Control & Guidance

2011

Enginyeria Tècnica d'Aeronàutica esp. en Aeronavegació

Escola d'Enginyeria de Telecomunicació i Aeroespacial de Castelldefels

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Classical control

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Classical control

1. Parametric estimation
2. Steady state error
3. Root locus
4. Controllers
5. Frequency response
6. Bode diagrams

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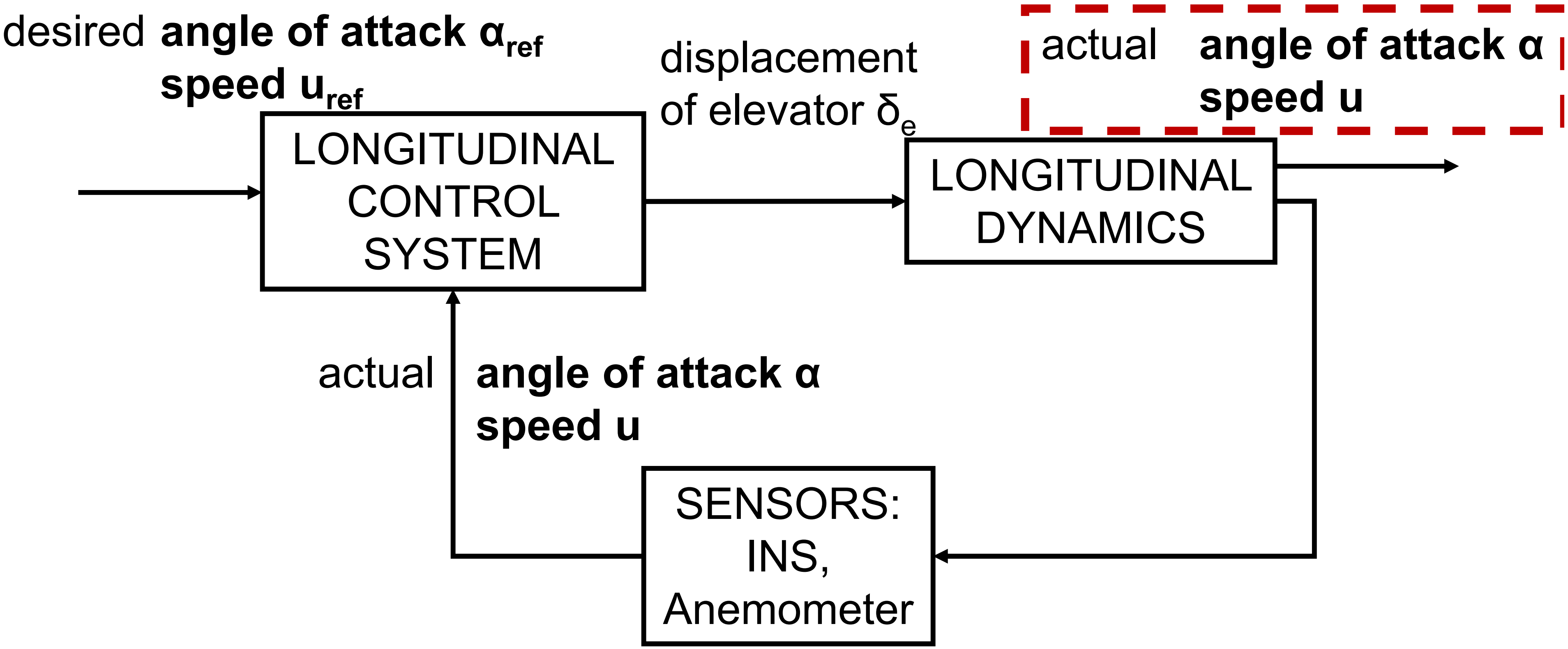
Study **properties** of the response of the system:



1- Parametric estimation

1- Parametric estimation

Temporal methods:

a. First-order systems

b. Second-order systems

c. Higher-order systems

a. First-order systems

A first-order system is defined by a first-order differential equation:

$$\tau \dot{y}(t) + y(t) = Kr(t) \xrightarrow{L} G(s) = \frac{Y(s)}{R(s)} = \frac{K}{1+\tau s}$$

τ: system time constant

K: gain

Electrical/mechanical examples

a. First-order systems

Impulse response

$$L[\delta(t)] = R(s) = 1 \rightarrow Y(s) = \frac{K}{1+\tau s}$$

using the inverse Laplace transform, the **impulse response** is:

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \qquad t \geq 0$$

1- Parametric estimation

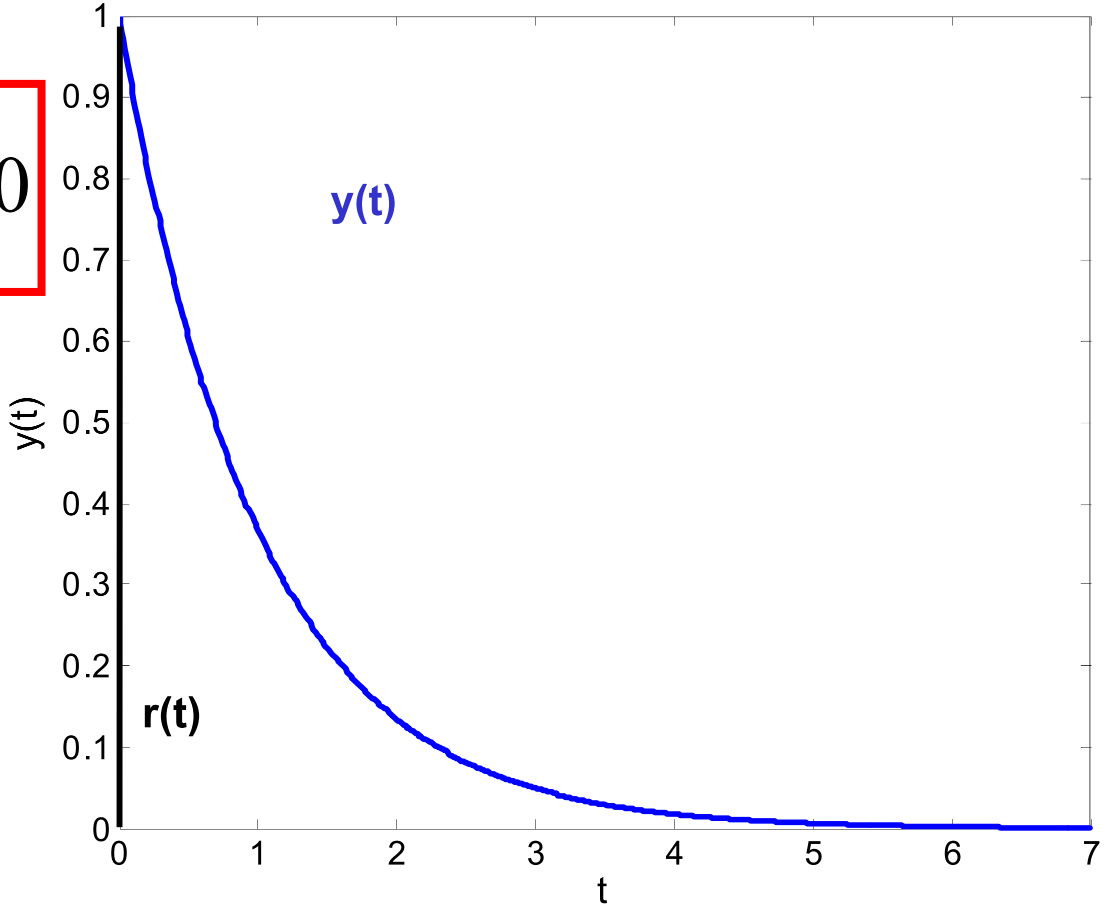
a. First-order systems

Impulse response

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \qquad t \geq 0$$

Tangent slope in 0:

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = -\frac{K}{\tau^2}$$



a. First-order systems

Step response: response to a unit step function

$$L[u(t)] = R(s) = \frac{1}{s} \rightarrow Y(s) = \frac{K}{(1+\tau s)s} = \frac{K}{s} - \frac{K\tau}{1+\tau s}$$

using the inverse Laplace transform, the **step response** or **indicial response** is:

$$y(t) = K\left(1 - e^{-t/\tau}\right) \qquad t \geq 0$$

a. First-order systems

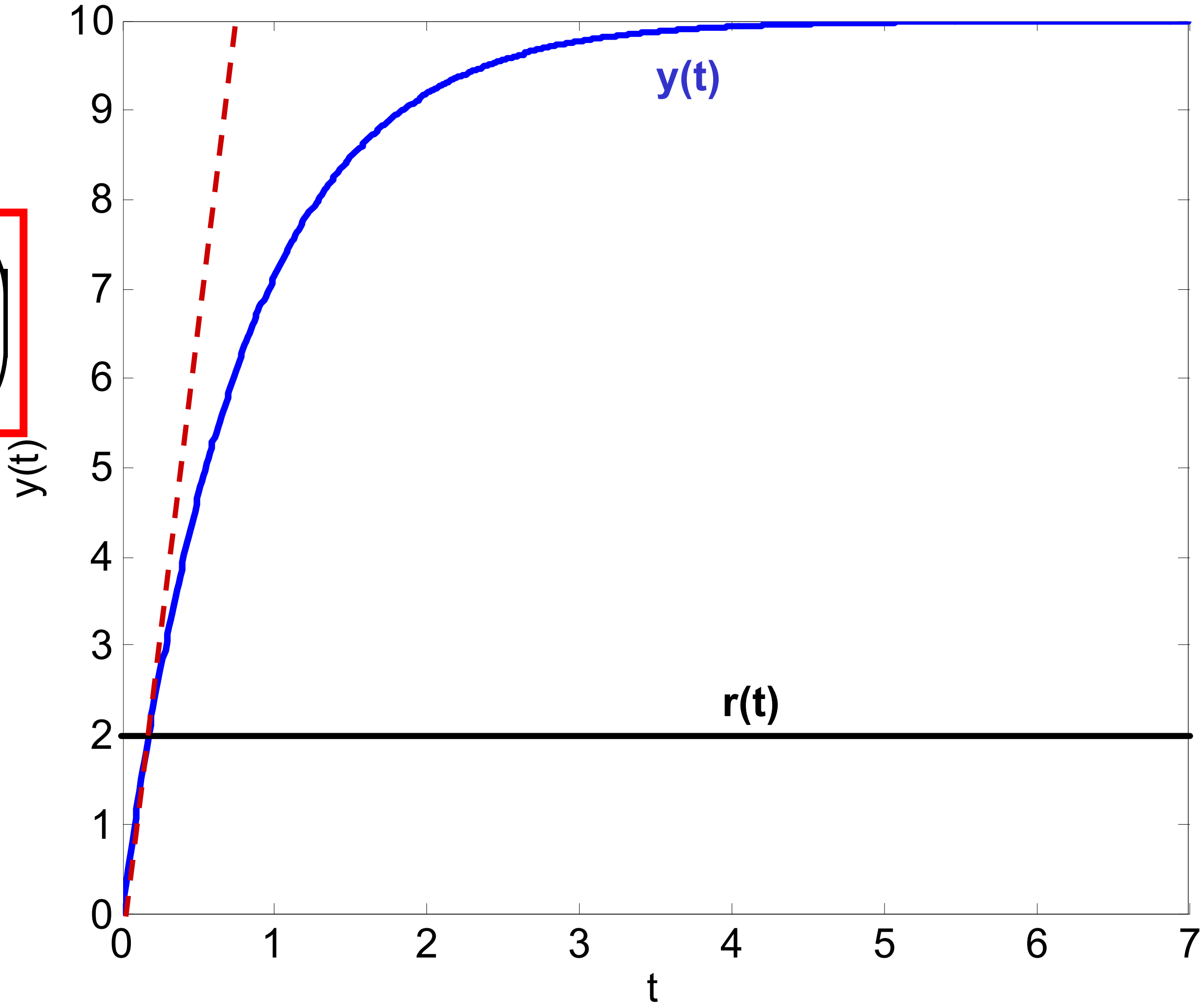
Step response:

$$y(t) = K\left(1 - e^{-t/\tau}\right)$$

Tangent slope in 0:

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = \frac{K}{\tau}$$

Example



b. Second-order systems

A second-order system is defined by a second-order differential equation:

$$b_2\ddot{y}(t)+b_1\dot{y}(t)+b_0y(t)=a_0r(t)$$

$$G(s)=\frac{Y(s)}{R(s)}=\frac{a_0}{b_2s^2+b_1s+b_0}$$

Electrical/Mechanical examples

b. Second-order systems

It can be factorized to emphasize particular parameters:

$$G(s) = \frac{Y(s)}{R(s)} = \frac{K}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with K: system gain (corresponds to final value for a unit step function)

ω_n: undamped natural frequency

ζ: damping factor (ζ>0)

b. Second-order systems

Step response: $$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Response depends on the poles of the transfer function

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\text{let } \Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2 = 4\omega_n^2\left(\zeta^2 - 1\right)$$

→ discriminant's sign depends on ζ value

→ **poles and response's properties depend on ζ value**

b. Second-order systems

ζ>1 Over-damped movement (non-oscillatory modes)

Real and negative poles: $s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$

$$Y(s) = \frac{1}{s} \times \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} \times \frac{K\omega_n^2}{(s - s_1) \times (s - s_2)}$$

Development in simple fractions:

$$Y(s) = K\left(\frac{1}{s} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{1}{s_1(s - s_1)} - \frac{1}{s_2(s - s_2)}\right)\right)$$

b. Second-order systems

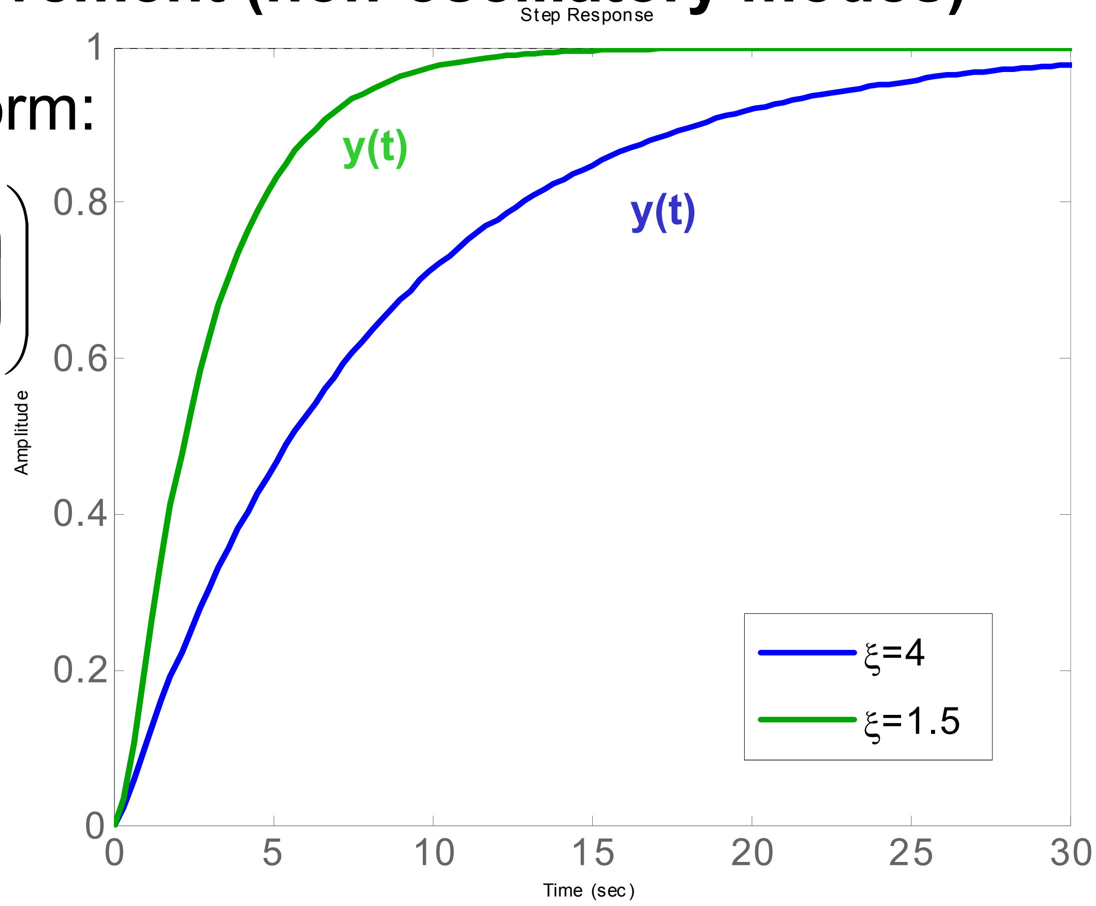
ζ>1 Over-damped movement (non-oscillatory modes)

Inverse Laplace transform:

$$y(t) = K\left(1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{s_1 t}}{s_1} - \frac{e^{s_2 t}}{s_2}\right)\right)$$

Tangent slope in 0:

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$$



b. Second-order systems

ζ=1 Critically damped movement (non-oscillatory modes)

Double real negative poles: $s_{1,2} = -\omega_n$

$$Y(s) = \frac{1}{s} \times \frac{K\omega_n^2}{(s+\omega_n)^2}$$

Development in simple fractions:

$$Y(s) = K\left(\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right)$$

b. Second-order systems

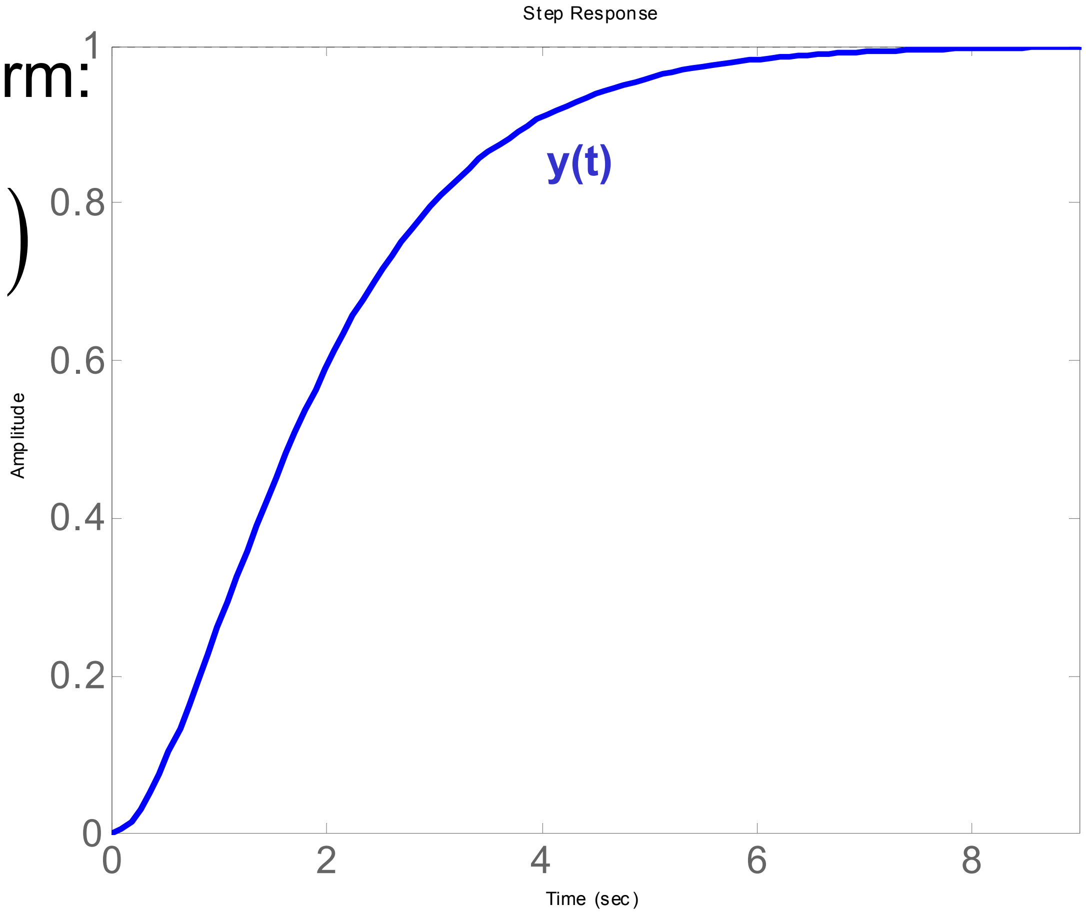
ζ=1 Critically damped movement (non-oscillatory modes)

Inverse Laplace transform:

$$y(t) = K\left(1-(1+\omega_n t)e^{-\omega_n t}\right)$$

Tangent slope in 0:

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$$



b. Second-order systems

ζ<1 Under-damped movement (oscillatory modes)

Conjugated complex poles: $$s_{1,2} = \omega_n\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)$$

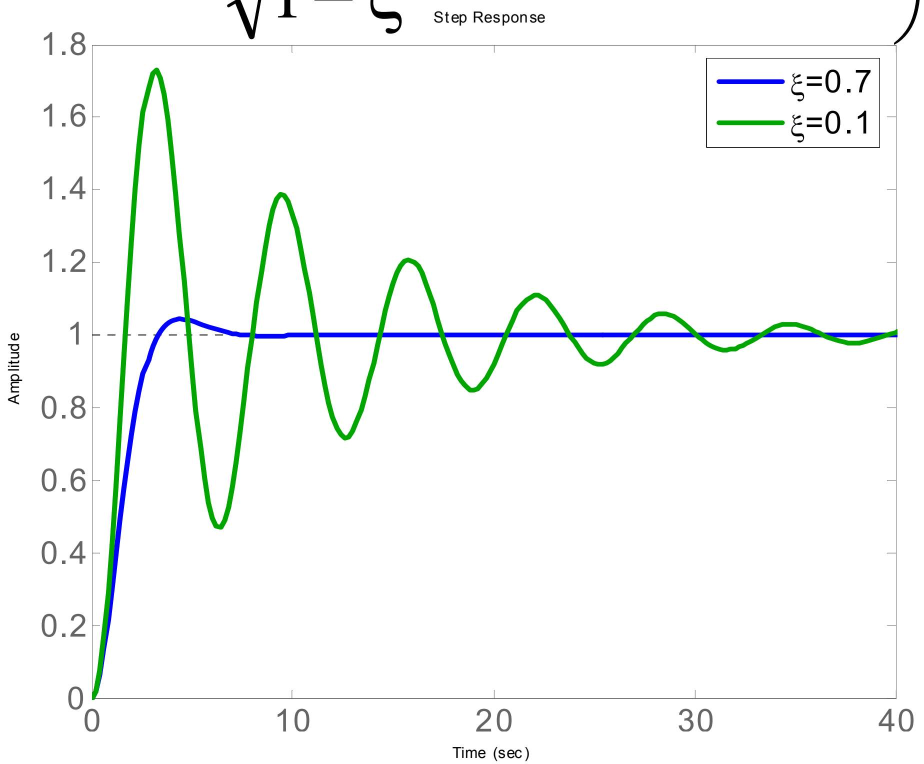
$$Y(s) = \frac{1}{s} \times \frac{K\omega_n^2}{s\left[(s+\zeta\omega_n)^2 + \omega_n^2\left(1-\zeta^2\right)\right]}$$

Development in simple fractions…

b. Second-order systems

ξ<1 Under-damped movement: Inverse Laplace transform:

$$y(t) = K\left(1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right)\right)$$



Tangent slope in 0:

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$$

b. Second-order systems

Characteristic parameters

K: gain $\dfrac{\text{output final value}}{\text{input final value}}$

M: maximum overshoot : represents the value of the highest peak of the system response measured with respect to the reference value (final value)

t_p: peak time: time needed for the response to arrive at its first peak

T: period

t_s: settling time

b. Second-order systems

Characteristic parameters: for second-order systems

K

$$M = e^{-\pi/\tan\varphi} = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

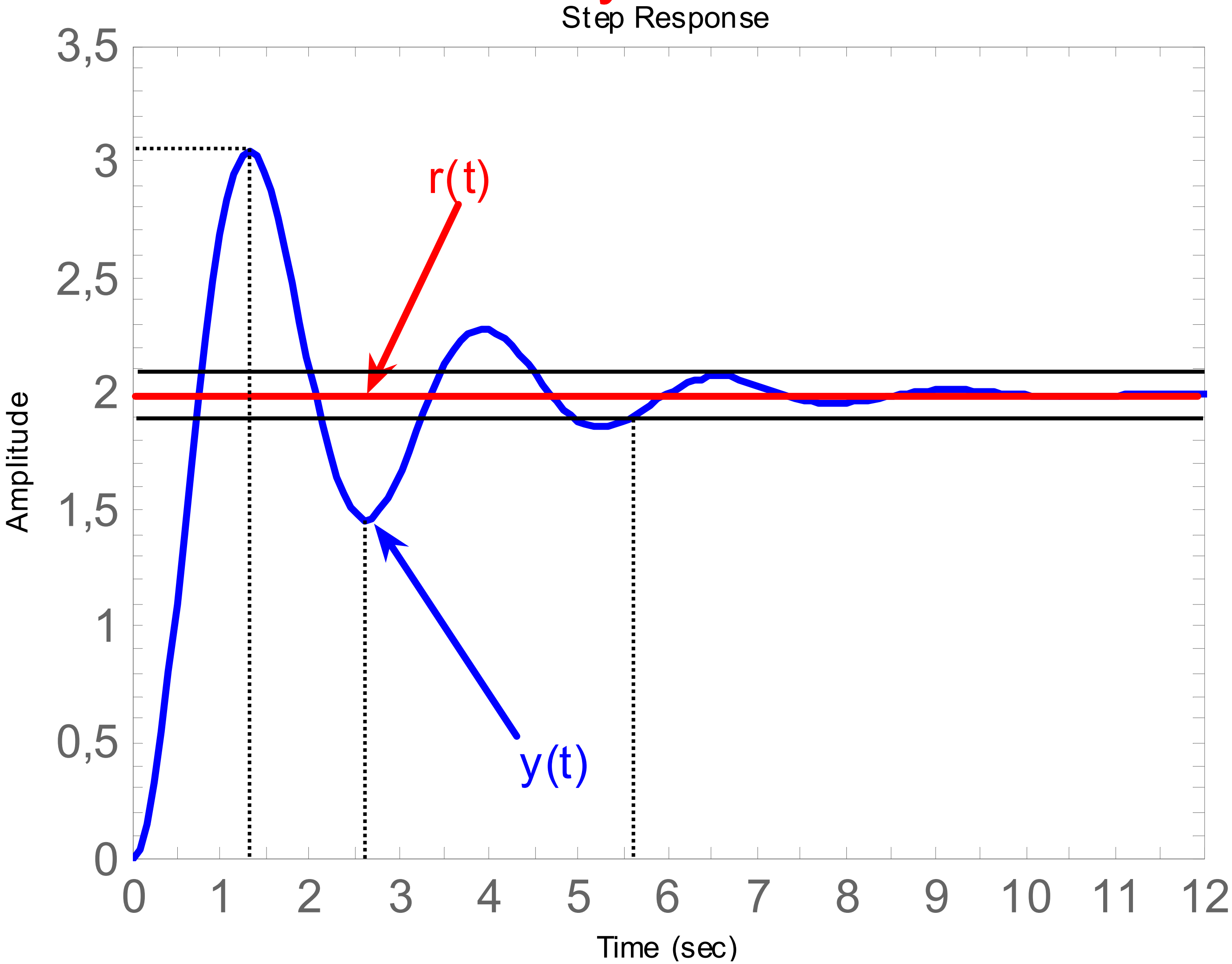
$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

(damped natural frequency)

1- Parametric estimation

b. Second-order systems



Obtain:

K

M

t_p

$t_{s5\%}$

T

Deduce:

ζ

ω_n

Calculate:

$G(s)=Y(s)/R(s)$

c. Higher-order systems

→ characterize the transitory state of any-order systems

generally *y(t)*= linear combination of elementary time functions defined by the nature (real or complex) of the characteristic equation roots: system modes:

- **real poles**: non- oscillatory modes, exponential term in the response
- **complex poles**: oscillatory modes, exponential term multiplied by sine or cosine

c. Higher-order systems

High order systems can be simplified using:

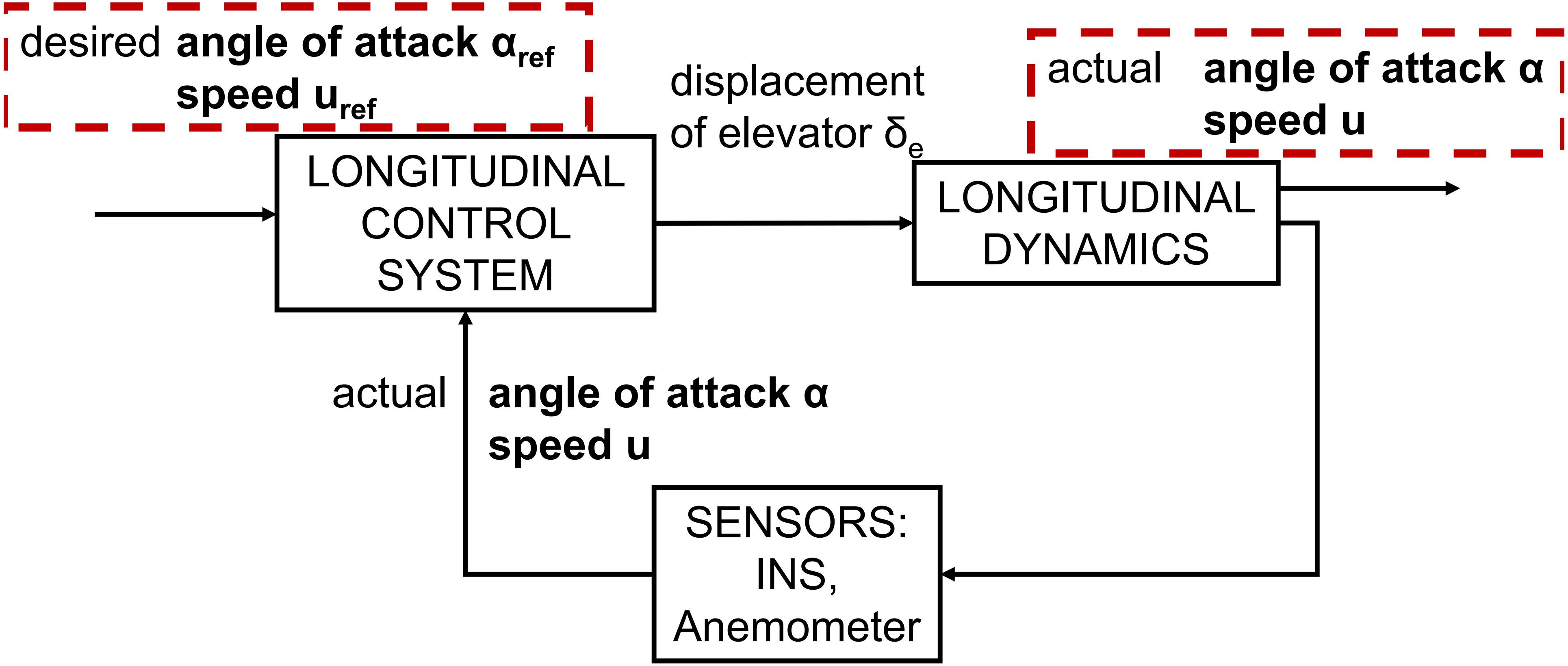
dominant poles

poles further from the imaginary axis have a weaker contribution

1 pole near 1 zero

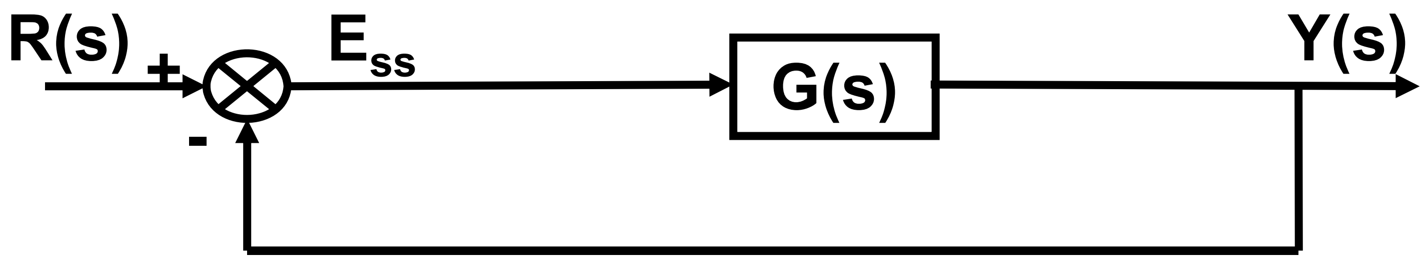
if there is a zero near a pole, this pole contribution will be weak

Study error of the response of the system:



2- Steady state error

2- Steady state error



Steady State error:

e_{ss}= difference between the entry signal and the exit signal

e_{ss} = “what we want minus what we get”

System's type:

Given the transfer function:

$$G(s) = K\frac{(1+as)(1+bs)...(1+cs+ds^2)...}{s^N(1+\alpha s)(1+\beta s)...(1+\chi s+\delta s^2)...}$$

with K: system gain,

and N: number of poles in the origin

$\rightarrow$ **N = system's type**

2- Steady state error

Definition of Steady State error:

$$e_{ss} = \lim_{t \to +\infty} \left[r(t) - y(t) \right]$$

$e_{ss} > 0$: exit signal has not reached the entry reference

$e_{ss} < 0$: exit signal is higher than the entry

2- Steady state error

$$e_{ss} = \lim_{t \to +\infty} [r(t) - y(t)]$$

Moving to the Laplace space: **Final value theorem**:

$$e_{ss} = \lim_{s \to 0} \left(s[R(s) - Y(s)] \right) = \lim_{s \to 0} \left(s \left[R(s) - \frac{G(s)}{1+G(s)} R(s) \right] \right)$$

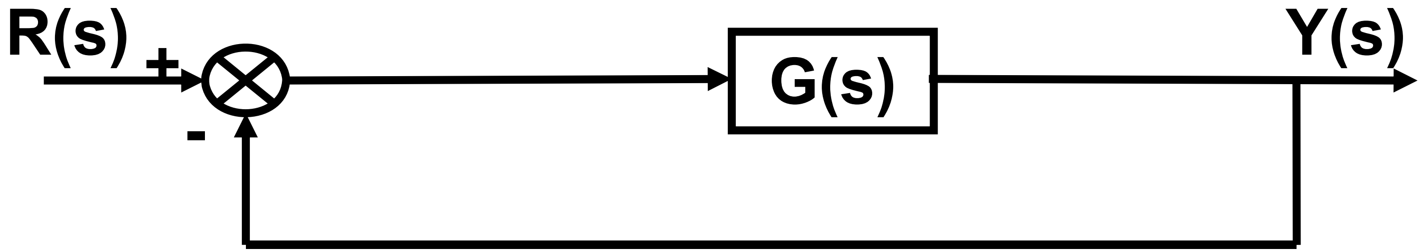
$$e_{ss} = \lim_{s \to 0} \left(s \times \frac{R(s)}{1+G(s)} \right)$$

Depends on the entry

+ on the system's type

2- Steady state error

$$G(s) = K\frac{(1+as)(1+bs)\ldots(1+cs+ds^2)\ldots}{s^N(1+\alpha s)(1+\beta s)\ldots(1+\chi s+\delta s^2)\ldots}$$



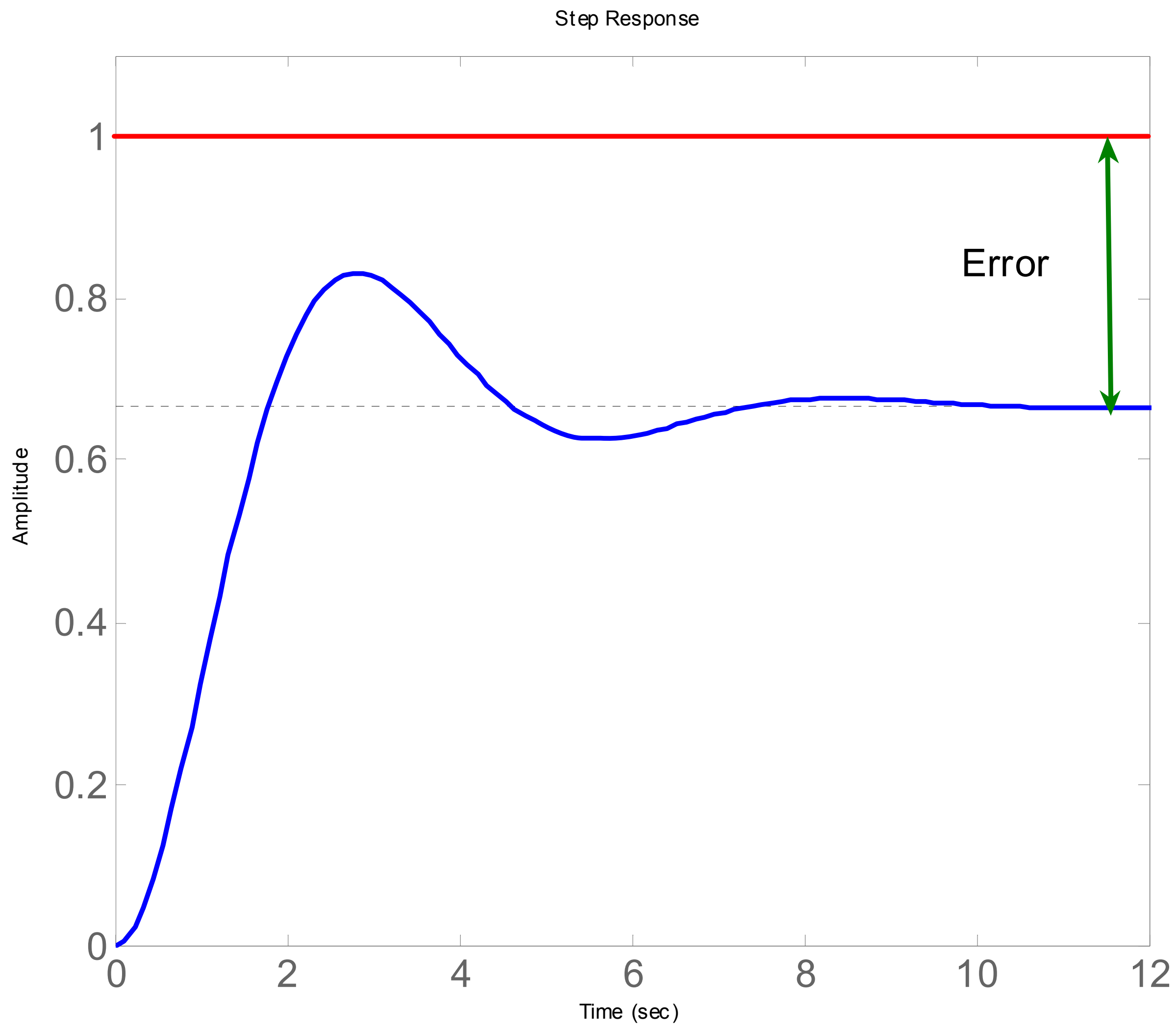
1. Position error: error for a step function entry: r(t)=u(t)

$$e_p = \lim_{s\to 0}\left(s \times \frac{1}{1+G(s)} \times \frac{1}{s} \right) = \lim_{s\to 0}\left(\frac{1}{1+G(s)} \right) = \frac{1}{1+\lim\limits_{s\to 0} G(s)}$$

$$= \begin{cases} \dfrac{1}{1+K} & \text{type } 0 \\ 0 & \text{type} \geq \mathrm{I} \end{cases}$$

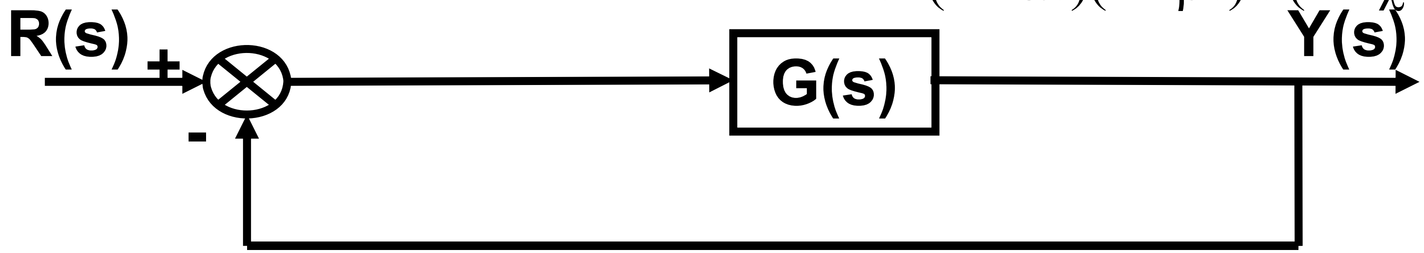
2- Steady state error

1. Position error: error for a step function entry: r(t)=u(t)



2- Steady state error

$$G(s) = K\frac{(1+as)(1+bs)\dots(1+cs+ds^2)\dots}{s^N(1+\alpha s)(1+\beta s)\dots(1+\chi s+\delta s^2)\dots}$$



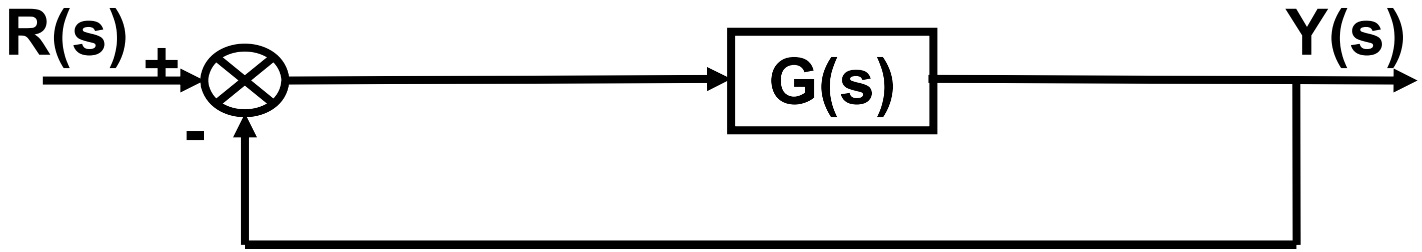
2. Speed error: error for a ramp function entry: r(t)= t

$$e_v = \lim_{s\to 0}\left(s\times\frac{1}{1+G(s)}\times\frac{1}{s^2}\right) = \lim_{s\to 0}\left(\frac{1}{s+sG(s)}\right) = \lim_{s\to 0}\left(\frac{1}{sG(s)}\right)$$

$$= \begin{cases} \infty & \text{type } 0 \\ \dfrac{1}{K} & \text{type I} \\ 0 & \text{type} \geq \text{II} \end{cases}$$

2- Steady state error

$$G(s) = K\frac{(1+as)(1+bs)\ldots(1+cs+ds^2)\ldots}{s^N(1+\alpha s)(1+\beta s)\ldots(1+\chi s+\delta s^2)\ldots}$$



3. Acceleration error: error for a parabolic entry: r(t)= t²

$$e_v = \lim_{s\to 0}\left(s\times\frac{1}{1+G(s)}\times\frac{1}{s^3}\right) = \lim_{s\to 0}\left(\frac{1}{s^2+s^2G(s)}\right) = \lim_{s\to 0}\left(\frac{1}{s^2G(s)}\right)$$

$$= \begin{cases} \infty & \text{type 0 or I} \\ \dfrac{1}{K} & \text{type II} \\ 0 & \text{type} \geq \text{III} \end{cases}$$

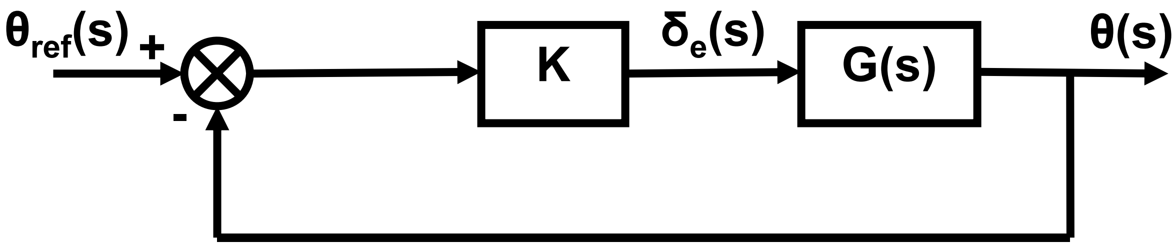
2- Steady state error

Error based on type + entry

Input: Type:	step	ramp	parabolic
0	constant	∞	∞
I	0	constant	∞
II	0	0	constant

2- Steady state error

Example:

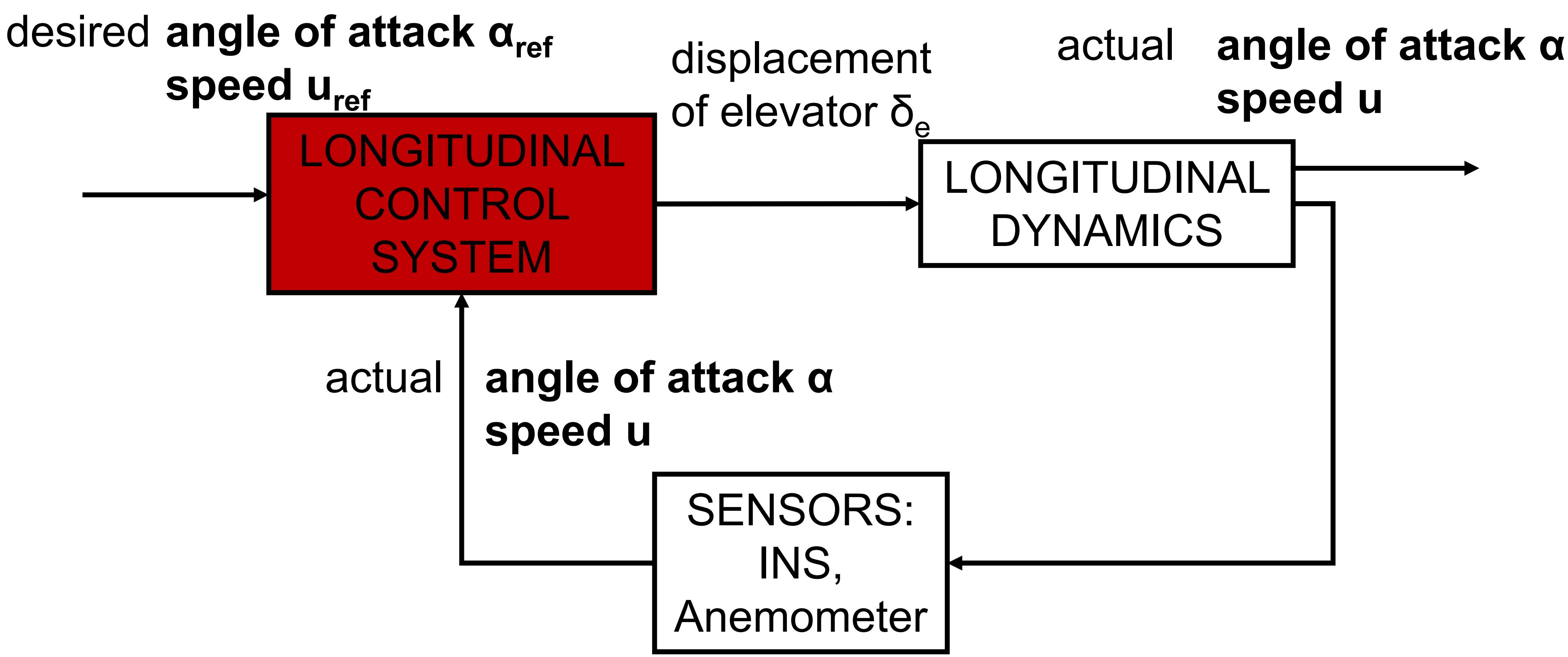


Compute the error in steady state for a unit step function entry and for a system with the following open loop transfer function:

$$\frac{\theta(s)}{\delta_e(s)} = \frac{2s + 0.1}{s^2 + 0.1s + 4}$$

- for K=1, K=10, K=100,
- for K=1 and $$\frac{\theta(s)}{\delta_e(s)} = \frac{2s + 0.1}{s\left(s^2 + 0.1s + 4\right)}$$

Design a simple proportional controller in order to satisfy some constraints on the response of the system



3- Root locus

3- Root locus

- Root locus technique
- Gain setting
- Effect of zeros and poles

Root locus technique

- Introduced by W. R. Evans in 1949: developed a series of rules that allow the control system engineer to quickly draw the **root locus diagram** = **locus of all possible roots of the characteristic equation**: **1+K G(s)= 0**

  = locus of all possible poles in closed loop

  as K varies from 0 to infinity

- The resulting plot helps us in selecting the best value of K

- Gives information for the **transitory** part of the response (**stability, damping factor, natural frequency**)

Root locus technique

Let
$$G(s)=\frac{\alpha(s+z_1)(s+z_2)(s+z_m)}{(s+p_1)(s+p_2)(s+p_n)}$$

And substitute it in the characteristic equation

$$1+\frac{k(s+z_1)(s+z_2)(s+z_m)}{(s+p_1)(s+p_2)(s+p_n)}=0 \quad \text{where} \quad k=K\alpha$$

Root locus technique

The characteristic equation is complex and can be written in terms of magnitude and angle as follows

$$\frac{|k||s+z_1||s+z_2||s+z_m|}{|s+p_1||s+p_2||s+p_n|}=1$$

$$\sum_{i=1}^{m}\angle(s+z_i)-\sum_{i=1}^{n}\angle(s+p_i)=(2q+1)\times 180$$

$$\text{for } q=0,1,2\ldots,(n-m-1)$$

Root locus technique: Rules

If we rearrange the magnitude criteria as

$$\frac{|s+z_1||s+z_2||s+z_m|}{|s+p_1||s+p_2||s+p_n|}=\frac{1}{|k|}$$

Rule 1: The number of separate branches of the root locus plot is equal to the number of poles of the transfer function (n)

Branches of the root locus **originate** at the poles of G(s) for k=0 and **terminate** at either the open-loop zeroes or at infinity for k=+∞

n separate branches, n-m infinite branches, m finite branches

Root locus technique: Rules

Rule 2: Because the complex poles are always "conjugated",

the root locus branches are **symmetric** with respect to the real axis

Root locus technique: Rules

Rule 3:

Segments of the real axis that are part of the root locus:

points on the real axis that have an **odd** number of poles and zeroes to their right

Root locus technique: Rules

Rule 4: Asymptotes

The root locus branches that approach the open-loop zeroes at infinity do so along straight-line asymptotes that intersect the real axis at the center of gravity of the finite poles and zeroes

$$\sigma = \frac{\left[\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i\right]}{n - m}$$

The angle that the asymptotes make with the real axis is given by

$$\phi_a = \frac{180^o\left[2q+1\right]}{n-m} \quad \text{for } q = 0, 1, 2 \ldots, (n-m-1)$$

Root locus technique: Rules

Rule 5: breakaway points

If a portion of the real axis is part of the root locus and a branch is between two poles the branch must break away from the real axis so that the locus ends on a zero as k approaches infinity. The breakaway points on the real axis are determined by solving

$$1+\frac{k(s+z_1)(s+z_2)(s+z_m)}{(s+p_1)(s+p_2)(s+p_n)}=0 \quad \text{for } k$$

and then finding the roots of the equation **dk/ds=0**

Only roots that lie on a branch of the locus are of interest

Root locus technique: Rules

Rule 6: **Intersection with the imaginary axis**

Solve the characteristic equation for **s=jω** (equation of the imaginary axis)

$$1+\frac{k(j\omega+z_1)(j\omega+z_2)(j\omega+z_m)}{(j\omega+p_1)(j\omega+p_2)(j\omega+p_n)}=0$$

Root locus technique: Rules

Rule 7: for **complex** poles and zeroes only:

The angle of departure of the root locus from a pole of G(s) or arrival angle at a zero of G(s) can be found by the following expression

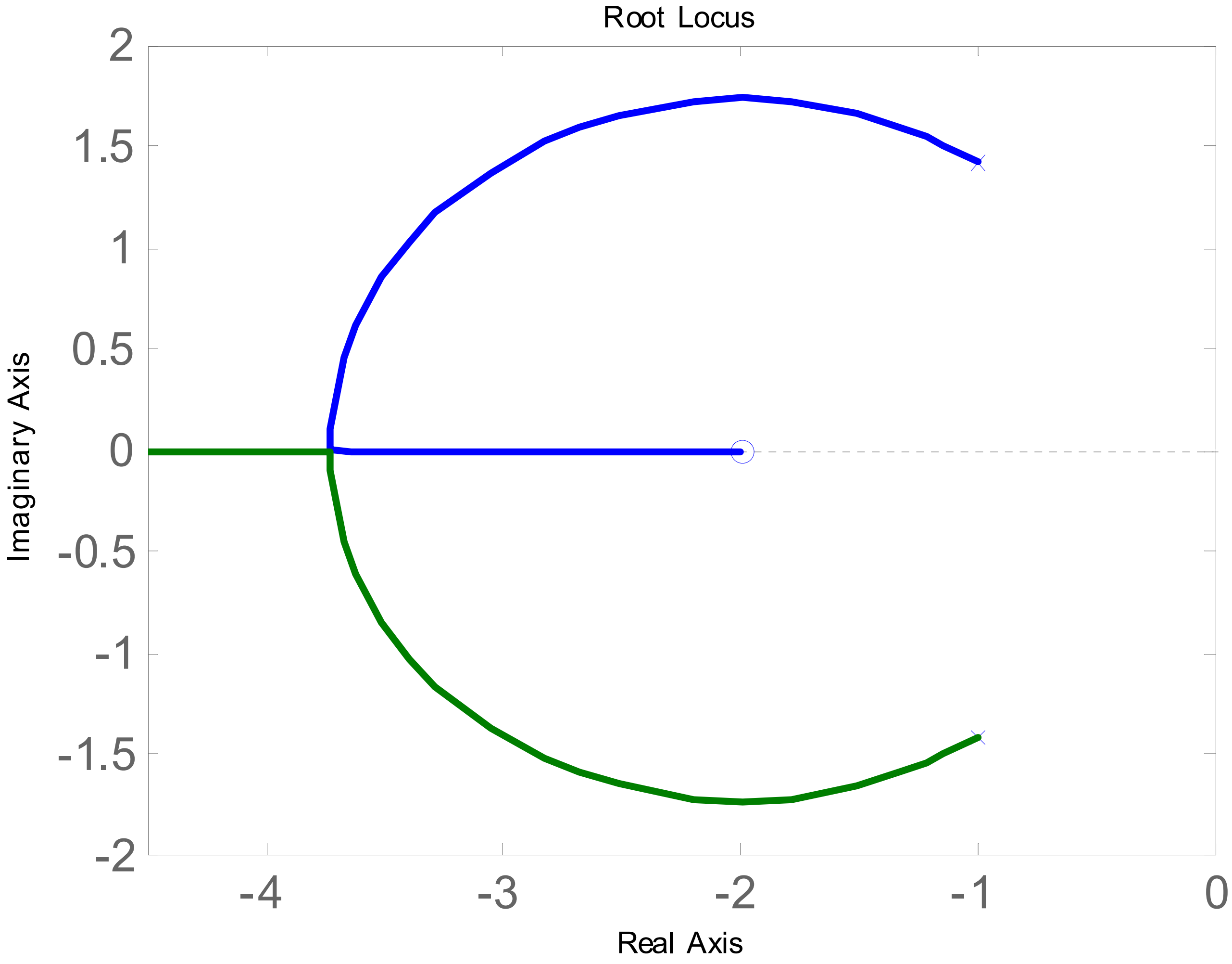
If you consider a test point t:

$$\sum_{i=1}^{m} \angle\left(t + z_i\right) - \sum_{i=1}^{n} \angle\left(t + p_i\right) = \pm 180$$

Root locus technique: examples

Example 1:

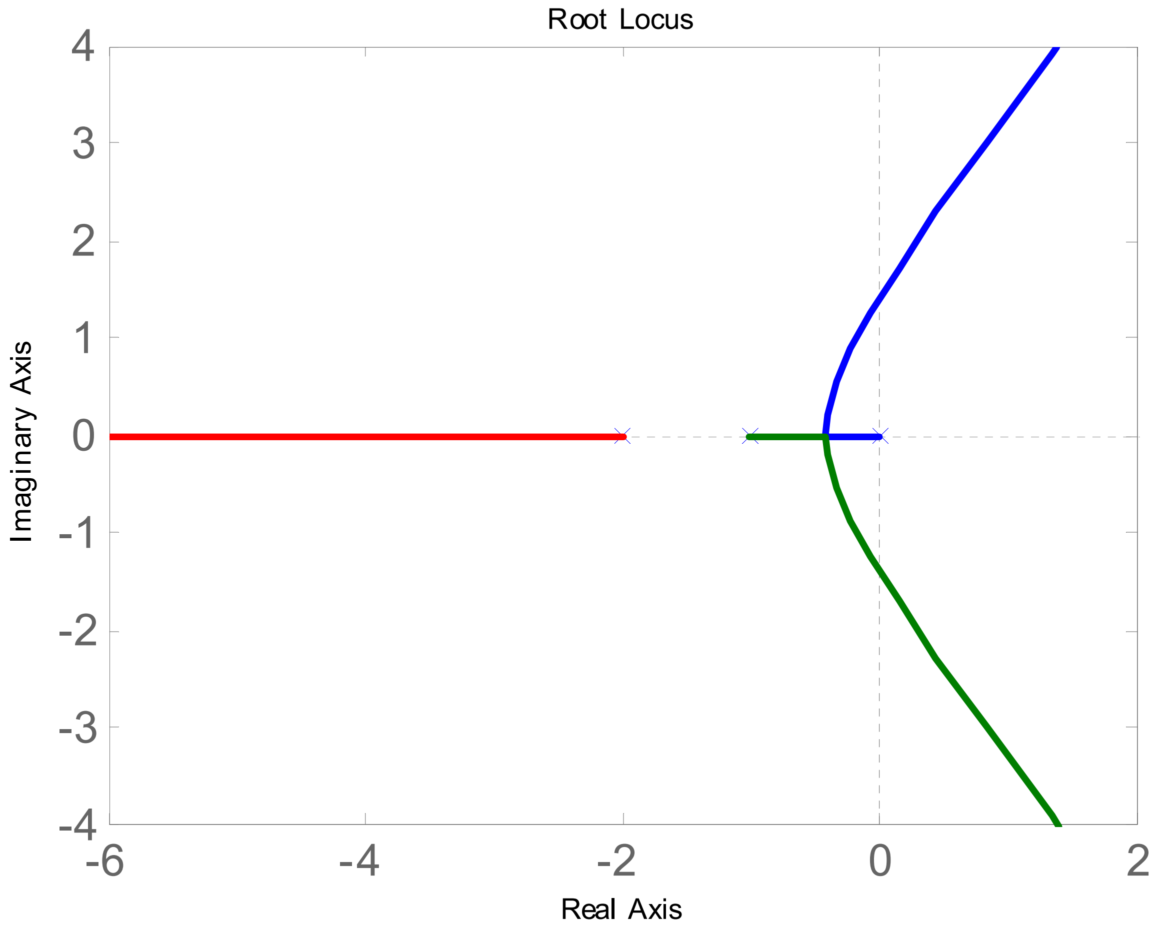
$$G(s) = \frac{s+2}{s^2+2s+3}$$



Root locus technique: examples

Example 2:

$$G(s)=\frac{1}{s(s+1)(s+2)}$$



3- Root locus

Setting of the gain and natural frequency

Basic operation: to adjust the **gain** K to obtain a **damping factor** given by the poles in closed-loop and fixed by the damping factor ζ

Cf second-order systems:

$$\zeta = \frac{|\mathrm{Re}(s_i)|}{|s_i|}$$

straight line doing an angle φ with the real axis (cos φ= ζ) sets an intersection point with the poles position, and k (and then K) is obtained solving the characteristic equation

Natural frequency for a second-order system: $\boldsymbol{\omega_n = |s_i|}$

Gain setting

$$1 + KG(s) = 0$$

$$1 + K\frac{\alpha(s+z_1)(s+z_2)(s+z_m)}{(s+p_1)(s+p_2)(s+p_n)} = 0$$

The system total gain is computed thanks to the module condition

$$k = K \times \alpha = \frac{\overline{p_1 s} \times \overline{p_2 s} \times \overline{p_m s}}{\overline{z_1 s} \times \overline{z_2 s} \times \overline{z_n s}}$$

“total gain” = product of the distances from the poles of G(s) to the intersection point (= target pole) divided by the product of the distances from zeros of G(s)

Gain setting

$$1 + KG(s) = 0$$

$$1 + K\frac{\alpha}{(s+p_1)(s+p_2)(s+p_n)} = 0$$

If there are no zeros:

$$K \times \alpha = \overline{p_1 s} \times \overline{p_2 s} \times \overline{p_m s}$$

“total gain” =product of the distances between the poles of G(s)

and the intersection point (= target pole)

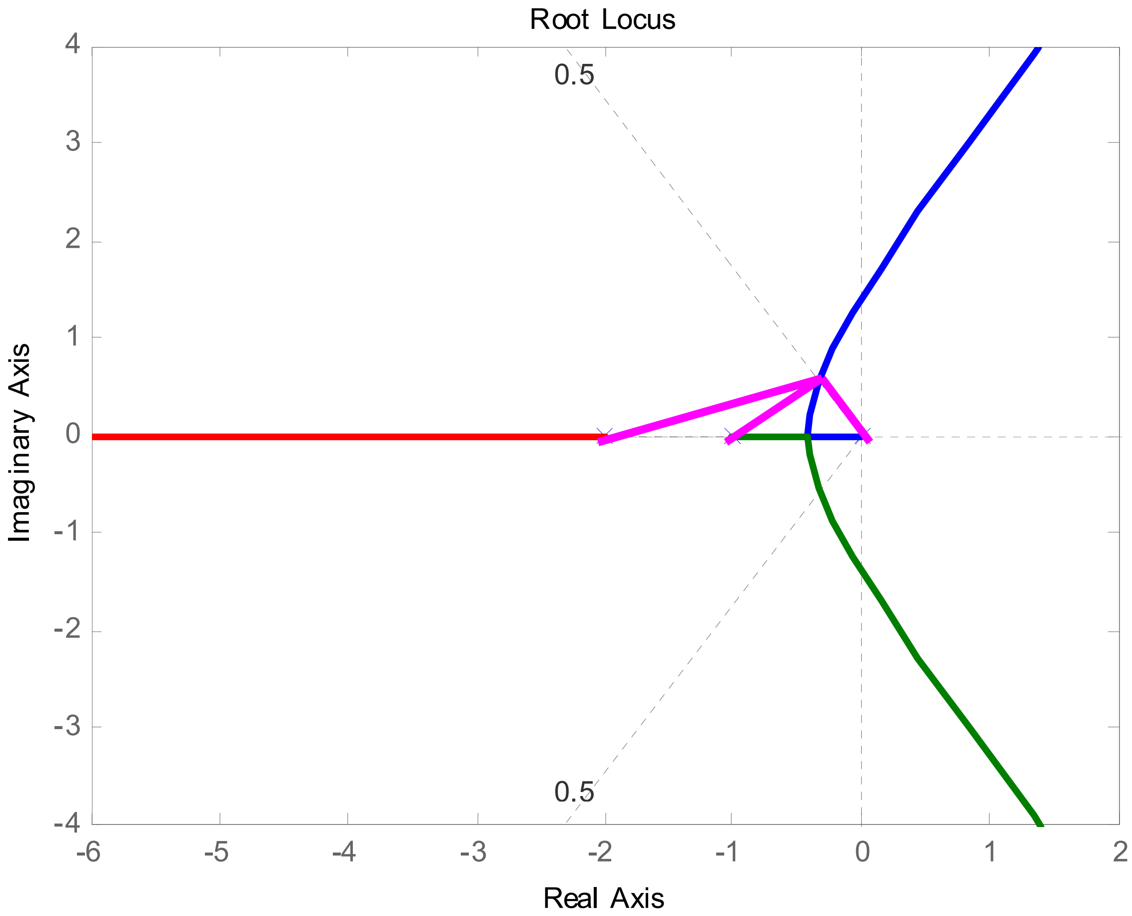
Examples

Gain setting

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Designer requirement:
we want ζ=0.5

ζ=0.5=cos φ → φ=60º



Examples

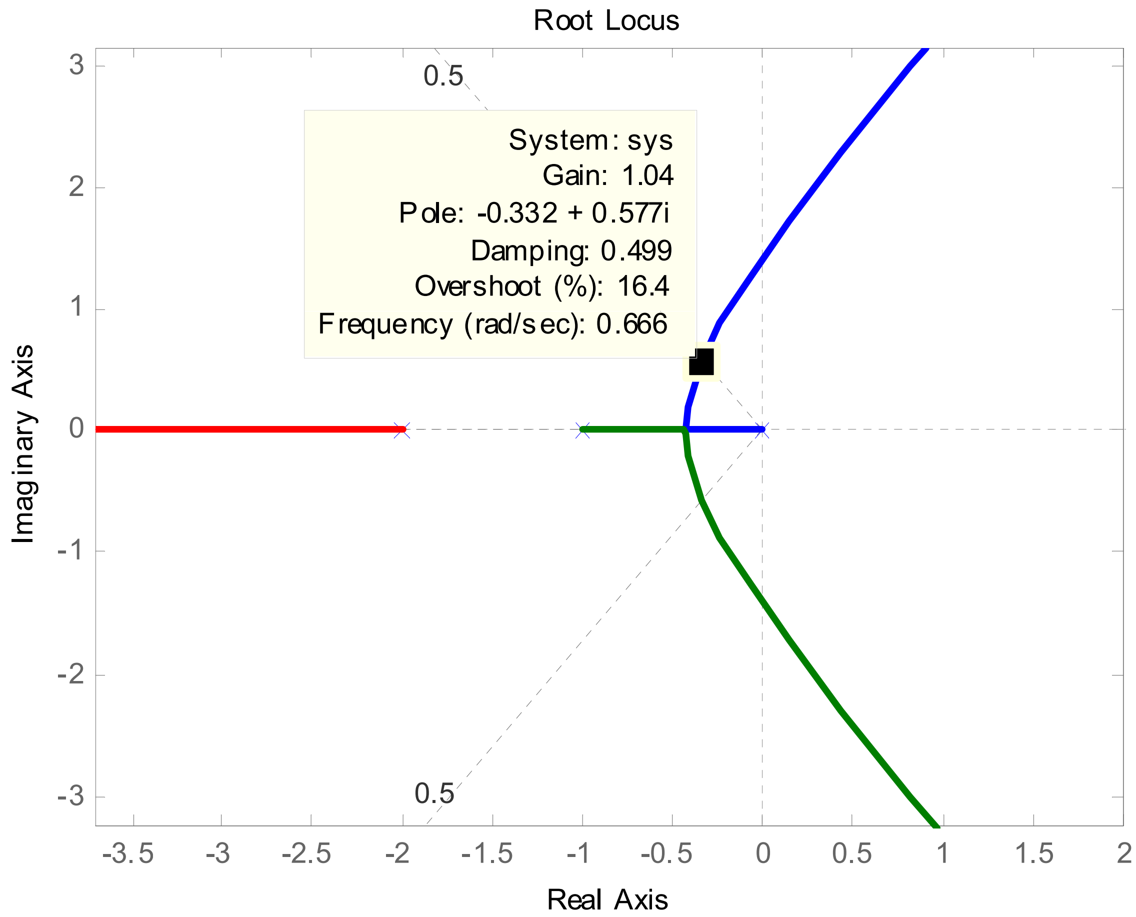
3- Root locus

Gain setting

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Designer requirement:
we want ζ=0.5

$\zeta = 0.5 = \cos\varphi \rightarrow \varphi = 60^\circ$



Examples

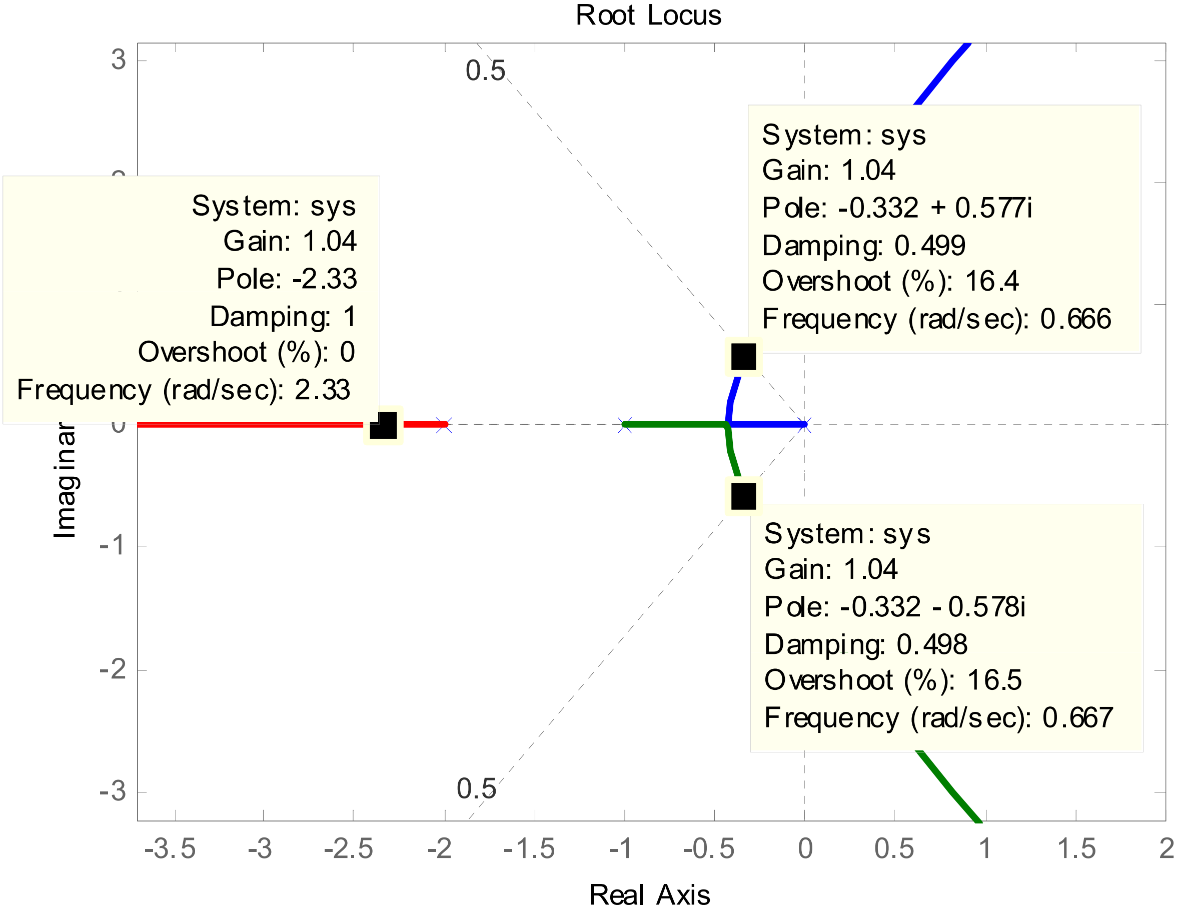
3- Root locus

Gain setting

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Designer requirement: we want ζ=0.5

This corresponds to k=1.04



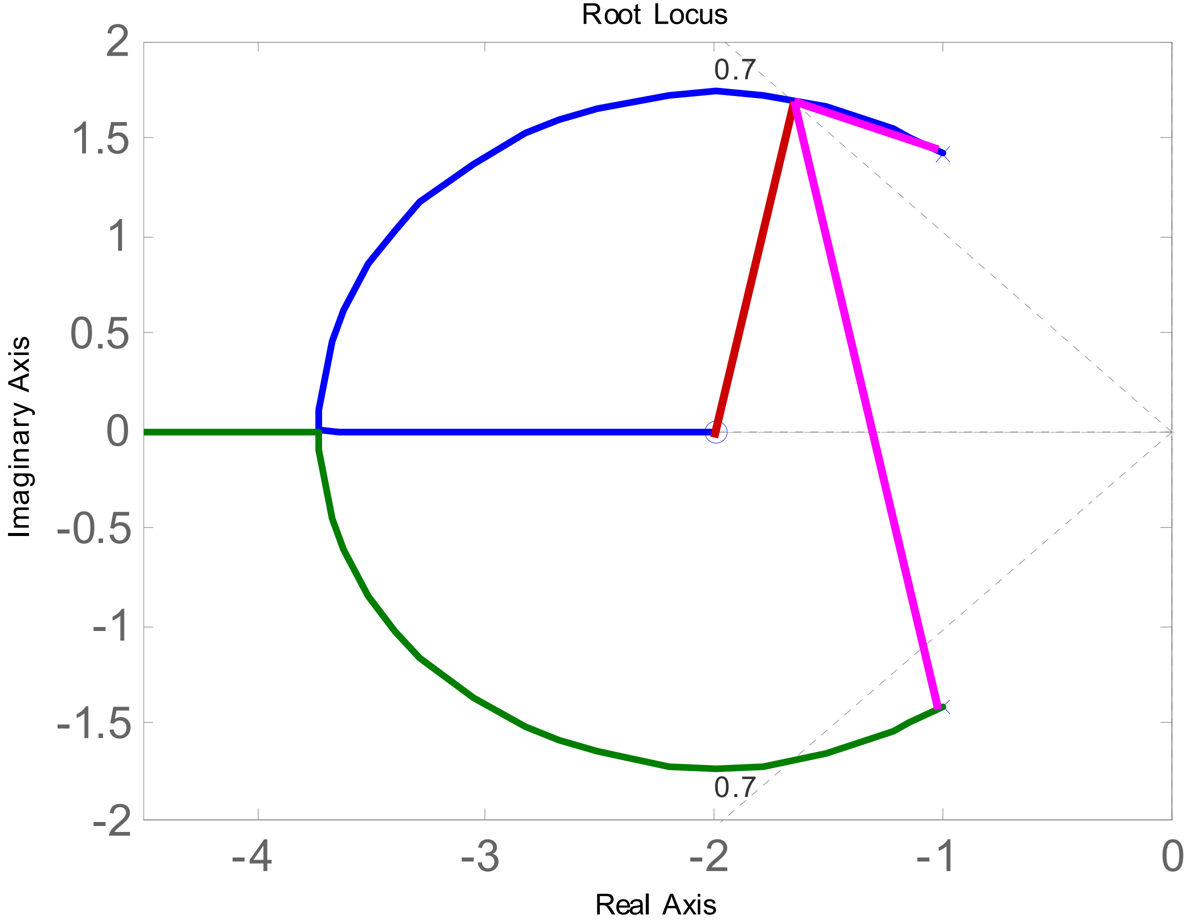
Examples

3- Root locus

Gain setting

$$G(s) = \frac{s+2}{s^2+2s+3}$$

Designer requirement:
we want ζ=0.7



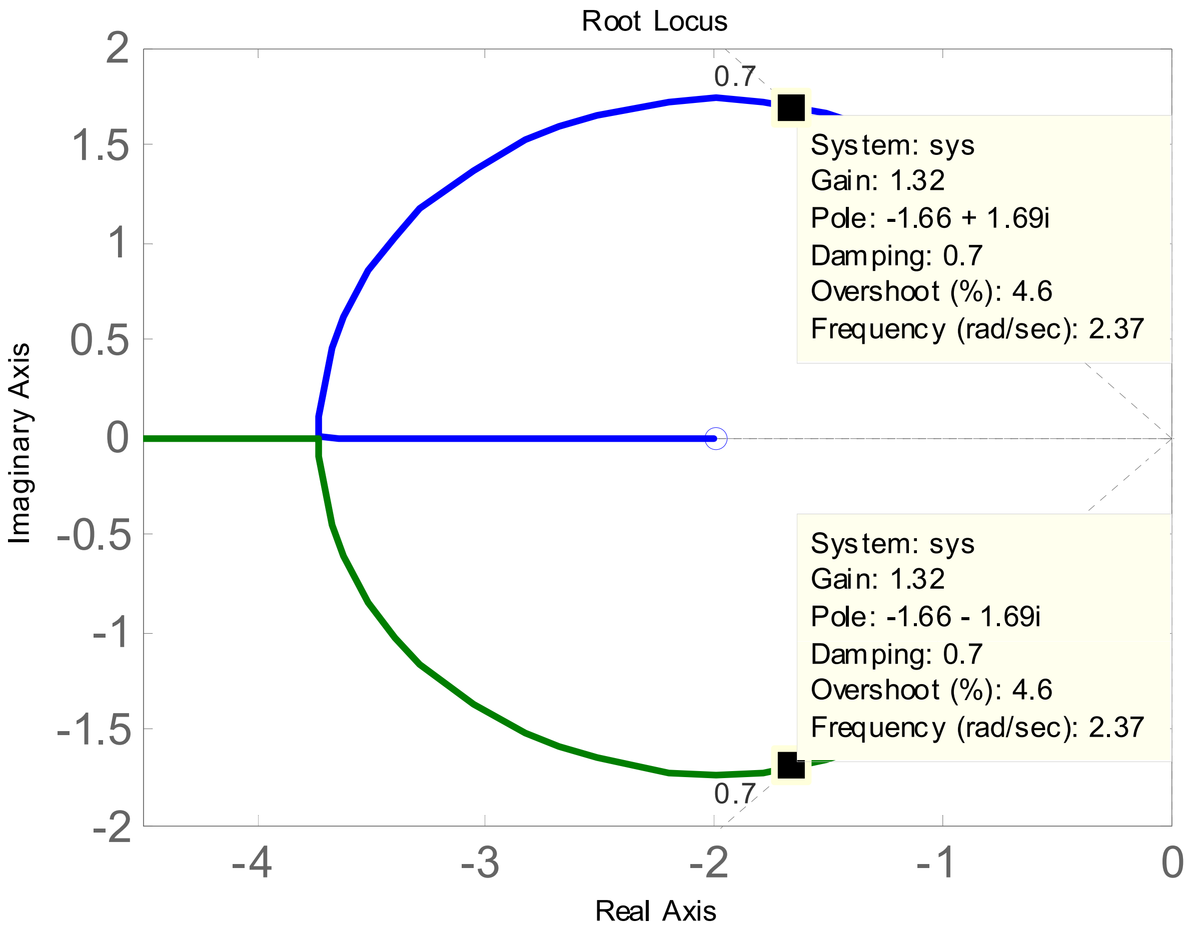
Examples

Gain setting

$$G(s) = \frac{s+2}{s^2+2s+3}$$

Designer requirement:
we want ζ=0.7

This corresponds to
k=1.32



Examples

Relative stability: gain margin

“Re(s)<0” criterion informs about the absolute stability of a system but it says nothing about its relative stability

= how far it is from the instability → system **strength**

Gain margin: maximum proportional factor that can be introduced into the control loop until the system becomes critically stable.

$$M_G = \frac{k_{Cr}}{k_{actual}}$$

Examples

Gain setting

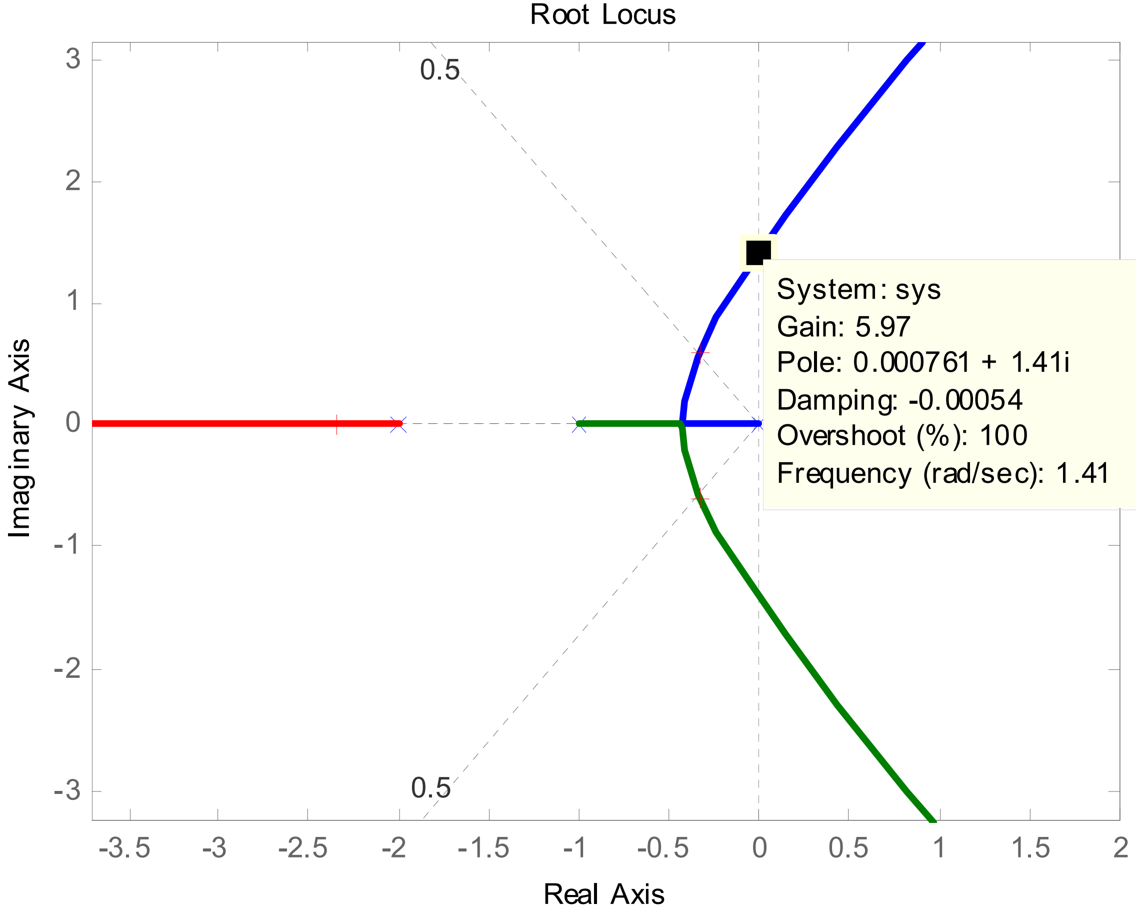
$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Designer requirement: we want $\zeta=0.5$

This corresponds to k=1

$k_{cr}=6$

$M_G=6$

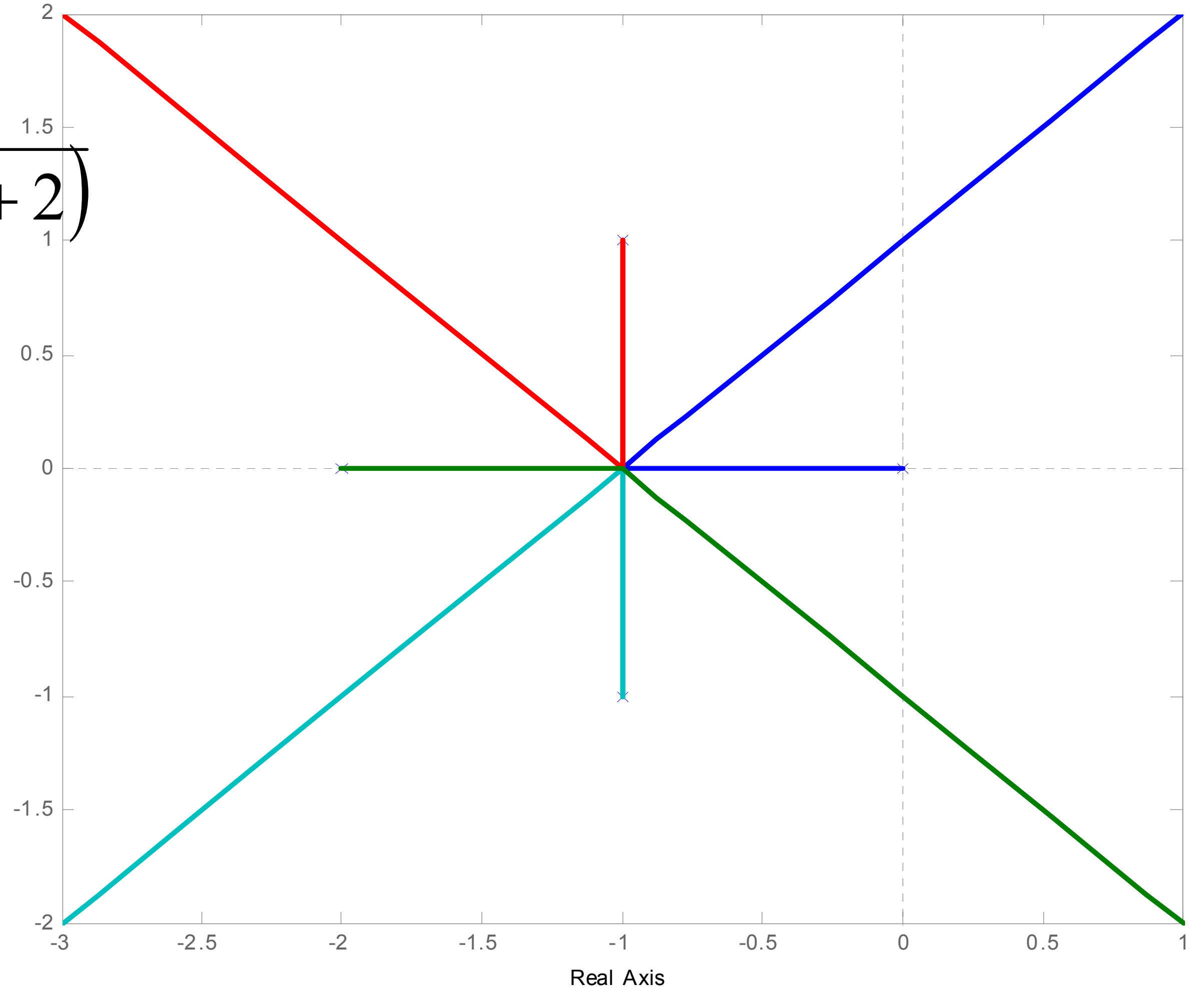


Examples

3- Root locus

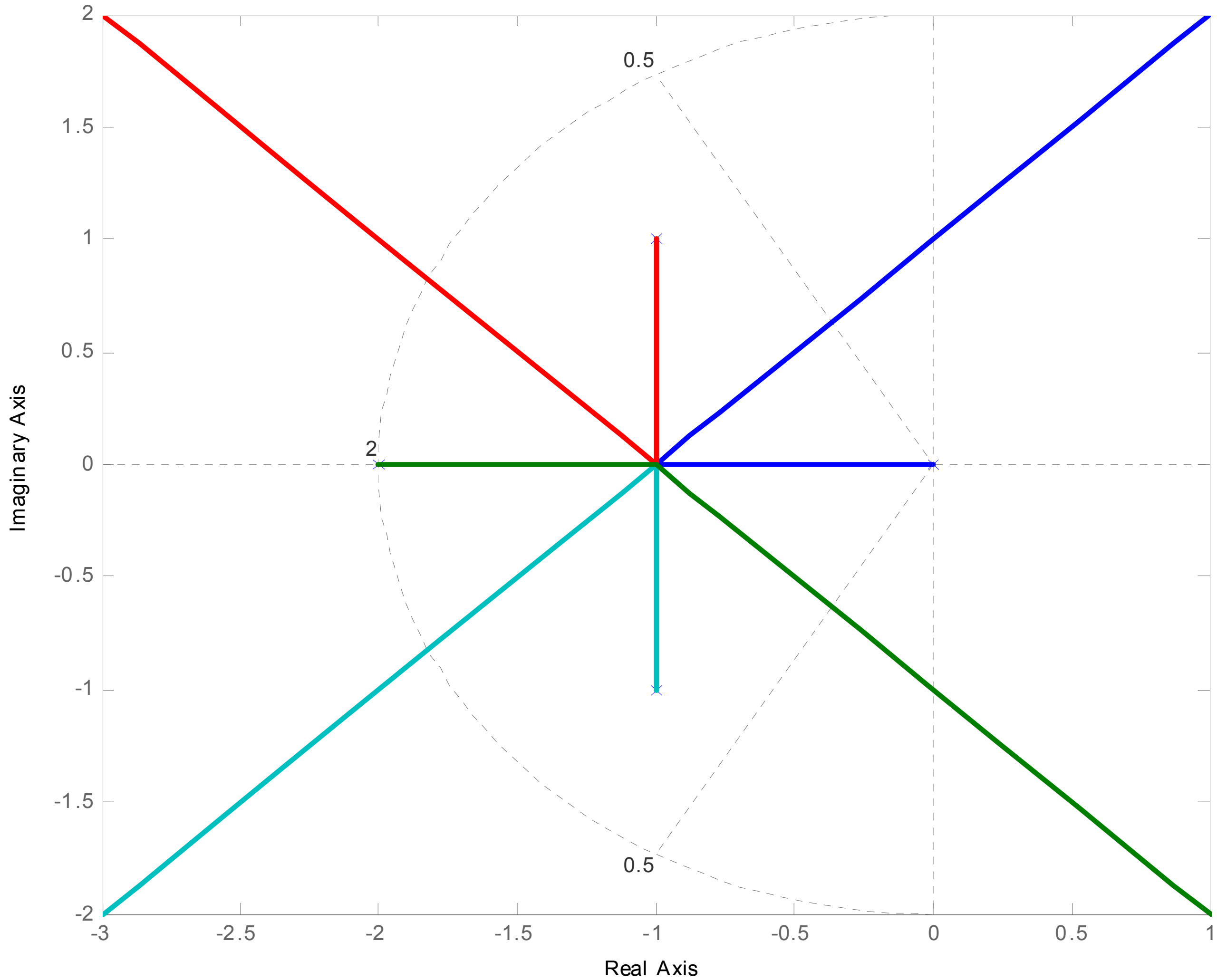
Roots locus exercise

$$G(s) = \frac{1}{s(s+2)(s^2+2s+2)}$$



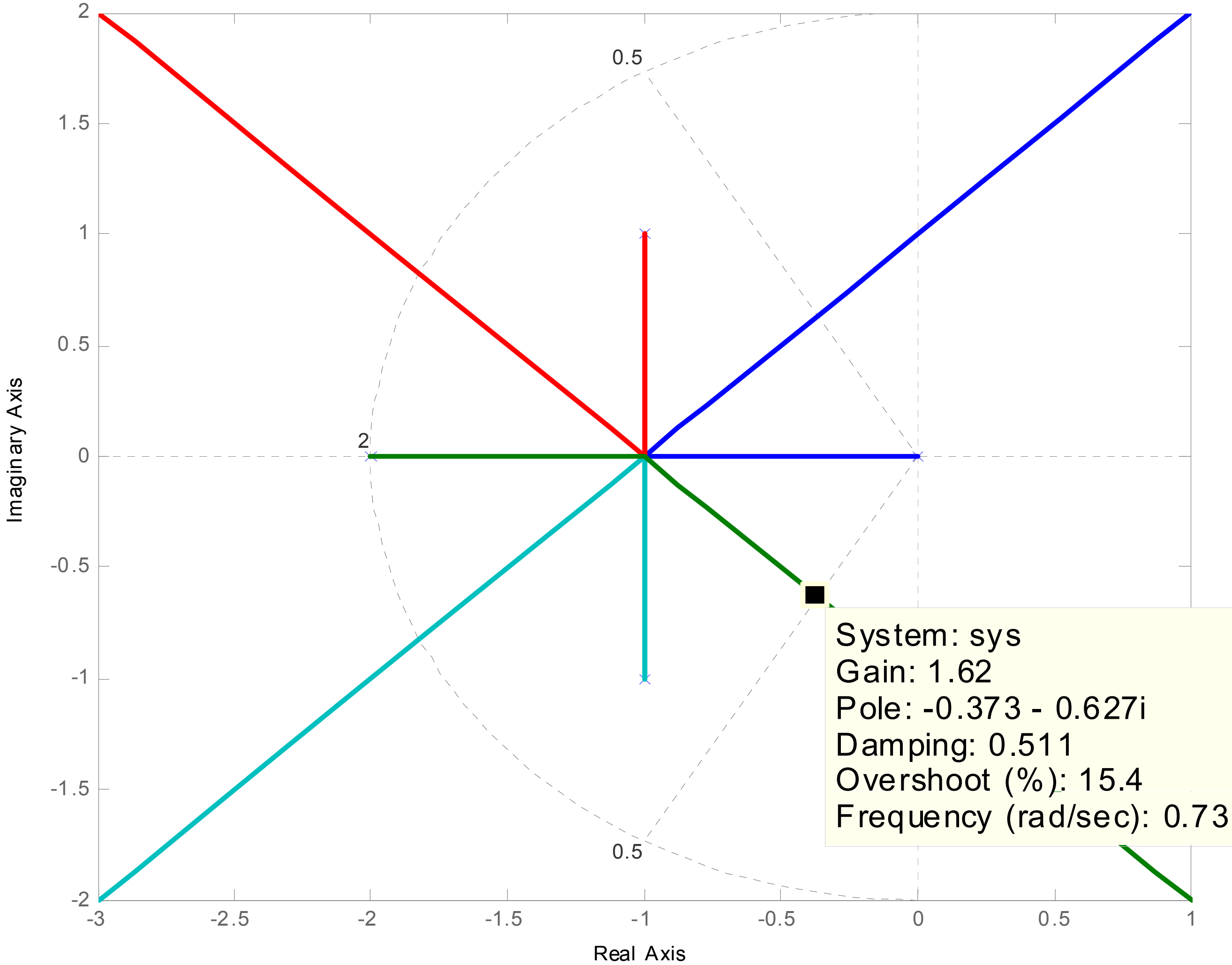
3- Root locus

Roots locus exercise



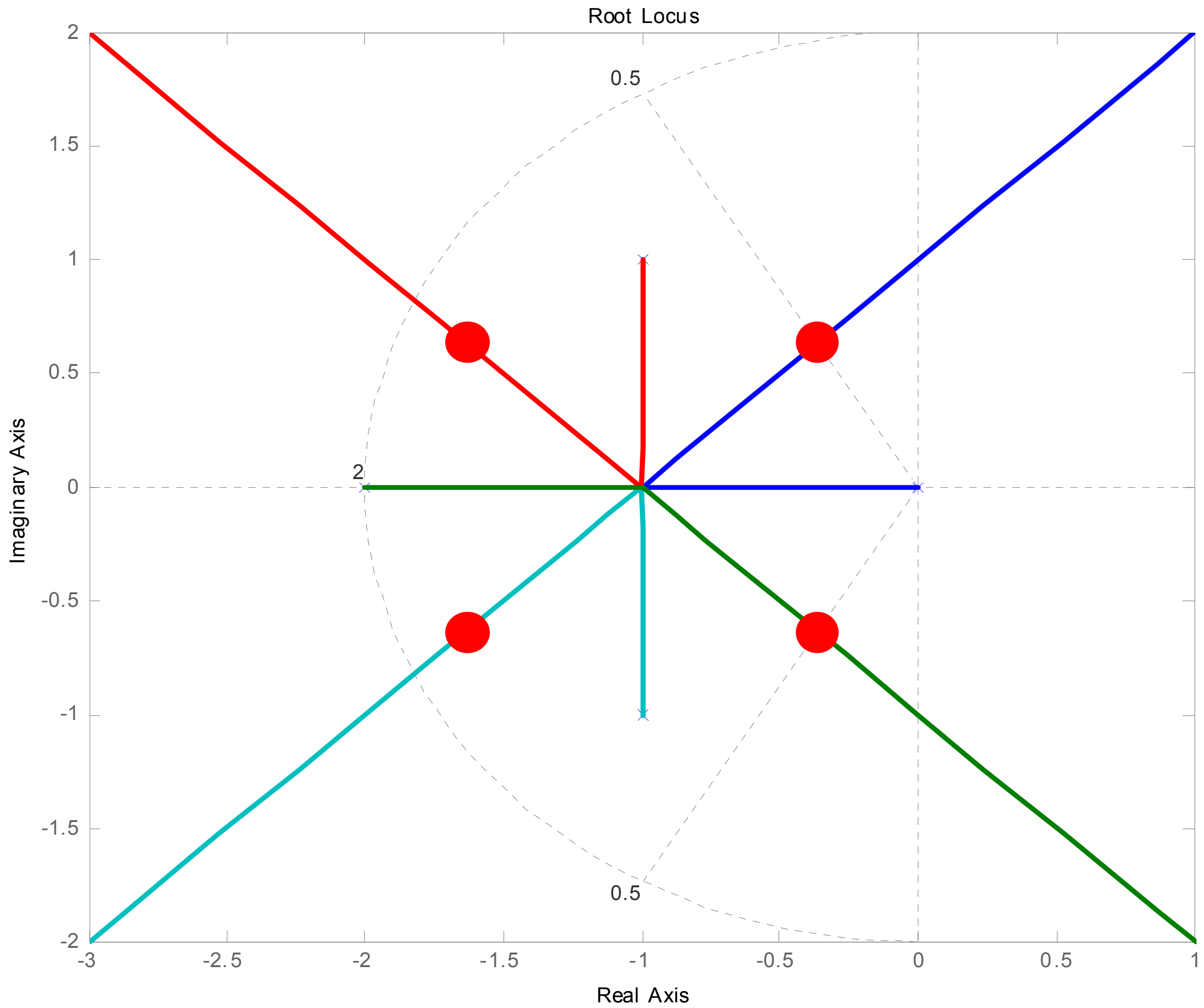
3- Root locus

Roots locus exercise



3- Root locus

Roots locus exercise



Gain setting

Note that even though the closed-loop poles have this value of damping factor, the transitory response is not exactly sub-damped with that characteristic, because the ζ formula has been used as if it was a 2^{nd} order system.

However, the approximation is valid to obtain a good ζ magnitude order, the influence of poles and zeros on the response is seen in the following study

Additional pole

1. A second-order system is considered

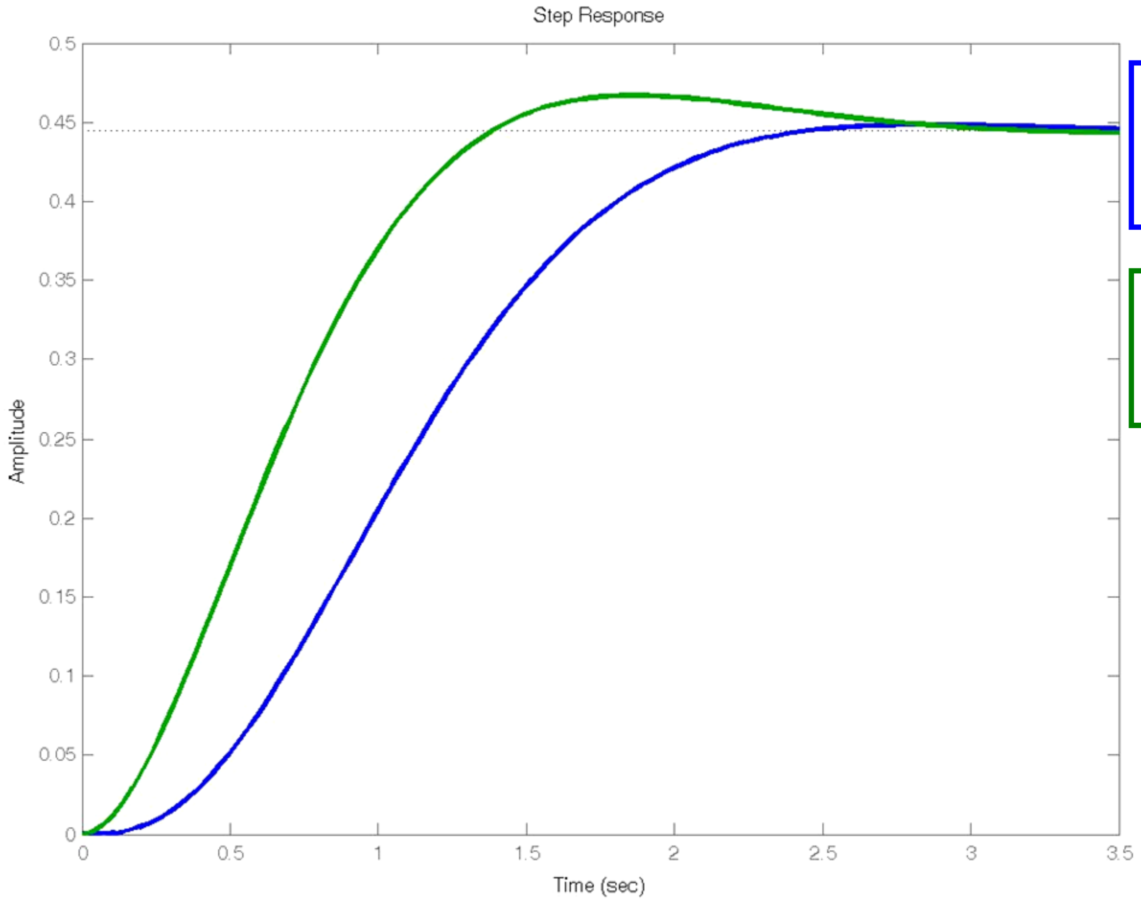
2. A pole is added in s=-p

• system reference signal first affected by a first-order system and then by a 2nd order one

• for a step function, signal attenuated by an exponential, which is the 2nd order system entry

→ **exit has less overshoot and it takes more time to reach its final value**

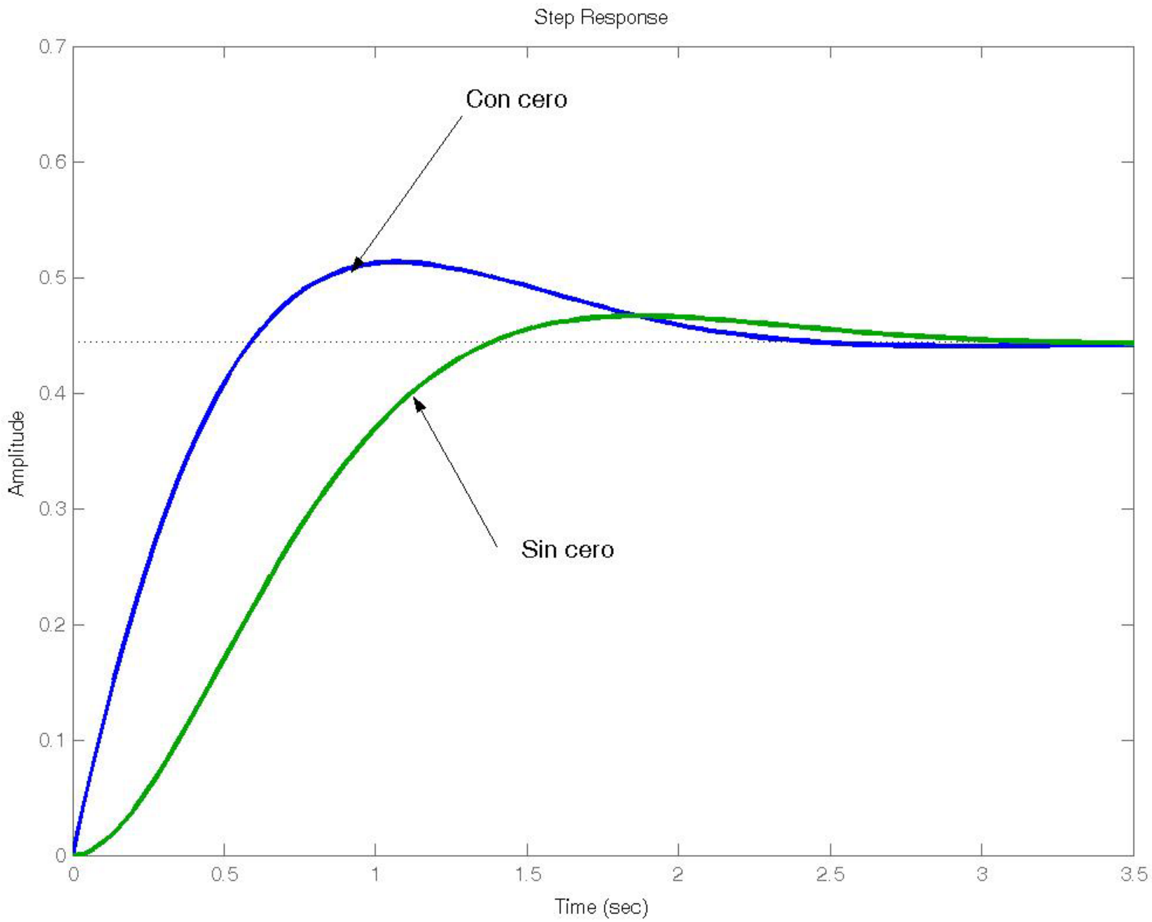
Additional pole



$$f_1 = \frac{2.4}{(0.5s+1)(s^2+3.2s+5.4)}$$

$$f_2 = \frac{2.4}{s^2+3.2s+5.4}$$

Additional zero



$$f_1 = \frac{1.2(s+2)}{s^2+3.2s+5.4}$$

$$f_2 = \frac{2.4}{s^2+3.2s+5.4}$$

Zeros (negative)

- increase the initial slope,
- make the system faster so it reaches its final value earlier,
- can produce overshoot

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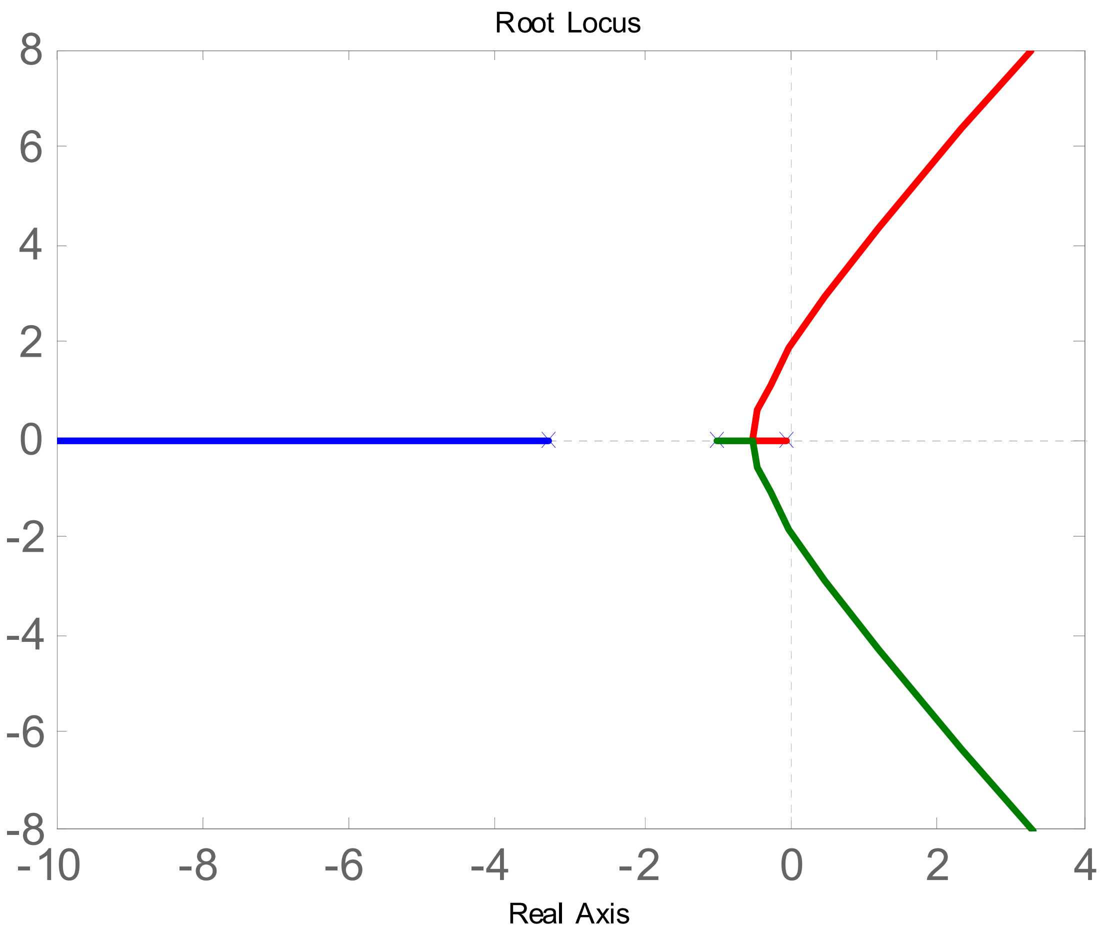
3- Root locus

Effect of an additional pole in the roots locus

Transfer function of a vehicle cruise-control system:

$$G(s) = \frac{2.48}{(s+0.06)(s+1)(s+3.33)}$$

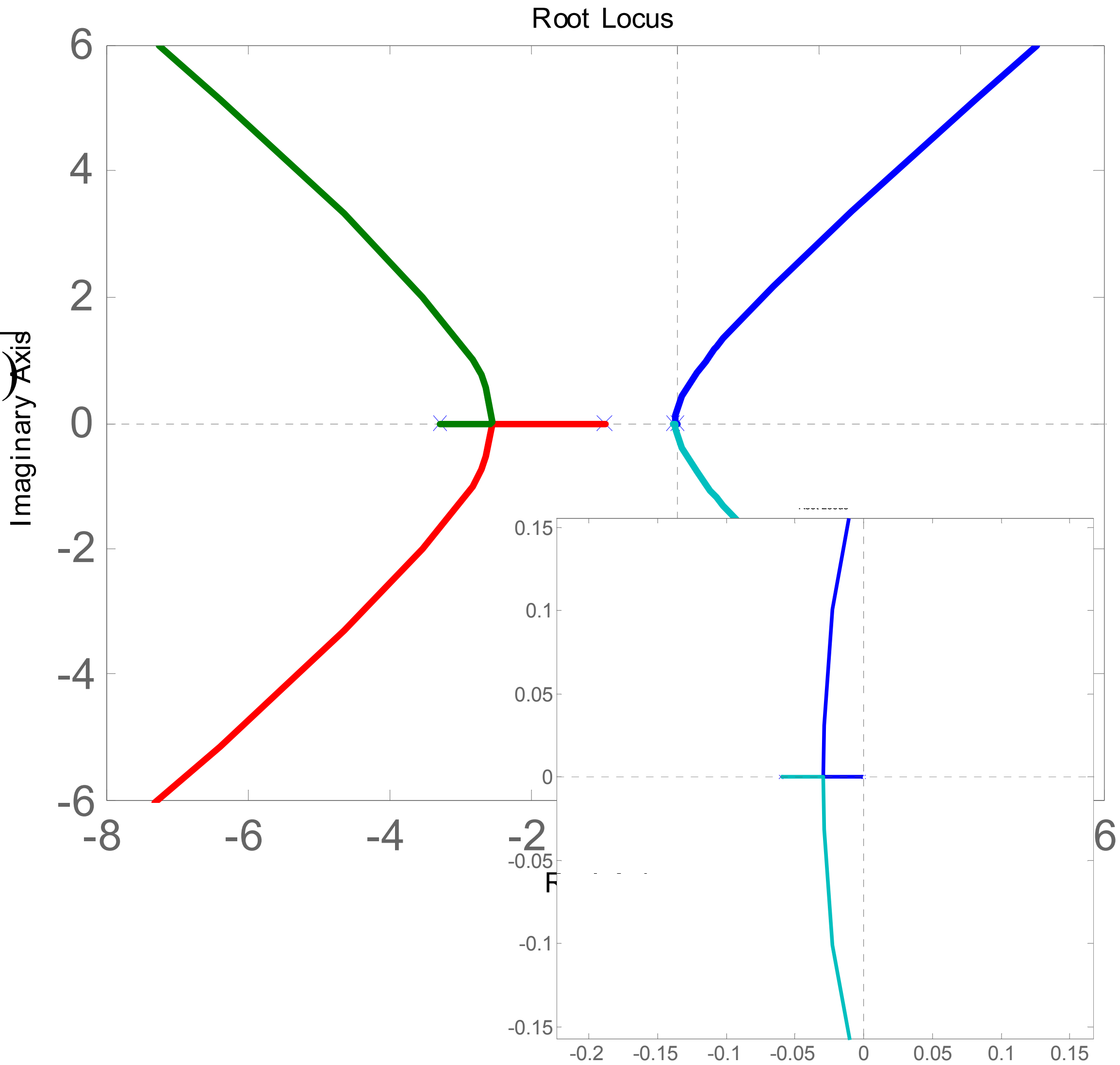


3- Root locus

Effect of an additional pole in the roots locus

A pole is added on 0 (integrator)

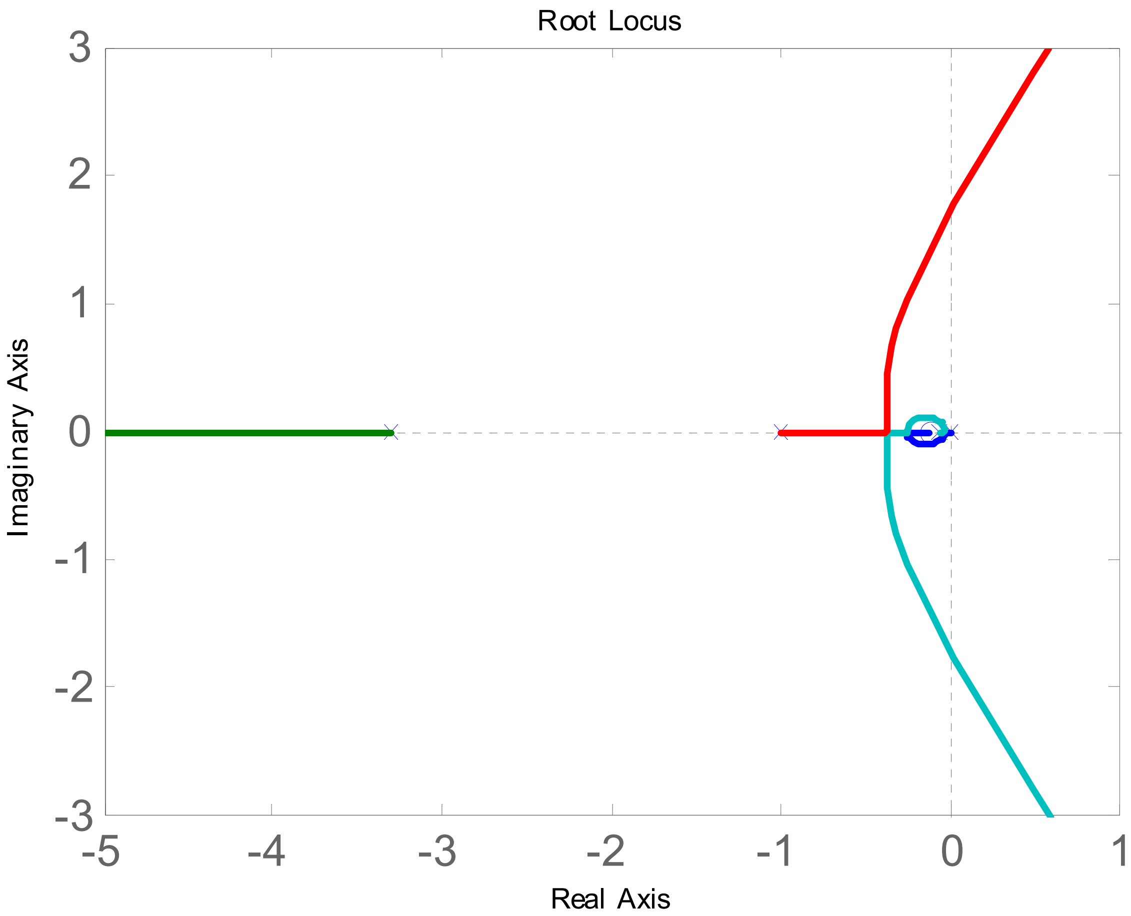
$$G(s) = \frac{2.48}{s(s+0.06)(s+1)(s+3.33)}$$



Roots locus moved to the **right**

→ **+ unstable**

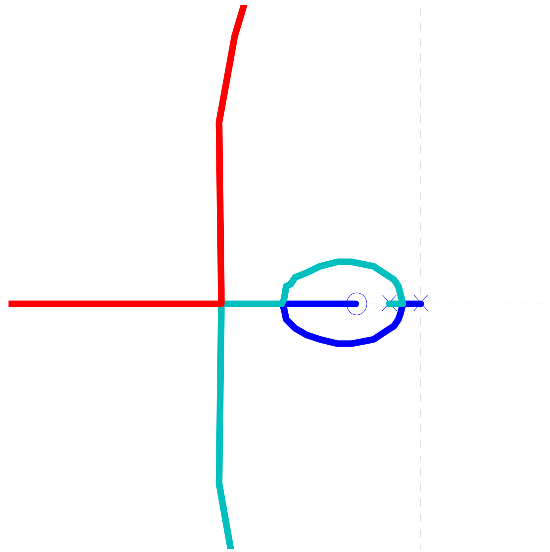
Effect of an additional zero in the roots locus



A zero is added

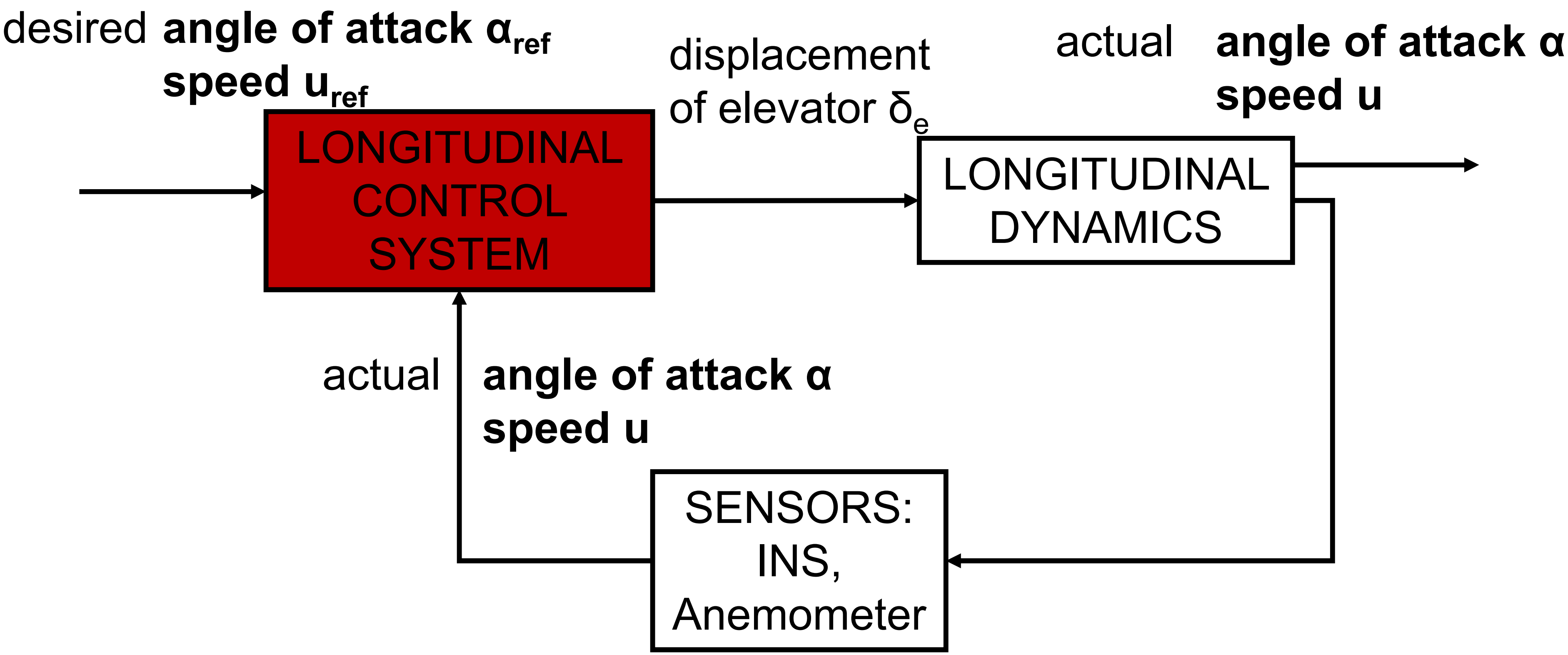
in -0.12

$$G(s) = \frac{2.48(s+0.12)}{s(s+0.06)(s+1)(s+3.33)}$$



Root locus moved to the **left → + stable, and faster**

Design a controller/compensator in order to satisfy some constraints on the response of the system



4- Controllers

4- Controllers

- **Proportional** controller: P: K_P
- **Integral** controller : I: $\frac{K_I}{s}$
- **Derivative** controller: D: $K_D s$

	t_p	M	t_s	steady-state error
P	decreases	increases	small changes	decreases
I	decreases	increases	increases	eliminates (=0)
D	small changes	decreases	decreases	small changes

- these correlations may not be exactly accurate, because K_P, K_I, and K_D are dependent of each other
- changing 1 of these variables can change the effect of the other 2

4- Controllers

• 2 kinds of controllers improve the **transitory response**:

Lead Compensator:

$$G_C(s) = \frac{s + z_0}{s + p_0} \quad \text{with} \; |p_0| > |z_0|$$

adds 1 zero and 1 pole, but zero is more important: it moves the root locus to the left: improves stability (system is faster and has less overshoot)

Proportional Derivative Compensator: $G_C(s) = K_P + K_D s$

adds 1 zero: improves stability

4- Controllers

• 2 kinds of controllers improve the **steady state response**:

Lag Compensator:

$$G_C(s) = \frac{s + z_0}{s + p_0} \quad \text{with } |z_0| > |p_0|$$

add 1 zero and 1 pole, but pole is more important: it moves the root locus to the right: decreases stability (system is slower and has more overshoot), but decreases the steady state error

Proportional Integral Compensator: $$G_C(s) = K_P + \frac{K_I}{s}$$

4- Controllers

• 2 kinds of controllers improve **both transitory and steady state response**:

Lead - Lag Compensator:

$$G_C(s) = \frac{s+z_0}{s+p_0} \times \frac{s+z_1}{s+p_1} \quad \text{with } |z_0| > |p_0| \text{ and } |p_1| > |z_1|$$

Proportional Integral Derivative (PID) Compensator:

$$G_C(s) = K_P + \frac{K_I}{s} + K_D s$$

5- Frequency response

1 Fourier transforms and properties

2 Frequency response

3 Examples

1 Fourier transforms and properties

The Fourier transform of a function x(t) is a function of the pulsation ω:

$$F[x(t)] = X(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} x(t)\,dt$$

→ **It transforms a signal from the time domain to the frequency domain**

1 Transforms and properties

The inverse Fourier transform recovers the original function x(t):

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t} X(\omega) d\omega$$

This is true for an absolutely integrable signal:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$$

1 Transforms and properties

Linearity:

$$F[\alpha x(t)+\beta y(t)]=\alpha X(\omega)+\beta Y(\omega)$$

Derivation:

$$F\left[\frac{dx(t)}{dt}\right]=j\omega X(\omega)$$

$$F\left[\frac{d^n x(t)}{dt^n}\right]=(j\omega)^n X(\omega)$$

1 Transforms and properties

Additional properties

$$F[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Duality

$$x(t) \xrightarrow{F} X(\omega)$$

$$X(t) \xrightarrow{F} 2\pi\, x(-\omega)$$

1 Transforms and properties

Convolution theorems

$$(f_1 * f_2)(t) \xrightarrow{F} F_1(\omega) \cdot F_2(\omega)$$

$$(f_1 \cdot f_2)(t) \xrightarrow{F} (F_1 * F_2)(\omega)$$

$$\text{where } (f_1 * f_2)(t) = \int_{-\infty}^{+\infty} f_1(s) f_2(t-s)\, ds$$

1 Transforms and properties

Time delay

$$F\left[x\left(t \pm T\right)\right] = e^{\pm j\omega T} X\left(\omega\right)$$

1 Transforms and properties

Important pairs of transforms

$f(t)$	$F(\omega)$
$\delta(t-t_0)$, unit impulse	$e^{-j\omega t_0}$
$e^{j\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
$u(t)$, unit step	$\delta(\omega)$
$e^{-at}u(t)$	$\dfrac{1}{\omega+a}$
$\cos(2\pi\omega_0 t)$	$\dfrac{1}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$
$\sin(2\pi\omega_0 t)$	$\dfrac{1}{2j}\left(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right)$

2 Frequency response

In previous examples we examined the free response of an airplane with step changes in control input

Other useful input function is the sinusoidal signal. Why?

1. Input to many physical systems takes the form or either a step change or sinusoidal signal
2. An arbitrary function can be represented by a series of step changes or a periodic function can be decomposed by means of Fourier analysis into a series of sinusoidal waves

→ if we know the response of a linear system to either a step or sinusoidal input then we can construct the system's response to an arbitrary input by the principle of superposition

2 Frequency response

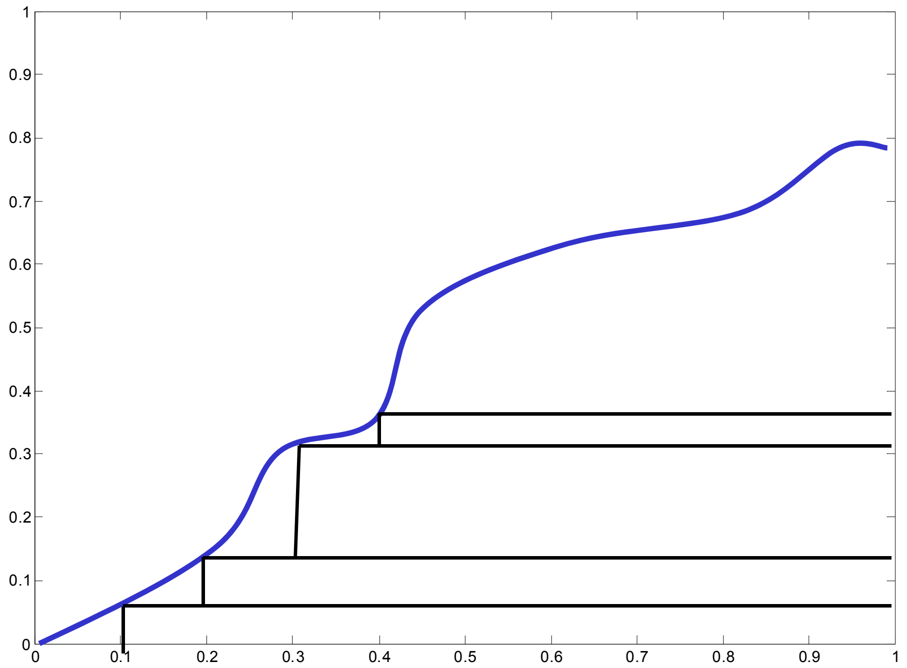
Example:

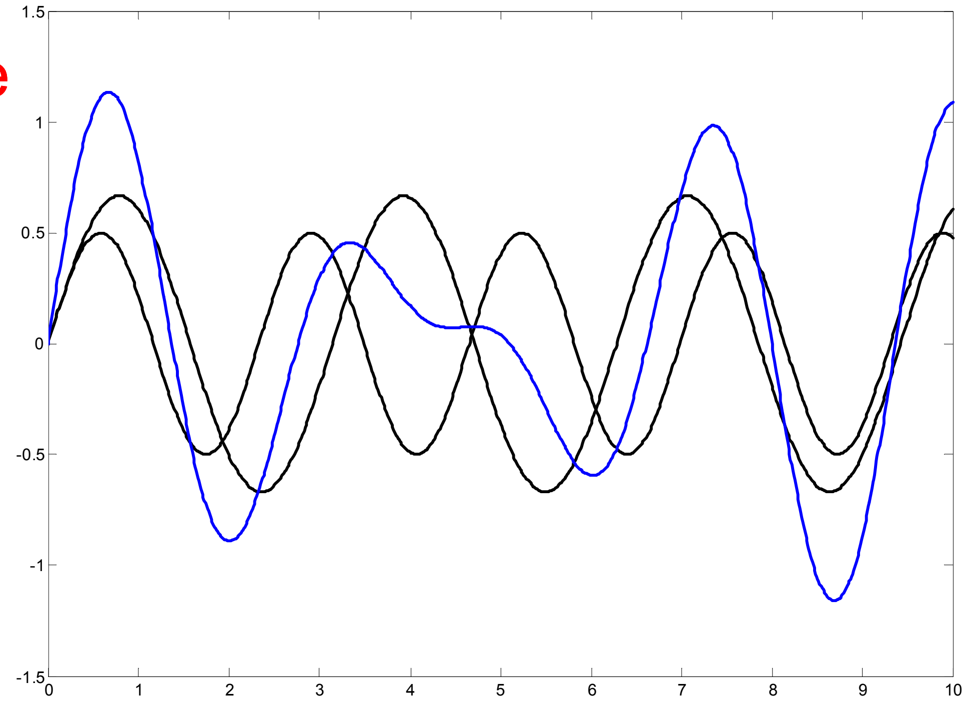
Examine the response of an airplane subjected to an external disturbance such as a wind gust

Wind gust can be a sharp edged profile or a sinusoidal profile (these 2 types of gust inputs occur quite often in nature) + arbitrary gust profile can be constructed by step and sinusoidal functions

2 Frequency response

Arbitrary wind gust profiles:





2 Frequency response

Definition of “ frequency response”:

Response in steady state to a sinusoidal input

We will demonstrate that the **steady state** response is another sinusoidal with the same frequency

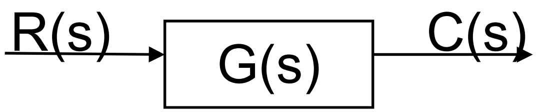
2 Frequency response

Similarity with Laplace functions with regard to the operational properties (ex: differentiation)

→ the transfer function models can be transformed from one method to the other replacing **jω** with **s** (or **s** with **jω**). (for **causal signals**: signals defined for positive time)

2 Frequency response

Given any system:



Hypothesis: **stable system**

Sinusoidal input

$$r(t) = \sin(\omega t) \rightarrow R(s) = L[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

2 Frequency response

It can be demonstrated that the **steady state** response is:

$$c(t) = |G(j\omega)| \sin(\omega t + \varphi)$$

$$\text{with } |G(j\omega)| = \sqrt{(\mathrm{Re}\{G(j\omega)\})^2 + (\mathrm{Im}\{G(j\omega)\})^2}$$

$$\text{and} \qquad \varphi = \arg(G(j\omega)) = \arctan\left(\frac{\mathrm{Im}\{G(j\omega)\}}{\mathrm{Re}\{G(j\omega)\}}\right)$$

Parametric estimation

a. First-order

Frequency response:

$$G(j\omega) = \frac{K}{1+\tau\omega j} = \frac{K(1-j\omega\tau)}{1+(\omega\tau)^2}$$

Gain:

$$|G(j\omega)| = \frac{K}{\sqrt{1+(\omega\tau)^2}}$$

Delay:

$$\varphi = -\arctan(\tau\omega)$$

b. Second-order

$$G(j\omega) = \frac{K\omega_n^2}{(\omega_n^2 - \omega^2) + 2\xi\omega_n j}$$

Gain:

$$|G(j\omega)| = \frac{K}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

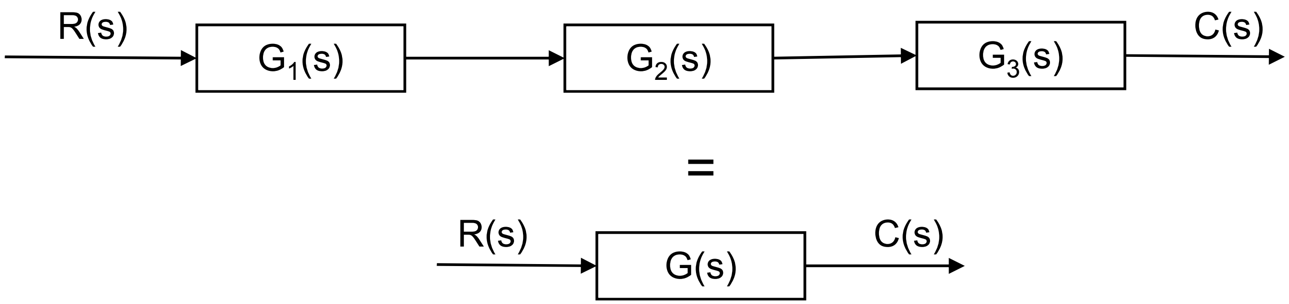
Delay:

$$\varphi = -\arctan\left(\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

5- Frequency response

c. Higher-order

For a system composed by series of blocks:



$$\text{with } G(s) = G_1(s) \times G_2(s) \times G_3(s)$$

$$G(j\omega) = G_1(j\omega) \times G_2(j\omega) \times G_3(j\omega)$$

$$|G(j\omega)| = |G_1(j\omega)| \times |G_2(j\omega)| \times |G_3(j\omega)|$$

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3$$

6- Bode diagram

1. Introduction

2. Construction rules

3. Stability

6- Bode diagrams

Goals of the Bode diagrams:

To show the frequency response characteristics in a graphical form

2 graphics for the frequency using a logarithmic scale:

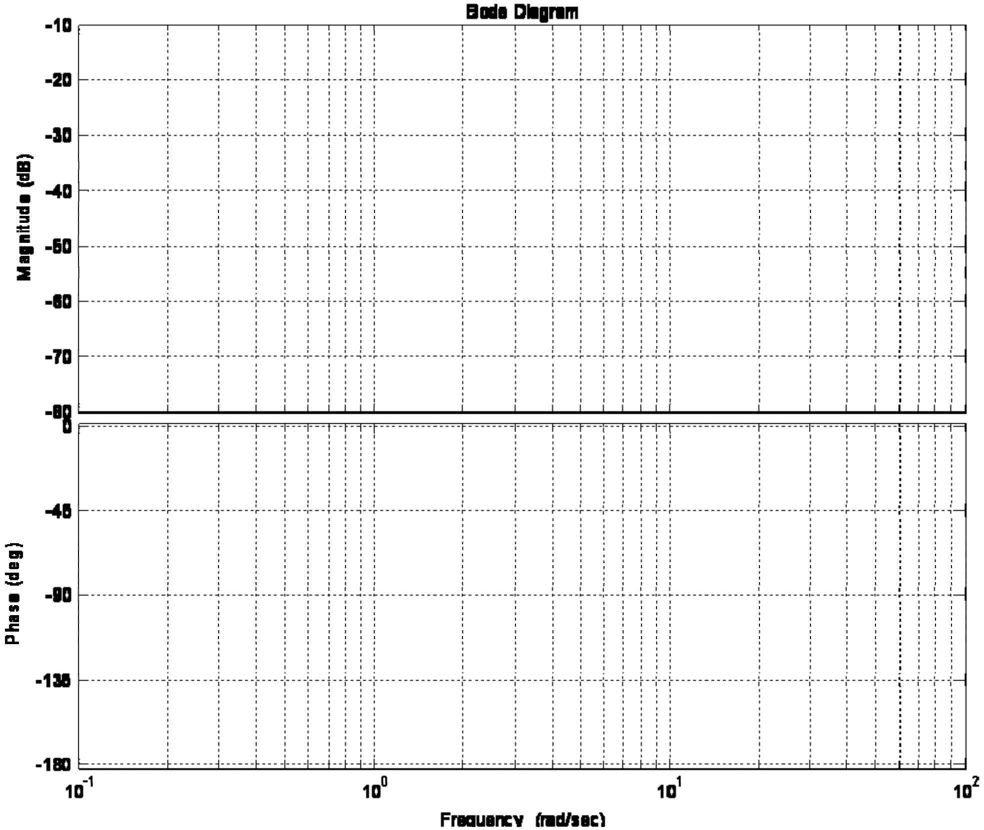
• one for the **logarithm of a function magnitude** (in decibels): $\left|G(j\omega)\right|_{dB}$

• one for the **phase angle** (in degrees): $\arg\{G(j\omega)\}$

The decibel is a unit measure used to compare a certain value with a reference one. It is basically used to measure a signal power, and it is defined as:

$$\left|G(j\omega)\right|_{dB} = 10\log\left\{\frac{P_{medida}}{P_{ref}}\right\} = 10\log\left\{\frac{\left|G(j\omega)\right|^2}{1}\right\} = 20\log\left\{\left|G(j\omega)\right|\right\}$$

6- Bode diagrams



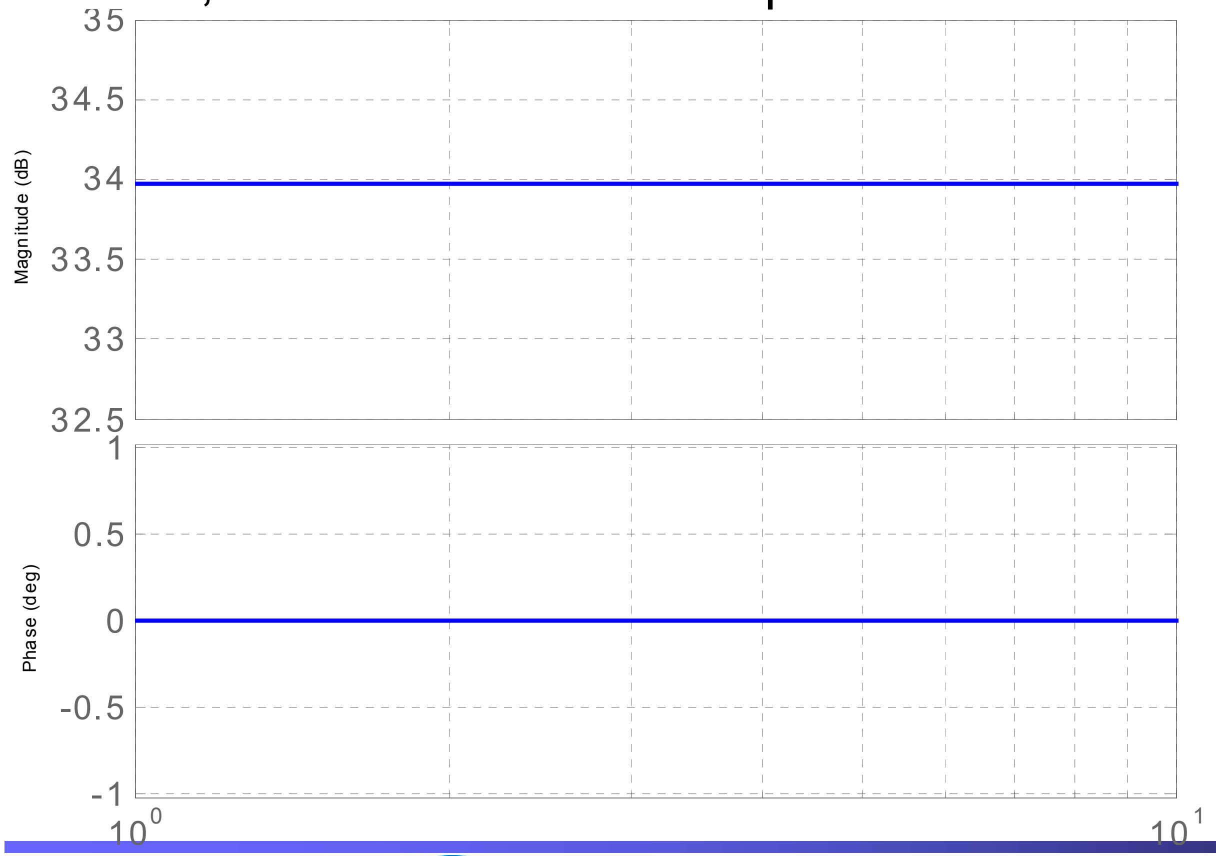
Semi-logarithmic axes: with lineal scale for the magnitude or the phase, and logarithmic for the frequency

Represents the complex transfer function adding each pole or zero effect, which compose this function (adding property of the log)

6- Bode diagrams

Gain

The gain is a factor that only modifies the magnitude and its angular value is 0º; that is, the gain value remains constant for any frequency value, because it does not depend on it.



Bode diagram for a K=50 gain

Integral and derivative factors

An **integral** factor or a pole centered in zero, has a transfer function of:

$$G(s)=\frac{1}{s}\Rightarrow G(j\omega)=\frac{1}{j\omega}=-\frac{j}{\omega}$$

Its **magnitude** is, therefore:

$$\left|G(j\omega)\right|_{dB}=20\log\left(\frac{1}{\omega}\right)=-20\log(\omega)$$

For a logarithmic frequency axis: it corresponds to a straight negative line of -20 dB per decade

The **phase** is:

$$\arg\{G(j\omega)\}=\arg\left\{\frac{1}{j\omega}\right\}=\arg\left\{\frac{-j}{\omega}\right\}=\arctan\left(\frac{-1/\omega}{0}\right)=-90^\circ$$

Integral and derivative factors

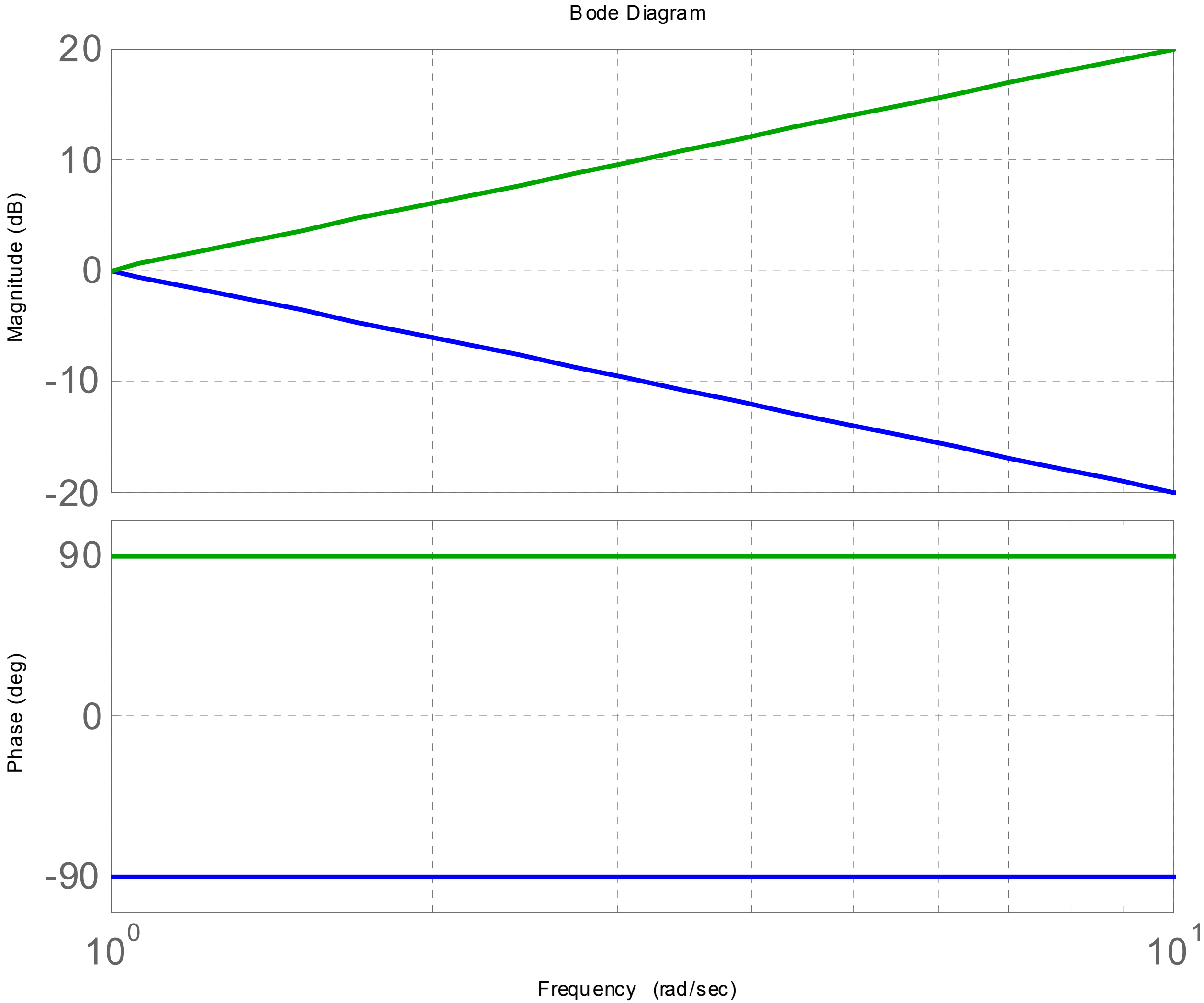
For a **derivative** factor or a zero centered in zero, the results are deduced using a similar development:

$$\left|G(j\omega)\right|_{dB} = 20\log(\omega)$$

$$\arg\{G(j\omega)\} = 90^{\circ}$$

6- Bode diagrams

Integral and derivative factors



Bode diagrams of the derivative and the integrator

First-order factors: first-order pole

$$G(j\omega) = \frac{1}{1 + j\omega\tau}$$ Its magnitude is: $$20\log\left[\frac{1}{\sqrt{1+\omega^2\tau^2}}\right]$$

It seems more complicated, but approximations are made:

$$\text{for} \quad \omega << \frac{1}{\tau} \Rightarrow \omega\tau << 1 \,, \left|G(j\omega)\right|_{dB} = -20\log\left(\sqrt{1+\omega^2\tau^2}\right) \approx -20\log(1) = 0$$

$$\text{for} \quad \omega >> \frac{1}{\tau} \Rightarrow \omega\tau >> 1 \,, \left|G(j\omega)\right|_{dB} = -20\log\left(\sqrt{1+\omega^2\tau^2}\right) \approx -20\log(\omega\tau)$$

- substitute the curve by its two asymptotes
- magnitude is 0 dB until it reaches the point where both asymptotes meet: $\omega\tau=1$, this point is called *cut frequency*
- from there: other asymptote, with a -20 dB per decade slope.
- point where approximation error is maximum corresponds to the cut frequency and the error is 3 dB.

First-order factors: first-order pole

similar for the phase, the phase real value is:

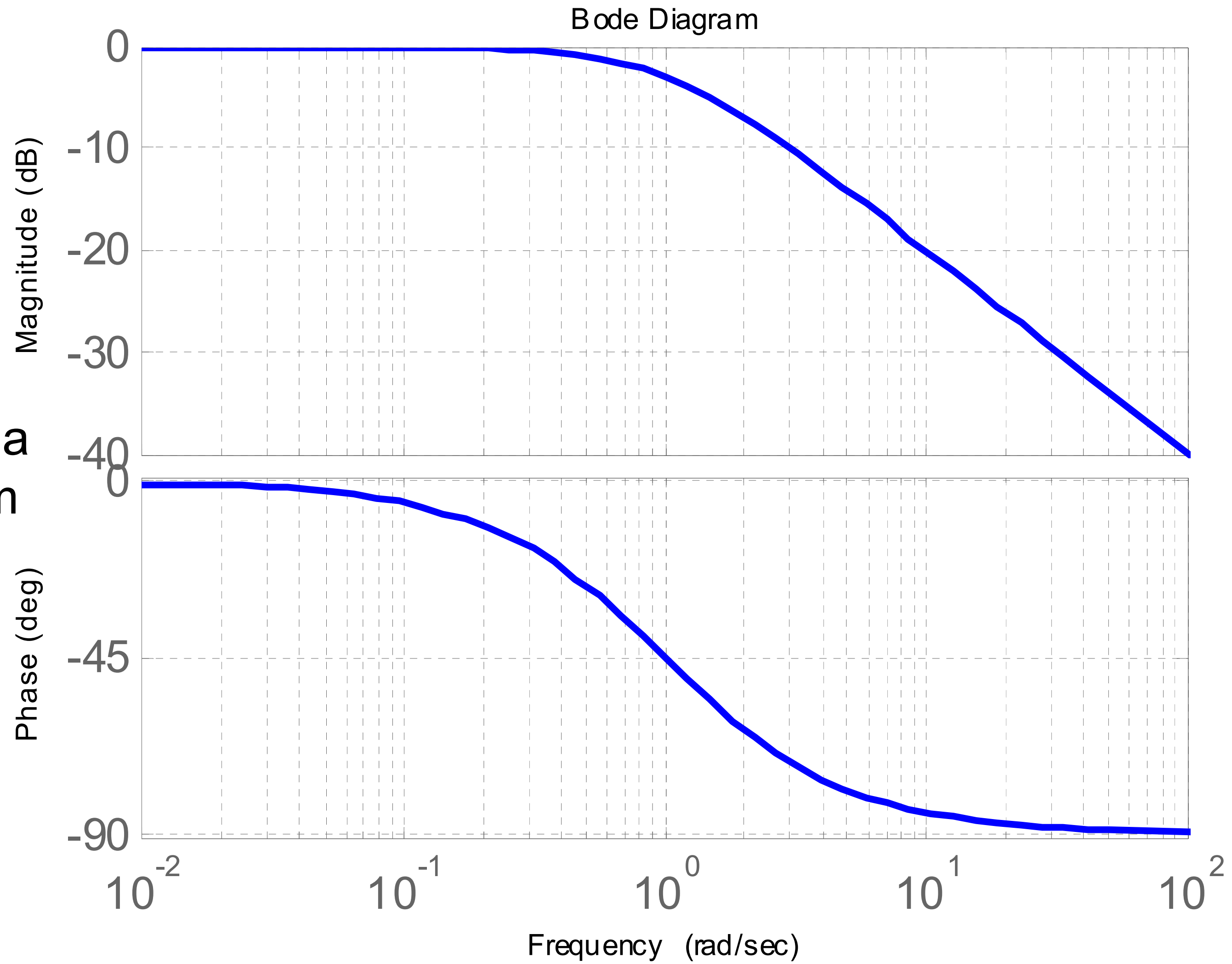
$$\arg\{G(j\omega)\} = \arg\left\{\frac{1}{1+j\omega\tau}\right\} = \arg\left\{\frac{1-j\omega\tau}{1+\omega^2\tau^2}\right\} = \arctan\left\{\frac{-\omega\tau}{1}\right\}$$

However the approximation in this case is:

- $$\begin{cases} \text{for} \quad \omega << \dfrac{1}{\tau} \Rightarrow \omega\tau << 1 \quad \arg\{G(j\omega)\} = 0^\circ \\ \text{for} \quad \omega >> \dfrac{1}{\tau} \Rightarrow \omega\tau >> 1 \quad \arg\{G(j\omega)\} = -90^\circ \end{cases}$$

First-order factors: first-order pole

$$G(s) = \frac{1}{1+s}$$



Bode diagram of a first-order system

First-order factors: first-order pole

For the study of the first-order zeros, similar development, there is only a sign change: $G(j\omega) = 1 + j\omega\tau$

Magnitude: $$20\log\left(\sqrt{1+\omega^2\tau^2}\right) = 10\log\left(1+\omega^2\tau^2\right)$$

Angular contribution:

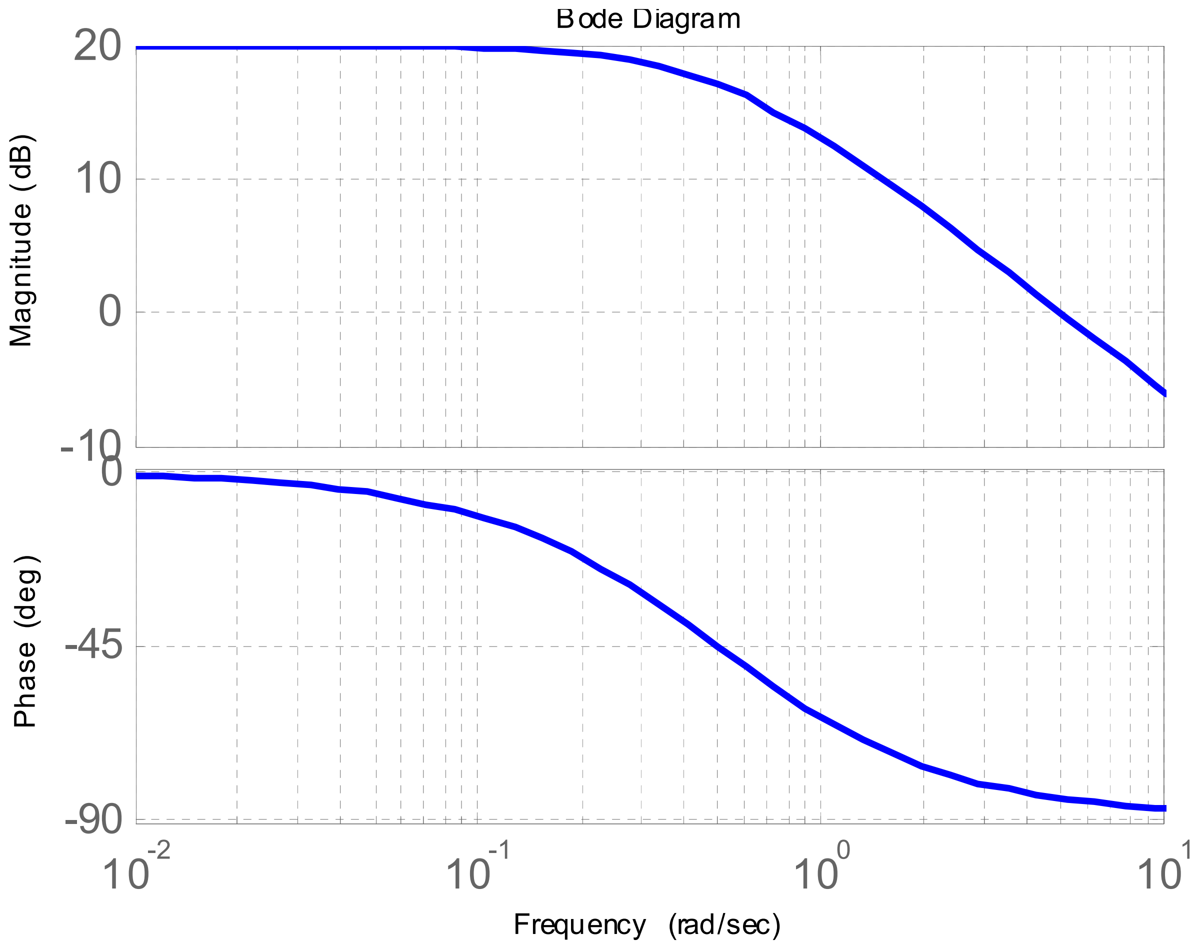
$$\arg\{G(j\omega)\} = \arg\{1 + j\omega\tau\} = \arctan\left\{\frac{\omega\tau}{1}\right\}$$

6- Bode diagrams

First-order factors: first-order pole

Example:

$$G(s) = \frac{10}{2s+1}$$



Second-order factors

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\Rightarrow G(j\omega)=\frac{\omega_n^2}{-\omega^2+j2\zeta\omega_n\omega+\omega_n^2}=\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j\left[2\zeta\frac{\omega}{\omega_n}\right]}$$

The magnitude is:

$$\left|G(j\omega)\right|=20\log\sqrt{\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}=-20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

6- Bode diagrams

Second-order factors

$$|G(j\omega)| = -20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}$$

it can be approximated this way:

$$\text{for } \frac{\omega}{\omega_n} << 1,\ |G(j\omega)| \approx -20\log(1) = 0\text{ dB}$$

$$\text{for } \frac{\omega}{\omega_n} >> 1,\ |G(j\omega)| \approx -20\log\left(\frac{\omega}{\omega_n}\right)^2$$

For low frequencies: straight line at 0dB

For high frequencies: straight line with a –40 dB per decade slope.

Both asymptotes cross on $\omega=\omega_n$.

However, in the second-order poles a resonance effect can appear.

In the frequency domain the resonance is shown as a peak close to the cut frequency; the resonance peak value is conditioned to the ζ value.

Second-order factors

For the phase: $$\tan\Phi = -\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\text{For } \frac{\omega}{\omega_n} << 1\text{, generally } \frac{\omega}{\omega_n} < 0.2\text{, } \Phi \approx -\text{Arc}\tan(0) = 0^\circ$$

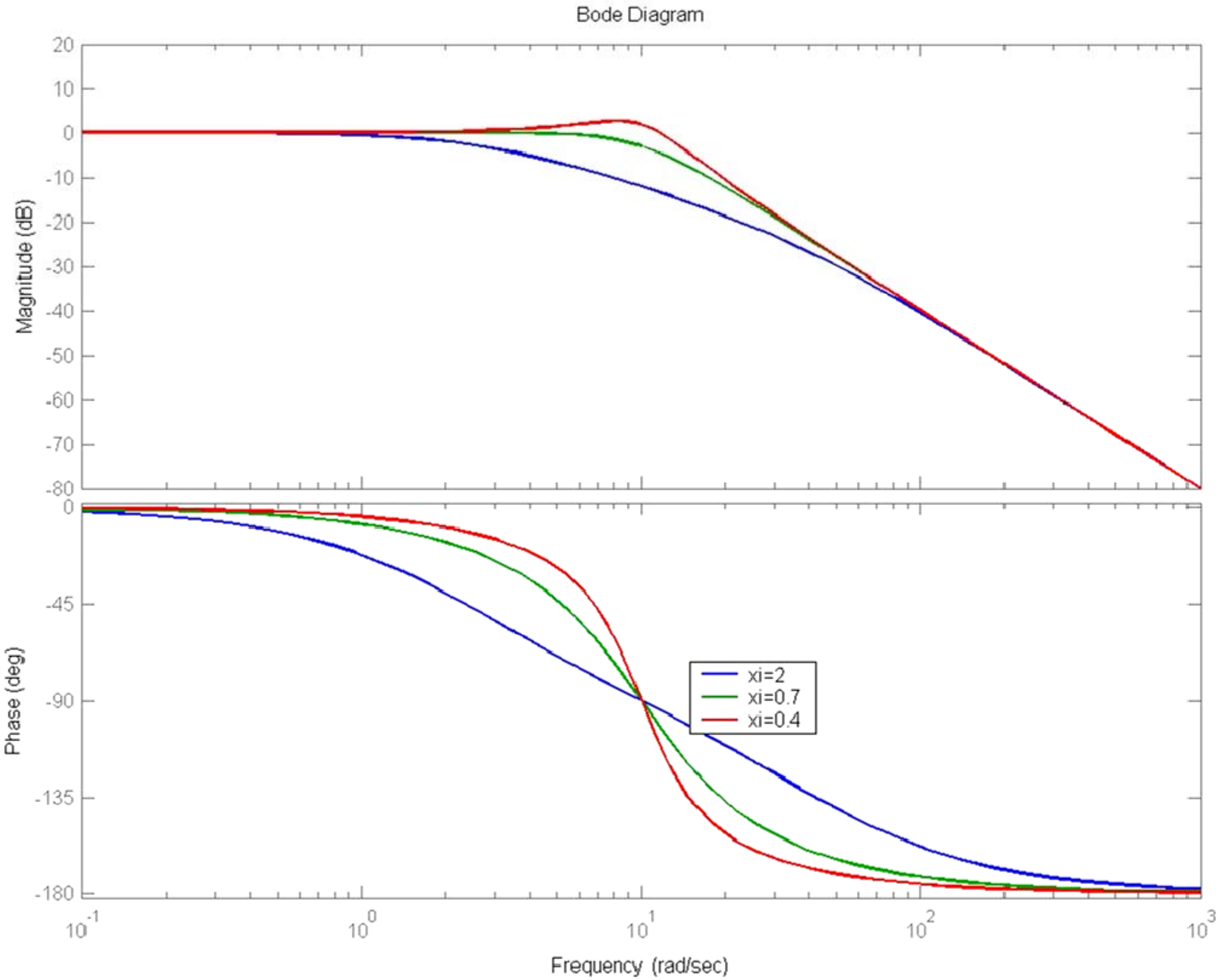
$$\text{For } \frac{\omega}{\omega_n} >> 1\text{, generally } \frac{\omega}{\omega_n} > 5\text{, } \quad \Phi \approx -\text{Arc}\tan(0^-) = -180^\circ$$

$$\text{For } \frac{\omega}{\omega_n} = 1\text{, } \quad \Phi = -\text{Arc}\tan(+\infty) = -90^\circ$$

The phase graphic form depends also on ζ

6- Bode diagrams

Second-order factors



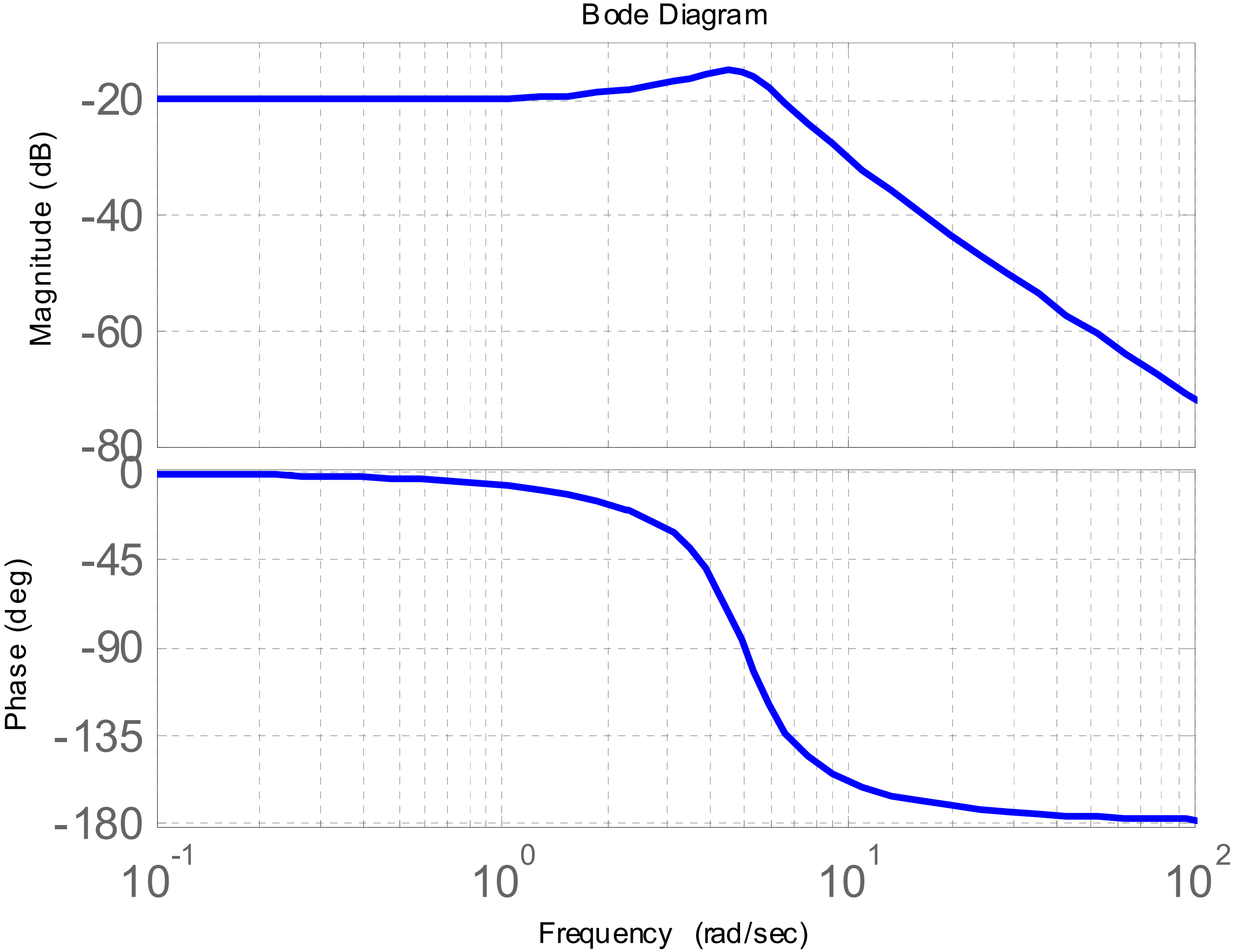
Bode diagram of a second-order pole for different ζ values

6- Bode diagrams

Second-order factors

Example:

$$G(s) = \frac{2.5}{s^2 + 3s + 25}$$



Stability condition

Notation: $G = 0 \text{ for } \omega = \omega_{(G0)}$ and $\Phi = -180^{o} \text{ for } \omega = \omega_{(G-180^{o})}$

The Bode diagram in open loop is studied

Stability condition:

If for $\omega = \omega_{(G0)}, \Phi > -180^{o}$

And for $\omega = \omega_{(-180^{o})}, G < 0$

THEN the system is STABLE in closed loop

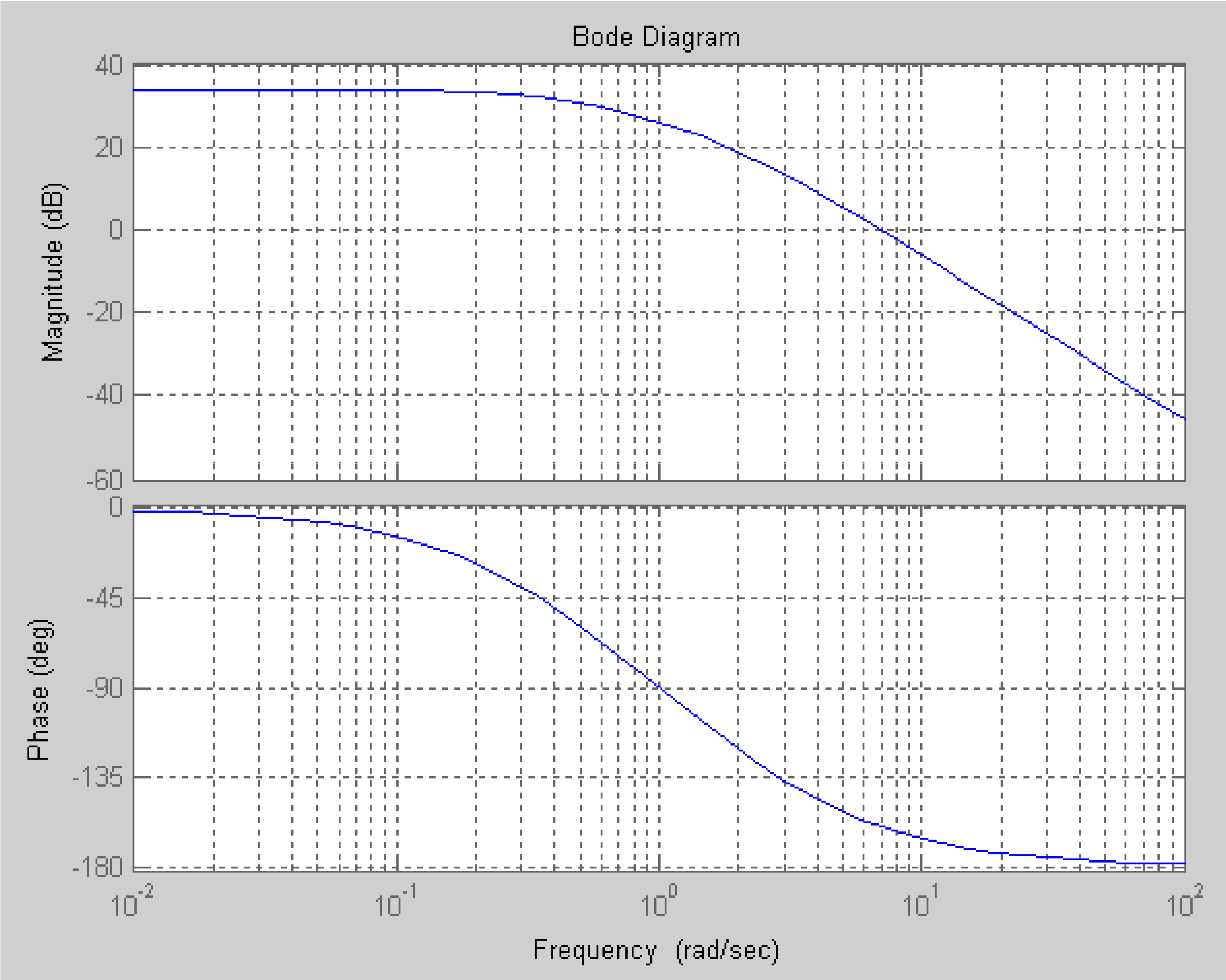
If for $\omega = \omega_{(G0)}, \Phi < -180^{o}$

Or for $\omega = \omega_{(-180^{o})}, G > 0$

THEN the system is UNSTABLE in closed loop

6- Bode diagrams

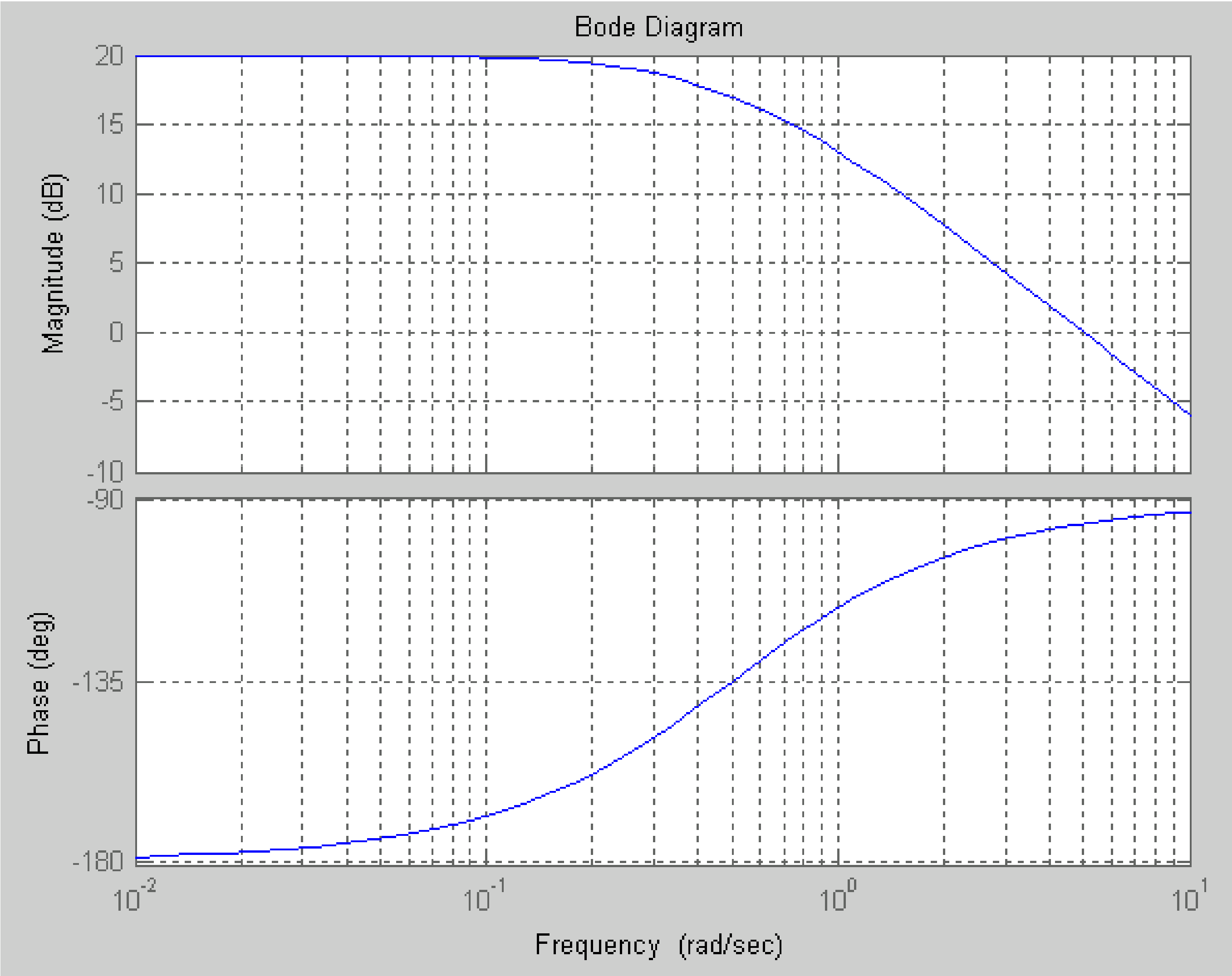
Stability condition



STABLE

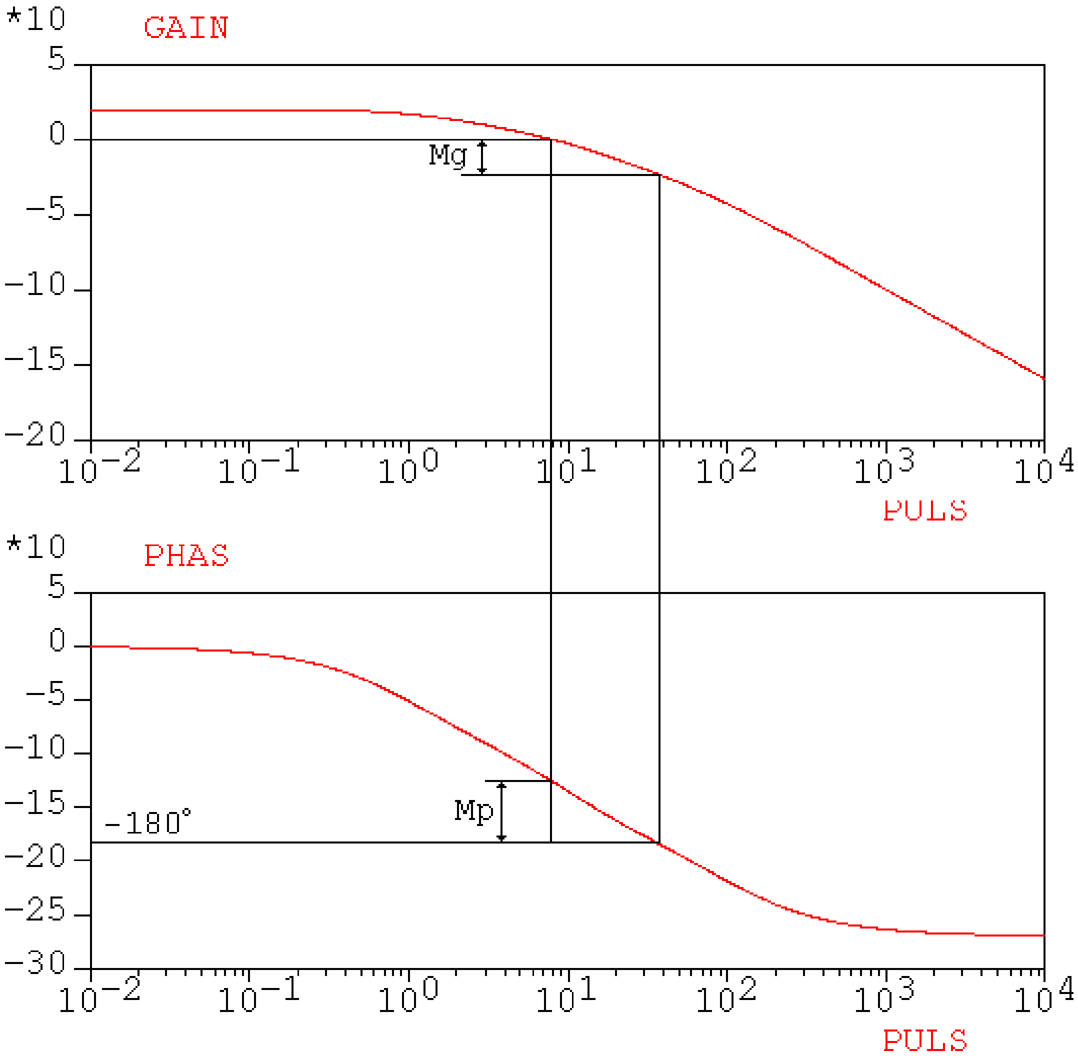
6- Bode diagrams

Stability condition



UNSTABLE

Stability margins



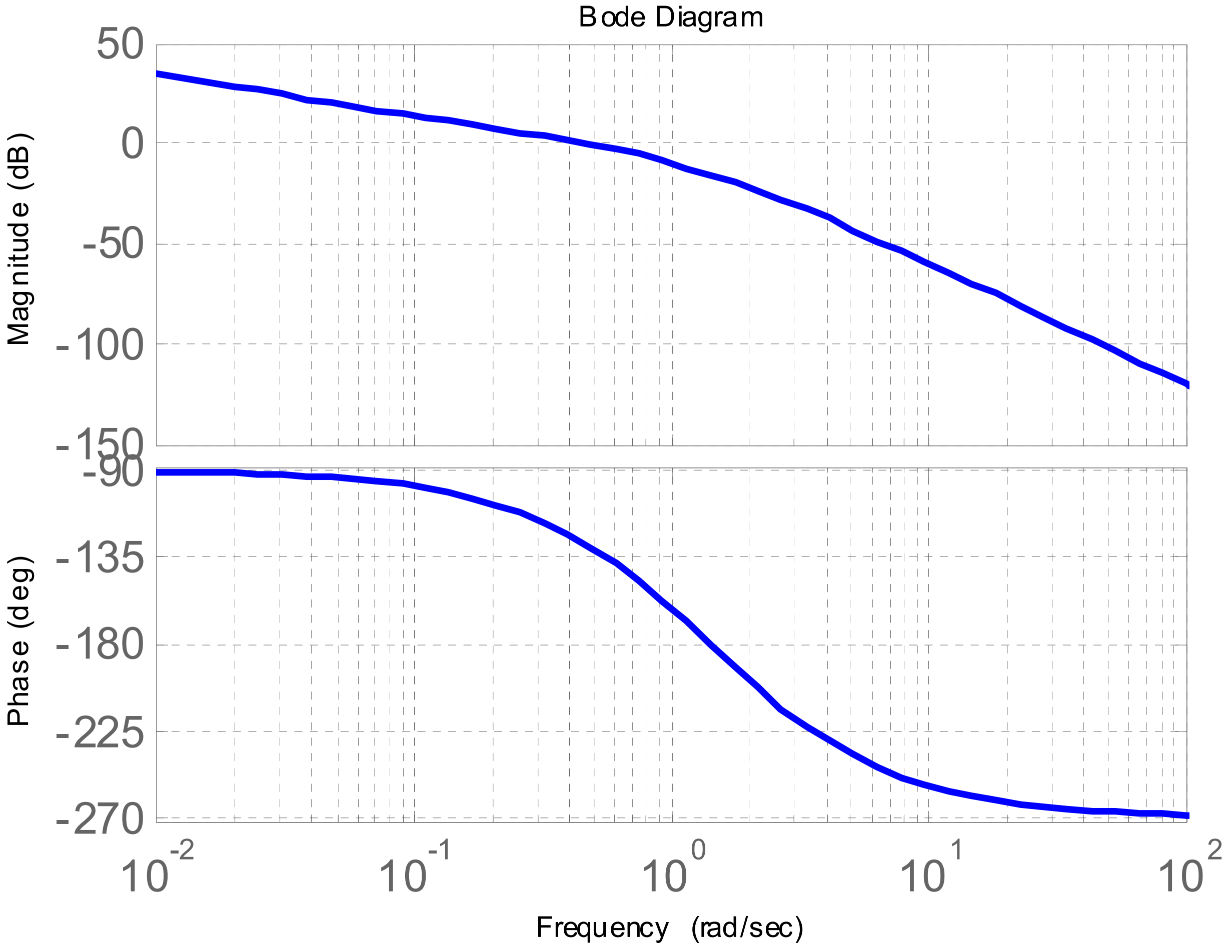
Common values:

Minimum **gain margin**: 10 a 12 dB,

Minimum **phase margin**: 45 a 50º

6- Bode diagrams

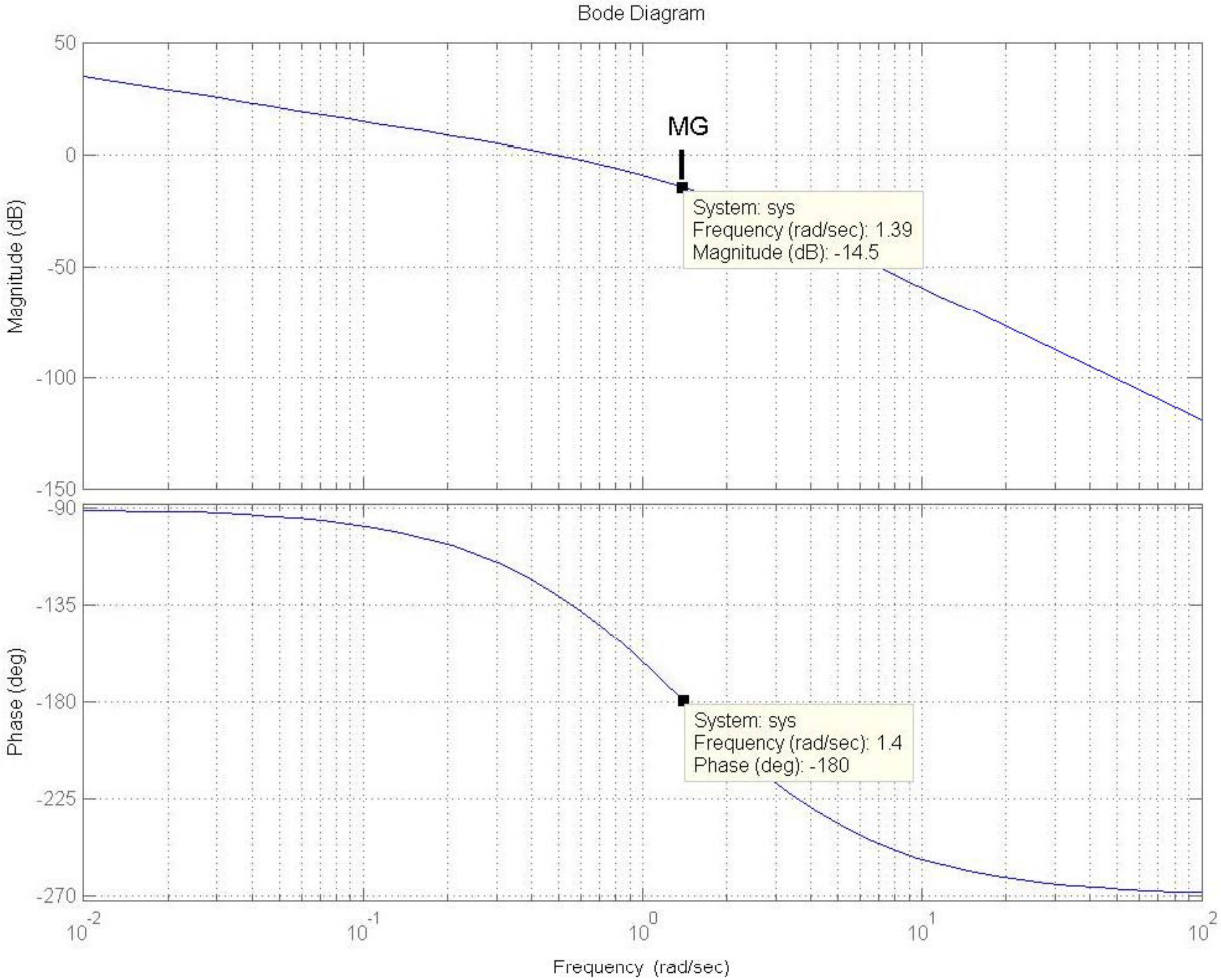
Stability margins



$$G(s) = \frac{1}{s(s+1)(s+2)}$$

6- Bode diagrams

Stability margins



$$G(s) = \frac{1}{s(s+1)(s+2)}$$

M_G=14.5dB

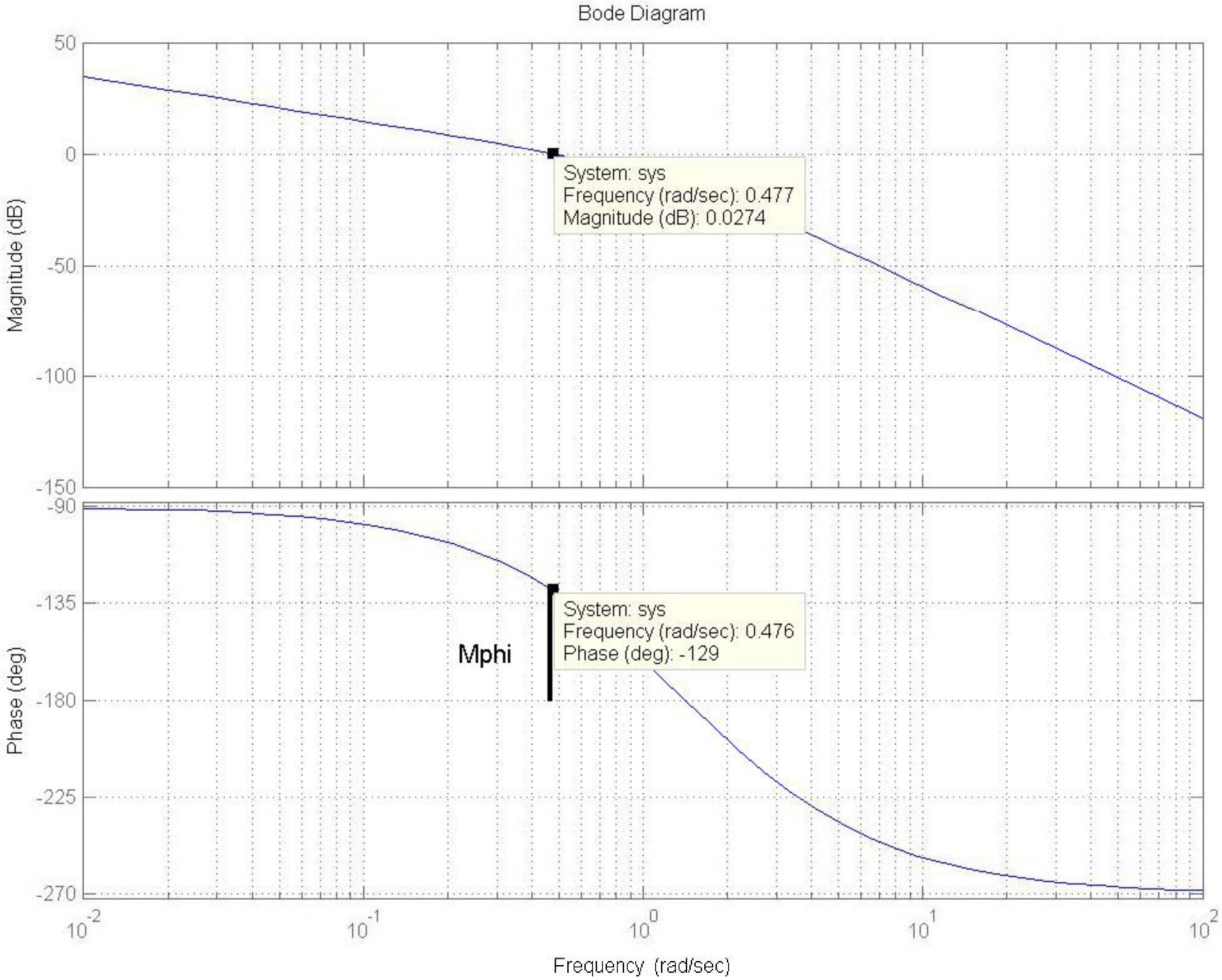
with roots locus:

M_G=6,

and note that:

20log(6)=15.5dB

Stability margins



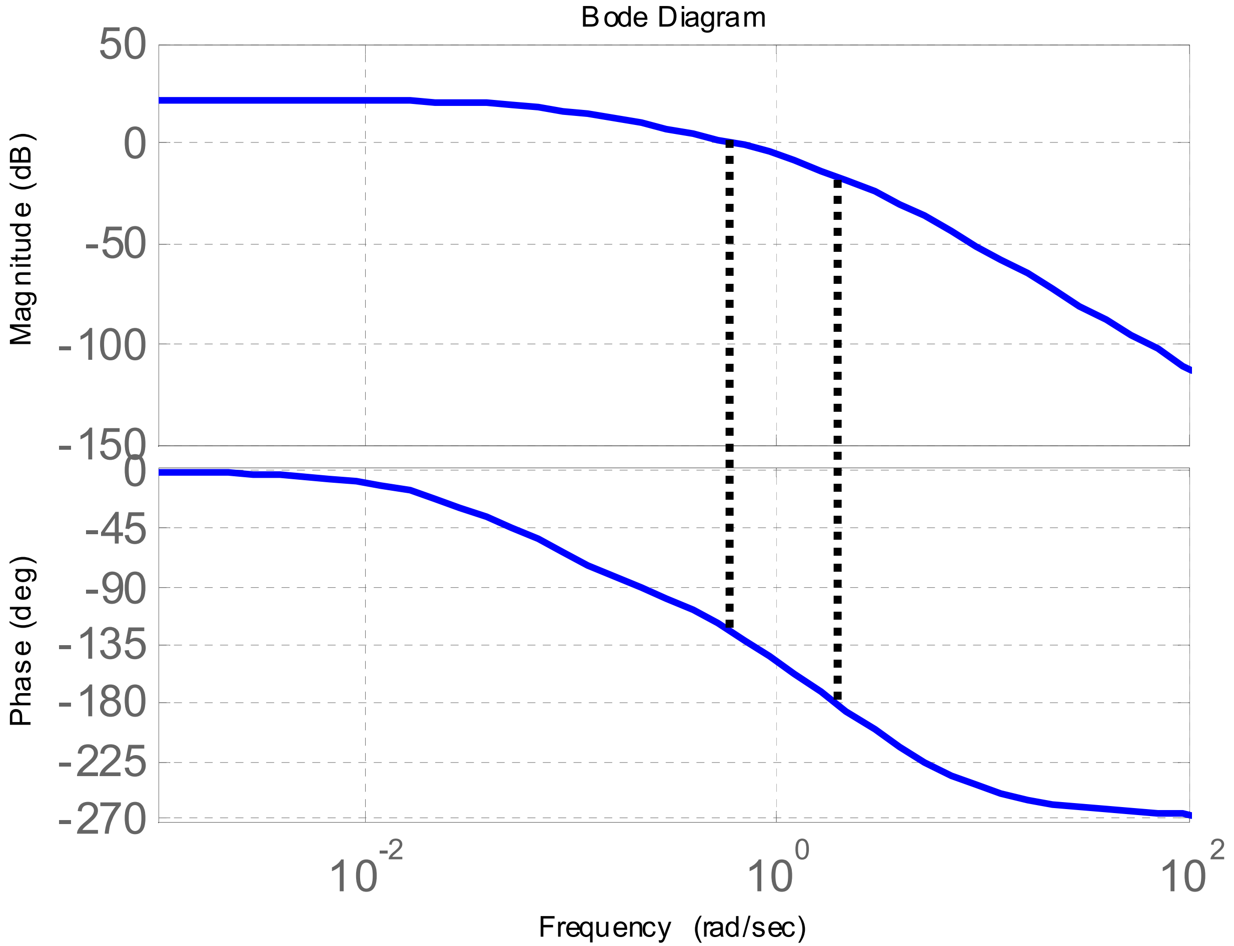
$$G(s) = \frac{1}{s(s+1)(s+2)}$$

M_{Φ}=51º

6- Bode diagrams

Stability margins

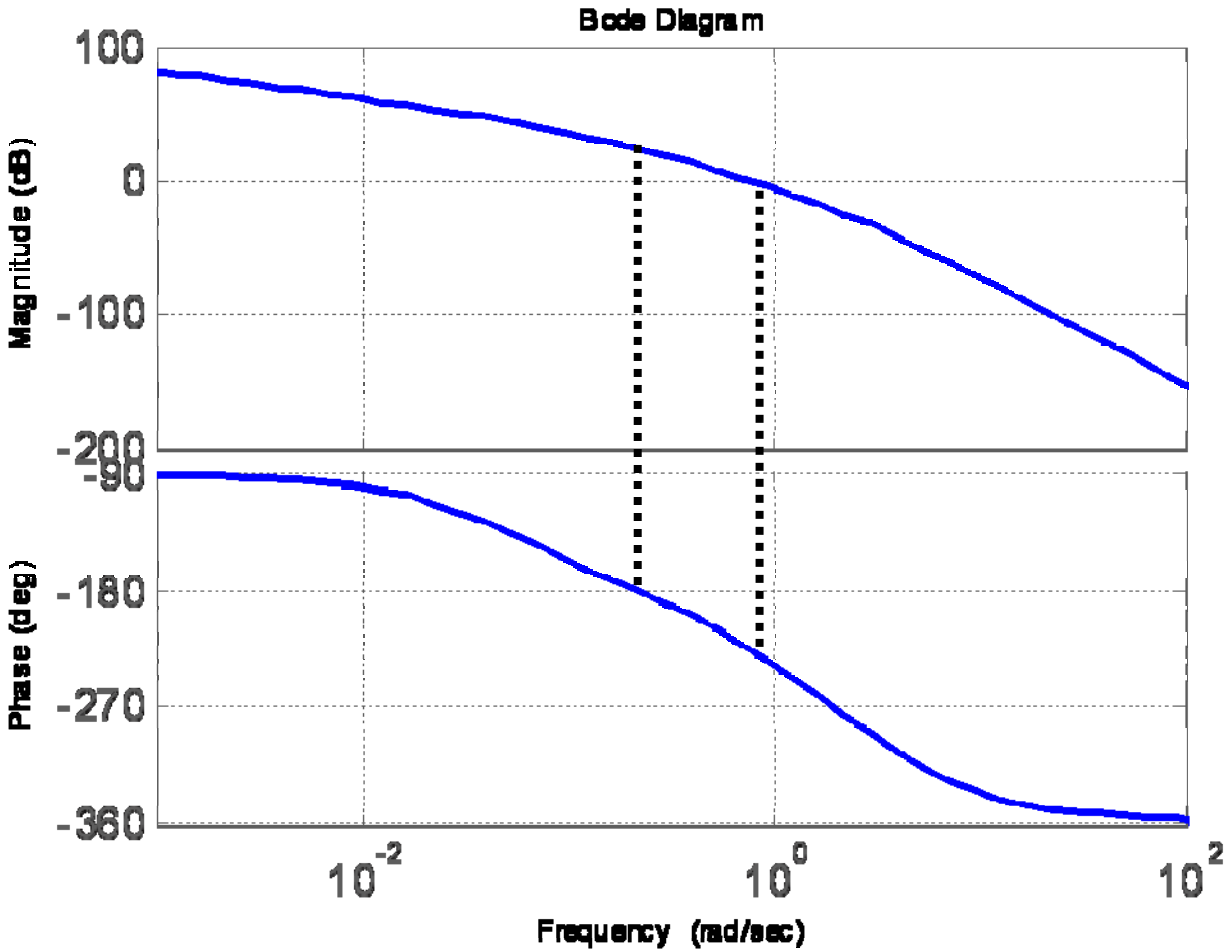
Transfer function of a vehicle cruise-control system $G(s) = \dfrac{2.48}{(s+0.06)(s+1)(s+3.33)}$



6- Bode diagrams

Stability margins

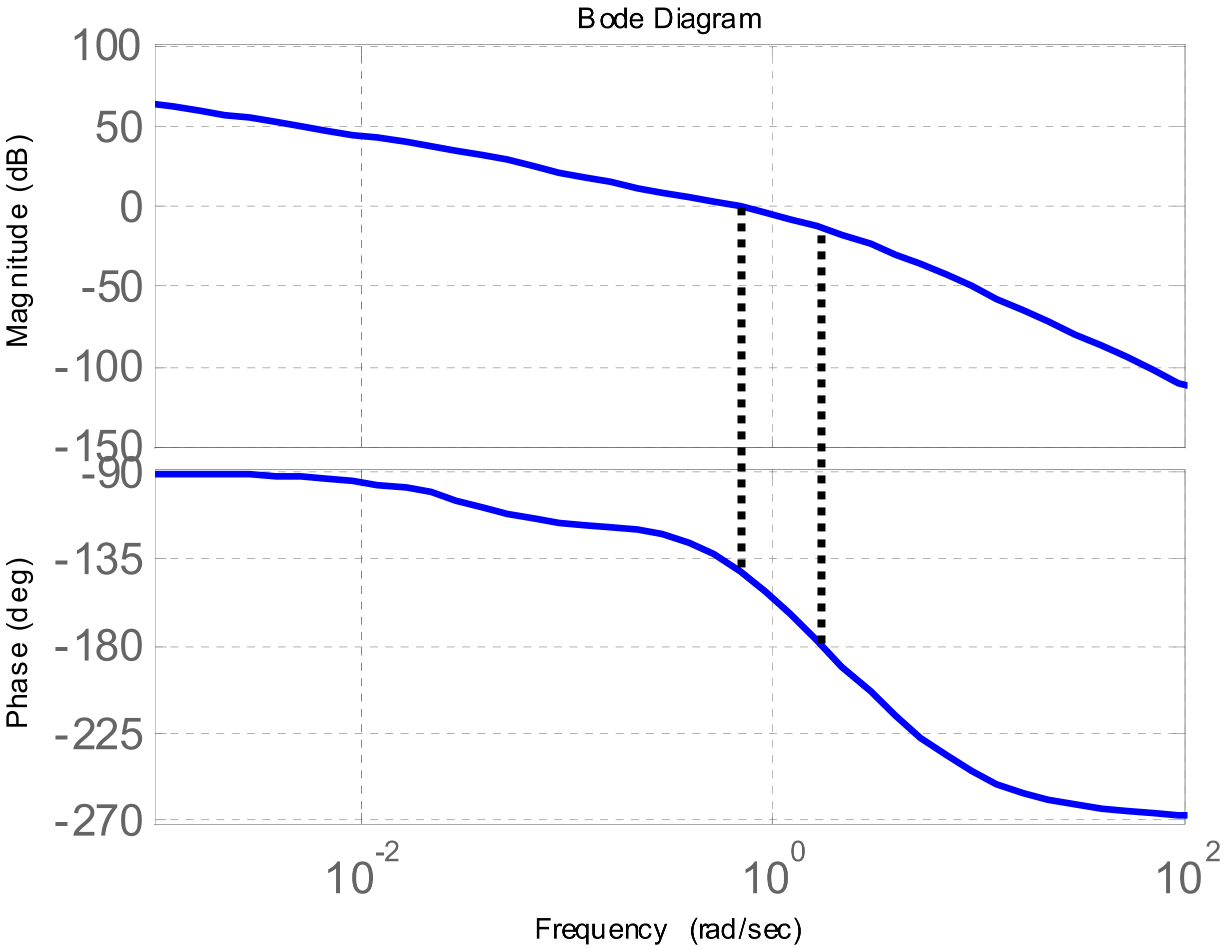
A pole is added on 0 (integrator): Bode diagram shifted downward : + unstable



6- Bode diagrams

Stability margins

A zero is added in -0.12: Bode diagram shifted upward : + stable



REFERENCES

G.F Franklin, J.D. Powell, A. Emani-Naeini, *Feedback Control of Dynamic Systems*, 4a Edición, Prentice-Hall, 2002

P. Lewis, *Sistemas de Control en Ingeniería*, Prentice Hall, 1999

W. Bolton, *Control Engineering*, 2ª Edición, Longman, 1998

D. Arzelier, D. Peaucelle, *Représentation et analyse des systèmes linéaires*, Tomes 1 et 2, Version 1, ENSICA, 1999