l_\alpha^2\nabla\cdot\left(\mathbf{H}_\alpha^{nloc}\nabla\dot{\lambda}\mathrm{m}_{Q_\alpha} + \dot{\lambda}\mathbf{H}_\alpha^{nloc}\cdot\nabla Q_\alpha\mathbf{m}_Q^2\right) \tag{22}$$

where $\mathbf{m}_Q^2 = \partial^2\Phi^*/\partial Q^2$. Thereby, local hardening/softening module $H_\alpha^{loc}$ have been introduced as well as the new non-local hardening/softening tensor $\mathbf{H}_\alpha^{nloc}$ as defined in [19]

$$H_\alpha^{loc} = \rho \frac{\partial^2 \Psi^{p,loc}}{\partial q_\alpha^2} \quad , \quad \mathbf{H}_\alpha^{nloc} = \rho \frac{1}{l_\alpha^2} \frac{\partial^2 \Psi^{p,nloc}}{\partial \nabla q_\alpha \partial \nabla q_\alpha} \tag{23}$$

with $\mathbf{H}_\alpha^{nloc}$ a second-order positive defined tensor. For the characteristic length $l_\alpha$ three alternative definitions can be given, see [16, 19, 20]. On the one hand, it can be defined as a convenient dimensional parameter so as $H_\alpha^{loc}$ and $\mathbf{H}_\alpha^{nloc}$ will get the same dimension. On the other hand, $l_\alpha$ can be interpreted as an artificial numerical stabilization mechanism for the non-local theory. Alternatively, as a physical entity that characterizes the material microstructure. In this last case, and for calibration porpuse, specific numerical analysis on the representative volume element (RVE) need to be performed at micro scale level.

## 4 MODIFIED CAM CLAY CONSTITUTIVE MODEL FOR GRADIENT PLASTICITY

The modified Cam Clay plasticity model was originally proposed by [17] for normally consolidated clays. However, due to accurate predictions of consolidated clay mechanical behavior obtained with this model and the reduced number of involved parameters it has been extended to a wide range of soils including unsaturated soils [1, 3].

The main characteristics of the modified Cam Clay plasticity model are:

**a-** The yield function is an ellipse on the $(\sigma', \tau)$ plane

**b-** The volumetric component of the plastic strain on the Critical State Line (CSL) is null while the plastic flow develops under constant volume

**c-** Associated plasticity is assumed

The yield function is defined by

$$\Phi(\sigma', \tau, Q_\alpha) = \left(\sigma' + \frac{\tau^2}{m^2 \sigma'}\right) - Q_\alpha \tag{24}$$

where $\sigma' = I_1/3 - \beta p$ is the effective hydrostatic stress, $\tau = \sqrt{3J_2}$ the shear stress, $m$ the CSL slope and $Q_\alpha$ thermodynamically consistent dissipative stress equivalent to the preconsolidation pressure $p_{co}$. Also $I_1$ and $J_2$ are the first and second invariants of the stress tensor and the deviator tensor, respectively.

To avoid overestimation of the volumetric compressibility coefficient $K_0$ by the conventional critical state model a non-associated flow rule was introduced by [11, 2]. Thereby, the following plastic potential function is proposed

$$\Phi^* \left(\sigma', \tau, Q_\alpha\right) = \eta \left(\sigma'^2 - \sigma' Q_\alpha\right) + \left(\frac{\tau}{m}\right)^2 \tag{25}$$

The $\eta$ coefficient is a restriction function limiting the influence of the volumetric pressure on the softening regime,

$$\eta = \eta_0 + \frac{a\left(1 + m\,\exp\left(\frac{-(\sigma+\beta p)}{\upsilon}\right)\right)}{1 + n\,\exp\left(\frac{-(\sigma+\beta p)}{\upsilon}\right)} \tag{26}$$

being $a$, $n$ and $m$ internal parameters of the exponential function, $\eta_0 = 1$ and $\upsilon = \mathrm{abs}\left(p_{co}/2\right)$.

The thermodynamic consistency is achieved by assuming the following expression for the dissipative part of the free energy in Eq. (14)

$$\rho\Psi^p\left(\varepsilon^p, \nabla\varepsilon^p\right) = \rho\Psi^{p,loc}\left(\varepsilon^p\right) + \rho\Psi^{p,nloc}\left(\nabla\varepsilon^p\right) = -\frac{1}{\chi} p_{co}^0 \exp\left(\chi\varepsilon^p\right) - \frac{1}{2} l_\alpha^2 \mathbf{H}^{nloc} \nabla^2 \varepsilon^p \tag{27}$$

where $\varepsilon^p$ is the volumetric plastic strain of the continuous solid skeleton expressed as a function of the internal variables which describe the plastic evolution of the porous and solid phases, in terms of the plastic porosity $\phi^p$ and the plastic volumetric strain of soil grain $\varepsilon_s^p$, respectively

$$\varepsilon^p = \phi^p + \left(1 - \phi_0\right) \varepsilon_s^p \tag{28}$$

From Eq. (13) the following expressions for local and non-local dissipative stresses are obtained

$$Q_\alpha^{loc}\left(\varepsilon^p\right) = -\rho\partial_{\varepsilon^p}\Psi = p_{co}^0 \exp\left(\chi\left(\phi^p + \left(1 - \phi_0\right)\varepsilon_s^p\right)\right) \tag{29}$$

$$Q_\alpha^{nloc}\left(\nabla\varepsilon^p\right) = -\rho\nabla\cdot\left(\partial_{\nabla\varepsilon^p}\Psi\right) = l_s^2 \mathbf{H}_s^{nloc} \nabla^2 \varepsilon_s^p + l_p^2 \mathbf{H}_p^{nloc} \nabla^2 \phi^p \tag{30}$$

where $l_s$ and $l_p$ are the characteristic length for solid and porous phase, respectively.

## 5 INSTABILITY ANALYSIS OF POROUS MEDIA

The global failure in a continuous media is generally preceded by local discontinuities taking place in areas or regions where the constituent material is subjected to a post-pick stress state. A large number of materials failure studies have been developed in the framework of continuous mechanics. Thereby, a succession of events that begins at microscopic scale and cause the progressive deterioration of the medium, which is initially treated as a continuous one, until transforming it in a discontinuous medium. Therefore, the following failure shapes are defined:

1. *Discrete failure*: this type of analysis lies beyond to the continuum mechanics and belongs the fracture mechanics. The discontinuity is presented in the displacement velocity field, i.e. $[[\dot{\boldsymbol{u}}]] \neq 0$ [1]

2. *Localized failure*: this analysis is characterized by the continuity in the displacement velocity field, while its gradient exhibits the discontinuity, i.e. $[[\dot{\boldsymbol{u}}]] = 0$ and $[[\dot{\boldsymbol{\varepsilon}}]] \neq 0$

3. *Diffuse failure*: this behaviour is generally presented in ductile materials. In this case the both velocity and deformation rate remains continuous, i.e. $[[\dot{\boldsymbol{u}}]] = 0$ and $[[\dot{\boldsymbol{\varepsilon}}]] = 0$.

These concepts of the solids mechanics can be appropriately extrapolated to the mechanics of porous media considering that the medium is composed by a solid skeleton surrounded, in the general case, by several fluids phases. The influence of these fluids phases is taken into account by its corresponding pore pressure.

Considering the Kuhn-Tucker complementary condition, the incremental constitutive equations Eq. (18) and Eq. (19), and the decomposition of the dissipative stress rate Eq. (21) and Eq. (22), the following expression for the plastic multiplier can be obtained

$$\dot{\lambda} = \left(\dot{\Phi}^e + \dot{\Phi}^{nloc}\right)/h \tag{31}$$

with

$$\dot{\Phi}^{nloc} = l_\alpha^2 \partial_{Q_\alpha}\Phi \left\{ \partial_{Q_\alpha}\Phi^* \left[ \mathbf{H}_\alpha^{nloc} \nabla^2 \dot{\lambda} + \nabla \mathbf{H}_\alpha^{nloc} \nabla \dot{\lambda} \right] + 2\partial^2_{Q_\alpha}\Phi^* \nabla Q_\alpha \mathbf{H}_\alpha^{nloc} \nabla \dot{\lambda} \right\} \tag{32}$$

$$\dot{\Phi}^e = \dot{\Phi}^e_s + \dot{\Phi}^e_p = \left(\partial_{\boldsymbol{\sigma}}\Phi \mathbf{C}\dot{\boldsymbol{\varepsilon}} - \mathrm{M}\partial_p \Phi \mathbf{B}\dot{\boldsymbol{\varepsilon}}\right) + \dot{m}/\rho_0^{fl}\left(\mathrm{M}\partial_p\Phi - \partial_{\boldsymbol{\sigma}}\Phi\mathbf{B}\right) \tag{33}$$

$$h = h_s + h_p + \bar{H} = \partial_{\boldsymbol{\sigma}}\Phi\mathbf{C}\partial_{\boldsymbol{\sigma}}\Phi^* + \mathrm{M}\left(-\partial_{\boldsymbol{\sigma}}\Phi\mathbf{B}\partial_p\Phi^* - \partial_p\Phi\mathbf{B}\partial^*_{\boldsymbol{\sigma}} + \partial_p\Phi\partial_p\Phi^*\right) + \bar{H}^{loc}_\alpha \tag{34}$$

with $\bar{H}^{loc}_\alpha = H^{loc}_\alpha \partial_{Q_\alpha}\Phi\partial_{Q_\alpha}\Phi^*$.

Since we shall only concerned with the possibility of bifurcations in the incremental solution, the difference between two possible solutions of $\dot{\boldsymbol{\sigma}}$ must satisfy the homogeneous equilibrium equations, i.e. $\nabla\dot{\boldsymbol{\sigma}} = 0$ .

An infinitive domain is considered and the solutions for the displacement rate $\dot{\boldsymbol{u}}$, the plastic multiplier $\dot{\gamma}$ and the mass content $\dot{\gamma}$ are expressed in terms of plane waves, as follows [19]

[1] $[[\bullet]]$ is the jump operator, defined by $[[\bullet]] = \bullet^+ - \bullet^-$

$$\dot{u}\left(\boldsymbol{x},t\right)=\dot{\mathcal{U}}\left(t\right)\exp\left(\frac{i2\pi}{\delta}\boldsymbol{n}\cdot\boldsymbol{x}\right) \tag{35}$$

$$\dot{\gamma}\left(\boldsymbol{x},t\right)=\dot{\mathcal{M}}\left(t\right)\exp\left(\frac{i2\pi}{\delta}\boldsymbol{n}\cdot\boldsymbol{x}\right) \tag{36}$$

$$\dot{\lambda}\left(\boldsymbol{x},t\right)=\dot{\mathcal{L}}\left(t\right)\exp\left(\frac{i2\pi}{\delta}\boldsymbol{n}\cdot\boldsymbol{x}\right) \tag{37}$$

being $\boldsymbol{x}$ the position vector, $\boldsymbol{n}$ is the normal direction of the wave and $\delta$ is the wave length. Also, $\dot{\mathcal{U}}$, $\dot{\mathcal{M}}$ and $\dot{\mathcal{L}}$ are spatially homogeneous amplitudes of the wave saolutions.

Upon introducing the Eq. (31) into the incremental constitutive equations, Eq. (18), satisfying the equilibrium equation on the discontinuity surface, and considering the assumed solutions given in Eqs. (35)-(37), it is concluded that this equation is satisfied for each $\boldsymbol{x}$ if

$$\left(\frac{2\pi}{\delta}\right)^2\mathbf{n}\cdot\left\{\mathbf{C}^0-\frac{\mathbf{C}^0\partial_{\boldsymbol{\sigma}}\Phi^*\otimes\partial_{\boldsymbol{\sigma}}\Phi\mathbf{C}^0}{h+\bar{h}^{nloc}}\right\}\cdot\mathbf{n}\,\dot{\mathcal{U}}=0 \tag{38}$$

for drained conditions ($\dot{p}=0$), and

$$\left(\frac{2\pi}{\delta}\right)^2\mathbf{n}\cdot\left\{\mathbf{C}-\frac{\mathbf{C}\partial_{\boldsymbol{\sigma}}\Phi^*\otimes\partial_{\boldsymbol{\sigma}}\mathbf{C}}{h+\bar{h}^{nloc}}-\mathrm{M}^2\frac{\partial_p\Phi^*\mathbf{B}\otimes\mathbf{B}\partial_p\Phi}{h+\bar{h}^{nloc}}+\right.$$
$$\left.\mathrm{M}\left(\frac{\mathbf{C}\partial_{\boldsymbol{\sigma}}\Phi^*\otimes\mathbf{B}\partial_p\Phi}{h+\bar{h}^{nloc}}+\frac{\partial_p\Phi^*\mathbf{B}\otimes\mathbf{C}\partial_{\boldsymbol{\sigma}}\Phi}{h+\bar{h}^{nloc}}\right)\right\}\mathbf{n}\,\dot{\mathcal{U}}=0 \tag{39}$$

for undrained conditions ($\dot{m}=0$), where $\bar{h}^{nloc}$ is the generalized gradient module.

$$\bar{h}^{nloc}=\bar{h}_s^{nloc}+\bar{h}_p^{nloc}=\mathbf{n}\cdot\left[l_s^2\left(\partial_{Q_s}\Phi\partial_{Q_s}\Phi^*\mathbf{H}_s^{nloc}\right)+l_p^2\left(\partial_{Q_p}\Phi\partial_{Q_p}\Phi^*\mathbf{H}_p^{nloc}\right)\right]\cdot\mathbf{n}\left(\frac{2\pi}{\delta}\right)^2 \tag{40}$$

By calling $\mathbf{A}^{d,nloc}$ and $\mathbf{A}^{u,nloc}$ to the expressions into the bracket of Eq. (38) and Eq. (39), respectively, the acoustic tensor for gradient-regularized plasticity under drained and undrained conditions are deduced. It is clear that these expressions differs from the local counterpart only by the additional term $\bar{h}^{nloc}$. Thereby, when $l_\alpha=0$ the acoustic tensor for local plasticity is recovered, $\mathbf{A}^{d,loc}=\mathbf{A}^{d,nloc}$ and $\mathbf{A}^{u,loc}=\mathbf{A}^{u,nloc}$.

## 6 CONCLUSIONS

In this work a general thermodynamically consistent gradient constitutive formulation to describe non-local behaviour of porous media is proposed. The proposal is an extension of the gradient-based thermodynamically consistent theories by [19] and [22] for non-porous continua particularized to the Modified Cam Clay constitutive model. Porous materials in this work are modelled from the macroscopic level of observation. They are considered to defined open thermodynamic systems characterized by the presence of occluded sub regions.

Discontinuous bifurcation theory to predict localized failure modes is consistently extended to porous media. As a result, the analytical expression of the localization tensor for gradient regularized plasticity in porous media is obtained. This failure indicator is particularized for both drained and undrained hydraulic conditions.

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