AN ANALYTICAL MODEL FOR
ELASTO-PLASTIC BUCKLING OF COLUMNS

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Abstract. The theory of buckling strength of compression members in the plastic range has been extensively studied, and numerical methods already exist which deal with such behaviour. However, there is significant research interest in developing analytical models for elasto-plastic buckling, largely driven by the need for simplified mechanics-based design tools, but also by the desire for enhanced understanding of this complex phenomenon. This paper is intended to illustrate the mechanics of the elasto-plastic buckling response of stocky columns by means of a simplified analytical model, starting from the point of buckling initiation and considering the post-buckling response. In this model, the Rotational Spring Analogy is used for formulating the geometric stiffness matrix, whereas the material stiffness matrix is obtained with due consideration for the spread of material plasticity. In addition to establishing some key features of elasto-plastic buckling, the imperfection sensitivity in the plastic range is also studied and as a result a threshold level of imperfection is identified.

1 INTRODUCTION

For more than a century, there has been a significant amount of research conducted on the elasto-plastic buckling analysis of structures, involving experimental, analytical and numerical methods. However, regardless of the major research topics and numerous investigations in this field, buckling of metal structures continues to attract research interest.

Inelastic buckling of columns was first investigated by Engesser in 1889 who proposed the use of a tangent-modulus (E_t) as an effective modulus in the Euler formula. However, at the same time Considère believed that the column buckling response is positively affected by unloading on the column convex side and that the value of effective modulus must be between (E) and (E_t). Almost two decades later von Karman suggested his well-known double-modulus (E_R) theory founded on Engesser's theory and Considère's idea. However, the extensive tests carried out showed that von Karman's reduced-modulus theory resulted in

considerably higher buckling stresses. Consequently in 1947 Shanley [1] stated that the tangent-modulus (E_t) is the correct effective modulus to be employed for buckling beyond the proportional limit, and that the unloading of one side of the column does not occur at buckling. Soon after, Duberg and Wilder [2] carried out a theoretical study on column behaviour in the plastic range allowing for initial imperfections and employing a realistic material model, where they also concluded that tangent-modulus load is the critical buckling load of the column (i.e. the load at which the column starts to bend).

On this topic, the first author [3] verified both theories analytically using the Rotational Spring Analogy [4] as well as numerically by means of a nonlinear analysis program ADAPTIC [5], where the influence of material nonlinearity was also taken into account. It was noted that the double-modulus theory consistently leads to an upper-bound solution whereas the tangent-modulus theory offers a lower-bound prediction of the true buckling stress.

Because of the highly nonlinear nature of elasto-plastic buckling, including both geometric and material nonlinearity, models based on the finite element method (FEM) are often used to either simulate or to verify the design of inelastic structures. While FEM is considered as the most powerful tool available for the solution of physical problems, it is still considered to be overly demanding for practical application. On the other hand, simplified analytical methods have the potential to yield enormous benefits in design, including ease of use, reasonable estimates of the buckling load, and importantly providing a better understanding of the main parameters influencing the elasto-plastic buckling response based on cause and effect.

The main objective of this paper is to develop a simplified analytical model for the elasto-plastic buckling of columns that would be much more practical for design application than nonlinear finite element analysis, while maintaining good levels of accuracy.

2 FUNDAMENTALS OF THE DEVELOPED APPAROACH

In this section, the background to the developed analytical model is briefly described for a perfectly straight column. The first part deals with formulating an equivalent modulus reflecting the material stiffness by allowing for the spread of material plasticity over the cross-section depth and along the length of column. The second part discusses the use of Rotational Spring Analogy (RSA) for formulating the geometric stiffness, while the last part establishes the buckling and post-buckling response.

2.1 Initially perfect column

Consider a simply-supported column subject to an axial compressive load at the ends, as shown in Figure 1. A bilinear elasto-plastic material model is considered, which is defined in terms of an elastic modulus E, yield strength σ_y and a strain-hardening parameter μ (Figure 2).
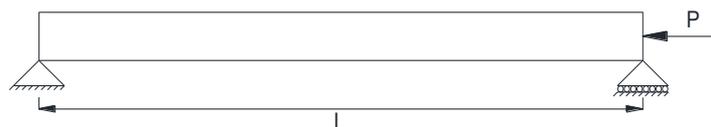

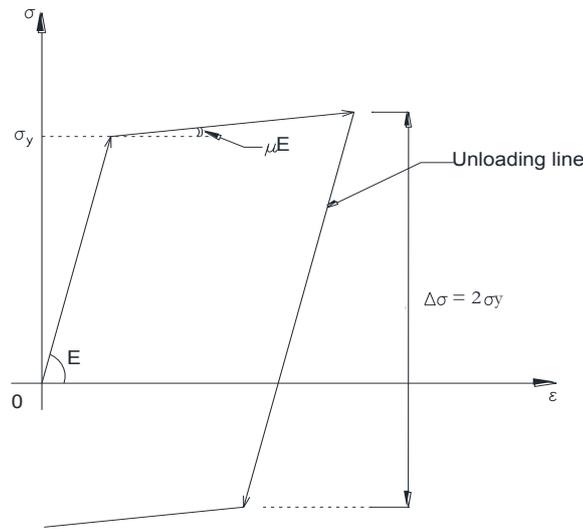**Figure 1:** Column under compressive load

Figure 2: Bilinear material model

The dimensions of column are chosen to provide a considerably low slenderness ratio, such that the stress corresponding to the tangent-modulus Engesser buckling load (P_t) exceeds the yield strength. Under a monotonically increasing load, it was acknowledged by Shanley's work that the column starts to buckle (i.e. deflect laterally) as the Engesser buckling load (P_t) is approached. Therefore, in order to capture the post-buckling response of a stocky column, the initial loading is assumed to be P_t at which the column is still straight. At the onset of buckling the stress distribution along the column is uniform and is equal to:

$$\sigma_t = \frac{P_t}{A} \text{ (in compression)} \tag{3}$$

At this point no unloading and deflection have yet occurred. However, once the Engesser load is reached, the load continues to increase and produces strain reversal on the convex side due to bending. As a result, the top fibre compressive stresses of the cross-section will start to decrease within an "x_e" distance from the mid-length of the column, as illustrated in Figure 3, where due to symmetry only half of the column is considered. In this zone, assuming a linear strain distribution over the cross-section, a von Karman reduced-modulus (E_R) is obtained as a function of depth of the unloading zone (y_n), while for the remaining part of length $L/2-x_e$ the tangent-modulus (E_t) is used. In this way, the spread of plasticity is accounted for over the cross-section depth and along the member length.

For an assumed buckling mode (w), in order to find the depth of unloading over the cross-section at various "x_e" in the range 0 to $L/2$, infinitesimal axial equilibrium is considered. Once y_n is found, the equivalent modulus ($\bar{E}$) over the cross-section along the column is obtained and so is the tangent material stiffness.
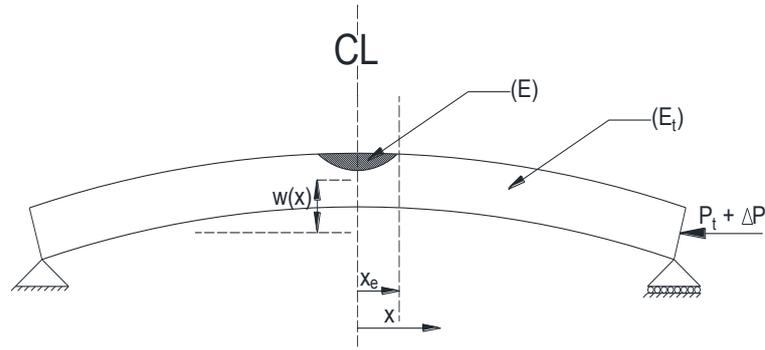
Figure 3: Deflected column

2.2 Buckling analysis using Rotational Spring Analogy

To predict the instantaneous elasto-plastic buckling load (P_c) of the column, the Rotational Spring Analogy [4] is employed. In this approach, when the trajectory of an axial force due to a buckling mode continuously varies over the structural domain, the geometric stiffness $[K_G]$ can be obtained as the integration of contributions from distributed equivalent rotational springs [6]. Together with the material stiffness $[K_E]$, the instantaneous buckling load is determined by solving a linear eigenvalue problem requiring the tangent stiffness matrix $[K_T] = [K_E] + [K_G]$ to singular. In this case, a single buckling mode w(x) is assumed, leading to the following instantaneous buckling load:

$$P_c = -\frac{k_e}{k_g} = \frac{\int_0^{\frac{L}{2}} \frac{\overline{EI}w''(x)^2}{w_0^2}dx}{\int_0^{\frac{L}{2}} \frac{w'(x)^2}{w_0^2}dx} = \frac{\int_0^{x_e} \frac{\overline{EI}w''(x)^2}{w_0^2}dx + \int_{x_e}^{\frac{L}{2}} \frac{E_t I w''(x)^2}{w_0^2}dx}{\int_0^{\frac{L}{2}} \frac{w'(x)^2}{w_0^2}dx} \tag{2}$$

It can be noticed that during buckling, the effective k_g associated with a unit axial load remains unchanged, while k_e varies due to elastic unloading.

So far, moment equilibrium as defined by:

$$M = F.w \tag{3}$$

has not been considered. This is addressed with the assumed mode by establishing incremental moment equilibrium, leading to the determination of the applied load P for a specific amplitude of the assumed deformation mode as follows:

$$P = P_c - w(\frac{\delta P}{\delta w}) \tag{4}$$

At each incremental step, equation (4) is used to trace the load-deflection curve, where the instantaneous P_c is obtained from equation (2). To interpret and compare the findings, results are obtained for an initially straight column with the following properties (length=0.6m, width=0.1m and depth=0.2m). The results are provided in Table 1 and Figure 4, where the column is assumed to reach its maximum elasto-plastic buckling resistance when tensile

yielding occurs at the extreme top fibre of the cross-section at x=0, defined with reference to Figure 2 by:

$$\Delta\sigma_{extreme\ top\ fibre} = 2\sigma_y \tag{5}$$

Figure 4a clearly demonstrates the initiation of the elasto-plastic buckling at the Engesser tangent-modulus load (P_t) and its subsequent increase towards the von Karman double-modulus load (P_R). However, due to tensile yielding at the outer fibre of column cross-section, the von Karman upper limit can is not realised (Figure 4b), and hence the maximum buckling load is attained within the range ($P_t < P_{max} < P_R$).

Table 1: An illustration of results from the analytical model

x_e(m)	w(m)	P(N)	P_c(N)	$\sigma_{extreme\ top\ fibre}$ (N/m^2)	$\Delta\sigma$(N/m^2)
0	0	7676359	7676359	3.84E+08	-5.00E-02
0.03	8.02E-06	7678190	7680014	3.84E+08	3.61E+04
0.06	0.000172	7714638	7752304	3.81E+08	2.49E+06
0.09	0.000894	7867179	8050646	3.60E+08	2.35E+07
0.12	0.002745	8227867	8739248	2.77E+08	1.07E+08
0.15	0.006506	8881062	9940469	4.67E+07	3.37E+08
0.18	0.013403	9902484	11716854	-4.87E+08	8.71E+08
0.21	0.02577	11370168	14064525	-1.64E+09	2.03E+09
0.24	0.049363	13401126	16914310	-4.21E+09	4.59E+09
0.27	0.106641	16296659	20139420	-1.13E+10	1.17E+10
0.3	2.214916	22931453	23568030	-3.10E+11	3.10E+11

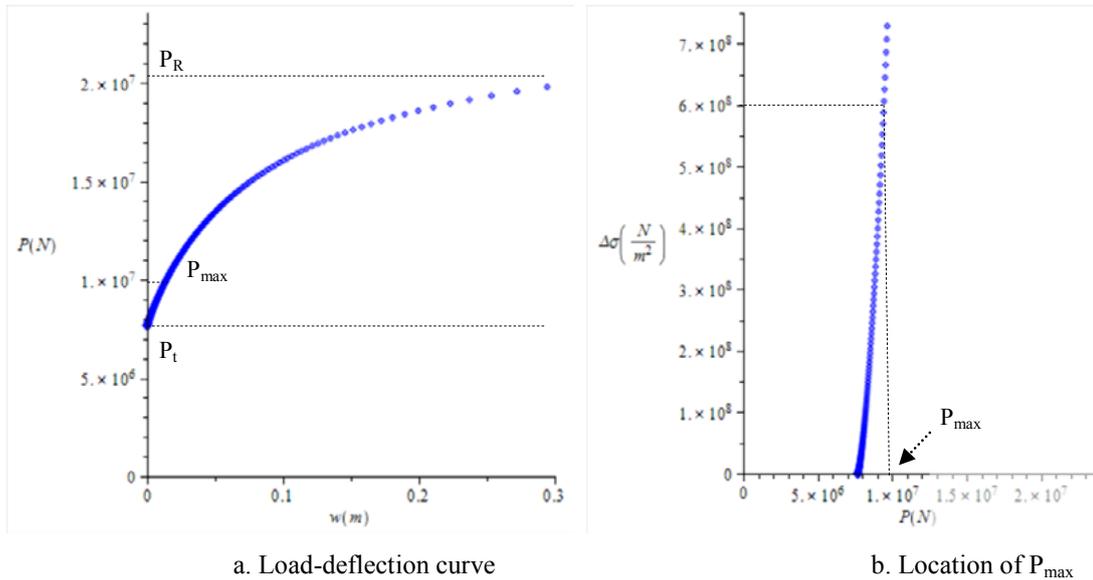a. Load-deflection curve   b. Location of P_{max}

Figure 4: Elasto-plastic buckling response of initially straight column

3 IMPERFECTION SENSITIVITY IN PLASTIC RANGE

Real structural members have imperfections in the form of out-of-straightness, which can influence the buckling and post-buckling response, as investigated hereafter.

3.1 Small imperfections

Consider an imperfect column with a relatively small initial imperfection similar to the assumed deformation mode $w(x)$ (Figure 3). An initial loaded configuration can be found such that the entire column has plastified (i.e. $\sigma \geq \sigma_y$) before strain reversal (i.e. $\frac{d\sigma}{dP} \geq 0$). In addition, Perry's rule [7] suggests a magnification factor of α for the deformation mode in relation to the initial imperfection due to the axial load:

$$\alpha = \frac{1}{\frac{P_{cr}}{P} - 1} \tag{6}$$

where in this case $P_{cr} = P_t$. With this information, a maximum threshold imperfection level w_{i_max} can be determined for a specific column based on the condition that stress reversal occurs for an extreme fibre stress equal to the yield stress. Imperfection values less than w_{i_max} are then identified as "small imperfections".
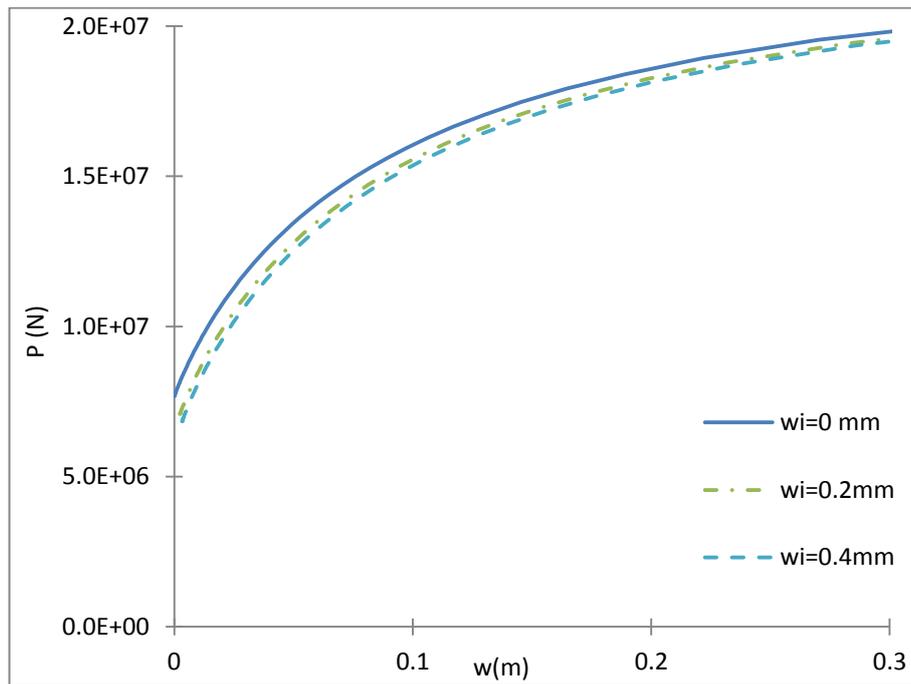

Figure 5: Effect of small imperfections

The procedure to calculate the maximum inelastic buckling load described in Sections 2.1 and 2.2 still applies for an imperfect column with small imperfections, except that at the onset

of buckling w starts from a magnified deformed configuration corresponding to an applied load P rather than from an initially straight configuration. Figure 5 illustrates the effect of small imperfections on the post-buckling behaviour of the column (length=0.6m, width=0.1m and depth=0.2m). It can be seen that initial imperfections smaller than the threshold level of imperfection (in this case w_{i_max}=0.45mm) can significantly reduce the buckling loads.
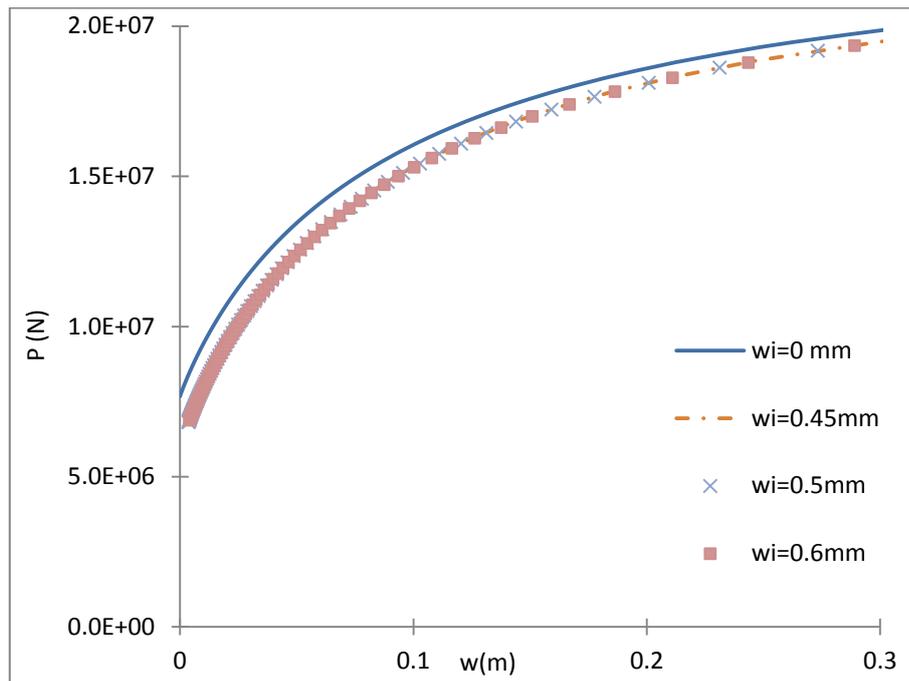
3.2 Large imperfections

When the initial out-of-straightness of the column exceeds the threshold level of imperfection (Figure 6a), a slightly different approach needs to be followed. In this case, full plastification of the column cross-section starts at the supports, and before it reaches the column mid-length unloading will take place due to the large initial imperfection. Therefore, some parts along the top extreme fibre of the column, within x_s distance from the column mid-length, remain elastic, other parts within x_e from x_s initially become plastic and then unload, while the remain cross-sections continue plastic loading in compression. For a given initial imperfection ($w_i > w_{i\_max}$), the conditions of total bending equilibrium at the point of strain reversal allow x_s, and the starting values of P and w to be obtained. Once these values are established, the post-buckling response can be traced in a similar way to the case of small imperfections. Figure 6b shows the influence of large imperfections on the inelastic buckling response of the same column considered previously. Unlike the case of small imperfections where imperfection sensitivity of the elasto-plastic buckling response was evident, the case of large imperfections exceeding the obtained threshold exhibits hardly any imperfection sensitivity.
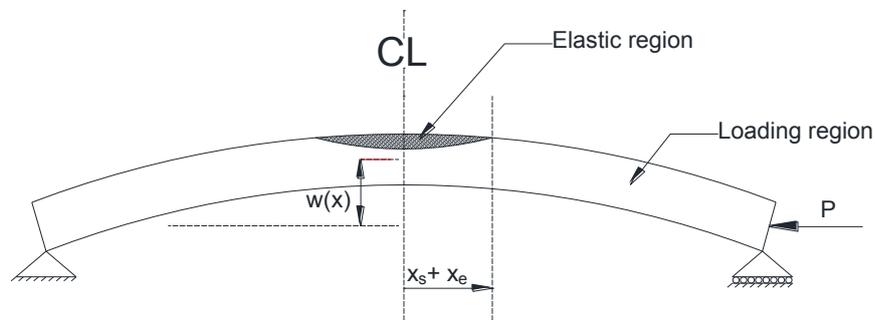
a. Column with large imperfection

Figure 6: Influence of large imperfection on column buckling response (Cont'd…)

b. Response of column with large imperfection

Figure 6: Influence of large imperfection on column buckling response

4 COMPARISON WITH FINITE ELEMENT RESULTS

The outcomes of the proposed analytical model are verified against the results of the nonlinear finite element analysis program ADAPTIC [5] and are depicted in Figure 7 for different levels of imperfection. For the purpose of this paper, a simply-supported column with a rectangular cross-section (width=0.1m and depth=0.2m) is considered. The strain-hardening value is taken as 2% with a yield strength of 300MPa. As an illustration, the buckling analysis of a 0.6m column length is considered.

In the course of analysis using ADAPTIC, it was found out that depending on the slenderness ratio and the level of strain-hardening, the column can exhibit significant axial shortening which would then in turn cause a notable increase in buckling loads. Accordingly, the effect of column shortening is also taken into account in the analytical model. Figure 7 demonstrates a good agreement between the analytical and numerical results, where P_{max} corresponds to the point at which the perfect column exhibits tensile yielding on the convex side at mid-length.
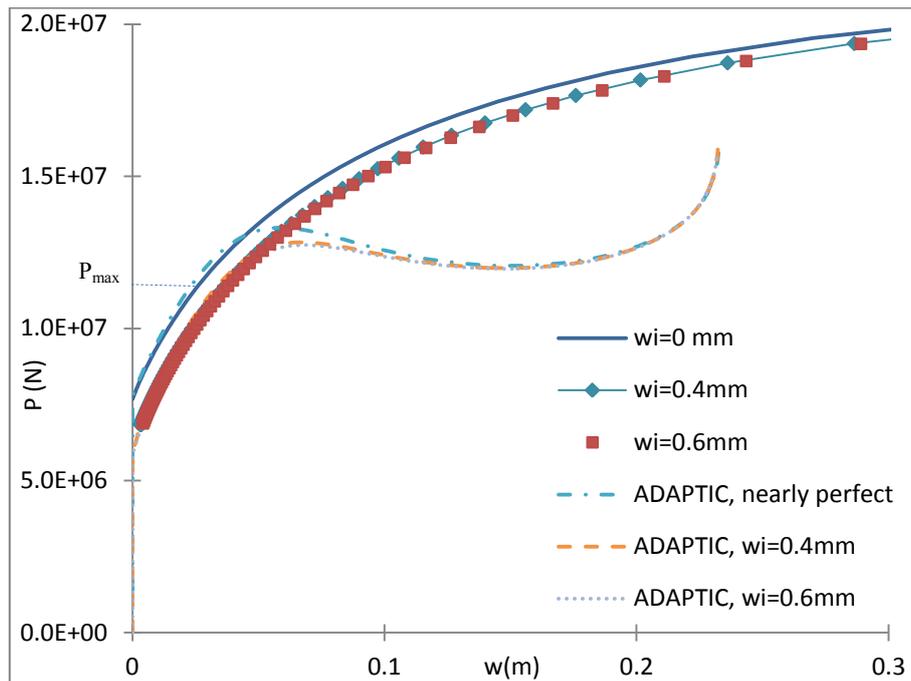
Figure 7: Comparison of results of analytical model and ADAPTIC

5 CONCLUSION

In this paper, a simplified analytical model is presented for the elasto-plastic buckling of columns which captures the inelastic behaviour of stocky columns, and considers the mechanics of the initiation of buckling and the post-buckling response. This analytical model employs the Rotational Spring Analogy for formulating the geometric stiffness matrix using an assumed mode, while the material stiffness matrix is obtained with due consideration for the spread of material plasticity. In addition, the model is derived for both cases of perfect and imperfect columns.

Besides ease of application, the developed analytical model provides insight into the initiation of elasto-plastic buckling at the tangent modulus/Engesser buckling load and its subsequent increase towards the double modulus/von Karman buckling load. It is also shown that the von Karman upper limit is typically not realised due to tensile yielding at the outer fibre of the column cross-section. Furthermore, it is observed that beyond a threshold level of imperfection, which can be directly evaluated from the developed model, the elasto-plastic post-buckling response is barely affected by a further increase in out-of-straightness. These findings are verified against the results of nonlinear finite element analysis using ADAPTIC, highlighting the important benefits of analytical models for direct application and enhanced understanding of elasto-plastic buckling.

Future work will utilise the outcomes from this analytical model for the elasto-plastic buckling analysis of plates.

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