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## *The Algebraization of Mathematics: Using Original Sources for Learning Mathematics<sup>2</sup>*

**Summary:** *The knowledge of the processes in the history of mathematics is very useful for a greater and complete comprehension of the foundations and nature of mathematics. The algebraization of mathematics was a key process in the transformation of the seventeenth century mathematics if we consider two of its relevant features: a new symbolic language and an analytic method for solving problems. The aim of this article, within the framework of the algebraization of mathematics, is to analyze the results of the implementation of a historical activity in a mathematics history course for the bachelor's degree in mathematics, which includes singular geometric constructions for solving equations. Furthermore, the article shows that students gain a better understanding of the role of the relationship between algebra and geometry in the development of mathematics, which helps them to improve their mathematical training.*

**Keywords:** *seventeenth-century, learning mathematics, algebra, geometry, algebraization of mathematics.*

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### Introduction<sup>3</sup>

The knowledge of the processes in the history of mathematics is essential for the mathematicians to get a greater comprehension of the foundations and nature of mathematics. Mathematicians who know the history of mathematics are able to understand better how and why the different branches of mathematics have taken shape: analysis, algebra, and geometry, their different interrelations and their connections with other disciplines (Bruckheimer & Arcavi, 2000; Roca-Rosell & Lusa, 2009; Massa Esteve, 2014).

The activities, based on the analysis of significant original sources, contribute to improving mathematicians' integral formation, giving them additional knowledge of the social and scientific context of the periods involved (Jahnke et al., 2000; Furinghetti et al., 2006; Massa Esteve et al., 2011; Herrero et al., 2017).

Reading ancient texts enables us to get a vision of mathematics not as a final and finished product, but as a useful, dynamic, human, interdisciplinary and heuristic science which has developed through efforts to solve the problems about the world around us that humanity has been faced with throughout history. History shows that societies develop because of the scientific activity and that mathematics is a fundamental part of this process. Indeed, mathematics can be presented as an intellectual activity for solving problems in each period (Massa Esteve, 2003; 2012b; Katz & Tzanakis, 2011).

This paper focuses on the process of algebraization of mathematics, which took place from the seventeenth century to the beginning of the eighteenth century, explained in a course of history of mathematics. The publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540-1603) represented a step forward in this process with the

development of the symbolic language and an analytic method for solving problems. Later, Pierre de Fermat (1601-1665) was among the mathematicians who used this algebraic analysis to solve geometric problems.<sup>4</sup> However, the most influential figure in this process of algebraization was René Descartes (1596-1650), who published *La Géométrie* in 1637 as an appendix to the *Discours de la Méthode* (Mancosu, 1996; Bos, 2001). In this process of algebraization, the creation of a symbolic language by Viète and Descartes for dealing with algebraic equations, geometric constructions and curves, was essential, as well as the introduction of a new analytic method, as we explain in the next section (Radford, 2006; Katz, 2007).

These original sources of the process of algebraization of mathematics are useful in an activity implemented in a classroom of the history of mathematics to clarify mathematical ideas. Indeed, the study of the origins of polynomials and their associated equations provides us with a history of geometrical constructions for the solution of equations with instructive and suggestive passages for students, whether college degree or high school ones. We ask ourselves and students about the relationship between the algebra and geometry in these geometrical constructions, whether because they are justifying and illustrating the algebraic rules or they are a necessary complement for solving the equation (Calinger, 1996; Massa Esteve, 2005; Allaire & Bradley, 2001).

Therefore, the aim of this paper is to analyze the results of the implementation of this historical activity in a mathematics history course for the bachelor's degree in mathematics. This activity contains singular geometric constructions for solving quadratic equations by Viète and Descartes in the process of algebraization of mathematics. As we will show, these analyses of the activity linking algebra to geometry and using original sources provide

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3 A first version of this document was presented at the symposium "History of Science for Science Education" of the 6<sup>th</sup> ESHS Conference, 4-6. September 2014, in Lisbon.

4 He did not publish during his lifetime, although his work circulated in the form of letters and manuscripts referred to in other publications (Mahoney, 1973).

students with a more comprehensive view of the development of mathematics and improve their mathematical training.

The course of History of Mathematics has been taught at the Faculty of Mathematics of the Universitat Politècnica de Catalunya, Barcelona-Tech since 2010.<sup>5</sup> It is not compulsory and students attend it in the last four months of their final year of mathematics studies. In this course, we try to work whenever possible with historical original sources or specialized secondary sources.<sup>6</sup> The goal of this course is to explore the past of mathematics, showing how some concepts, theorems, and axiomatic methods have emerged and developed over time, which are now presented in the texts within the framework of the pragmatic conception the teaching of which often does not match the historical order in which they were invented or discovered. Indeed, through the course, students develop a panoramic view of the development of the mathematical concepts throughout history.<sup>7</sup> The sessions have a structure based on monographic presentations,

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5 This course of history of mathematics is shared with Monica Blanco, she teaches the last three topics of the course.

6 The course follows the line of historical research that attempts to understand the formation processes of mathematical concepts in their own context, in terms of mathematical knowledge and of the aims with which they worked more than in terms of what could have happened then. The circulation of contributions from different authors shows the path for learning the history of mathematics.

7 The aim of the course is broken down into four specific objectives, which fall within different facets of this development: a) Learning about the original sources on which knowledge of mathematics in the past is based. This involves reading and interpreting a selection of classical mathematical texts, as well as learning how to locate and use historical literature or historical online resources. b) Recognizing the most significant changes in the discipline of Mathematics; those, which have influenced its structure and classification, its methods, its fundamental concepts and its relation to other sciences. c) Showing the socio-cultural relations of mathematics with politics, religion, philosophy and culture in each period, and perhaps the most important, d) encouraging students to reflect on the development of mathematical thought and the transformations of natural philosophy.

some by the students, and others by the teacher, with a part of reflection and debates for clarifying ideas. We present the most important historical periods in six topics: 1) Mathematics in antiquity; 2) From Arabic science to the Renaissance; 3) The birth of modern mathematics; 4) Contributions prior to calculus; 5) Conceptual Development of Calculus in the 18<sup>th</sup> century and, finally, 6) Arithmetization and rigorous formulation of the Calculus. Every week in the classroom, we carry out an activity using original sources following a script of questions on the subject, with the intention of clarifying the doubts and problems that may have arisen and reflect together on the development of mathematical thought in each historical age.

In the next section, we shall emphasize the significance of the process of algebraization of mathematics in the development of mathematics and then we shall analyze the historical activity reproducing the implementation of these materials on algebraization of mathematics and students' reasoning and comments, after analyzing and comparing the singular geometrical constructions by Viète and Descartes.

### **The significance of algebraization of mathematics**

The process of algebraization in mathematics was mainly the result of the introduction of algebraic procedures for solving geometrical problems that enabled two fundamental transformations in mathematics to occur: the creation of what is now called analytic geometry and the emergence of infinitesimal calculus (Mahoney, 1980; Mancosu, 1996). These disciplines became exceptionally powerful when the connections between algebraic expressions and curves, and between algebraic operations and geometric constructions, were established.

Viète in his *In Artem analyticen Isagoge* (1591) used symbols to represent both known and unknown quantities and was thus able to investigate equations in a completely general form. Viète intro-

duced specious logistic, a method of calculation with “species”, the kinds or classes of elements, so that the symbols of this analytic art (or algebra) could be used to represent not just numbers, but also the values of any abstract magnitude. Moreover, he introduced a new analytic algebraic method that allowed problems of any magnitude, numerical or geometric, to be dealt with, and he used as a tool his new symbolic language. Viète divided his new analytical procedures into three stages; the first stage consisted of transforming the problem into one equation composed of known and unknown quantities (*zetetics*). In the second, he tested the correctness of any stated theorem by means of equations (*poristics*). The third stage, which was the most important to him, consisted of displaying the value of the unknown quantity, thus solving the equation (*exegetics*). The goal of his analytic art (also called algebra) was to solve all problems. Viète sums up his ideas at the end of the *Isagoge*:

Finally, the analytic art, endowed with its three forms of Zetetics, Poristics and Exegetics, claims for itself the greatest problem of all, which is TO LEAVE NO PROBLEM UNSOLVED.<sup>8</sup>

Viète showed the usefulness of algebraic procedures for solving problems in arithmetic, geometry and trigonometry (Viète, 1591; Giusti, 1992; Bos, 2001; Massa Esteve, 2008; 2012a; Oaks, 2018).

As Viète’s and Descartes’ work came to prominence at the beginning of the seventeenth century, some authors began to consider the utilization of algebraic procedures for solving all kind of problems. According to Mahoney (1980), this new algebraic thought had three relevant features: the use of a new symbolism that not only abbreviated notation, but it acted as a powerful tool for operations; an emphasis on mathematical relations, rather than on mathematical objects; and abstraction, which meant the freedom from a mere phys-

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<sup>8</sup> Denique fastuosum problema problematum ars Analytice, triplicem Zeteticis, Poristicis & Exegeticis formam tandem induct, jure sibi ad rogat, quod est, NULLUM NON PROBLEMA SOLVERE (Viète, 1591: 9r).

ical representation. In fact, this passage from a geometrical mode of thought to this algebraic one was characterized by Mahoney (1980: 141) as “the most important and basic achievement of mathematics at the time”.

However, as mathematicians used different and complicated notations, the new analytical methods derived from the algebraic analysis of Viète were not considered a well-founded new science, even though they constituted a very powerful tool for calculation in comparison with classical geometry. Therefore, this process of algebraization of mathematics, which involved a change from a predominantly geometrical way of thinking to a more algebraic or analytical approach, was slow and irregular. In that period, Latin translations of Greek texts recovered classical mathematical thought. At the same time, in some mathematical works authors introduced algebraic procedures. These procedures were very productive, although their significance sometimes contradicted the understanding of classical techniques. Not all mathematicians adopted algebraic procedures during this period: some considered these new techniques and symbolism as “art” and tried to justify them according to a more “classical” form of mathematics; others disregarded algebra because their research evolved along other paths. Eventually, a few mathematicians accepted these new techniques as an additional tool in their mathematical procedures. (Pycior, 1997; Massa Esteve, 2001).

In spite of these few tendencies, with the dissemination of the works by Viète, Fermat and Descartes during the seventeenth and eighteenth centuries, the symbolic language and algebraic procedures were applied in different fields and the ways for resolving problems and the process of algebraization of mathematics became practically complete (Bashmakova & Smirnova, 2000; Katz & Parshall, 2014).

**Using original sources in the classroom:  
the algebraization of mathematics**

Now we present the activity implemented in the classroom of undergraduate students of mathematics, within the framework of the algebraization of the seventeenth-century mathematics, which contains singular geometric constructions used for solving equations. This activity consists of three parts: first, a brief resume of the previous historical sessions of the theme two of the course. Second, the presentation and comparison of the geometrical justifications developed by Viète and Descartes in their works using original sources, and, finally, the analysis of students' reflections and debates on the meaning of the algebraization of mathematics in answering some prepared questions by the teacher.

*Precedents. First Geometrical Justifications*

We begin the activity in the theme 2, "From Arabic Science to the Renaissance" introducing the rhetorical treatment of equations by Arabic Science. This part is a necessary reminder for students in order to understand the relevance of the algebraization of mathematics. This is followed by a brief overview.

We then remind the students of the treatise *Al-kitab a-Imukhtasar fi hisab al-jabr wa'l-muqabala* (c. 813) by Mohamed Ben-Musa Alkharizmi (780-850), which describes different kinds of equations using rhetorical explanations without symbols. His geometrical justifications of the solutions of equations with a numerical example use squares, rectangles, and bases in classical geometry (Alkharizmi, 1986; Parshall, 1988; Romero et al., 2015). We also remind the students that later, when Leonardo Pisano (1170-1240) (known as Fibonacci) expressed rhetorically these Arabic rules in his *Liber Abaci* (1202), he used the "radix" to represent "the thing" or an unknown quantity (also called "res" by other authors) and the word "census" to represent the square power of the unknown. Fibonacci also made the same justification of quadratic equations as did the Arabic algebras with squares and rectangles (Sigler, 2002).

This rhetorical language continued to be used in the early Italian Renaissance, in several algebraic works such as the *Summa de Arithmetica, Geometria, Proportioni e Proportionalita* (1494) by Luca Pacioli (1445-1514) (Hoyrup, 2010; Parshall, 2017). Later, we remind the students of other Renaissance algebras, such as *Artis Magnae Sive de Regulis Algebraicis* (1545) by Girolamo Cardano (1501-1576) and *Quesiti et Inventioni Diversae* (1546) by Niccolò Tartaglia (1500-1557) (Heffer, 2006). Almost all these writers made similar representations or justifications of the quadratic equations using squares, rectangles, and cubes (Stedall, 2011). For example, see the geometrical justification using numerical coefficients by Cardano, in his *Artis Magnae* quoting Euclid's *Elements* (Figure 1).

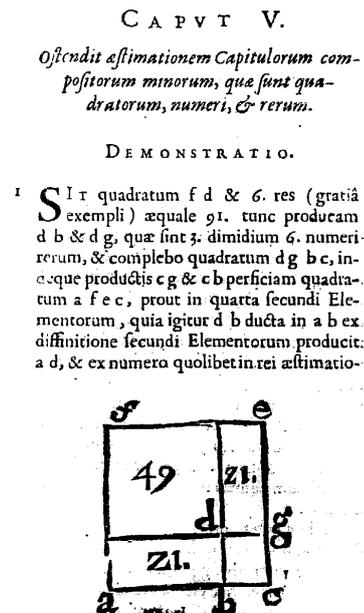


Figure 1. *Artis Magnae* (Cardano, 1545: 229).

We emphasize to students that one of the first authors to question these geometrical justifications of squares and rectangles was Pedro Núñez (1502-1578) in his book *Libro de algebra en arithmética y geometría* (1567). After showing the same classic geometrical justifications, Núñez claims:

“While these demonstrations of the last three rules are very clear, the adversary will be able to prevent, saying that in the demonstration of the first one we presuppose that a ‘censo’ ( $x^2$ ) with the ‘things’ ( $bx$ ) in any number that they be, can be equal to any number ( $n$ ), taken number as we have defined at the beginning of this book, and that this assumption is not true. Therefore, it will be necessary to demonstrate it.”<sup>9</sup>

After this statement, we explain that Núñez proceeds by introducing new geometrical constructions of the solution of the quadratic equation (Massa Esteve, 2010). The difference with the classical demonstrations is remarkable. In these new constructions, the solution is represented by a line that is expressed and constructed from the other lines and that finally is identified with a number (a value) by means of the rule of resolution of the quadratic equation. In fact, in classical demonstrations the solution of the equation is part of the reasoning of the construction, without questioning whether this solution exists, which leads us to conclude that it is rather a justification of this rule of resolution. On the other hand, Núñez justifies the construction with Pythagorean Theorem from Euclides’ *Elements*, I.47 (Heath, 1956: 349) and binomial theorem, II.4 (Heath, 1956: 379-382) and proceeds to prove the exactness of the resolution rule with a numerical example, quoting explicitly Euclides. Although Núñez was a pioneer in introducing new geometrical constructions, we draw attention to students that the more singular ones will occur later, as we will analyze below in the second part of the activity implemented in the classroom.

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9 Más estas demostraciones de las tres reglas postreras puesto que sean muy claras, podrá el adversario impedir, diciendo, que en la demostración de la primera presuponemos que un censo con las cosas en cualquier número que ellas sean, pueden ser iguales a cualquier número, tomado número como habemos definido en principio de este libro y que este presupuesto no es cierto. Por lo cual será necesario demostrarlo. (Núñez, 1567: 14r).

### *Comparing Geometrical Demonstrations from Viète’s and Descartes’ work*

We continue with the second part, already in the theme 3, “The birth of modern mathematics”, introducing first Viète’s work. We emphasize that the publication in 1591 of *In Artem Analyticen Isagoge* by Viète constituted a step forward in the development of a new symbolic language. Viète uses the vowels (A, E, I, O, U) to represent the unknowns and the consonants to represent the known quantities. Moreover, we explain to students how Viète introduced a new analytical method for solving problems in the context of Greek analysis. We reflect with students about this algebraic method of analysis that allowed for solving the problems of any magnitude dealing with lines, planes, volumes or angles.

In fact, we explain to students that Viète solved equations geometrically using the Euclidean idea of proportion: proportions can be converted into equations by setting the product of the medians equal to the product of the extremes (Viète, 1591: 2). This Viète’s principle was taken directly from Euclid’s *Elements* VII.19. (Heath, 1956: 318-320). Viète affirms in the chapter 2 of *Isagoge*: “And so, a proportion can be called the composition (*constitutio*) of an equation (aequalitatis), an equation the resolution (*resolutio*) of a proportion”.<sup>10</sup>

In 1593, Viète published *Effectionum Geometricarum canonica recensio*, in which he geometrically constructed the solutions of the second- and fourth-degree equations and used this principle to solve a geometrical problem.<sup>11</sup> In the classroom, we present Viète’s claims relating to the quadratic equa-

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10 Ita que proportio potest dici constitutio aequalitatis. Aequalitas, resolutio proportionis (Viète 1591: 4v).

11 The idea of constructing and founding the root as a line to prove the solution of a quadratic equation can be found in few algebraic texts of the sixteenth century. Bombelli, for instance, gave a geometrical construction in the fourth book of his *Algebra* (1572), using rhetorical language, with explicit reference to an arithmetic problem solved in the third book (Bombelli, 1966; Massa Esteve, 2006). The main differences between Bombelli’s and Viète’s procedure is that the former used known quanti-

tion ( $x^2 + bx = d^2$ ) and how Viète solved a geometrical problem with a singular construction (see Figure 2). In this construction, Viète set up the quadratic equation *A quadratum plus B in A, aequari D quadrato* by means of a proportion  $(A + B) : D = D : A$ , using Viète's principle.

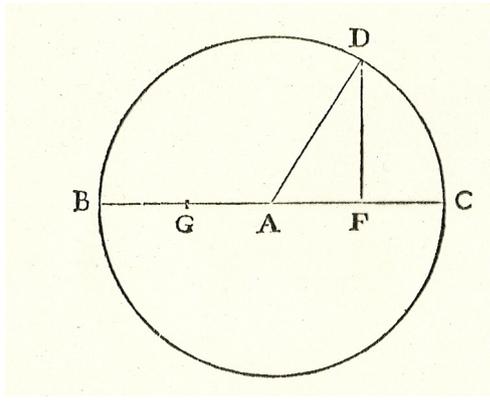


Figure 2. Viète's construction of three proportional (Viète, 1646: 234).

Viète's geometric construction of the lines  $B$ ,  $D$  satisfying this equality is set out in Figure 2. Viète used the height theorem from Euclid's *Elements*, VI.13 and a similar figure to the figure for this theorem in Euclid's *Elements*, without explicitly quoting; however, we remind the students of this theorem and prove it.<sup>12</sup> Viète draws  $FD = D$  and  $GF = B$ , making a right angle, and divides  $B$  by half  $AF = B/2$ . Viète applies the Pythagorean Theorem I.47 (Heath, 1956) for expressing the hypotenuse of the triangle. He describes a circle whose radius is equal to  $AC$ , which we can identify with the hypotenuse of the triangle formed by  $B/2$  and  $D$ ,  $AD = AC = [(B/2)^2 + D^2]^{1/2}$ . The solutions are then the segments (unknown)  $FC = AC - AF$  and  $BF = BA + AF$ , which take  $BA = AC = \text{radius}$  (Massa Esteve, 2008). In Viète's words:

es (numbers) in the geometric construction and the latter used symbolic language and also Euclid's theorems in the proof.

12 Theorem VI.13. The two given straight lines to find a mean proportion (Heath, 1956: 216).

“Proposition XII. Given the mean of three proportional magnitudes and the difference between the extremes, find the extremes. [This involves] the geometrical solution of a square affected by a [plane based on a] root [ $A^2 + BA = D^2$ ]. Let  $FD$  be the mean of three proportional [=  $D$ ] and let  $GF$  be the difference between the extremes [=  $B$ ]. The extremes are to be found. Let  $GF$  and  $FD$  stand at right angles and let  $GF$  be cut in half at  $A$ . Describe a circle around the center  $A$  at the distance  $AD$  and extend  $AG$  and  $AF$  to the circumference at the points  $B$  and  $C$ . I say that what was to be done has been done, for the extremes are found to be  $BF$  [ $A+B$ ] and  $FC$  [=  $A$ ], between which  $FD$  [=  $D$ ] is the mean proportional. Moreover,  $BF$  and  $FC$  differ by  $FG$ , since  $AF$  and  $AG$  are equal by construction and  $AC$  and  $AB$  are equal by construction. Thus, subtracting the equals  $AG$  and  $AF$  from the equals  $AB$  and  $AC$ , there remain the equals  $BG$  and  $FC$ .  $GF$ , in addition, is the difference between  $BF$  and  $BG$  or  $FC$ , as was to be demonstrated.”<sup>13</sup>

Then we emphasize that the basis of Viète's geometrical construction procedures is the identification of terms of an equation, both known and unknown quantities, as terms of a proportion, or proportional lines through the height theorem. After analyzing Viète's geometrical construction, we pose some questions to the students to clarify their ideas and help them to reflect on the meaning of this geometrical construction (see the third part below).

The other singular example that we analyze in this activity is the geometrical construction in a quadratic equation found in *La Géométrie* (1637) by Descartes. In the classroom, first, we explain the significance of Descartes' work and we describe the contents of the three books in *La Géométrie*. We be-

13 Propositio XII. Data media trium proportionalium & differentia extremarum, invenire extremas. Sit data  $FD$  media trium proportionalium, data quoque  $GF$  differentia extremarum. Oportet invenire extremas. Inclinentur  $GF$ ,  $FD$  ad angulos rectos, & secetur  $GF$  bifariam in  $A$ . Centro autem  $A$  intervallo  $AD$ , describatur circulus, ad cuius circumferentiam producantur  $AG$ ,  $AF$ , in punctis  $B$ ,  $C$ . Dico factum esse quod oportuit. Extremas enim inveniundas esse  $BF$ ,  $FC$  inter quas media proportionalis est  $FD$ . (Viète, 1646: 234).

gin Book I describing the creation of an *algebra* of segments by Descartes and showing how Descartes adds, multiplies, divides and calculates the square root of segments with geometrical constructions. We emphasize the use of Tales Theorem for the product of segments, the introduction of the segment unity for the operations between segments and the height theorem for the extraction of the square root.

Next, we show how a quadratic equation ( $x^2 = bx + cc$ ) may have been solved geometrically by Descartes, reproducing the singular geometric construction (see Figure 3):

“For example, if I have  $z^2 = az + bb$ , I construct a right triangle NLM with one side LM, equal to  $b$ , the square root of the known quantity  $b^2$ , and the other side, LN equal to  $\frac{1}{2} a$ , that is, to half the other known quantity which was multiplied by  $z$ , which I assumed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line  $z$ . This is expressed in the following way:  $z = \frac{1}{2} a + (\frac{1}{4} aa + bb)^{1/2}$ ”<sup>14</sup>

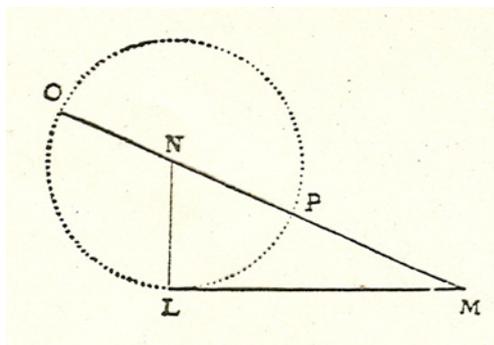


Figure 3. Descartes' geometrical construction (Descartes, 1637: 302).

14 Car si i'ay par exemple  $z^2 \propto az + bb$  ie fais le triangle rectangle NLM, dont le costé LM est esgal a  $b$  racine quarrée de la quantité connue  $bb$ , & l'autre LN est  $1/2a$ , la moitié de l'autre quantité connue, qui estoit multipliée par  $z$  que ie suppose estre la ligne inconnue puis prolongeant MN la baze de ce triangle, iusques a O, en sorte qu'NO soit esgale a NL, la toute OM est  $z$  la ligne cherchée. Et elle s'exprime en cete sorte  $z \propto 1/2 a + (\frac{1}{4} aa + bb)^{1/2}$  (Descartes, 1637: 302-303).

In the classroom, we could make some reflections with the students after analyzing Descartes' geometrical construction. Note that the symbolic formula appears explicitly in Descartes' work. His geometrical construction corresponds to the construction of an unknown line in terms of some given lines without numerical coefficients. Therefore, the solution of the equation is given by the sum of a line and a square root, which has been obtained using the Pythagorean Theorem. However, Descartes ignores the second root, which is negative, and he does not claim that this geometrical construction could be justified by Euclid's *Elements* III. 36 where the power of a point with respect to a circumference is shown (Heath, 1956: 75-77).

#### Questionnaire and Students' Reflections

The third part of this activity is more interesting for the formation of the mathematicians' thought. We make questions for comparing the two geometrical constructions and reflect on the relationship between algebra and geometry.

Reproduce Viète's geometrical construction and explain the procedure; could this geometrical construction be used for any quadratic equation? Give reasons.

What about negative solutions?

How are the Pythagorean and the height theorem used?

Explain their relationship with the solution of the equation.

What is the main difference between Viète's geometrical construction and the Euclidean one?

The questions posed to students for the geometrical construction by Descartes are similar to those of Viète. Moreover, students may also reflect on the meaning of both constructions.

The differences with Viète are relevant because Descartes explicitly writes in the margin

“how to solve” the equations, referring to an algebraic expression (equality), but making a geometrical construction for solving the equation. Viète, by contrast, solves a geometric problem on proportions with a geometric figure in which the proportion is identified with an equation. Another relevant subject to consider with students is the analytical and/or synthetic approach used in each construction. Other questions that we have dealt with in this activity are:

What geometrical reasoning did the author use?

What is the role of the Pythagorean Theorem in solving the quadratic equation?

What is the relation between this construction and the algebraic solution of the quadratic equation?

All these questions enable students to consider the solution of quadratic equations from a geometrical point of view.

Nevertheless, in our activity we focus on other students’ reflections answering questions about the relationship between algebra and geometry. In fact, we reflect on the introduction and systematic use of abstract symbols, which had a profound effect on the development of mathematics. Mathematicians began to do the work that focused on the patterns and objects exhibited in the symbols themselves, as well as in the objects that had originally been entered to symbolize. Somewhat previously separate research objects could be grouped together when they could be managed by the same symbols. The symbolic representation made it easier to develop general arguments and apply them to broad classes of problems. Therefore, we ask our students: Can we say that geometric reasoning reaches its full potential by relating algebra to geometry? Here is the answer of one student:

“Thus, the tool that emerges from the fusion of algebra and geometry made it possible to select the

best properties of both sciences. From the first (algebra), the optimization of the treatment of mathematical concepts, dissipating the need to represent the respective procedures and results of a demonstration, and at the same time, have more information, intrinsic in the proper symbolism. From the second (geometry), the possibility of visualizing in a particular case the object studied algebraically, and at the same time have a large number of properties that could use as an axis or complement to a proof. But this is not all, this combination not only allowed for the construction of the mentioned method, but also catalyzed a much more effervescent development of both sciences and, consequently, the creation (to be constructed later) of new fields of study within mathematics, as would be the case of analytical geometry or the convulsion that trigonometry triggered in the seventeenth century” (student 1).

The second example is by another student that commented on these questions: What did this fusion of algebra and geometry represent to the mathematicians? What made this fusion possible? More analytical methods? The generalization of the results? More “rigorous” proofs? The student’s answer was as follows:

“First, it (the fusion of algebra and geometry) allows a passage from a particular case to a general one. Understanding the six equation cases as simply the first or second degree equation with positive, negative or zero coefficients, which in some cases can be viewed geometrically but in others not, and have positive, negative solutions, and sometimes not in any case (complex case), makes the process of abstraction and generalization very important. Second, the appearance of new problems without geometrical relation (negative lengths, even negative roots...). Algebra is the overcoming of mathematics as a science that can only make sense in nature, the first path to abstract thinking that develops rapidly with the analytic method and the emergence of the symbol and notation as a way of working” (student 2).

These examples show that students can understand the fusion of algebra and geometry, with

the advantages of this combination for the development of mathematics, as well as prompting reflection on the relationship between algebra and geometry through history.

### **Concluding Remarks**

Regarding the students' reflections, we can affirm that our students are able to comprehend the influence of linking algebra and geometry in the transformations of mathematics in the seventeenth century, represented by the introduction of negative numbers, trigonometry, logarithms, and the use of the infinite. Therefore, we can continue the research asking ourselves: What conceptual changes helped to produce the algebraization of mathematics? How could other disciplines have influenced these transformations? What was the situation with regard to the foundations of mathematics?

We can also conclude that these kinds of activities are very rich in terms of the competency-based learning, as they allow students to apply their knowledge (algebra and geometry) in different situations and from different points of view, rather than reproduce exactly what they have learned. In addition, they help students to appreciate the contribution of different cultures to the knowledge of and the transformation of mathematics.

An important part of this activity is devoted to geometry which is a relevant subject for the mathematical formation at the bachelor's degree level of mathematics. Geometry has a great visual and aesthetic value and offers a beautiful way of understanding the world. The elegance of its constructions and proofs makes it a part of mathematics that is very appropriate for developing the process of reasoning of students and providing proof, as well as for incorporating geometrical constructions as a part of the heuristic in solving problems.

Geometric proofs have a great potential for relating geometrical and numerical reasoning in some of the activities proposed, and geometrical and alge-

braic reasoning in some others. In this way, we can establish connections among numbers, figures, and formulas, i.e., calculations, geometric constructions and algebraic expressions. Algebra provides the rules for solving the equations and geometry contributes with singular geometrical constructions to the justification of the exactness of these rules.

In addition, there are more good reasons to study these geometric solutions of the second-degree equation. For students who tend to focus visually, rather than symbolically, studying the demonstrations of these geometric solutions by comparing them to geometric justifications by completing a square, can make the involved algebraic procedures more meaningful.

The activity is structured to be suitable for classroom use, with attention to original sources on geometrical constructions offering lessons in both historical and mathematical interpretation. As for mathematical ideas put to practice in this activity, we have to mention the use of the height theorem, Pythagoras' theorem, Thales' theorem and the power of a point with respect to a circumference. Once again, we emphasize the importance of knowing some geometric theorems from the year 300 BC., either by underlying mathematical ideas or by their usefulness in geometric constructions and in the justification of algebraic expressions of an equation. Making and comparing these geometrical constructions helps our students to develop both analytic and synthetic reasoning and to improve their mathematic knowledge. In addition, the students' analysis and reflections on these historical geometrical constructions using algebraic expressions show us a rich source of ideas on the relationship between algebra and geometry throughout history.

Finally, we wish to add that through this activity students learn that at the end of the process of the algebraization of mathematics, algebra and geometry became complementary and that it was from the coordination and conjunction of both branches that new fields of mathematics developed on the path to modern mathematics.

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### АЛГЕБАРИЗАЦИЈА МАТЕМАТИКЕ: КОРИШЋЕЊЕ ОРИГИНАЛНИХ ИЗВОРА У НАСТАВИ МАТЕМАТИКЕ

Познавање процеса који су се одвијали у историји математике врло је корисно за боље и потпуније разумевање основа математике и саме њене природе. Активношћу засноване на анализи важних историјских извора доприносе свеобухватнијој припреми студентима будућих наставника математике, иако што им уручају додатна знања у вези са друштвеним и научним контекстима периода који су предмети проучавања, док се математика може сматрати интелектуалном активношћу за решавање проблема својствених сваком периоду. Историја нам показује да се друштва развијају захваљујући деловању науке, а математика је суштински део тог процеса. Чињање стarih математичких текстова омогућава нам да боље схватимо како математика није тогов производ, већ је динамична, корисна, хумана, интердисциплинарна и хеуристичка наука која се развијала кроз најоре да се реше проблеми са којима се човечанство суочавало кроз историју.

Циљ овог рада је да анализира, у оквирима алгебаризације математике, резултате примене једне наставне активношћу на часовима историје математике за студенте основних студија математике у којој се поједине геометријске конструкције користе за решавање једначина.

Алгебаризација математике била је кључни процес у трансформацији математике у 17. веку, пошто ако узмемо у обзир њене две бивне карактеристике: стварање новој симболичкој језика и увођење аналитичкој метода за решавање математичких проблема. Заправо, објављивање 1591. године дела Франсоа Вијета (1540-1603) под називом *In Artem Analyticen Isagoge*, представљало је корак напред у процесу развоја симболичкој језика и аналитичкој метода за решавање математичких проблема. У каснијем периоду, Пјер Ферма (1601-1665) био је један од математичара који су користили ову алгебарску анализу за решавање геометријских проблема. На процес алгебраизације највише је утицао Рене Декарт (1596-1650) који је дело под називом *La Géométrie* објавио 1637. године, као додатак другом делу, *Discours de la Méthode*. Проучавањем порекла полинома и једначина које се на њих односе у оквиру историје геометријских конструкција за решавање једначина студентима математике, али и ученици средњих школа, наћи ће на делове у којима ће сазнати нешто ново и поучно.

У овом раду представљена је једна наставна активношћу, смештена у оквире алгебаризације математике из 17. века, која садржи појединачне геометријске конструкције за решавање једначина. Активношћу је примењена на часовима основних студија математике. Прво описујемо и поредимо, користећи оригиналне историјске изворе, образложења за решавање геометријских задатака која су у својим делима изложили Вијет и Декарт, а потом

анализирамо сџавове сџуденаџа о значењу алџедаризације маџемаџике које су изнели у одџоворима на џиџања која је џриџремџо насџавник.

Осим џоџа, у раду џоказујемо да сџуденџи џако боље разумеју улоџу коју однос између алџедре и џеометџрије има у развоју маџемаџике, џџо им џомаже да унаџреде своја маџемаџичка знања. Најзначајнији резулџаџи доџијени су из одџовора сџуденаџа. Сасџављање и џоређење ових џеометџријских консџрукција џакође џомаже сџуденџима да развију аналиџичко и синџетџичко резоновање, као и да унаџреде своја маџемаџичка знања. Поред џоџа, сџроведена анализа и сџавови сџуденаџа о исџоријским џеометџријским консџрукцијама насџалим коришћењем алџедарских израза џружају нам доџаџ извор идеја у вези са односом између алџедре и џеометџрије џоком исџорије. Кроз ову акџивносџ сџуденџи уче да су алџедра и џеометџрија на крају џроцеса алџедаризације џосџале комџлементџарне, џе да су џовезивањем и координацијом ове две џране маџемаџике насџале нове обласџи које су довеле до развоја модерне маџемаџике.

**Кључне речи:** седамнаесџи век, учење маџемаџике, алџедра, џеометџрија, алџедаризација маџемаџике.