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Overlapping Guaranteed Cost Control for Time-Varying Discrete-Time Uncertain Systems

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Abstract

The paper presents results on the inclusion principle for uncertain, nominally linear, time-varying, discrete-time systems. The systems under consideration have time-varying norm-bounded parameter uncertainties in both state and input matrices. Robust controllers are assumed to be available by using guaranteed cost control approach. The main contribution is in the derivation of explicit block structured conditions on complementary matrices of systems and controllers within the expansion-contraction scheme. A particular selection procedure for complementary matrices is included.

1 Introduction

In this paper, the expansion-contraction relations considered within the inclusion principle are specialized for a class of uncertain, nominally linear, time-varying, and discrete-time systems with parametric norm-bounded uncertainties. The inclusion principle defines a framework for two dynamic systems with different dimensions, in which solutions of the system with larger dimension include solutions of the system with smaller dimension [1], [2]. Robust controllers for each systems are supposed to be available by using guaranteed cost control approach. The notion of guaranteed cost control has been used for control design for uncertain, nominally time-invariant, discrete-time systems by using the Riccati equation approach [3], [4], [5], [6].

The expansion-contraction scheme must include the relations between both systems, performance criteria and controllers. The paper is primarily focused on the derivation of the expansion-contraction scheme for this class of systems by expressing the desired relations in the form of explicit conditions on complementary matrices in both systems and controllers. This approach is called a generalized selection of complementary matrices [7]. It essentially simplifies their choice when comparing it with standard forms such as aggregations and restrictions [1], [2], [8]. Up to now, is has been specialized only on certain overlapping LQ design for continuous-time LTI systems [9] as well as for certain discrete-time LTV systems [10]. Note that while the expansion-contraction relations for the continuous-time LTI systems can be directly applied to their discrete-time LTI counterpart, this is not the case of the LTV systems [10],[11]. Finally, we present the procedure for particular selection of complementary matrices. Structural example showing a possible elimination of uncertainties in the expansion is supplied.

2 Problem formulation

Consider the uncertain system as follows:

$$\mathbf{S}: \; x(k+1) = (A + \Delta A)(k)x(k) + (B + \Delta B)(k)u(k). \tag{1}$$

Associated with this system is the cost function

$$J\big(x(k_0), u(k)\big) = x^T(k_f)\Pi\, x(k_f) + \\ + \sum_{k=k_0}^{k_f-1} \left[x^T(k)Q^*(k)x(k) + u^T(k)R^*(k)u(k)\right]. \tag{2}$$

Further, consider another uncertain system

$$\tilde{\mathbf{S}}: \; \tilde{x}(k+1) = \left(\tilde{A} + \Delta\tilde{A}\right)(k)\tilde{x}(k) + \left(\tilde{B} + \Delta\tilde{B}\right)(k)\tilde{u}(k). \tag{3}$$

Associated with this system is the cost function

$$\tilde{J}\big(\tilde{x}(k_0), \tilde{u}(k)\big) = \tilde{x}^T(k_f)\tilde{\Pi}\, \tilde{x}(k_f) + \\ + \sum_{k=k_0}^{k_f-1} \left[\tilde{x}^T(k)\tilde{Q}^*(k)\tilde{x}(k) + \tilde{u}^T(k)\tilde{R}^*(k)\tilde{u}(k)\right], \tag{4}$$

where k_0 is the initial time, k_f is the final time and integers $k \in [k_0, k_0+1, ..., k_f]$. The vectors $x(k) \in \mathbb{R}^n$,

$u(k) \in \mathbb{R}^m$ and $\tilde{x}(k) \in \mathbb{R}^{\tilde{n}}$, $\tilde{u}(k) \in \mathbb{R}^{\tilde{m}}$ are the states and inputs of $\mathbf{S}$ and $\tilde{\mathbf{S}}$ at time k for $k \in [k_0, k_f]$, resp. Suppose $n \leqslant \tilde{n}$.$A(k)$, $B(k)$, $Q^*(k)$, $R^*(k)$ and $\tilde{A}(k)$, $\tilde{B}(k)$, $\tilde{Q}^*(k)$, $\tilde{R}^*(k)$ are matrices of appropriate dimensions satisfying standard assumptions on the LQ design. $\Delta A(k)$, $\Delta B(k)$, $\Delta\tilde{A}(k)$ and $\Delta\tilde{B}(k)$ denote time-varying matrices of uncertain parameters as follows:

$$\begin{aligned}(\Delta A\ \ \Delta B)(k) &= D(k)\,\Delta_k\,(E_1\ \ E_2)(k),\\ (\Delta\tilde{A}\ \ \Delta\tilde{B})(k) &= \tilde{D}(k)\,\Delta_k\,(\tilde{E}_1\ \ \tilde{E}_2)(k),\end{aligned} \tag{5}$$

where D, E_1, E_2, $\tilde{D}$, $\tilde{E}_1$, $\tilde{E}_2$ are known time-varying matrices, Δ_k is an unknown time-varying matrix with Lebesgue measurable elements and satisfying $\|\Delta_k\| \leqslant 1$ for all k. $x(k) = x\,(k; x(k_0), u(k))$, $\tilde{x}(k) = \tilde{x}\,(k; \tilde{x}(k_0), \tilde{u}(k))$ denote the solutions of (1), (3) for given initial states $x(k_0)$, $\tilde{x}(k_0)$ and inputs $u(k)$, $\tilde{u}(k)$ defined for all $k \in [k_0, k_f]$, resp. These solutions are unique and satisfy

$$x(k) = \Phi\,(k, k_0)\; x(k_0) + \sum_{j=k_0}^{k-1} \Phi\,(k, j+1)\,(B + \Delta B)\,(j) u(j), \tag{6}$$

$$\tilde{x}(k) = \tilde{\Phi}\,(k, k_0)\; \tilde{x}(k_0) + \sum_{j=k_0}^{k-1} \tilde{\Phi}\,(k, j+1)\left(\tilde{B} + \Delta\tilde{B}\right)(j)\tilde{u}(j). \tag{7}$$

Suppose the sum is zero if $k = k_0$. Φ, $\tilde{\Phi}$ are *discrete-time transition matrices* [12]. Specifically, and only for the ordering of arguments corresponding to forward iteration, denote

$$\Phi(k, j) = (A + \Delta A)\,(k-1)\,(A + \Delta A)\,(k-2)\cdots \cdots (A + \Delta A)\,(j),\ \ k \geqslant j, \tag{8}$$

by adopting the convention that an empty product is the identity, i.e. $\Phi(k, j) = I$ if $k = j$. The matrix Φ satisfies $\Phi(k+1, j) = (A + \Delta A)\,(k)\Phi(k, j)$. Analogously for the matrix $\tilde{\Phi}$.

$\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the transformations $\tilde{x}(k) = Vx(k)$, $x(k) = U\tilde{x}(k)$, $\tilde{u}(k) = Ru(k)$, $u(k) = Q\tilde{u}(k)$, where V, U, R and Q are constant full-rank matrices of appropriate dimensions.

Consider the controllers $u(k) = -K(k)x(k)$ and $\tilde{u}(k) = -\tilde{K}(k)\tilde{x}(k)$ designed by using the guaranteed cost control approach [3], [4], [5], [6]. This means that the controllers guarantee bounds for the performance criteria in (2) and (4) in the form $J \leqslant \Omega$ and $\tilde{J} \leqslant \tilde{\Omega}$, resp. A simple way of relating the performances of both closed-loop systems is to impose the equality of both criteria when the states and the inputs are related by the above transformations, that is, $J(x(k_0), u(k)) = \tilde{J}(Vx(k_0), Ru(k))$. This means that relations between the systems and the controllers must be satisfied as follows.

Definition 1 *We say that a pair $(\tilde{\mathbf{S}}, \tilde{J})$ includes a pair $(\mathbf{S}, J)$, that is $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$, if there exist a quadruplet of constant matrices (U, V, Q, R) such that $UV = I_n$, $QR = I_m$ and for any initial state $x(k_0)$ and any fixed $u(k)$ of $\mathbf{S}$, $x(k; x(k_0), u(k)) = U\tilde{x}(k; Vx(k_0), Ru(k))$ for all $k \in [k_0, k_f]$; and $J(x(k_0), u(k)) = \tilde{J}(Vx(k_0), Ru(k))$.*

Definition 2 *A control law $\tilde{u}(k) = -\tilde{K}(k)\,\tilde{x}(k)$ for $\tilde{\mathbf{S}}$ is contractible to the control law $u(k) = -K(k)\,x(k)$ for $\mathbf{S}$ if the choice $\tilde{x}(k_0) = Vx(k_0)$ and $\tilde{u}(k) = Ru(k)$ implies $K(k)x(k; x(k_0), u(k)) = Q\tilde{K}(k)\tilde{x}(k; Vx(k_0), Ru(k))$ for all $k \in [k_0, k_f]$, any initial state $x(k_0)$ and any fixed input $u(k)$ of $\mathbf{S}$.*

Suppose given the pairs of matrices (U, V) and (Q, R). Then the matrices $\tilde{A}(k)$, $\Delta\tilde{A}(k)$, $\tilde{B}(k)$, $\Delta\tilde{B}(k)$, $\tilde{\Pi}$, $\tilde{Q}^*(k)$, $\tilde{R}^*(k)$ and $\tilde{K}(k)$ can be described as

$$\begin{aligned}\tilde{A}(k) + \Delta\tilde{A}(k) &= VA(k)U + V\Delta A(k)U + M(k),\\ \tilde{B}(k) + \Delta\tilde{B}(k) &= VB(k)Q + V\Delta B(k)Q + N(k),\\ \tilde{\Pi} &= U^T\Pi U + M_\Pi,\\ \tilde{Q}^*(k) &= U^TQ^*(k)U + M_{Q^*}(k),\\ \tilde{R}^*(k) &= Q^TR^*(k)Q + N_{R^*}(k),\\ \tilde{K}(k) &= RK(k)U + F(k),\end{aligned} \tag{9}$$

where $M(k)$, $N(k)$, M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ and $F(k)$ are *complementary matrices.*

The problem. The specific goals are as follows: (i) To derive explicit conditions on complementary matrices satisfying $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ and contractibility conditions for the class of discrete LTV uncertain systems under consideration; (ii) To present a systematic procedure for their selection.

3 Main results

Theorem 1 *Consider $\mathbf{S}$, $\tilde{\mathbf{S}}$ given by (1), (3), resp. $\tilde{\mathbf{S}} \supset \mathbf{S}$ if and only if*

$$U\left[\tilde{\Phi}(k, k_0)\,\tilde{x}(k_0) + \sum_{j=k_0}^{k-1} \tilde{\Phi}(k, j+1)\left(\tilde{B} + \Delta\tilde{B}\right)(j)\tilde{u}(j)\right] = \Phi(k, k_0)\,x(k_0) + \sum_{j=k_0}^{k-1} \Phi(k, j+1)\,(B + \Delta B)\,(j)u(j) \tag{10}$$

for all $k \in [k_0, k_f]$.

We must impose some conditions on complementary matrices in (9) to satisfy $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$. Define for integers $r, s \in [k_0, k_f]$ and any square matrix $\mathcal{M}$ the matrix product $\mathcal{M}[r, s]$ as follows:

$$\begin{aligned} &\mathcal{M}[r,s] = \mathcal{M}(r)\mathcal{M}(r-1)\cdots\mathcal{M}(s), \;\; r > s \\ &\mathcal{M}[r,s] = \mathcal{M}(r), \;\; r = s. \end{aligned} \tag{11}$$

$\mathcal{M}[r, s]$ is undefined for $r < s$. Now, we are ready to formulate the inclusion principle for discrete–time LTV uncertain systems in terms of complementary matrices.

Theorem 2 *Consider (1)-(4).* $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ *if and only if*

$$\begin{aligned} UM[r,s]V = 0, &\qquad UN(r)R = 0, \\ UM[p,q]N(q-1)R = 0, &\qquad V^T M_{\Pi} V = 0, \\ V^T M_{Q^*}(k) V = 0, &\qquad R^T N_{R^*}(k) R = 0 \end{aligned} \tag{12}$$

hold for all fixed $k \in [k_0, k_f]$, *for all* r, s *such that* $k_0 \leqslant s \leqslant r \leqslant k-1$ *and all* p, q *such that* $k_0 + 1 \leqslant q \leqslant p \leqslant k-1$.

The following theorems give the conditions on complementary matrices to satisfy Definition 2.

Theorem 3 *A control law* $\tilde{u}(k) = -\tilde{K}(k)\,\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k) = -K(k)\,x(k)$ *for* $\mathbf{S}$ *if and only if*

$$\begin{aligned} &QF(k)\big[\tilde{\Phi}(k,k_0)Vx(k_0) + \\ &\quad + \sum_{j=k_0}^{k-1} \tilde{\Phi}(k, j+1) + \left(\tilde{B} + \Delta\tilde{B}\right)(j)\tilde{u}(j)\big] = 0 \end{aligned} \tag{13}$$

hold for all fixed $k \in [k_0, k_f]$.

Theorem 4 *A control law* $\tilde{u}(k) = -\tilde{K}(k)\,\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k) = -K(k)\,x(k)$ *for* $\mathbf{S}$ *if*

$$\begin{aligned} QF(k)V &= 0, \\ QF(k)M[k-1,r]V &= 0, \\ QF(k)N(k-1)R &= 0, \\ QF(k)M[k-1,p]N(p-1)R &= 0 \end{aligned} \tag{14}$$

hold for all fixed $k \in [k_0, k_f]$, *all* $r \in [k_0, k-1]$ *and all* $p \in [k_0+1, k-1]$.

3.1 Expansion–contraction process

3.1.1 Change of basis: The change of basis in the expansion-contraction process introduced in [2], [13] represents $\tilde{\mathbf{S}}$ in a canonical form. Since the inclusion principle does not depend on the specific basis used in the state, input and output spaces, we may introduce convenient changes of basis in $\tilde{\mathbf{S}}$ for a prespecified purpose [7]. The expansion-contraction process between $\mathbf{S}$ and $\tilde{\mathbf{S}}$ can be illustrated in the form

$$\begin{array}{ccccccccc} \mathbf{S} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \bar{\tilde{\mathbf{S}}} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \mathbf{S}, \\ \mathbb{R}^n & \xrightarrow{V} & \mathbb{R}^{\tilde{n}} & \xrightarrow{T_A^{-1}} & \bar{\mathbb{R}}^{\tilde{n}} & \xrightarrow{T_A} & \mathbb{R}^{\tilde{n}} & \xrightarrow{U} & \mathbb{R}^n, \\ \mathbb{R}^m & \xrightarrow{R} & \mathbb{R}^{\tilde{m}} & \xrightarrow{T_B^{-1}} & \bar{\mathbb{R}}^{\tilde{m}} & \xrightarrow{T_B} & \mathbb{R}^{\tilde{m}} & \xrightarrow{Q} & \mathbb{R}^m, \end{array} \tag{15}$$

where $\bar{\tilde{\mathbf{S}}}$ denotes the expanded system in a new bases. Given V and R, we define $U = (V^TV)^{-1}V^T$, $Q = (R^TR)^{-1}R^T$ as their pseudoinverses, resp. Let us consider $T_A = (V \;\; W_A)$, $T_B = (R \;\; W_B)$, where W_A, W_B are chosen such that Im W_A=Ker U, Im W_B=Ker Q. By using these transformations, the conditions $\bar{U}\bar{V} = I_n$, $\bar{V}\bar{U} = \left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\bar{Q}\bar{R} = I_m$, $\bar{R}\bar{Q} = \left(\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can be easily verified, where $\bar{V} = T_A^{-1}V = \left(\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right)$, $\bar{U} = UT_A = (\, I_n \;\; 0 \,)$ and $\bar{R} = T_B^{-1}R = \left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)$, $\bar{Q} = QT_B = (\, I_m \;\; 0\,)$. Note that the motivating factor for defining T_A and T_B is the fulfillment of these conditions. They play a crucial role in deriving explicit block structured complementary matrices (with zero blocks) including a general strategy for their selection.

3.1.2 Expansion-contraction in the new basis: Consider the system $\mathbf{S}$ given in (1) partitioned as follows:

$$A(k) = \left(\begin{array}{c|c|c} A_{11}(k) & A_{12}(k) & A_{13}(k) \\ \hline A_{21}(k) & A_{22}(k) & A_{23}(k) \\ \hline A_{31}(k) & A_{32}(k) & A_{33}(k) \end{array}\right) \tag{16}$$

and similarly for the matrices $\Delta A(k)$, $B(k)$, $\Delta B(k)$, where $A_{ii}(k)$, $B_{ii}(k)$, $i = 1,2,3$, are $n_i \times n_i$, $n_i \times m_i$ matrices, resp. Analogously for the uncertain matrices $\Delta A_{ii}(k)$, $\Delta B_{ii}(k)$, $i = 1,2,3$. This structure has been adopted as a prototype structure for overlapping decompositions [2], [8],[14].

Now, consider the uncertain system $\bar{\tilde{\mathbf{S}}}$ as follows:

$$\bar{\tilde{\mathbf{S}}} : \bar{\tilde{x}}(k+1) = \left(\bar{\tilde{A}} + \Delta\bar{\tilde{A}}\right)(k)\bar{\tilde{x}}(k) + \left(\bar{\tilde{B}} + \Delta\bar{\tilde{B}}\right)(k)\bar{\tilde{u}}(k). \tag{17}$$

Associated with this system is the cost function

$$\begin{aligned} &\bar{\tilde{J}}(\bar{\tilde{x}}(k_0), \bar{\tilde{u}}(k)) = \bar{\tilde{x}}^T(k_f)\bar{\tilde{\Pi}}\,\bar{\tilde{x}}(k_f) + \\ &\quad + \sum_{k=k_0}^{k_f-1} \left[\bar{\tilde{x}}^T(k)\bar{\tilde{Q}}^*(k)\bar{\tilde{x}}(k) + \bar{\tilde{u}}^T(k)\bar{\tilde{R}}^*(k)\bar{\tilde{u}}(k)\right], \end{aligned} \tag{18}$$

where $\bar{\tilde{A}}(k)$, $\Delta\bar{\tilde{A}}(k)$, $\bar{\tilde{B}}(k)$, $\Delta\bar{\tilde{B}}(k)$, $\bar{\tilde{\Pi}}$, $\bar{\tilde{Q}}^*(k)$ and $\bar{\tilde{R}}^*(k)$ are matrices of appropriate dimensions. The vectors $\bar{\tilde{x}}(k)$ and $\bar{\tilde{u}}(k)$ are defined as $\bar{\tilde{x}}(k) = T_A^{-1}Vx(k) = \bar{V}x(k)$, $\bar{\tilde{u}}(k) = T_B^{-1}Ru(k) = \bar{R}u(k)$. $\Delta\bar{\tilde{A}}(k)$, $\Delta\bar{\tilde{B}}(k)$

are norm bounded uncertainties with the similar structure as in (5).

Now, the relations between $\tilde{\tilde{\mathbf{S}}}$ and $\mathbf{S}$ are defined as $\tilde{\tilde{A}}(k) = \bar{V}A(k)\bar{U} + \bar{V}\Delta A(k)\bar{U} + \bar{M}(k)$, $\tilde{\tilde{B}}(k) = \bar{V}B(k)\bar{Q} + \bar{V}\Delta B(k)\bar{Q} + \bar{N}(k)$, $\tilde{\tilde{\Pi}} = \bar{U}^T\Pi\bar{U} + \bar{M}_\Pi$, $\tilde{\tilde{Q}}^*(k) = \bar{U}^TQ^*(k)\bar{U} + \bar{M}_{Q^*}(k)$, $\tilde{\tilde{R}}^*(k) = \bar{Q}^TR^*(k)\bar{Q} + \bar{N}_{R^*}(k)$, where new complementary matrices are

$$\begin{aligned} \bar{M}(k) &= T_A^{-1}M(k)T_A, & \bar{N}(k) &= T_A^{-1}N(k)T_B, \\ \bar{M}_\Pi &= T_A^TM_\Pi T_A, & \bar{M}_{Q^*}(k) &= T_A^TM_{Q^*}(k)T_A, \\ \bar{N}_{R^*}(k) &= T_B^TN_{R^*}(k)T_B. \end{aligned} \tag{19}$$

First, we analyze the structure of the matrices $\bar{M}(k)$, $\bar{N}(k)$, $\bar{M}_\Pi$, $\bar{M}_{Q^*}(k)$ and $\bar{N}_{R^*}(k)$ in the expanded system. Consider the complementary matrices of $\tilde{\tilde{\mathbf{S}}}$ in the form $M(k) = M_{ij}(k)$, $N(k) = N_{ij}(k)$, $M_\Pi = M_{\Pi_{ij}}$, $M_{Q^*}(k) = M_{Q^*_{ij}}(k)$, $N_{R^*}(k) = N_{R^*_{ij}}(k)$ for $i,j = 1,\dots,4$, where $M_{\Pi_{ij}} = M^T_{\Pi_{ji}}$, $M_{Q^*_{ij}}(k) = M^T_{Q^*_{ji}}(k)$, $N_{R^*_{ij}}(k) = N^T_{R^*_{ji}}(k)$ and each matrix dimensions correspond to (16). Consider the matrices $\bar{M}(k) = \begin{pmatrix} \bar{M}_{11}(k) & \bar{M}_{12}(k) \\ \bar{M}_{21}(k) & \bar{M}_{22}(k) \end{pmatrix}$, $\bar{N}(k) = \begin{pmatrix} \bar{N}_{11}(k) & \bar{N}_{12}(k) \\ \bar{N}_{21}(k) & \bar{N}_{22}(k) \end{pmatrix}$, $\bar{M}_\Pi = \begin{pmatrix} \bar{M}_{\Pi_{11}} & \bar{M}_{\Pi_{12}} \\ \bar{M}^T_{\Pi_{12}} & \bar{M}_{\Pi_{22}} \end{pmatrix}$, $\bar{M}_{Q^*}(k) = \begin{pmatrix} \bar{M}_{Q^*_{11}}(k) & \bar{M}_{Q^*_{12}}(k) \\ \bar{M}^T_{Q^*_{12}}(k) & \bar{M}_{Q^*_{22}}(k) \end{pmatrix}$, $\bar{N}_{R^*}(k) = \begin{pmatrix} \bar{N}_{R^*_{11}}(k) & \bar{N}_{R^*_{12}}(k) \\ \bar{N}^T_{R^*_{12}}(k) & \bar{N}_{R^*_{22}}(k) \end{pmatrix}$, where $\bar{M}_{11}(k)$, $\bar{M}_{22}(k)$ are $n\times n$, $n_2\times n_2$ matrices, resp. $\bar{N}_{11}(k)$, $\bar{N}_{22}(k)$ are $n\times m$, $n_2\times m_2$ matrices, resp. $\bar{M}_{\Pi_{11}}$, $\bar{M}_{\Pi_{22}}$ are $n\times n$, $n_2\times n_2$ matrices, resp. $\bar{M}_{Q^*_{11}}(k)$, $\bar{M}_{Q^*_{22}}(k)$ are $n\times n$, $n_2\times n_2$ matrices, resp. $\bar{N}_{R^*_{11}}(k)$, $\bar{N}_{R^*_{22}}(k)$ are $m\times m$, $m_2\times m_2$ matrices, resp. We need to know the form of these submatrices. This is presented by the following propositions.

Proposition 1 *Consider* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given by (1) and (17) satisfying* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$*, resp. Then* $\bar{M}(r) = \begin{pmatrix} 0 & \bar{M}_{12}(r) \\ \bar{M}_{21}(r) & \bar{M}_{22}(r) \end{pmatrix}$*, where (0) denotes a null matrix of order n and the other blocks satisfy* $\bar{M}_{12}(p)\bar{M}_{21}(p-1) = 0$ *and* $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{M}_{21}(j-1) = 0$ *for all fixed* $k \in [k_0, k_f]$*, all* $r \in [k_0, k-1]$*, all* $p \in [k_0+1, k-1]$ *and all* $j \in [k_0+1, p-1]$.

Proposition 2 *Consider* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given by (1) and (17) satisfying* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$*, resp. Then* $\bar{N}(r) = \begin{pmatrix} 0 & \bar{N}_{12}(r) \\ \bar{N}_{21}(r) & \bar{N}_{22}(r) \end{pmatrix}$*, where (0) is an* $n\times m$ *matrix and the other blocks satisfy* $\bar{M}_{12}(p)\bar{N}_{21}(p-1) = 0$ *and* $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{N}_{21}(j-1) = 0$ *for all fixed* $k \in [k_0, k_f]$*, all* $r \in [k_0, k-1]$*, all* $p \in [k_0+1, k-1]$ *and all* $j \in [k_0+1, p-1]$.

Theorem 5 *Consider* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given by (1) and (17), resp. The pair* $(\tilde{\tilde{\mathbf{S}}}, \tilde{\tilde{J}}) \supset (\mathbf{S}, J)$ *if* $\bar{M}_\Pi = \begin{pmatrix} 0 & \bar{M}_{\Pi_{12}} \\ \bar{M}^T_{\Pi_{12}} & \bar{M}_{\Pi_{22}} \end{pmatrix}$*,* $\bar{M}_{Q^*}(k) = \begin{pmatrix} 0 & \bar{M}_{Q^*_{12}}(k) \\ \bar{M}^T_{Q^*_{12}}(k) & \bar{M}_{Q^*_{22}}(k) \end{pmatrix}$*,* $\bar{N}_{R^*}(k) = \begin{pmatrix} 0 & \bar{N}_{R^*_{12}}(k) \\ \bar{N}^T_{R^*_{12}}(k) & \bar{N}_{R^*_{22}}(k) \end{pmatrix}$ *and either*

$$\begin{aligned} &a)\ \bar{M}(p) = \begin{pmatrix} 0 & 0 \\ \bar{M}_{21}(p) & \bar{M}_{22}(p) \end{pmatrix},\ \bar{N}(p) = \begin{pmatrix} 0 & \bar{N}_{12}(p) \\ \bar{N}_{21}(p) & \bar{N}_{22}(p) \end{pmatrix} \ \text{or} \\ &b)\ \bar{M}(p) = \begin{pmatrix} 0 & \bar{M}_{12}(p) \\ 0 & \bar{M}_{22}(p) \end{pmatrix},\ \bar{N}(p) = \begin{pmatrix} 0 & \bar{N}_{12}(p) \\ 0 & \bar{N}_{22}(p) \end{pmatrix} \end{aligned} \tag{20}$$

hold for all fixed $k \in [k_0, k_f]$ *and all* $p \in [k_0+1, k-1]$.

3.1.3 Contractibility: The idea is to design a control law for $\tilde{\mathbf{S}}$ so that it can be contracted and implemented into $\mathbf{S}$. Now, we want to determine the conditions under which a control law designed for $\tilde{\tilde{\mathbf{S}}}$ can be contracted into $\mathbf{S}$ in terms of complementary matrices.

Denote matrices appearing in the contraction process as follows. The complementary matrix $F(k)$ has the form $F(k) = (F_{ij}(k))$, $i,j = 1,\dots,4$, where $F_{11}(k)$, $F_{22}(k)$, $F_{33}(k)$ and $F_{44}(k)$ are $m_1\times n_1$, $m_2\times n_2$, $m_2\times n_2$ and $m_3\times n_3$ matrices, resp. Define $\bar{F}(k) = \begin{pmatrix} \bar{F}_{11}(k) & \bar{F}_{12}(k) \\ \bar{F}_{21}(k) & \bar{F}_{22}(k) \end{pmatrix}$, where $\bar{F}_{11}(k)$ and $\bar{F}_{22}(k)$ are $m\times n$ and $m_2\times n_2$ matrices, resp. Similarly, denote the gain matrix $K(k) = (K_{ij}(k))$, $i,j = 1,2,3$, where $K_{ii}(k)$ are $m_i\times n_i$ matrices, resp. The gain matrix $\tilde{\tilde{K}}(k)$ for $\tilde{\tilde{\mathbf{S}}}$ has the form $\bar{\tilde{K}}(k) = \bar{R}K(k)\bar{U} + \bar{F}(k)$, where $\bar{\tilde{K}}(k) = T_B^{-1}\tilde{\tilde{K}}(k)T_A$ and $\bar{F}(k) = T_B^{-1}F(k)T_A$. So far we do not know the form of the complementary matrix $F(k)$ and the corresponding contractibility conditions. The following theorem solves the problem.

Theorem 6 *Consider* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given by (1) and (17) satisfying* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$*, resp. A control law* $\tilde{\tilde{u}}(k) = -\tilde{\tilde{K}}(k)\tilde{\tilde{x}}(k)$ *for* $\tilde{\tilde{\mathbf{S}}}$ *is contractible to the control law* $u(k) = -K(k)x(k)$ *of* $\mathbf{S}$ *if* $\bar{F}(k) = \begin{pmatrix} 0 & \bar{F}_{12}(k) \\ \bar{F}_{21}(k) & \bar{F}_{22}(k) \end{pmatrix}$ *satisfies*

$$\begin{aligned} \bar{F}_{12}(k)\bar{M}_{21}(k-1) &= 0, \\ \bar{F}_{12}(k)\bar{N}_{21}(k-1) &= 0, \\ \bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{M}_{21}(p-1) &= 0, \\ \bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{N}_{21}(p-1) &= 0 \end{aligned} \tag{21}$$

for all fixed $k \in [k_0, k_f]$ *and all* $p \in [k_0+1, k-1]$.

3.2 Selection of complementary matrices

The above results are general, i.e. they do not depend on the selection of the matrices V and R. Therefore, they can be applied to any expansion-contraction process. Specific transformation matrices V and R must be selected to expand a given problem (1) when considering the control design. We select the following expansion transformation matrices:

$$V = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}, \quad R = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}. \tag{22}$$

$x_2(k)$ and $u_2(k)$ appear in a repeated form in $\tilde{x}(k) = \left(x_1^T(k), x_2^T(k), x_2^T(k), x_3^T(k)\right)^T$ and $\tilde{u}(k) = \left(u_1^T(k), u_2^T(k), u_2^T(k), u_3^T(k)\right)^T$ by using (22), resp. The change of basis results in

$$T_A = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 & -I_{n_2} \\ 0 & 0 & I_{n_3} & 0 \end{pmatrix},$$

$$T_A^{-1} = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_{n_2} & \frac{1}{2}I_{n_2} & 0 \\ 0 & 0 & 0 & I_{n_3} \\ 0 & \frac{1}{2}I_{n_2} & -\frac{1}{2}I_{n_2} & 0 \end{pmatrix}. \tag{23}$$

T_B, T_B^{-1} have an analogous structure. The following theorems present the structure of the complementary matrices $M(k)$, $N(k)$, M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ and $F(k)$ in the initial bases.

Theorem 7 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (3), resp.* $\tilde{\mathbf{S}} \supset \mathbf{S}$ *if and only if* $M(r)$ *has the following structure:*

$$M(r) = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}(r) \tag{24}$$

and their blocks satisfy

$$\begin{aligned}
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(p-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}[p-1,j] \\
&\qquad \begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(j-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(p-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}[p-1,j] \\
&\qquad \begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(j-1) = 0
\end{aligned} \tag{25}$$

for all fixed $k \in [k_0, k_f]$, *all* $r \in [k_0, k-1]$, *all* $p \in [k_0+1, k-1]$ *and all* $j \in [k_0+1, p-1]$. *The matrix* $N(r)$ *has the same structure as* $M(r)$.

Theorem 8 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (3), resp.* $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ *if the matrices* M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ *have the following structure:*

$$M_\Pi = \begin{pmatrix} 0 & M_{\Pi_{12}} & -M_{\Pi_{12}} & 0 \\ M_{\Pi_{12}}^T & -M_{\Pi_{23}}-M_{\Pi_{23}}^T-M_{\Pi_{33}} & M_{\Pi_{23}} & M_{\Pi_{24}} \\ -M_{\Pi_{12}}^T & M_{\Pi_{23}}^T & M_{\Pi_{33}} & -M_{\Pi_{24}} \\ 0 & M_{\Pi_{24}}^T & -M_{\Pi_{24}}^T & 0 \end{pmatrix},$$

$$M_{Q^*} = \begin{pmatrix} 0 & M_{Q_{12}^*} & -M_{Q_{12}^*} & 0 \\ M_{Q_{12}^*}^T & -M_{Q_{23}^*}-M_{Q_{23}^*}^T-M_{Q_{33}^*} & M_{Q_{23}^*} & M_{Q_{24}^*} \\ -M_{Q_{12}^*}^T & M_{Q_{23}^*}^T & M_{Q_{33}^*} & -M_{Q_{24}^*} \\ 0 & M_{Q_{24}^*}^T & -M_{Q_{24}^*}^T & 0 \end{pmatrix}(k),$$

$$N_{R^*} = \begin{pmatrix} 0 & N_{R_{12}^*} & -N_{R_{12}^*} & 0 \\ N_{R_{12}^*}^T & -N_{R_{23}^*}-N_{R_{23}^*}^T-N_{R_{33}^*} & N_{R_{23}^*} & N_{R_{24}^*} \\ -N_{R_{12}^*}^T & N_{R_{23}^*}^T & N_{R_{33}^*} & -N_{R_{24}^*} \\ 0 & N_{R_{24}^*}^T & -N_{R_{24}^*}^T & 0 \end{pmatrix}(k),$$

and either

a) $$M(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -M_{22} & -M_{23} & -M_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}(p),$$

$$N(p) = \begin{pmatrix} 0 & N_{12} & -N_{12} & 0 \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{21} & -(N_{22}+N_{23}+N_{33}) & N_{33} & -N_{24} \\ 0 & N_{42} & -N_{42} & 0 \end{pmatrix}(p)$$

or

b) $$M(p) = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ 0 & M_{22} & -M_{22} & 0 \\ 0 & M_{32} & -M_{32} & 0 \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}(p),$$

$$N(p) = \begin{pmatrix} 0 & N_{12} & -N_{12} & 0 \\ 0 & N_{22} & -N_{22} & 0 \\ 0 & N_{32} & -N_{32} & 0 \\ 0 & N_{42} & -N_{42} & 0 \end{pmatrix}(p)$$

hold for all fixed $k \in [k_0, k_f]$ *and all* $p \in [k_0+1, k-1]$.

Theorem 9 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (3) satisfying* $\tilde{\mathbf{S}} \supset \mathbf{S}$, *resp. A control law* $\tilde{u}(k) = -\tilde{K}(k)\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k) = -K(k)x(k)$ *of* $\mathbf{S}$ *if* $F(k) = \begin{pmatrix} 0 & F_{12} & -F_{12} & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} \\ -F_{21} & -(F_{22}+F_{23}+F_{33}) & F_{33} & -F_{24} \\ 0 & F_{42} & -F_{42} & 0 \end{pmatrix}(k)$ *and satisfies*

$$\begin{aligned}
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(k-1) = 0, \\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(k-1) = 0, \\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}[k-1,p] \\
&\qquad \begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(p-1) = 0, \\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}[k-1,p] \\
&\qquad \begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(p-1) = 0
\end{aligned} \tag{26}$$

for all fixed $k \in [k_0, k_f]$ *and all* $p \in [k_0+1, k-1]$.

4 Example

Consider the system $\mathbf{S}$ defined by the matrices $A(k)$, $\Delta A(k)$, $B(k)$ and $\Delta B(k)$ given in (16). For instance, suppose that the complementary matrices $M(k)$ and $N(k)$ have the structure given in the case *a)* of Theorem 8. Consider $M(k)$ satisfying the following relations: $M_{21}(k) = A_{21}(k) + \Delta A_{21}(k)$, $M_{22}(k) = \frac{1}{2}A_{22}(k) + \frac{1}{2}\Delta A_{22}(k)$, $M_{23}(k) = -\frac{1}{2}A_{22}(k) - \frac{1}{2}\Delta A_{22}(k)$, $M_{24}(k) = -A_{23}(k) - \Delta A_{23}(k)$. Then, by using (9), the expanded matrix $\tilde{A}(k) + \Delta\tilde{A}(k)$ has the form:

$$\left(\begin{array}{cc|cc} A_{11}+\Delta A_{11} & \frac{1}{2}(A_{12}+\Delta A_{12}) & \frac{1}{2}(A_{12}+\Delta A_{12}) & A_{13}+\Delta A_{13} \\ 2(A_{21}+\Delta A_{21}) & A_{22}+\Delta A_{22} & 0 & 0 \\ \hline 0 & 0 & A_{22}+\Delta A_{22} & 2(A_{23}+\Delta A_{23}) \\ A_{31}+\Delta A_{31} & \frac{1}{2}(A_{32}+\Delta A_{32}) & \frac{1}{2}(A_{32}+\Delta A_{32}) & A_{33}+\Delta A_{33} \end{array}\right)(k).$$

Observe that the choice of the complementary matrix $M(k)$ can eliminate some elements of $\tilde{A}(k) + \Delta\tilde{A}(k)$ in the expanded space to get more zero blocks in the interconnected subsystems when considering the decentralized control design. $M(k)$ offers variety of possibilities to do this. Similar observation holds for the complementary matrix $N(k)$. It is possible to eliminate some submatrices $B_{ij}(k)$ and $\Delta B_{ij}(k)$, $i, j = 1, 2, 3$, of the matrix $\tilde{B}(k) + \Delta\tilde{B}(k)$ in the expanded space by using $N(k)$. In this case, $N(k)$ offers more flexibility than $M(k)$ because its structure is less restrictive.

5 Conclusion

In this paper, we have presented the expansion-contraction relations for a class of uncertain, nominally linear, time-varying, discrete-time systems associated with quadratic cost functions. These relations enable to design overlapping robust decentralized state feedback controllers by using guaranteed cost control approach. The solution is based on generalized selection of complementary matrices. A procedure has been proposed to select the complementary matrices in a constructive way.

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