Control & Guidance

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Enginyeria Tècnica d'Aeronàutica
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Escola d'Enginyeria de Telecomunicació
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Aircraft Dynamics

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1 Laplace transform

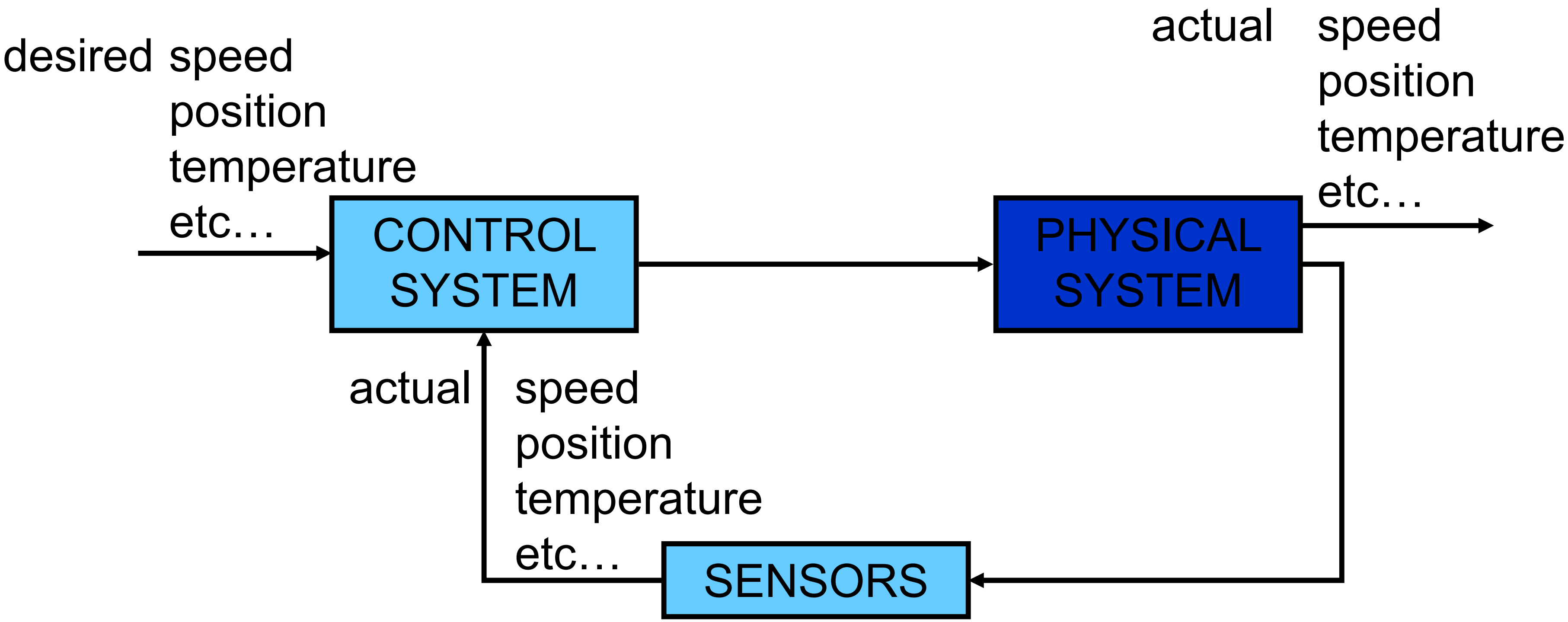
2 System modeling

3 Aircraft dynamics

1 Laplace transform

1. Transforms and properties
2. Transfer Functions

1. Laplace transform: MOTIVATION



Physical system usually modelized by differential equations (electrical systems, mechanical systems with application of Newton laws, etc…)

→ use of Laplace transforms to solve differential equations

1. Transforms and properties

When system models are made from lineal differential equations with constraint coefficients, Laplace transform methods can be used with great advantage

Laplace transform of a function is:

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

1. Transforms and properties

Inverse transform recovers the original function and returns 0 for time prior to t=0.

$$L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

$$= \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= f(t)u(t)$$

1. Transforms and properties

Linearity:

$$L[ax(t) + by(t)] = aX(s) + bY(s)$$

$$\forall (a,b) \in \Re^2$$

1. Transforms and properties

Derivation:

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$$

Can be generalized as:

$$L\left[\frac{d^{(n)}x(t)}{dt^n}\right] = s^n X(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0)\ldots - \frac{d^{(n-1)}x(0)}{dt^{n-1}}$$

Integration:

$$L\left[\int_0^t x(\tau)d\tau\right] = \frac{X(s)}{s}$$

1. Transforms and properties

Initial value theorem:

$$\lim_{t\to 0^+} x(t) = \lim_{s\to +\infty} sX(s)$$

Final value theorem, for stable systems:

$$\lim_{s\to 0} sX(s) = \lim_{t\to +\infty} x(t)$$

1. Transforms and properties

Important transforms

$f(t)$	$F(s)$
$\delta(t)$, unitary impulse	1
$u(t)$, unitary step	$1/s$
$\alpha.t.u(t)$	α/s^2
t^n	$\dfrac{n!}{s^{n+1}}$
e^{-at}	$\dfrac{1}{s+a}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}$	$\dfrac{1}{(s+a)^n}$
$\sin(\omega t)$ or $\cos(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$ or $\dfrac{s}{s^2+\omega^2}$

1. Transforms and properties

Solving differential equations using Laplace transform

1. apply Laplace transform to linear differential equations with constraint coefficients →linear algebraic equations
2. solve system of equations
3. get the solution of differential equations by inverse Laplace transform

Initial conditions may be included when using Laplace transform

Example 1 $$\frac{dy(t)}{dt} + 4y(t) = 6e^{2t} \quad \text{with} \quad y(0) = 3$$

1. Transforms and properties

Decomposing into simple fractions:

When calculating inverse transform: often have to develop a fraction in simpler fractions

1- If polynomial of numerator is of smaller order than the one of denominator and it has no repeated roots, it is possible to determine constants K_1, K_2, …, called residues that lead to:

$$Y(s) = \frac{q(s)}{p(s)} = \frac{\text{polynomial numerator}}{(s+a)(s+b)\ldots} = \frac{K_1}{s+a} + \frac{K_2}{s+b} + \ldots$$

1. Transforms and properties

Decomposition en simple fractions:

Note that individual terms in the development represent exponential functions for t>0:

$$y(t) = K_1 e^{-at} + K_2 e^{-bt} + K_3 e^{-ct} + \ldots \qquad t \geq 0$$

Coefficients can be obtained through the following expression:

$$\boxed{K_i = \lim_{s \to s_i} \frac{(s - s_i)q(s)}{p(s)}}$$

Example 1

1. Transforms and properties

Decomposition in simple fractions:

2- If polynomial in numerator is of bigger order than the one in denominator: there is a **quotient polynomial** and a **remainder polynomial**.

$$Y(s) = \text{quotient polynomial} + \frac{\text{remainder polynomial}}{(s+a)(s+b)(s+c)\ldots}$$

$$= \text{quotient polynomial} + \frac{K_1}{s+a} + \frac{K_2}{s+b} + \frac{K_3}{s+c} + \ldots$$

1. Transforms and properties

Decomposition in simple fractions:

3- If roots or factors in denominator are repeated, corresponding terms in the partial fraction development are:

$$\frac{\text{Numerator}}{(s+a)^n} = \frac{K_1}{s+a} + \frac{K_2}{(s+a)^2} + \ldots + \frac{K_n}{(s+a)^n}$$

Inverse Laplace transform for a repeated root:

$$L^{-1}\left[\frac{K_n}{(s+a)^n}\right] = \frac{K_n}{(n-1)!} t^{n-1} e^{-at} u(t)$$

Example 2

$$F(s) = \frac{s^2+2}{s^3-s^2-5s-3} = \frac{s^2+2}{(s+1)^2(s-3)}$$

1. Transforms and properties

Decomposition in simple fractions:

4- If there is a complex number root:

$$Y(s) = \frac{\text{Numerator}}{(s+a)\left(s^2+bs+c\right)} = \frac{K_1}{s+a} + \frac{K_2 s + K_3}{s^2+bs+c}$$

The inverse transform for a repeated root has the form of a sine or a cosine

Example 3 $F(s) = \dfrac{2s+1}{s^3+2s^2+s+2}$

2. Transfer functions (TF)

One of the most powerful tools to design control systems

For a simple in & out system, with x(t) input and y(t) output, transfer function that links the output with the input is defined as the following quotient

$$T(s) = \frac{Y(s)}{X(s)}$$

where Y(s): Laplace transform of output

X(s): Laplace transform of input

with initial conditions equal to zero

2. Transfer functions

Given a described system for the following differential equation relating output $y(t)$ with input $x(t)$:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x$$

Applying Laplace transform on this equation, **with zero initial conditions**:

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) =$$

$$b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \ldots + b_1 s X(s) + b_0 X(s)$$

2. Transfer functions

Can be factorized as:

$$Y(s)\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) =$$
$$X(s)\left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right)$$

The following transfer function is obtained:

$$T(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

2. Transfer functions

Block diagram

- describes systems schematically

- describes internal functions of a system (amplifiers, control engines, filters, etc.)

- offers a simpler alternative to directly study the equations

2. Transfer functions

Block Diagram

original system of equations can be replaced by a diagram formed by:

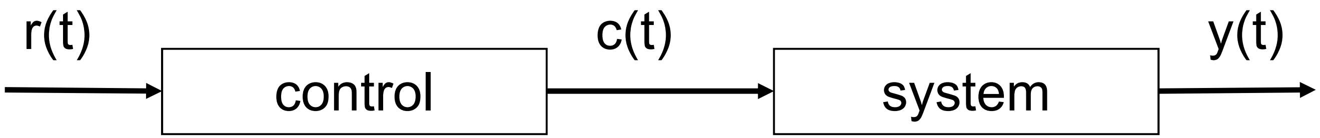
- **branches** (arrows) representing variables,
- **blocks** showing proportionality between 2 Laplace transform signals, inside of which TF relating input and output is shown,
- **sums** used to show signal sums or subtractions,
- **unions** showing that the same signal parts in two different ways

Schematics + Example 4

2. Transfer functions

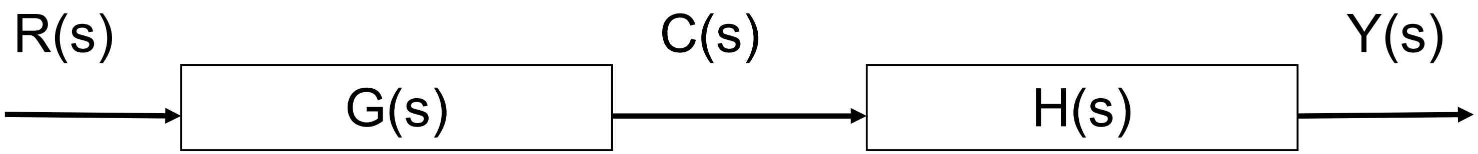
How to calculate TF?

Transfer function in direct transmittance or open-loop systems



- no perturbation intakes
- input not influenced by output results

$$\frac{Y(s)}{R(s)} = G(s) \times H(s)$$



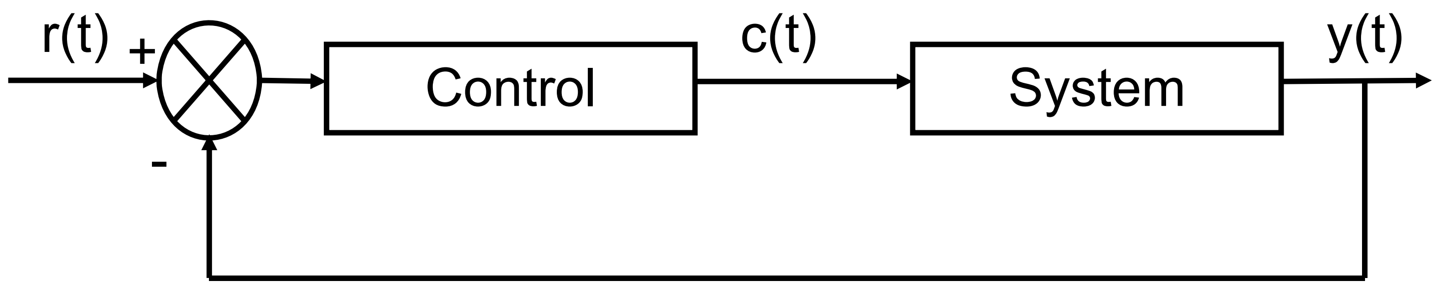
2. Transfer functions

How to calculate TF?

Transfer function in a unitary closed loop system (with feedback):

• perturbation exists,

• system not fully known: output information needed



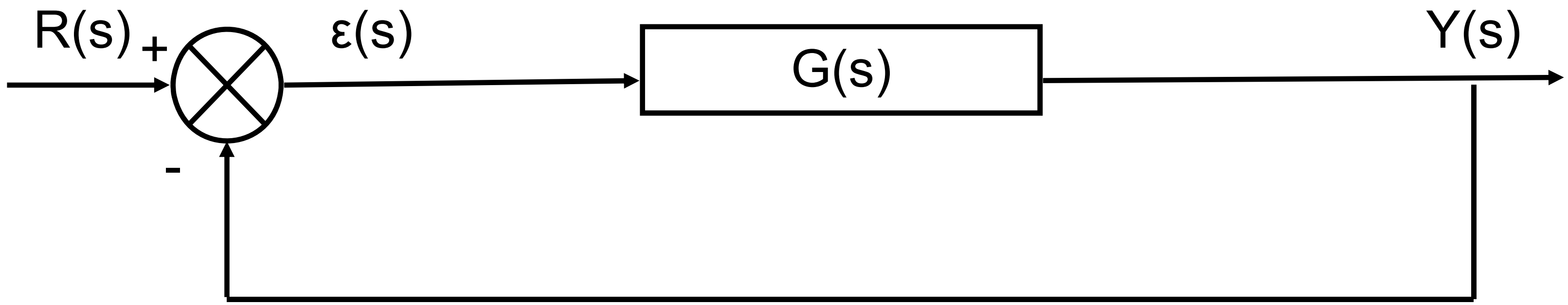
• verifies that the system output corresponds to the reference input

• unstability is created

2. Transfer functions

How to calculate TF?

Transfer function for unitary closed loop system:



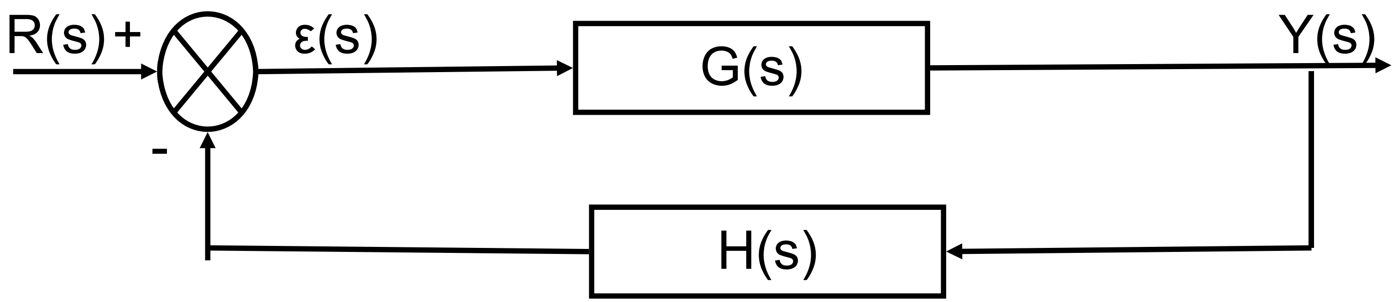
- R(s): desired response
- Y(s): actual response
- ε(s): system error

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

2. Transfer functions

How to calculate TF?

Transfer function for non-unitary closed loop system:



- R(s): desired response
- Y(s): actual response
- ε(s): system error
- H(s): observation

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + H(s) \times G(s)}$$

Proof + Example 4

2. Transfer functions

Poles and zeros: definition

Function's zeros = values of a variable for which function is equal to zero

Function's poles = values of the variables for which function goes infinite

In a transfer function:

zeros = roots of numerator

poles = roots of denominator

2. Transfer functions

Poles & zeros locus

• when zeros and poles of a function are shown in the complex plane → poles and zeros locus

• important properties of the function can be deduced

• zeros are shown as O in the graph

• poles are shown as X in the graph

Example 4

2. Transfer functions

Dynamic stability

A system is **asymptotically stable** if its response for all the possible inputs is zero or tends to it

A linear system, with transfer function *T(s)*, has a different response for each root of *T(s)*'s denominator (each pole of *T(s)*).

→ **each response is called a mode of the system**

2. Transfer functions

Dynamic stability

A mode increases or decreases with time depending if the pole is in the right semi-plane (RSP) or left semi-plane (LSP).

So, the given system will be asymptotically stable only if all its poles belong to the LSP

Ejemplo 4

2. Transfer functions

Speed

The asymptotic stability condition ensures that a response tends to zero with time, but does not give any indication of the qualitative evolution of the signal

response *s(t)* is formed by the linear combination of elementary functions called **modes**

real poles correspond to **aperiodic modes**

conjugated complex poles correspond to **oscillatory modes**

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2. Transfer functions

Speed

time of disappearance of a transitory mode defines mode's speed

$$\tau_i = -\frac{1}{\mathrm{Re}\{p_i\}}$$

Faster modes are associated to poles further away from the imaginary axis

Examples 5

2. System modeling

Introduction

Basic prerequisite in the development of almost any control strategy:

obtain a new mathematical model for the system part to control

model is formulated as a system of differential equations

3. Aircraft dynamics

1. Longitudinal dynamics
2. Transfer function for longitudinal models
3. Lateral dynamics
4. Crossed coupling

Ref: *Automatic control of Aircraft and Missiles*, 2nd edition, John H. Blakelock

1. Longitudinal dynamics

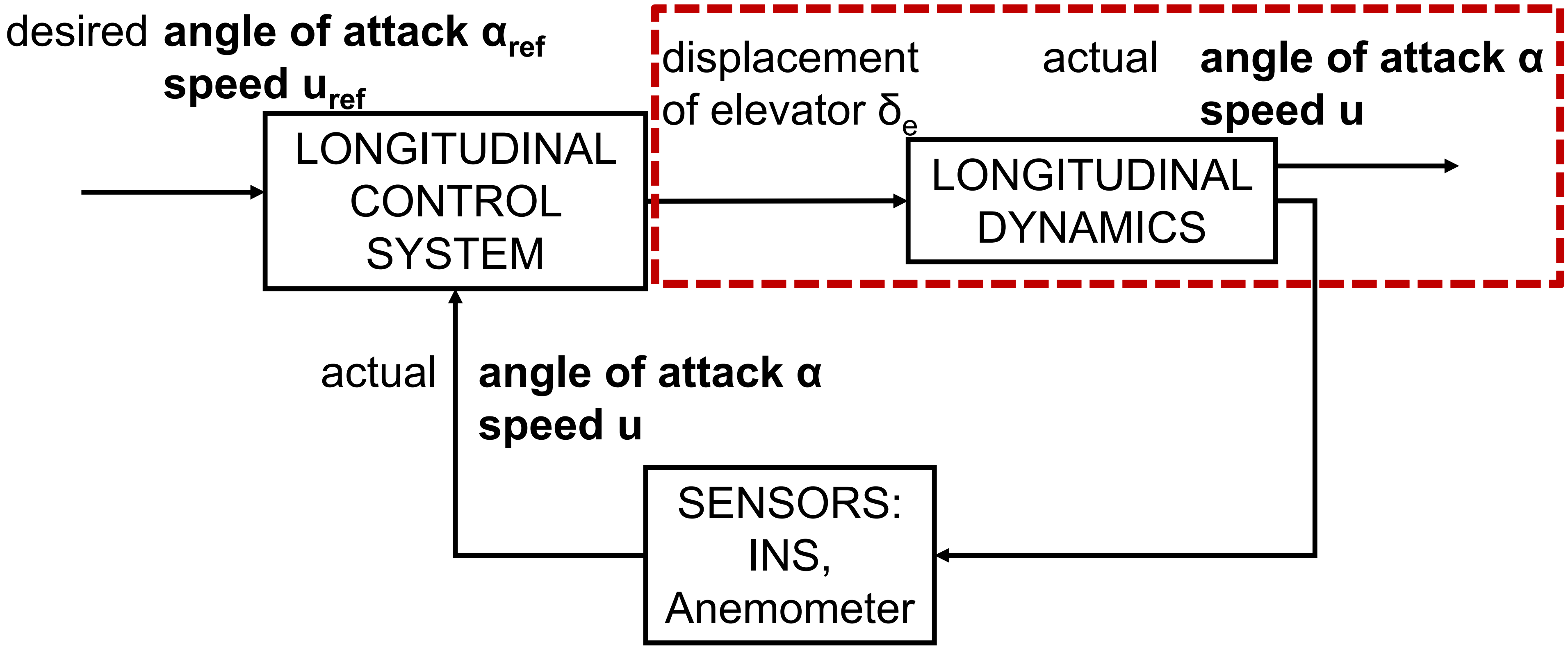
Objective: obtain differential equations for airplane longitudinal movements, based on a slight perturbation (displacement of the elevator),

and then obtain transfer functions

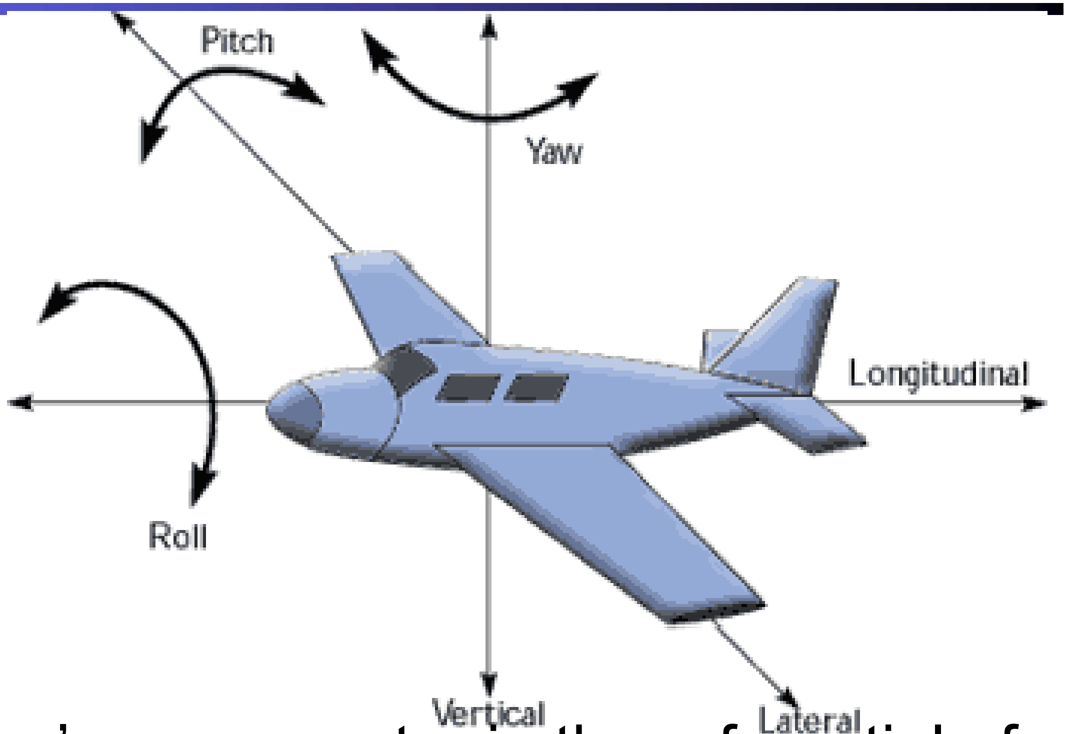
(for ex. between displacement of the elevator and angle of attack, …)

→ First step: apply Newton laws in the defined axis system

1. Longitudinal dynamics



1. Longitudinal dynamics



(U, V, W) speed of airplane's mass center in the referential of the airplane with respect to the referential of the ground

(P, Q, R) angular speed in the referential of the airplane with respect to the referential of the ground

(L, M, N) roll, pitch and yaw momentum

1. Longitudinal dynamics

Hypothesis # 1: ***X*** and ***Z*** axis are in the airplane's symmetrical axis and center of gravity = origin of the axis system

Inertia tensor:
$$\begin{bmatrix} I_x & 0 & J_{xz} \\ 0 & I_y & 0 \\ J_{xz} & 0 & I_z \end{bmatrix} \quad \text{because } J_{xy} \text{ and } J_{yz} = 0$$

Remember:
$$I_x = \iint_S \left(y^2 + z^2\right) dm$$
$$J_{xy} = \oiint xy \, dm$$

1. Longitudinal dynamics

Newton Law:

$$\sum \vec{F}_{Ext} = \frac{d\left(m\,\vec{V}_T\right)}{dt} = \sum \vec{F}_0 + \sum \Delta \vec{F}$$

$$\sum \vec{M}_{Ext} = \frac{d\,\vec{H}}{dt} = \sum \vec{M}_0 + \sum \Delta \vec{M}$$

Where ***H*** is the angular momentum.

Airplane is considered in equilibrium before perturbation occurs, thus $\sum \vec{F}_0 = 0$

$$\sum \vec{M}_0 = 0$$

1. Longitudinal dynamics

Hypothesis # 2: Constant airplane mass

$$\frac{d\left(m\vec{V}_T\right)}{dt} = m\frac{d\vec{V}_T}{dt}$$

Hypothesis # 3: Airplane = rigid body

Hypothesis # 4: Ground = inertial referential (a free particle has a rectilinear uniform translation movement)

1. Longitudinal dynamics

Vectorial derivation: takes into account: changes in the linear velocity V_T and in ω, total angular velocity of the aircraft with respect to the Earth

$$\left.\frac{d\vec{V}_T}{dt}\right|_{Tierra} = I_{V_T}\frac{dV_T}{dt} + \vec{\omega}\wedge\vec{V}_T$$

$$= \dot{U}\vec{i} + \dot{V}\vec{j} + \dot{W}\vec{k} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P & Q & R \\ U & V & W \end{vmatrix}$$

$$= \dot{U}\vec{i} + \dot{V}\vec{j} + \dot{W}\vec{k} + \vec{i}(QW - VR) - \vec{j}(PW - UR) + \vec{k}(PV - UQ)$$

1. Longitudinal dynamics

Under these hypothesis:

$$\begin{cases} \sum \Delta F_x = \left(\dot{U} + QW - RV \right) m \\ \sum \Delta F_y = \left(\dot{V} + UR - PW \right) m \\ \sum \Delta F_z = \left(\dot{W} + PV - UQ \right) m \end{cases}$$

$$\begin{cases} \sum \Delta L = \dot{P} \times I_x - \dot{R} \times J_{xz} + QR \times \left(I_z - I_y \right) - PQ \times J_{xz} \\ \sum \Delta M = \dot{Q} \times I_y + PR \times \left(I_x - I_z \right) + \left(P^2 - R^2 \right) \times J_{xy} \\ \sum \Delta N = \dot{R} \times I_z - \dot{P} \times J_{xy} + PQ \times \left(I_y - I_x \right) + QR \times J_{xy} \end{cases}$$

1. Longitudinal dynamics

Hypothesis # 5: Leveled flight, non turbulent and non-accelerated

In case of **longitudinal** study:

→ there is only pitch movement **/Oy**

→ there is variation in $\mathbf{F_x}$ and $\mathbf{F_z}$ but not in $\mathbf{F_y}$ (speed **V=0**)

→ there is no roll nor yaw momentum → angular speed **P=R=0**

1. Longitudinal dynamics

Simplified longitudinal equations:

$$\sum \Delta F_x = m\left(\dot{U} + QW\right)$$

$$\sum \Delta F_z = m\left(\dot{W} - UQ\right)$$

$$\sum \Delta M = \dot{Q} \times I_y$$

1. Longitudinal dynamics

Exterior forces:

- Weight → $\mathbf{F_x}$ and $\mathbf{F_z}$
- Thrust
- Aerodynamic forces (lift + drag)

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1. Longitudinal dynamics

Notation (cf. Schematics)

$\mathbf{U=U_0+u}$, $\mathbf{W=W_0+w}$, $\mathbf{Q=Q_0+q}$

$\mathbf{U_0}$, $\mathbf{W_0}$, $\mathbf{Q_0}$ values in equilibrium

u, **w**, **q** changes due to perturbation.

Hypothesis # 6: small equilibrium perturbations

compared to equilibrium values

$\mathbf{u \ll U_0}$, $\mathbf{w \ll W_0}$, $\mathbf{q \ll Q_0}$ → **linearization**

1. Longitudinal dynamics

- Since **OX_0** is lined up with the longitudinal airplane axis: **$W_0=0$**

  → **$U=U_0+u$, $W=w$**
- Airplane initially non accelerated: **$Q_0=0$** → **$Q=q=$** $\dot{\theta}$

$$\sum \Delta F_x = m(\dot{u} + wq)$$

$$\sum \Delta F_z = m(\dot{w} - U_0 q - uq)$$

1. Longitudinal dynamics

With the hypothesis of **small perturbations**, the product of the perturbations (product of 2 smalls terms) is negligible in front of a simple term:

$$\sum \Delta F_x = m\dot{u}$$

$$\sum \Delta F_z = m\left(\dot{w} - U_0 q\right)$$

$$\sum \Delta M = \dot{q} \times I_y = I_y \ddot{\theta}$$

1. Longitudinal dynamics

Eventually, we write the **variations** of the parameters with respect to the equilibrium as

$$'u = \frac{u}{U}$$

$$'\alpha = \frac{w}{U}$$

$$'\dot{\alpha} = \frac{\dot{w}}{U}$$

1. Longitudinal dynamics

$$\left(\frac{mU}{Sq}\dot{u}' - C_{X_u} u'\right) + \left(-\frac{c}{2U} C_{X_{\dot{\alpha}}} \dot{\alpha}' - C_{X_\alpha} \alpha'\right) + \left(-\frac{c}{2U} C_{X_q} \dot{\theta} - C_\omega \cos(\Theta)\, \theta\right) = C_{F_{X_a}}$$

$$\left(-C_{Z_u} u'\right) + \left[\left(\frac{mU}{Sq} - \frac{c\, C_{Z_{\dot{\alpha}}}}{2U}\right)\dot{\alpha}' - C_{Z_\alpha} \alpha'\right] + \left[\left(-\frac{mU}{Sq} - \frac{c}{2U} C_{Z_q}\right)\dot{\theta} - C_\omega \sin(\Theta)\, \theta\right] = C_{F_{Z_a}}$$

$$\left(-C_{m_u} u'\right) + \left(-\frac{c\, C_{m_{\dot{\alpha}}}}{2U}\dot{\alpha}' - C_{m_\alpha} \alpha'\right) + \left(\frac{I_Y}{Sqc}\ddot{\theta} - \frac{c}{2U} C_{m_q} \dot{\theta}\right) = C_{m_a}$$

1. Longitudinal dynamics

With: **S**: wing span

q: dynamic pressure $\left(\frac{1}{2}\rho U^2\right)$

c: average aerodynamic chord

C...: non-dimensional coefficients (examples: variation of drag and thrust with u, lift and drag variations along X, gravity, downwash effect on drag, effect of pitch rate on drag, etc…)

all angles in **radians**

2. Transfer functions for the longitudinal model

Consider a transport airplane, with 4 engines flying straight and leveled at 40,000ft with a constant speed of 600ft/sec (=355 knots)

$\Theta=0$

Mach=0.62

M=5800 slugs (lb.s^2/ft 1slug=14.594kg)

U= 600ft/sec

S=2400 sq.ft

c=20.2ft (1ft=0.3048m)

…

2. Transfer functions for the longitudinal model

1. With a fixed elevator:

Differential system of equations is

$$\begin{cases} 13.78\,'\,\dot{u}(t) + 0.088\,'\,u(t) - 0.392\,'\,\alpha(t) + 0.74\,\theta(t) = 0 \\ 1.48\,'\,u(t) + 13.78\,'\,\dot{\alpha}(t) + 4.46\,'\,\alpha(t) - 13.78\,\dot{\theta}(t) = 0 \\ 0.0552\,'\,\dot{\alpha}(t) + 0.619\,'\,\alpha(t) + 0.514\,\ddot{\theta}(t) + 0.192\,\dot{\theta}(t) = 0 \end{cases}$$

2. Transfer functions for the longitudinal model

1. With a fixed elevator:

Applying the Laplace transform (initial conditions being zero):

$$
\begin{array}{llll}
(13.78s + 0.088)\ 'u(s) & -0.392\ '\alpha(s) & +0.74\ \theta(s) & = 0 \\
1.48\ 'u(s) & +(13.78s + 4.46)\ '\alpha(s) & -13.78s\ \theta(s) & = 0 \\
0 & (0.0552s + 0.619)\ '\alpha(s) & +(0.514s^2 + 0.192s)\theta(s) & = 0
\end{array}
$$

2. Transfer functions for the longitudinal model

1. With a fixed elevator:

The only solution different from **(0, 0, 0)** needs the system determinant to be zero:

$$\begin{vmatrix} 13.78s + 0.088 & -0.392 & +0.74 \\ 1.48 & 13.78s + 4.46 & -13.78s \\ 0 & 0.0552s + 0.619 & 0.514s^2 + 0.192s \end{vmatrix} = 0$$

2. Transfer functions for the longitudinal model

1. With a fixed elevator:

Equivalent to:

$$\left(13.78s+0.088\right)\begin{vmatrix} 13.78s+4.46 & -13.78s \\ 0.0552s+0.619 & 0.514s^2+0.192s \end{vmatrix}$$

$$-1.48\begin{vmatrix} -0.392 & +0.74 \\ 0.0552s+0.619 & 0.514s^2+0.192s \end{vmatrix}=0$$

2. Transfer functions for the longitudinal model

1. With a fixed elevator:

We obtain the system determinant:

$$\nabla = 97.5s^4 + 79s^3 + 128.9s^2 + 0.998s + 0.677$$

And after simplifying it we obtain the following characteristic equation:

$$s^4 + 0.811s^3 + 1.32s^2 + 0.0102s + 0.00695 = 0$$

2. Transfer functions for the longitudinal model

2. With a displacement of the elevator:

$\boldsymbol{\delta_e}$: elevator deviation (rad), $\boldsymbol{\delta_e > 0}$: elevator goes down

$$(13.78s + 0.088)\;{}'u(s) - 0.392\;{}'\alpha(s) + 0.74\,\theta(s) = 0$$

$$1.48\;{}'u(s) + (13.78s + 4.46)\;{}'\alpha(s) - 13.78s\,\theta(s) = -0.246\,\delta_e(s)$$

$$(0.0552s + 0.619)\;{}'\alpha(s) + (0.514s^2 + 0.192s)\theta(s) = -0.710\,\delta_e(s)$$

2. Transfer functions for the longitudinal model

2. With a displacement of the elevator :

Remember: use determinant to solve algebraic equations (Cramer):

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 2y - z = 3 \\ 3x + 2y + z = 2 \end{cases} \Rightarrow \begin{cases} x = \dfrac{\begin{vmatrix} 6 & 2 & 3 \\ 3 & -2 & -1 \\ 2 & 2 & 1 \end{vmatrix}}{\nabla} \\ y = \dfrac{\begin{vmatrix} 1 & 6 & 3 \\ 2 & 3 & -1 \\ 3 & 2 & 1 \end{vmatrix}}{\nabla} \end{cases}$$

Where ∇ is the determinant of the system of homogeneous equations

2. Transfer functions for the longitudinal model

2. With a displacement of the elevator :

$$\frac{{}^{'}u(s)}{\delta_e(s)} = \frac{\begin{vmatrix} 0 & -0.392 & 0.74 \\ -0.246 & 13.78s + 4.46 & -13.78s \\ -0.710 & 0.055s + 0.619 & 0.514s^2 + 0.192s \end{vmatrix}}{\nabla}$$

Where:

$$\nabla = 97.5s^4 + 79s^3 + 128.9s^2 + 0.998s + 0.677$$

2. Transfer functions for the longitudinal model

$$\frac{{}'u(s)}{\delta_e(s)} = \frac{-0.0494s^2 + 3.3691s + 2.223}{97.5s^4 + 79s^3 + 128.9s^2 + 0.998s + 0.677}$$

The determinant of the system (=denominator of the transfer functions) has 4 complex conjugated roots:

$$s = -0.4032 \pm 1.0717j$$

and

$$s = -0.0023 \pm 0.0728j$$

Remember: **real roots of the denominator (= poles of the transfer function)** associated to **non-oscillatory modes**, and **complex poles** to **oscillatory modes**

2. Transfer functions for the longitudinal model

Note: $s_i = \sigma_i + j\omega_i$

We define the **time constant**: $$\tau = -\frac{1}{\mathrm{Re}(s_i)}$$

And the **damping factor**: $$\zeta = \left|\frac{\mathrm{Re}(s_i)}{s_i}\right| = \frac{|\sigma_i|}{\sqrt{\sigma_i^2 + \omega_i^2}}$$

2. Transfer functions for the longitudinal model

From the 2 pairs of conjugated roots we can identify 2 periodic modes:

Mode 1: $$\tau = \frac{-1}{-0.4032} = 2.48\text{s}$$

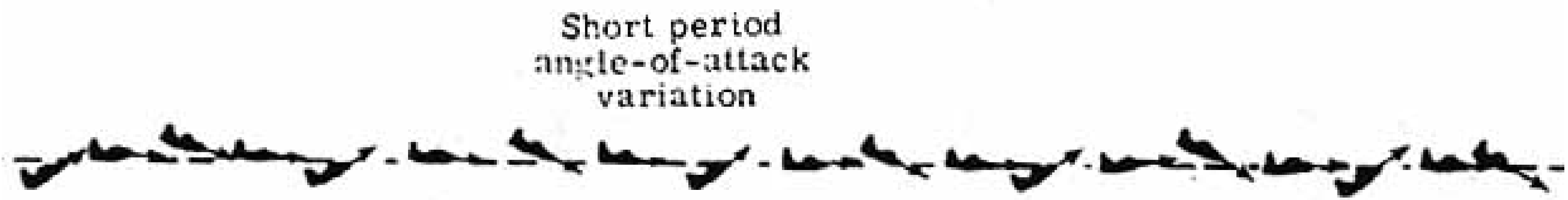
$$\zeta = \frac{0.4032}{\sqrt{0.4032^2 + 1.0717^2}} = 0.352$$

→ high frequency: **short period oscillation mode**

2. Transfer functions for the longitudinal model

• Variations of ´α y θ, with little change of speed ´u

• If ζ is too low, we need a feedback system (closed loop) to increase the damping factor ζ



(b) Short-period longitudinal oscillation.

2. Transfer functions for the longitudinal model

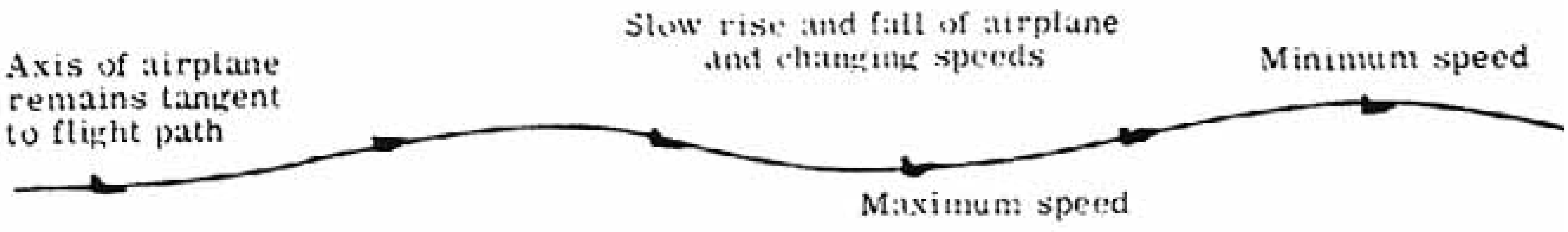
Mode 2: $\tau = \dfrac{-1}{-0.0023} = 434.8\text{s}$

$$\zeta = \frac{0.0023}{\sqrt{0.0023^2 + 0.0728^2}} = 0.032$$

→ low frequency: **phugoid mode**

2. Transfer functions for the longitudinal model

- variations of ´u and θ, with ´α nearly constant
- kinetic and potential energy exchange
- airplane tends to a sinusoidal flight
- values of period and ζ depend on the airplane and its flight conditions



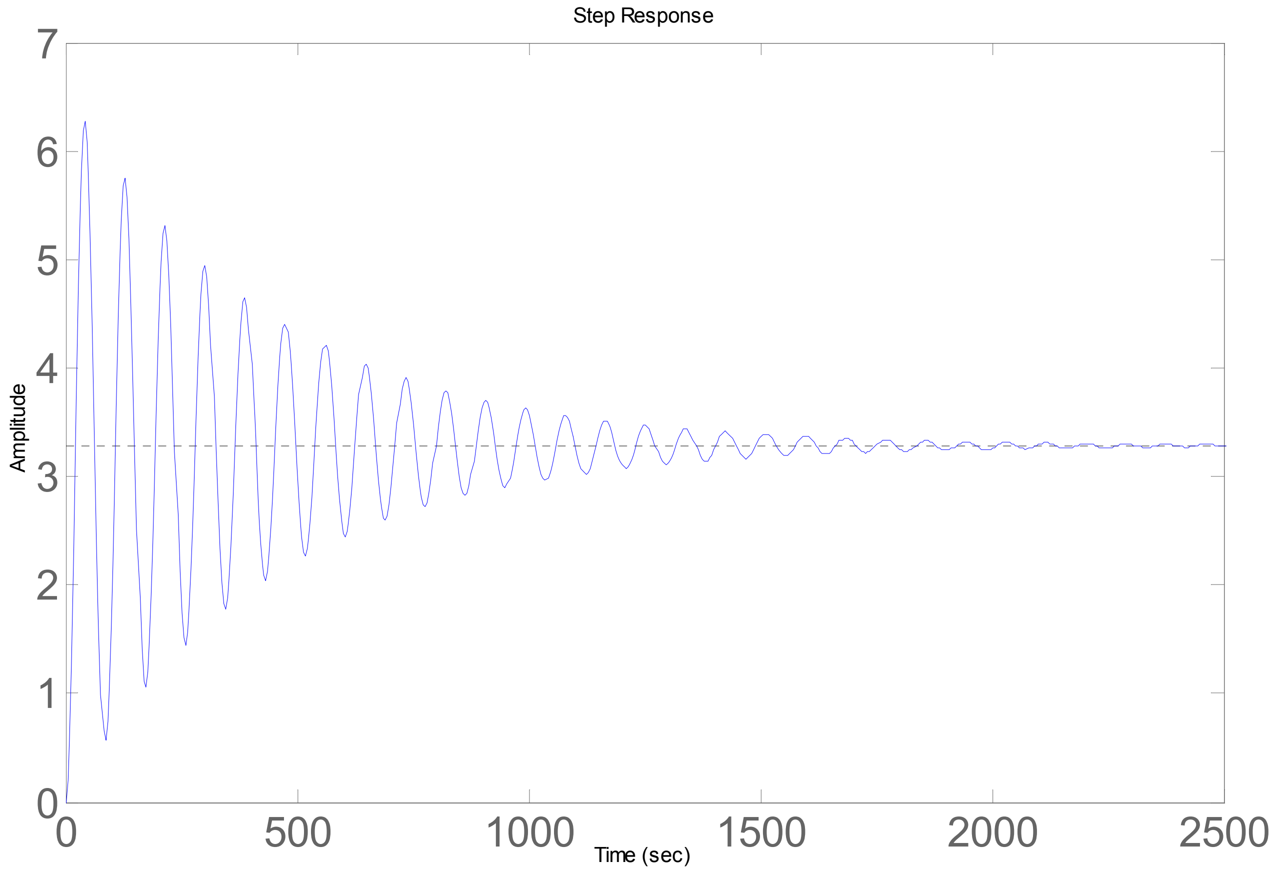
(a) Phugoid longitudinal oscillation.

Linear Speed

We obtain:

$$\frac{u(s)}{\delta_e(s)} = \frac{-0.0494s^2 + 3.3691s + 2.223}{97.5s^4 + 79s^3 + 128.9s^2 + 0.998s + 0.677}$$

Response to a step input using Matlab



Linear Speed

To obtain a u value for the step input δ_e we use the **final value theorem** (*system is stable*):

$$\lim_{t\to\infty} {}'u(t) = \lim_{s\to 0}\left(s\times {}'u(s)\right) \quad \text{for } \delta_e(t) = 1 \rightarrow \delta_e(s) = \frac{1}{s}$$

$$\lim_{t\to\infty} {}'u(t) = \lim_{s\to 0}\left(s\times\frac{1}{s}\times\frac{-0.0494s^2 + 3.3691s + 2.223}{97.5s^4 + 79s^3 + 128.9s^2 + 0.998s + 0.677}\right)$$

$${}'u_\infty = 3.28 \text{ for } \delta_e = 1 \text{ rad}$$

$$\text{and} \quad u = {}'u_\infty \times U \quad \text{with } U = 600\, {}^{ft}\!/_{sec}$$

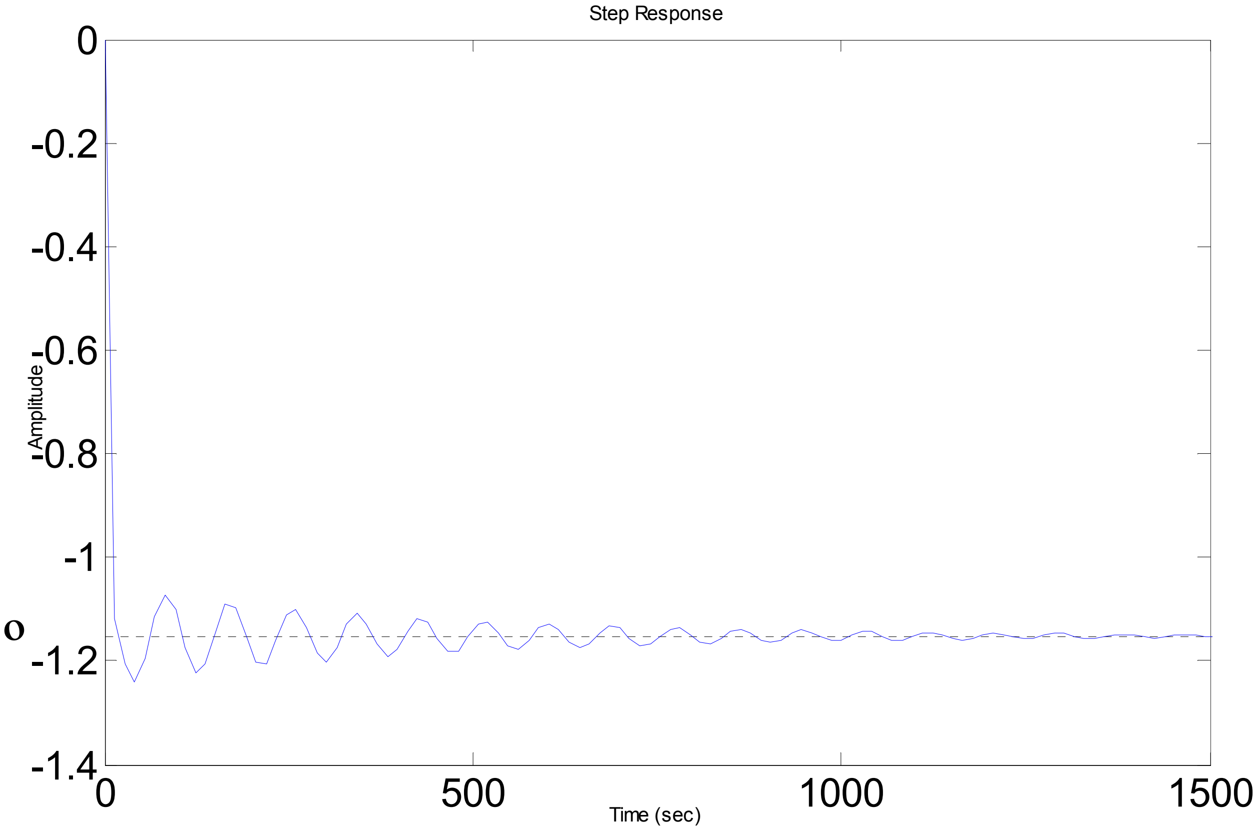
$$u = 1969\, {}^{ft}\!/_{sec} \quad \text{for } \delta_e = 1 \text{ rad}$$

$$\boxed{u = \frac{1969}{180/\pi} = 34.36\, {}^{ft}\!/_{sec} \quad \text{for } \delta_e = 1^o}$$

Angle of Attack

$$\frac{\alpha'(s)}{\delta_e(s)} = \frac{-0.0179s^3 - 1.3887s^2 - 0.0089s - 0.0080}{(s^2 + 0.00466s + 0.0053)(s^2 + 0.806s + 1.311)}$$

Response to a step input using Matlab:



Can also be obtained using the final value theorem:

$$\alpha'_\infty = -1.14^\circ \text{ for } \delta_e = 1^\circ$$

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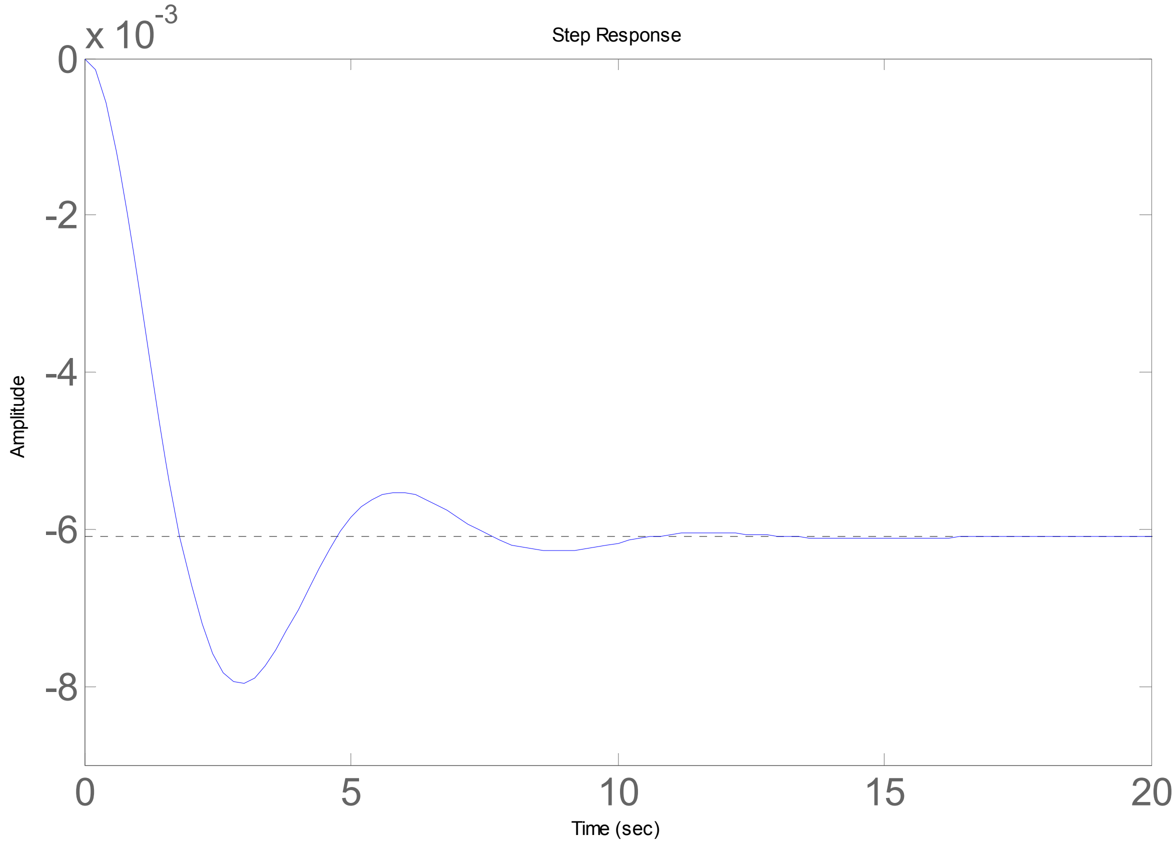
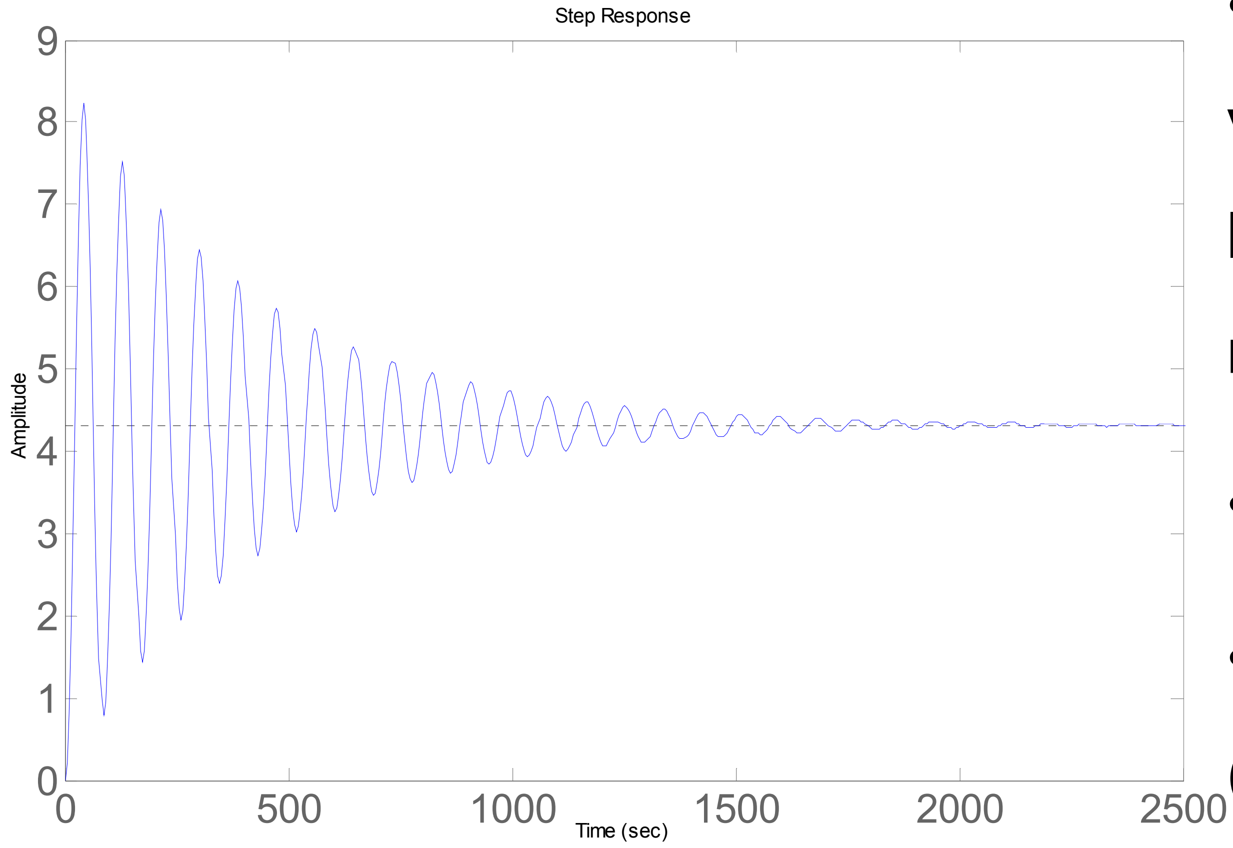
Low period oscillation mode



Low period oscillation mode only: angle of attack ’α

- low period: de 0.6 a 6s
- difficult to know its existence: cause can be a wind burst or a sudden activation of flight controls
- fast damping without effort from the pilot

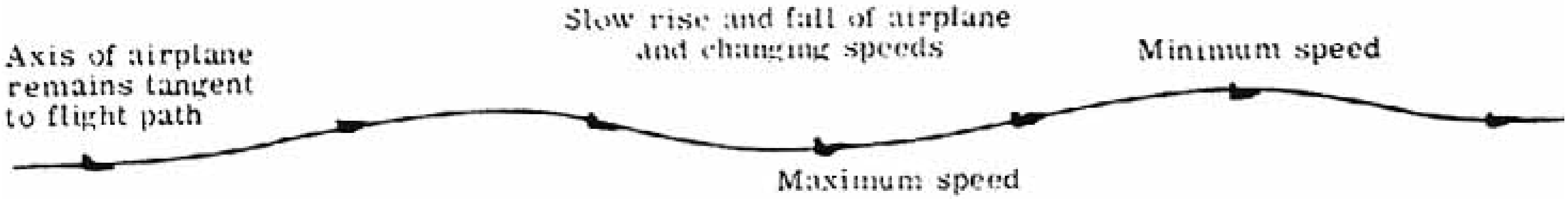
Phugoid Mode



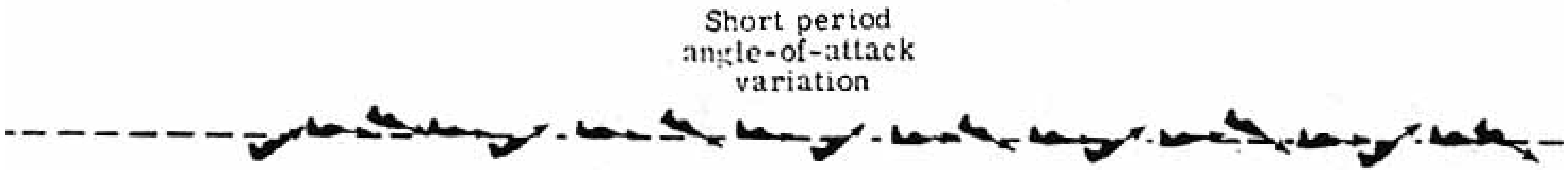
Phugoid mode only: linear speed 'u

- phugoid's period varies between 25s at low speed to several minutes at high speeds
- low damping
- easy to control by pilot (high period → more time to react and activate flight controls)

Longitudinal Modes



(a) Phugoid longitudinal oscillation.



(b) Short-period longitudinal oscillation.

Longitudinal Modes

Amplitude, oscillation period and damping depend on

- aircraft (C coefficients…)

- altitude (air density)

- airspeed

• phugoid period increases with speed, and decreases with altitude at fixed Mach number

• short-period oscillation mode does the opposite: decreases with speed and increases with altitude

3. Lateral dynamics

Using the same hypothesis for longitudinal mode:

$$\sum \Delta F_Y = m\left(\dot{V} + UR - WP\right)$$

$$\sum \Delta L = \dot{P}\, I_X - \dot{R}\, J_{XZ} + QR\,(I_Z - I_Y) - PQ\, J_{XZ}$$

$$\sum \Delta M = \dot{R}\, I_Z - \dot{P}\, J_{XZ} + PQ\,(I_Y - I_X) + QR\, J_{XZ}$$

3. Lateral dynamics

Under the same airplane model we obtain the characteristic equation:

$$\nabla = 0.00748\,s^5 + 0.01827\,s^4 + 0.01876\,s^3 + 0.0275\,s^2 - 0.0001135\,s = 0$$

Can be factorized:

$$s(s^2 + 0.38s + 1.813)(s + 2.09)(s - 0.004) = 0$$

3. Lateral dynamics

- **solution** s=0

  once disturbed, airplane recovers its original flight path

- s= -2.09 **roll subsidence mode**:

  airplane's response to an aileron movement

- s=0.004 **spiral divergence mode:**

  long time constant : easily controlled by pilot

3. Lateral dynamics

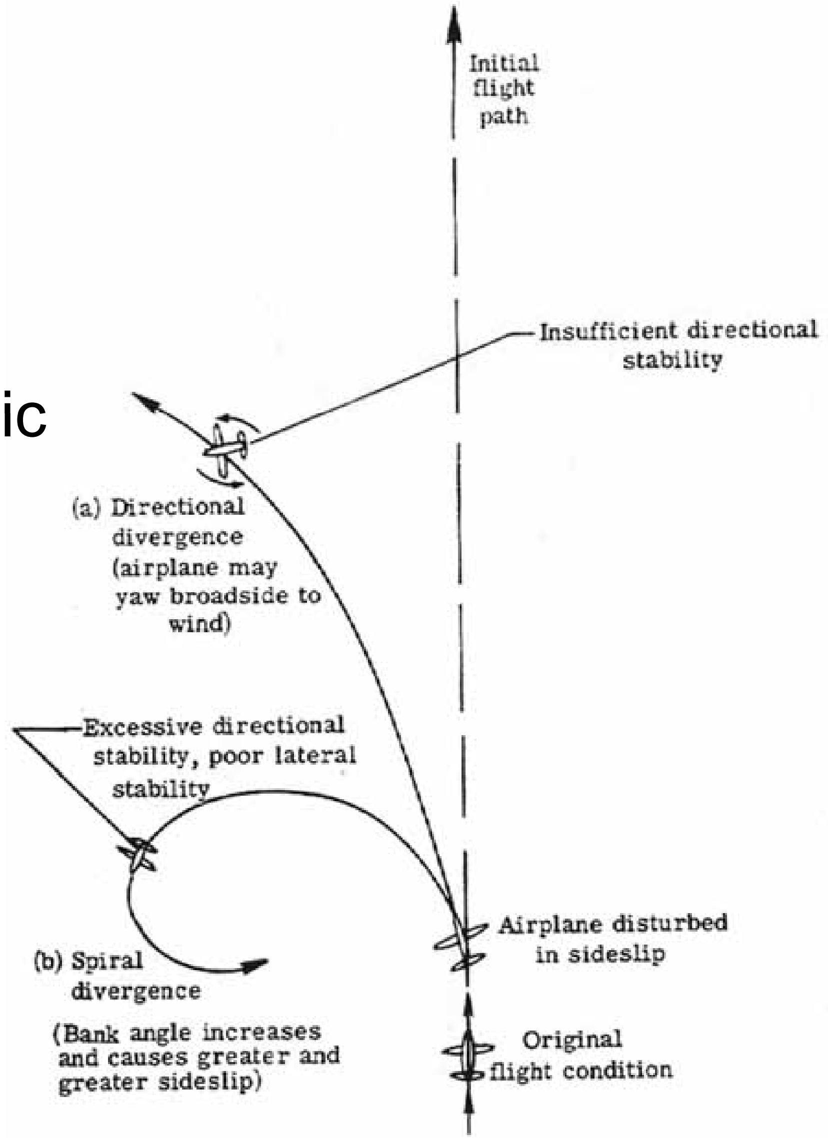
Directional and spiral divergence:

Aircraft has much directional static stability and small dihedral

Perturbation turns downward the left wing and turns left

Dihedral: left wing goes up

If dihedral is too small no time to recover horizontal position



3. Lateral dynamics

$$s^2 + 0.38s + 1.813 = 0$$

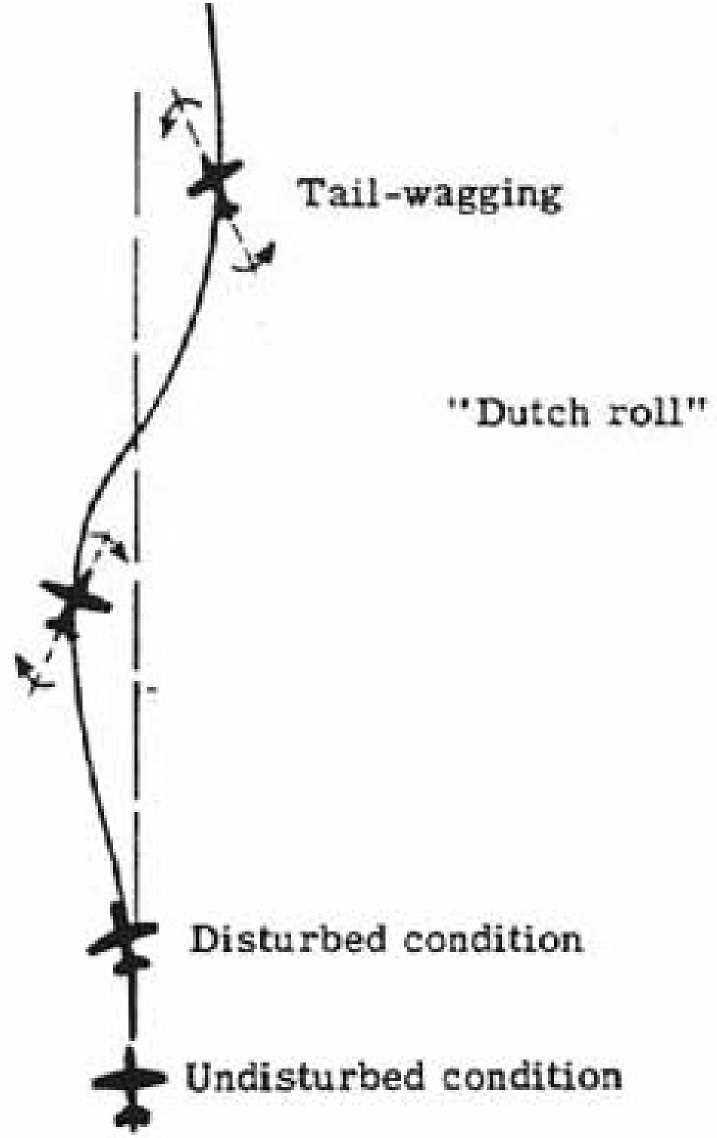
Dutch roll

characteristics of both divergences:

• strong lateral stability

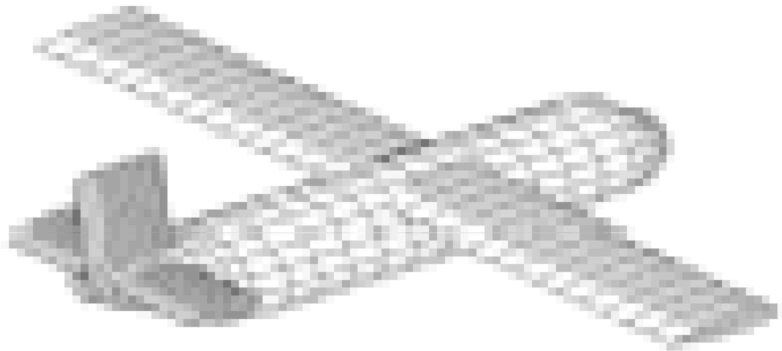
• low directional stability

Needs artificial damper if natural damper is too low (yaw damper)



3. Lateral dynamics

Dutch roll Mode



If slip occurs, airplane has a yaw movement in a given direction and a roll movement in the opposite direction

3. Lateral dynamics

β: slip angle (between relative wind and roll axis)

ψ: yaw angle (between roll axis at equilibrium and actual roll axis)

Φ: lateral inclination angle (between yaw axis at equilibrium and actual yaw axis)

3. Lateral dynamics:

Transfer functions for rudder variation

$$\frac{\Phi(s)}{\delta_r(s)} = \frac{0.485(s+1.53)(s-2.73)}{s(s^2+0.38s+1.813)(s+2.09)(s-0.004)}$$

$$\frac{\Psi(s)}{\delta_r(s)} = \frac{-1.38(s+2.07)(s^2+0.005s+0.066)}{s(s^2+0.38s+1.813)(s+2.09)(s-0.004)}$$

$$\frac{\beta(s)}{\delta_r(s)} = \frac{0.0364(s-0.01)(s+2.06)(s+37.75)}{s(s^2+0.38s+1.813)(s+2.09)(s-0.004)}$$

3. Lateral dynamics:

Transfer functions for aileron variation

$$\frac{\Phi(s)}{\delta_a(s)} = \frac{22.1(s^2 + 0.4s + 1.67)}{s(s^2 + 0.38s + 1.813)(s + 2.09)(s - 0.004)}$$

$$\frac{\Psi(s)}{\delta_a(s)} = \frac{-0.171(s - 1.14)(s + 9.29)(s + 1.45)}{s(s^2 + 0.38s + 1.813)(s + 2.09)(s - 0.004)}$$

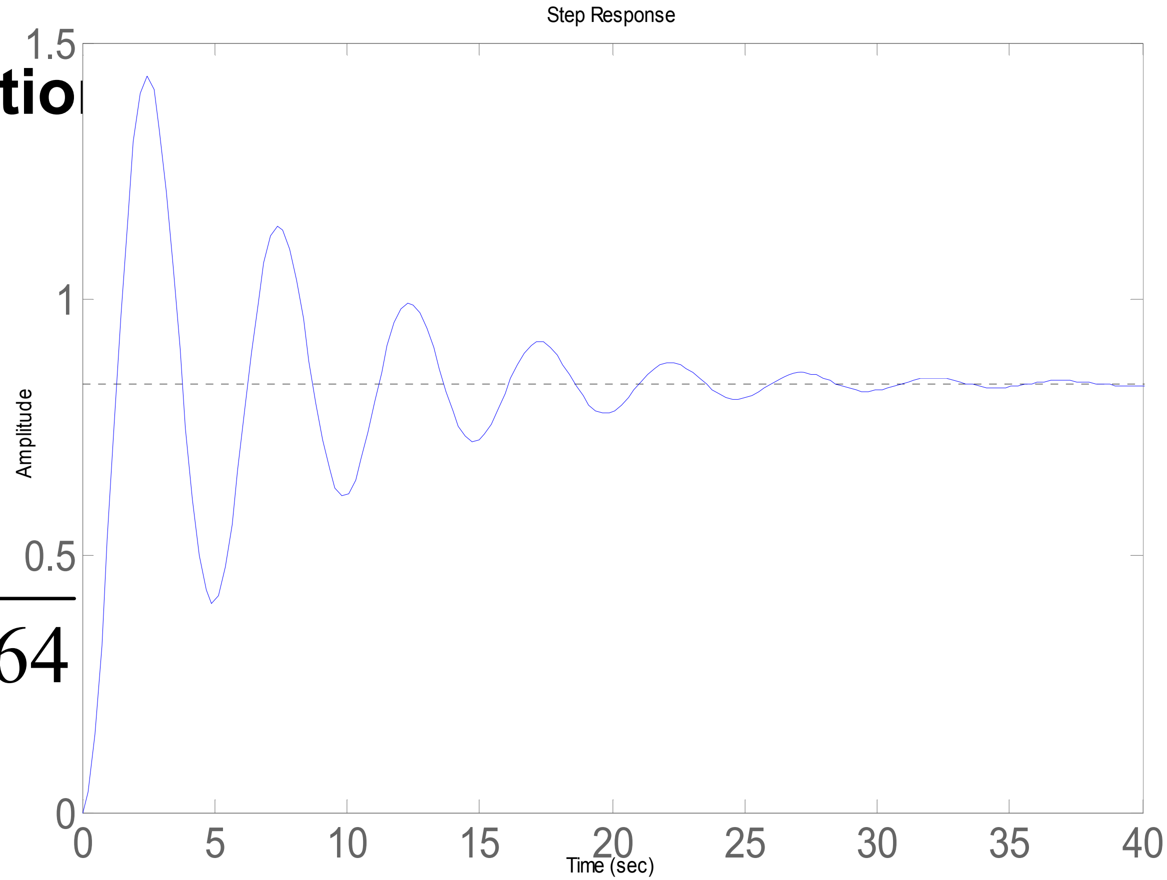
$$\frac{\beta(s)}{\delta_a(s)} = \frac{0.171(s + 18.75)(s + 0.15)}{s(s^2 + 0.38s + 1.813)(s + 2.09)(s - 0.004)}$$

3. Lateral dynamics

Dutch roll approximation

only slip and yaw:

$$\frac{\beta(s)}{\delta_r(s)} = \frac{1.37}{s^2 + 0.27s + 1.64}$$



4. Crossed coupling

= when a turn movement or a maneuver over an axis produces movement over a different axis

Under hypothesis of small perturbations: movement can be separated, the only coupling is lateral/directional:

- rudder movement → lateral turn

- elevator deflection → pitch only

4. Crossed coupling

With higher angles of attack,

- pitch can generate roll and yaw (and the opposite)

- roll maneuver → pitch and yaw (divergent)

→ pilot training

→ installation of roll speed limiters and mechanism that increases angular damping