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AN EFFICIENT NUMERICAL METHOD FOR SHAKEDOWN ANALYSIS

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Abstract. The algorithm proposed in [9] for incremental elastoplasticity is extended and applied to shakedown analysis. Using the three field mixed finite element proposed in [22] a series of mathematical programming problems or steps, obtained from the application of the proximal point algorithm to the static shakedown theorem, are obtained. Each step is solved by an Equality Constrained Sequential Quadratic Programming (EC-SQP) technique that allows a consistent linearization of the equations improving the computational efficiency.

1 INTRODUCTION

Directs methods are largely used for shakedown analysis of elastic-plastic structure under variable [1, 2] or cyclic loading [3] as an efficient alternative to time consuming incremental time-stepping calculations. In recent papers [4, 5], the finite element shakedown analysis has been effectively carried out by means of an iterative algorithm that in [6] has been shown to be obtained from a mathematical programming problem, consisting in the application of the proximal point algorithm to the static shakedown theorem and in the solution of this problem by means of dual decomposition methods. Each proximal point step define a pseudo elasto-plastic problem, coincident with that defined by using incremental iterative algorithms in the elastoplastic case [7, 8] while the optimization subproblems, deriving from decomposition techniques, exactly correspond to the standard return mapping by closest point projection scheme (CPP). The major advantage of the dual decomposition approach is that the inequality constraints arising from the constitutive laws are eliminated from the step equations using the CPP scheme at the local level (Gauss point or finite element) while the stresses and the plastic multipliers are implicitly defined in terms of the displacement. The finite step equations are so transformed into a nonlinear system of equations, without inequalities, easily solved by means of standard

arc–length strategies. The global description of the algorithm is always performed in terms of displacement variables alone.

The use of descriptions based on displacement variables alone can not be the best choice, potentially more efficient and robust analysis algorithms can be obtained by directly solving the proximal point step equations maintaining all the variables of the problems at the same level. In this work the algorithm already successfully used for incremental elastoplastic analyses in [9, 10], is applied to shakedown (see also [8]). Adopting the mathematical programming point of view an equality constraints sequential quadratic programming (EC-SQP) formulation is proposed to solve the problem. The algorithm is subdivided in two phases: i) a suitable estimate of the active constraints at the current iteration is performed employing the closest point projection scheme; ii) the solution of a quadratic programming that retains only the active constraints is performed. In this way the solution of each QP problem is easier than the general case and it also makes it possible to deal with very large dimension problems. In particular the solution of the QP subproblem can be performed after condensation of the locally defined quantities (stresses and plastic multipliers) so maintaining, at the global level of analysis, a pseudo compatible system with a smaller computational cost. The algorithm has then the same organization as standard strain driven elastoplastic algorithm based on closest point projection return mapping schemes and only a few modifications of the existing codes are required to implement the present proposal.

The proposed framework can gain a further improvement if the use of high performance finite elements is advocated, see for example [11, 12, 13, 14, 15] for applications in the linear and in the nonlinear field, [16, 17, 18, 19] for applications to inverse problems and [20, 21] for fracture related contexts. In the present case the finite elements used are of mixed type, see [22], and plastically enriched in order to work well both in the elastic and in the elastoplastic fields as mandatory for shakedown problems [5, 23]. They are based on a three field interpolation and are so well suited for the application of the proposed algorithm. The numerical results show how a great improvement in terms of robustness is achieved with respect to previous proposals.

2 A MATHEMATICAL PROGRAMMING APPROACH FOR SHAKEDOWN ANALYSIS

In the following, the static shakedown theorem is rewritten as a mathematical programming problem by introducing the finite element interpolation. The chosen FEM format is based on the general three field interpolation presented in [22] but any other kind of finite element could be used.

2.1 The shakedown equations in FEM format

We consider an elastoplastic body Ω subjected to bulk load $\boldsymbol{b}$ and tractions $\boldsymbol{f}$, that can varying in time inside a given load domain. The load domain will be amplified by a

load parameter λ.

Following [22] where more details can be found, we adopt a finite element formulation based on the interpolation of three fields: displacement $\boldsymbol{u}$, stress $\boldsymbol{\sigma}$ and plastic multiplier γ. These interpolations can be expressed as:

$$\boldsymbol{u} := \boldsymbol{N}\boldsymbol{d}_e \qquad \boldsymbol{\sigma} := \boldsymbol{S}\boldsymbol{\beta}_e \qquad \gamma := \boldsymbol{G}\boldsymbol{\kappa}_e, \tag{1}$$

where $\boldsymbol{N}$, $\boldsymbol{S}$ and $\boldsymbol{G}$ are the matrices containing the interpolation functions and $\boldsymbol{d}_e$, $\boldsymbol{\beta}_e$ and $\boldsymbol{\kappa}_e$ are the vectors collecting the finite element parameters. The interpolation functions $\boldsymbol{G}$ are assumed to be non–negativeness so allowing the condition $\gamma \geq 0$ to be easily expressed by making $\boldsymbol{\kappa}_e \geq \boldsymbol{0}$, where, from now on, vector inequality will be considered in a component-wise fashion. Moreover an important aspect regards the continuity order of the assumed interpolations, the displacement field has to be capable of assuring the inter-element continuity while $\boldsymbol{\sigma}$ and γ can be defined locally inside the element.

The interpolation allows to write the discrete form of the equilibrium equations as

$$\mathcal{A}_e \left\{ \boldsymbol{Q}_e^T \boldsymbol{\beta}_e - \lambda \boldsymbol{p}_e \right\} = 0 \tag{2}$$

$\mathcal{A}_e$ being the standard assembling operator which takes into account the inter–element continuity conditions on the displacement field and

$$\boldsymbol{Q}_e := \int_{\Omega_e} \boldsymbol{S}^T \boldsymbol{D} \boldsymbol{N} \quad , \quad \boldsymbol{p}_e := \int_{\Omega_e} \boldsymbol{N}^T \boldsymbol{b} + \int_{\partial\Omega_e} \boldsymbol{N}^T \boldsymbol{f} \tag{3}$$

are the element equilibrium matrix and load vector while $\boldsymbol{D}$ is the compatibility operator. For the sake of the following discussion eqs.(2) can be rewritten as

$$\boldsymbol{Q}^T \boldsymbol{\beta} - \lambda \boldsymbol{p} = \boldsymbol{0} \tag{4}$$

where $\boldsymbol{\beta}$, $\boldsymbol{d}$ and $\boldsymbol{p}$ denote the global vectors collecting all the stress parameters $\boldsymbol{\beta}_e$, the displacement parameters $\boldsymbol{d}_e$ and the applied loads $\boldsymbol{p}_e$, while $\boldsymbol{Q}^T$ the related global equilibrium matrix. From now on a subscript e denotes finite element representation of a global quantity.

2.2 The elastic envelope of the stresses

We assume that the external actions $\boldsymbol{p}$ are expressed as a combination of basic loads $\boldsymbol{p}_i$ belonging to the admissible closed and convex *load domain*

$$\mathbb{P} := \left\{ \boldsymbol{p} \equiv \sum_{i=1}^{p} a_i \boldsymbol{p}_i : \; a_i^{min} \leq a_i \leq a_i^{max} \right\} \tag{5}$$

Denoting with $\hat{\boldsymbol{\beta}}_i$ the elastic stress solution for $\boldsymbol{p}_i$, the *elastic envelope* $\hat{\mathbb{S}}$

$$\hat{\mathbb{S}} := \left\{ \hat{\boldsymbol{\beta}} \equiv \sum_{i=1}^{p} a_i \hat{\boldsymbol{\beta}}_i : \; a_i^{min} \leq a_i \leq a_i^{max} \right\} \tag{6}$$

defines the set of the elastic stresses $\hat{\boldsymbol{\beta}}$ produced by each load path contained in $\mathbb{P}$.

By construction $\hat{\mathbb{S}}$ and $\mathbb{P}$ are convex polytopes and each $\hat{\boldsymbol{\beta}} \in \hat{\mathbb{S}}$ can be expressed as a convex combination of the N_v elastic envelope vertexes $\hat{\boldsymbol{\beta}}^{\alpha}$ that can be usefully referred to the reference stress $\hat{\boldsymbol{\beta}}^{0}$ so obtaining:

$$\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^{0} + \sum_{\alpha=1}^{N_v} t^{\alpha} \hat{\boldsymbol{\beta}}^{\alpha} \qquad t^{\alpha} \geq 0 \qquad \sum_{\alpha=1}^{N_v} t^{\alpha} = 1 \tag{7}$$

If the external loads increase by a real number λ the elastic envelope becomes $\lambda\hat{\mathbb{S}} := \left\{ \lambda\hat{\boldsymbol{\beta}}_e : \ \hat{\boldsymbol{\beta}}_e \in \hat{\mathbb{S}} \right\}$. Note that the vertexes of the stress envelope could be a subset of the 2^p vertexes of $\mathbb{P}$.

2.3 The shakedown elastic domain

Following [12, 22, 9], the plastic admissibility condition is rewritten, in a weak form, as

$$\int_{\Omega} \delta\gamma\phi[\boldsymbol{\sigma}] \equiv \Delta\boldsymbol{\kappa}_e^T \boldsymbol{\phi}_e[\boldsymbol{\beta}_e] = 0 \quad \forall \delta\gamma \geq 0, \tag{8}$$

where $\boldsymbol{\phi}_e[\boldsymbol{\beta}_e] := \int_{\Omega_e} \boldsymbol{G}^T \phi[\boldsymbol{\beta}_e]$ and ϕ is the yield function.

Eq. (8) allows to control plastic admissibility in the N_e element so that $\boldsymbol{\beta}$ will be plastically admissible if

$$\boldsymbol{\phi}[\boldsymbol{\beta}] \leq \mathbf{0} \iff \boldsymbol{\phi}_e[\boldsymbol{\beta}_e] \leq \mathbf{0}, \ \forall e = 1 \dots N_e \, . \tag{9}$$

Finally it is useful to express the plastically admissible condition for all the stresses contained in the amplified elastic envelope $\lambda\hat{\mathbb{S}}$ translated by $\bar{\boldsymbol{\beta}}$. Due to the convexity of $\boldsymbol{\phi}_e$ and $\hat{\mathbb{S}}$ this can be easily expressed in terms of the plastic admissibility of all vertex stresses $\boldsymbol{\beta}^{\alpha} = \lambda(\hat{\boldsymbol{\beta}}^{\alpha} + \hat{\boldsymbol{\beta}}^{0}) + \bar{\boldsymbol{\beta}}$

$$\boldsymbol{\phi}[\lambda\hat{\boldsymbol{\beta}} + \bar{\boldsymbol{\beta}}] \leq \mathbf{0}, \ \forall \hat{\boldsymbol{\beta}} \in \hat{\mathbb{S}} \iff \boldsymbol{\phi}[\boldsymbol{\beta}^{\alpha}] \leq \mathbf{0}, \ \forall \alpha \tag{10}$$

From now on we denote with a Greek superscript vertex quantities.

2.4 The static theorem in discrete format and the mathematical programming point of view

The Bleich–Melan static theorem states that a load domain multiplier λ_s will be safe if there exists a time–independent self-equilibrated stress field $\bar{\boldsymbol{\beta}}$ so that each stress in $\lambda_s\hat{\mathbb{S}} + \{\bar{\boldsymbol{\beta}}\}$ is plastically admissible. The multiplier λ_a can be evaluated as the maximum of these safe multipliers recasting the static theorems in terms of total stress, instead of self–equilibrated ones, making possible a unified notation for shakedown and limit

analysis, i. e.

$$\begin{aligned} &\text{maximize} && \lambda_s \\ &\text{subject to} && \boldsymbol{Q}^T\boldsymbol{\beta} = \lambda_s \boldsymbol{p} \\ & && \boldsymbol{\phi}[\boldsymbol{\beta} + \lambda_s \hat{\boldsymbol{\beta}}^{\alpha}] \leq \mathbf{0}, \quad \alpha = 1 \cdots N_v \end{aligned} \tag{11}$$

with $\boldsymbol{p} \equiv \boldsymbol{Q}^T \hat{\boldsymbol{\beta}}^0$ and $\boldsymbol{\beta} \equiv \bar{\boldsymbol{\beta}} + \lambda_s \hat{\boldsymbol{\beta}}^0$. When $\hat{\boldsymbol{\beta}}^0 = \mathbf{0}$ we have the classic form in terms of the self–equilibrated stress. Furthermore, without any loss in generality, we can set $\hat{\boldsymbol{\beta}}^0$ as a generic vertex of $\hat{\mathbb{S}}$ so $\boldsymbol{\beta}$ becomes the total stress of this vertex. We assume $\hat{\boldsymbol{\beta}}^1 \equiv \mathbf{0}$ and in the follow we also put $\boldsymbol{\phi}^{\alpha}[\boldsymbol{\beta}, \lambda] \equiv \boldsymbol{\phi}[\boldsymbol{\beta} + \lambda_s \hat{\boldsymbol{\beta}}^{\alpha}]$. When the external load domain collapses in a single point ($a_i^{min} = a_i^{max}$) eqs.(11) directly transform into the standard form of the static theorem of limit analysis.

The actual solution strategy can be implemented on the basis of a proximal point method applied to (11) by definineg a sequences of subproblems or *steps* by adding a quadratic terms to the objective function of Eq. (11)

$$\begin{aligned} &\text{maximize} && \Delta\xi^{(n)}\lambda^{(n)} - \frac{1}{2}\Delta\boldsymbol{\beta}^T \boldsymbol{H} \Delta\boldsymbol{\beta} \\ &\text{subject to} && \boldsymbol{Q}^T\boldsymbol{\beta}^{(n)} - \lambda^{(n)}\boldsymbol{p} = \mathbf{0} \\ & && \boldsymbol{\Phi}[\boldsymbol{\beta}^{(n)}, \lambda^{(n)}] \leq \mathbf{0}, \end{aligned} \tag{12}$$

where the superscript $(\cdot)^{(n)}$ will denote quantities evaluated in the nth step, the symbol $\Delta(\cdot) = (\cdot)^{(n)} - (\cdot)^{(n-1)}$ is the increment of a quantity from the previous step and $\Delta\xi^{(k)} > 0$ is a real positive number and to simplify the notation we collect all $\boldsymbol{\phi}^{\alpha}[\boldsymbol{\beta}, \lambda]$ in the global vector $\boldsymbol{\Phi}[\boldsymbol{\beta}, \lambda] = \{\boldsymbol{\phi}^1, \cdots, \boldsymbol{\phi}^{N_v}\}$ while $\boldsymbol{\Phi}_e[\boldsymbol{\beta}, \lambda]$ is its counterpart at the element level.

$\boldsymbol{H}$ is the compliance matrix and is defined by the following equivalence

$$\sum_e \boldsymbol{\beta}_e^T \boldsymbol{H}_e \boldsymbol{\beta}_e = \boldsymbol{\beta}^T \boldsymbol{H} \boldsymbol{\beta} \quad \text{with} \quad \boldsymbol{H}_e := \int_{\Omega_e} \boldsymbol{S}^T \boldsymbol{\mathcal{C}}^{-1} \boldsymbol{S} \tag{13}$$

Note that, due to the local nature of the stress interpolation $\boldsymbol{H}$ has a block diagonal structure that couples only the local finite element stress variables.

2.5 The pseudo elastoplastic step equations

Introducing the dual multiplier $\Delta\boldsymbol{d}$ and $\Delta\boldsymbol{\kappa}$ associated to the equalities and inequalities constraints respectively, each finite step is defined by the first order conditions of (12). In the following we subdivide the step equations in the *local equations*, that is those defined on the element, and in the *global equations*, that is those coming from the assemblage of the contributions of all the elements. In the case of limit analysis these equations exactly corresponds to a step of an arc-length algorithm used to solve the incremental elastoplastic problem [4, 5]. For this reason it will be called the *pseudo elastoplastic step*.

In order to simplify the notation the superscript (n) will be omitted.

2.5.1 First order conditions

From the stationary condition of the Lagrangian of (12) with respect to $\boldsymbol{\beta}$ and $\Delta\boldsymbol{\kappa}$ we obtain the plastic admissibility and plastic consistence conditions

$$\begin{cases} \boldsymbol{r}_\sigma \equiv \boldsymbol{H}_e \Delta\boldsymbol{\beta}_e - \boldsymbol{Q}_e \Delta\boldsymbol{d}_e + \boldsymbol{A}_e[\boldsymbol{\beta}_e, \lambda]\Delta\boldsymbol{\kappa}_e = \boldsymbol{0} \\ \boldsymbol{r}_\mu \equiv \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e, \lambda] \leq 0, \quad \Delta\boldsymbol{\kappa}_e \geq 0, \quad \Delta{\boldsymbol{\kappa}_e}^T \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e, \lambda] = 0, \end{cases} \tag{14a}$$

where $\boldsymbol{A}_e[\boldsymbol{\beta}_e, \lambda] := \left(\frac{\partial \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e,\lambda]}{\partial \boldsymbol{\beta}_e}\right)^T$. Eq. (14a) are expressed in terms of quantities defined only at the element level, for this reason they constitutes the *local equations* of the problem.

In the same fashion the stationary condition with respect to $\Delta\boldsymbol{d}$ and λ furnish the equilibrium equations and the normalization condition, coupling all the variables of the problem and define the *global level* of the analysis,

$$\begin{cases} \boldsymbol{r}_u \equiv \boldsymbol{Q}^T\boldsymbol{\beta} - \lambda\boldsymbol{p} = \boldsymbol{0} \\ r_\lambda \equiv \Delta\xi - \Delta\boldsymbol{d}^T\boldsymbol{p} - \Delta\boldsymbol{\kappa}^T\boldsymbol{\Phi}_{,\lambda} = 0 \end{cases} \tag{14b}$$

where $\boldsymbol{\Phi}_{,\lambda} := \left(\frac{\partial \boldsymbol{\Phi}_\beta[\boldsymbol{\beta},\lambda]}{\partial\lambda}\right)$ has zero first components due to the choice of $\hat{\boldsymbol{\beta}}^0 = \boldsymbol{0}$.

Eqs. (14) for $\boldsymbol{H}\Delta\boldsymbol{\beta} = \boldsymbol{0}$, apart from the inessential scaling for $\Delta\xi^{(k)}$, are the primal–dual conditions of the shakedown theorems. We refer to [6, 9] to prove this occurrence at convergence.

3 A NEW SOLUTION SCHEME FOR THE PSEUDO-ELASTOPLASTIC STEP

We will present now an application of the SQP method to solve Eq. (14). The algorithm exploits the problem structure allowing its solution at the global level by means of a Newton (Riks) scheme which is characterized by slight differences with respect to standard SD-CPP formulations.

3.1 The linearized equations for the elastoplastic step and the sequential quadratic programming (SQP) formulation

The estimate of the unknowns relative to the new step, $\boldsymbol{z}^{(n)} = \{\lambda^{(n)}, \boldsymbol{\beta}^{(n)}, \boldsymbol{d}^{(n)}, \boldsymbol{\kappa}^{(n)}\}$, will be denoted by $\boldsymbol{z}^{j+1} = \boldsymbol{z}^j + \dot{\boldsymbol{z}}$ where, in order to make the notation simpler, the superscript relative to the step number has been dropped leaving only the indication for the current j–th iteration. The starting point for the new algorithm is the linearization of the finite step equation (14) which yields for the local equations (14a)

$$\begin{cases} -\boldsymbol{H}_{et}\dot{\boldsymbol{\beta}}_e + \boldsymbol{Q}_e\dot{\boldsymbol{d}}_e - \boldsymbol{A}_e^j\dot{\boldsymbol{\kappa}}_e - \dot{\lambda}\boldsymbol{a}_\lambda^j = -\boldsymbol{r}_\sigma^j, \\ {\boldsymbol{\Phi}_e}^{j+1} \leq \boldsymbol{0}\ ,\ {\boldsymbol{\kappa}_e}^{j+1} \geq \boldsymbol{0}\ ,\ ({\boldsymbol{\kappa}_e}^{j+1})^T{\boldsymbol{\Phi}_e}^{j+1} = 0. \end{cases} \quad \forall e \tag{15a}$$

where $\boldsymbol{a}_\lambda^j = \boldsymbol{A}_e{}_{,\lambda}^{\,j}\,\Delta\boldsymbol{\kappa}_e{}^j$ and

$$\boldsymbol{H}_{et} \equiv \boldsymbol{H}_e + \sum_k \Delta\kappa_{ek}\frac{\partial\Phi_{ek}}{\partial\boldsymbol{\beta}_e}\bigg|_{\boldsymbol{\beta}_e=\boldsymbol{\beta}_e^j} \quad , \quad \boldsymbol{\Phi}_e{}^{j+1} \equiv \boldsymbol{\Phi}_e{}^j + \boldsymbol{A}_e^{jT}\dot{\boldsymbol{\beta}}_e + \boldsymbol{\Phi}_e{}^j{}_{,\lambda}\,\dot{\boldsymbol{\beta}}_e.$$

κ_{ek}^j and Φ_{ek} are the kth components of $\Delta\boldsymbol{\kappa}_e^j$ and $\boldsymbol{\Phi}_e{}^j$ respectively, Moreover, the linearization of the global finite step equations (4) gives:

$$\begin{cases} \boldsymbol{Q}^T\dot{\boldsymbol{\beta}} - \dot{\lambda}\boldsymbol{p} = -\boldsymbol{r}_u^j \\ \dot{\boldsymbol{d}}^T\boldsymbol{p} + \dot{\boldsymbol{\kappa}}^T\boldsymbol{\Phi}_{,\lambda}^{\,j} + \boldsymbol{a}_\lambda^{jT}\dot{\boldsymbol{\beta}} + a_{\lambda\lambda}^j\dot{\lambda} = r_\lambda^j \end{cases} \tag{15b}$$

where $a_{\lambda\lambda}^j = \Delta\boldsymbol{\kappa}^T\boldsymbol{\Phi}_{,\lambda\lambda}$.

Eq. (15) could also be obtained by applying a sequential quadratic programming (SQP) approach to (14). However the solution of a QP sub-problems with a standard SQP algorithm requires a great computational effort due to the coupling action exerted by the equilibrium constraints. A method to efficiently solve Eq. (15) is that proposed in [9] and briefly described in the following section.

3.2 The EC-SQP formulation

In the present proposal the SQP equations in (15) are solved by using an equality constraint sequential quadratic programming (EC-SQP) approach [9]. Each iteration of the EC-SQP approach consists of two phases: i) estimation of the active set of constraints; ii) solution of an equality constrained quadratic program that imposes the apparently active constraints and ignores the apparently inactive ones. The idea is to identify the active constraints for the actual estimate of the solution using information available at a point near to $\mathbf{z}^{j+1}$, a point which in the sequel will be denoted by $\bar{\mathbf{z}}^{j+1}$.

3.2.1 The detection of the active set of constraints

The estimation of the active constraints is performed by advocating the decomposition point of view, i. e. solving an optimization problem obtained by the original ones (15a) for a fixed, properly assumed, value of the global variables: $\bar{\boldsymbol{d}}^{j+1} = \boldsymbol{d}^j$ and $\bar{\lambda}^{j+1} = \lambda^j$. The series of decoupled problems obtained in this way have the same form as a standard CPP scheme and it can be easily solved at the local element level in a way as efficient as, or also more, than the standard SD-CPP approach.

At the iteration $j+1$ then the active set of constraints is obtained by solving (15a) assuming $\dot{\boldsymbol{d}} = \boldsymbol{0}$ and $\dot{\lambda} = \boldsymbol{0}$. The result is a problem that is now decoupled at the local level, i. e.

$$\begin{cases} -\boldsymbol{H}_{et}\dot{\bar{\boldsymbol{\beta}}}_e - \boldsymbol{A}_e^j\dot{\bar{\boldsymbol{\kappa}}}_e = -\boldsymbol{r}_\sigma^j, \\ \bar{\boldsymbol{\Phi}}_e{}^{j+1} \le \boldsymbol{0}\ ,\ \bar{\boldsymbol{\kappa}}_e{}^{j+1} \ge \boldsymbol{0}\ ,\ (\bar{\boldsymbol{\kappa}}_e{}^{j+1})^T\bar{\boldsymbol{\Phi}}_e{}^{j+1} = 0. \end{cases} \quad \forall e \tag{16}$$

where the symbols with a bar denote the estimates of the new quantities. In particular Eqs.(16) are the first order conditions of the following QP problem:

$$\begin{cases} \min\limits_{(\dot{\boldsymbol{\beta}}_e)} : \dfrac{1}{2}(\dot{\bar{\boldsymbol{\beta}}}_e)^T \boldsymbol{H}_{et} \dot{\bar{\boldsymbol{\beta}}}_e + (\dot{\bar{\boldsymbol{\beta}}}_e)^T \mathbf{g}^j, \\ \text{subj.: } \boldsymbol{A}_e^{jT} \dot{\bar{\boldsymbol{\beta}}}_e + \boldsymbol{\Phi}_e^j \leq \mathbf{0}, \end{cases} \quad \forall e \tag{17}$$

where $\mathbf{g}^j = \boldsymbol{H}_e(\bar{\boldsymbol{\beta}}_e^{\,j} - \boldsymbol{\beta}_e^*)$ and $\boldsymbol{\beta}_e^* = \boldsymbol{\beta}_e^{(n-1)} + \boldsymbol{H}_e^{-1}\boldsymbol{Q}_e \Delta \boldsymbol{d}_e^j$. The decoupled QP problems (17) can be efficiently solved at local level, by using the Goldfarb-Idnani active set method [5]. The evaluation of the set of active constraints is then continuously updated with the iterations and if $\Delta \boldsymbol{d}^j$ converges to $\Delta \boldsymbol{d}^{(n+1)}$ the active set converges to that of the nonlinear problem.

3.2.2 The solution of the QP equality constraint scheme

After the detection of the set of active constraints, and assuming that this set is not void, we have to solve Eqs. (15) by means of the following system of equations in which only the residuals of the active constraints are considered:

$$\begin{bmatrix} \cdot & \boldsymbol{A}_e^{jT} & \cdot & \boldsymbol{\Phi}_{e,\lambda}^j \\ -\boldsymbol{A}_e^j & -\boldsymbol{H}_{et} & \boldsymbol{Q}_e & -\boldsymbol{a}_\lambda^j \\ \cdot & \boldsymbol{Q}_e^T & \cdot & -\boldsymbol{p}_e \\ -(\boldsymbol{\Phi}_{e,\lambda}^j)^T & -(\boldsymbol{a}_\lambda^j)^T & -\boldsymbol{p}_e^T & -a_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\kappa}}_e \\ \dot{\boldsymbol{\beta}}_e \\ \dot{\boldsymbol{d}}_e \\ \dot{\lambda} \end{bmatrix} = - \begin{bmatrix} \boldsymbol{r}_\mu^j \\ \boldsymbol{r}_\sigma^j \\ \boldsymbol{r}_u^j \\ r_\lambda^j \end{bmatrix} \quad , \quad \boldsymbol{z}^{j+1} = \boldsymbol{z}^j + \dot{\boldsymbol{z}}, \tag{18}$$

where the further condition $\boldsymbol{\kappa}_{j+1} \geq 0$ needs to be imposed.

System (18) is easily solved by static condensation of the local defined quantities. In particular, recalling that the QP scheme in (17) solves the first two equations of (18) zeroing the global variables we obtain

$$\begin{cases} \dot{\boldsymbol{\beta}}_e = \dot{\bar{\boldsymbol{\beta}}}_e + \boldsymbol{H}_{et}^{-1}(\boldsymbol{Q}_e \dot{\boldsymbol{d}}_e - \dot{\lambda} \boldsymbol{a}_\lambda^j) \\ \dot{\boldsymbol{\kappa}}_e = \dot{\bar{\boldsymbol{\kappa}}}_e + \boldsymbol{W}\boldsymbol{A}_j^T \boldsymbol{H}_{et}^{-1}(\boldsymbol{Q}_e \dot{\boldsymbol{d}}_e - \dot{\lambda} \boldsymbol{a}_\lambda^j), \end{cases} \tag{19}$$

where $\boldsymbol{W} = \left[\boldsymbol{A}_j^T \boldsymbol{H}_{et}^{-1} \boldsymbol{A}_j\right]^{-1}$. At the global level then we have to assemble the condensed element contribution as

$$\mathcal{A}_e \left(\boldsymbol{Q}_e^T \boldsymbol{E}_t \boldsymbol{Q}_e\right) \dot{\boldsymbol{d}} - \dot{\lambda} \boldsymbol{y}^j = -\mathcal{A}_e \left(\tilde{\boldsymbol{r}}_u^j\right) \quad , \quad -\boldsymbol{p}^T \dot{\boldsymbol{d}}_e - h_t \dot{\lambda} = -\tilde{r}_\lambda^j, \tag{20}$$

where $\boldsymbol{y}^j = \boldsymbol{p} + \mathcal{A}_e \left(\boldsymbol{Q}_e^T \boldsymbol{E}_t \boldsymbol{a}_\lambda^j\right)$ and

$$\tilde{\boldsymbol{r}}_u^j = \boldsymbol{r}_u^j + \boldsymbol{Q}_e^T(\boldsymbol{E}_t \boldsymbol{r}_\sigma^j - \boldsymbol{H}_{et}^{-1} \boldsymbol{A}_j \boldsymbol{W} \boldsymbol{r}_\mu^j) \quad \text{and} \quad \boldsymbol{E}_t = \boldsymbol{H}_{et}^{-1} - \boldsymbol{H}_{et}^{-1} \boldsymbol{A}_j \boldsymbol{W} \boldsymbol{A}_j^T \boldsymbol{H}_{et}^{-1}.$$

$\boldsymbol{E}_t$ has the same expression as the algorithmic tangent matrix evaluated by standard SD-CPP formulation and h_t is obtained from the Gauss elimination process.

System (20) is coincident with the iteration scheme proposed in [6] except for the new definition of quantities $\tilde{\boldsymbol{r}}_u^j$. Note that $\boldsymbol{E}_t$ and $\boldsymbol{H}_{et}^{-1}\boldsymbol{A}_j\boldsymbol{W}$ are evaluated at each step of the QP problem, by the optimization algorithm used, so only the evaluation of $\tilde{\boldsymbol{r}}_u^j$ and $\boldsymbol{u}^j$ is required. In the case of an element with zero active constraints the solution is obtained from previous scheme by deleting the first row and column from system (18).

4 NUMERICAL RESULTS

To evaluate the performance of the proposed algorithm, several numerical tests have been performed by analyzing 2D problems in plane stress/strain conditions and for different material. For each test a series of equilibrium paths at increasing values of the first arc–length parameter are evaluated using the strategies better described in [9, 24, 5]).

The results are compared with the interior point algorithm IP-M proposed in [6, 23] and with the algorithm in [6] (SD-CPP). In order to highlight the efficiency and the robustness of each nonlinear strategy the following indicators are compared: *(I)* the number of points furnished by the Riks analysis strategy in the evaluation of the equilibrium path (stps), the number of false steps (when convergence is not reached within a maximum number of loops) are reported in brackets; *(II)* the total number of the iterations required for each step to converge (lps); *(III)* the total cpu time (CPU).

The finite elements used in the numerical tests are those proposed in [12, 22]. They are four-node elements with a bi-linear interpolation of the displacement field and a 5-parameter stress field interpolation and a piecewise-constant interpolation over some sub-areas into which the internal area of the element is divided: one for the FC_1 element and four for the FC_4 element. A comparison with a compatible four node element with a bilinear interpolation for the displacements and 4 Gauss integration points, denoted by Q_4, is also presented.

The only test presented here regards a classical limit analysis problem for which the geometry, the material, the applied loads and one of the meshes used in the analyses are shown in Fig. 1. Plane stress condition is assumed and the collapse load multiplier is compared with the value $\lambda_c = 0.8006$ obtained by several authors [5, 1].

Tab. 1 reports the results obtained with the three nonlinear algorithms for different initial step size and the three finite elements. As can be observed the robustness of the algorithm is good showing a smooth decrease in the number of required steps and iterations according to the assigned step size. For the greatest initial step size it performs the evaluation of the collapse state with a single step and without any loss in accuracy. On the contrary the standard SD-CPP algorithm is adversely affected by the increase in the step size registering occurrences of step failure already from the second size of the first step increment. The IP-M confirms its robustness but also its unsuitability when used for small step sizes.

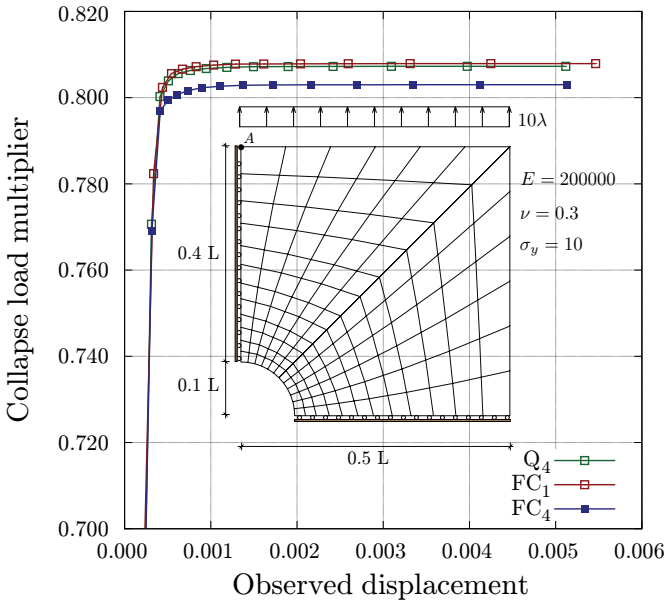

Figure 1: Plate with a circular hole.

5 CONCLUSIONS

In this paper the method presented in [9] for the incremental elastoplastic analysis has been extended to shakedown. The method is based on a proximal point approximation of the Melan static theorem that retains, at each iteration, all the variables of the problems. In the solution process, based on the equality constrained approach, the set of active constraints is obtained by solving a simple quadratic programming problem which has the same structure and variables of a standard return mapping by closest point projection scheme, i.e. it is decoupled and it can be solved at a local level. The solution of the equality constraint problems is performed by means of a static condensation of the locally defined variables, stress and plastic multiplier parameters, for which the inter element continuity is not required so obtaining, at the global level, a pseudo-compatible scheme of analysis that has the same structure as classic path following arc-length methods.

The numerical results are performed adopting the finite element interpolation proposed in [22]. This finite element uses a three field interpolation with a good accuracy with respect to both the elastic and elastoplastic response. This makes the proposed numerical framework particularly suitable for shakedown analysis. The numerical results show a great improvement in robustness and efficiency with respect to previous proposals.

The presentation and the application are limited to the perfect plasticity case but its extension to other more complex associated cases is simple [2].

	Q_4			FC_1			FC_4			
	stps	lps	CPU	stps	lps	CPU	stps	lps	CPU	incr.
SD-CPP	15 (-)	54	2.23	15 (-)	54	1.63 (1.16)	16 (-)	68	2.69	5e-5
	9 (1)	40	2.55	12 (2)	46	1.81	15 (3)	63	4.28	5e-4
	11 (5)	45	4.19	11 (5)	43	2.60	14 (6)	60	8.26	5e-3
	10 (8)	41	6.04	10 (8)	42	3.67	16 (10)	66	12.35	5e-2
EC-SQP	15 (-)	57	1.46	15 (-)	55	1.17	16 (-)	71	1.56	5e-5
	7 (-)	36	0.93	7 (-)	36	0.77	9 (-)	78	1.72	5e-4
	2 (-)	21	0.56	2 (-)	22	0.49	2 (-)	48	1.07	5e-3
	1 (-)	19	0.51	1 (-)	19	0.42	1 (-)	38	0.85	5e-2
IP-M				20 (-)	118	2.58	20 (-)	114	2.46	5e-5
				8 (-)	56	1.29	8 (-)	62	1.35	5e-4
				2 (1)	32	1.95	2 (-)	32	1.03	5e-3
				1 (1)	28	1.80	1 (-)	26	0.87	5e-2

Table 1: Plate with circular hole, mesh 12×12 (338 dofs). Analysis report, $v_{A,max} = 5e-3$, $toll = 1e-4$, $desired = 12$, $max = 50$. Computed collapse multiplier and relative error: $\lambda_c = 0.8073\,(0.84\%)$ Q_4; $\lambda_c = 0.8079\,(0.91\%)$ FC_1; $\lambda_c = 0.8030\,(0.30\%)$ FC_4.

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