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MODELING THE BEHAVIOR OF ELASTIC MATERIALS WITH STOCHASTIC MICROSTRUCTURE

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Abstract. Even in the simple linear elastic range, the material behavior is not deterministic, but fluctuates randomly around some expectation values. The knowledge about this characteristic is obviously trivial from an experimentalist's point of view. However, it is not considered in the vast majority of material models in which "only" deterministic behavior is taken into account.

One very promising approach to the inclusion of stochastic effects in modeling of materials is provided by the Karhunen-Loève expansion. It has been used, for example, in the stochastic finite element method, where it yields results of the desired kind, but unfortunately at drastically increased numerical costs.

This contribution aims to propose a new ansatz that is based on a stochastic series expansion, but at the Gauß point level. Appropriate energy relaxation allows to derive the distribution of a synthesized stress measure, together with explicit formulas for the expectation and variance. The total procedure only needs negligibly more computation effort than a simple elastic calculation. We also present an outlook on how the original approach in [7] can be applied to inelastic materials

1 INTRODUCTION

The real behavior of materials is influenced by many different aspects. Examples are grain size and grain size distributions, dislocations, segregations, crystal orientation, defects, inclusions, and many more. Since these phenomena cannot be predicted for engineering materials, they have to regarded as random. Thus, the respective material

behavior, which is realized e.g. in terms of stress/strain diagrams, shows stochastic fluctuations. For a material prediction with increased accuracy, material models are desired which account for the stochastic properties of materials: a simulation yielding the expectation value of important quantities as elastic constants and stresses along with an error estimate or, to be more precise, along with with the variance would be of major interest.

There exist different strategies to include stochastic information to material modeling. A prominent example is the use of the so-called Chaos Polynomial Expansion and the derived Karhunen-Loève expansion. It has been successfully applied in sensitivity analysis [2], nonlinear random vibration [10] the analysis of human faces [9] and selection and ordering [5]. The key idea is to approximate the stochastic quantities like elastic constants by a broken series expansion while the stochastic dependence is expressed in terms of a stochastic vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{k_{\max}})$ with $k_{\max}$ the maximum number of terms considered. The space-dependent stochastic elastic tensor is thus approximated by

$$\mathbb{E}(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{E}_0(\boldsymbol{x}) + \sum_{k=1}^{k_{\max}} \xi_k \, \mathbb{E}_k(\boldsymbol{x}) \tag{1}$$

with the spatial coordinate $\boldsymbol{x}$ and the expectation value $\mathrm{E}[\mathbb{E}(\boldsymbol{x}, \boldsymbol{\xi})] = \mathbb{E}_0(\boldsymbol{x})$ [6]. For scalar random fields, the coefficients in the series are obtained from an eigenvalue decomposition of the covariance operator, see [13] and [1]. An analogous series expansion for the displacements $\boldsymbol{u}$ yields the stochastic finite element method (see e.g. [4, 11, 3]). Here, the expected value $\boldsymbol{u}_0$ together with the series terms $\boldsymbol{u}_k$ are the unknowns in a coupled algebraic equation system. It is obvious that the calculation of both the eigenfunctions $\mathbb{E}_k$ and the vector of unknown displacements $(\boldsymbol{u}_0, \boldsymbol{u}_1, ..., \boldsymbol{u}_{k_{\max}})$ is of high numerical effort compared to the simple elastic simulation. Of course, the benefit of the increased computation time is a stochastic displacement field from which the strains and thus stresses may be derived. The level of accuracy is increased with increasing length of the series expansion, i.e. with higher values for $k_{\max}$ resulting in higher computational costs.

In this contribution, we recall a novel approach for the calculation of stochastic information for the elastic constants and stresses at the Gauß point level, which was presented in [7]. In this approach, also a stochastic series expansion as in (1) is applied. However, since there is no mathematical theory for the the general expansion of tensor-valued fields, we start with a general representation as in (1), where we assume that the ξ_k are independent random variables with

$$\mathrm{E}[\xi] = \mathrm{E}[\xi^3] = 0, \qquad \mathrm{E}[\xi^2] = 1, \tag{2}$$

and the coefficients $\mathbb{E}_k(\boldsymbol{x})$ satisfy the symmetry conditions $\mathbb{E}_{ijkl} = \mathbb{E}_{klij} = \mathbb{E}_{jikl} = \mathbb{E}_{ijlk}$ and such that the expectation and variance

$$\mathrm{E}[\mathbb{E}(\boldsymbol{x}, \boldsymbol{\xi})] = \mathbb{E}_0(\boldsymbol{x}) \quad \text{and} \quad \mathrm{Var}(\mathbb{E}(\boldsymbol{x}, \boldsymbol{\xi})) = \sum_{k=1}^{k_{\max}} \mathbb{E}_k(\boldsymbol{x}) : \mathbb{E}_k(\boldsymbol{x}) \tag{3}$$

are stationary, i.e., they do not depend on the location. This allows to model a wide range of random distributions and is not restricted to Gaussians like in the case of the stochastic

finite elements. Furthermore, it turns out that in the new approach we may choose $k_{\max}$ very large without a substantial increase in the numerical cost. We present the derivation based on [7] and recall several numerical results which compare the analytical solution with Monte Carlo calculations.

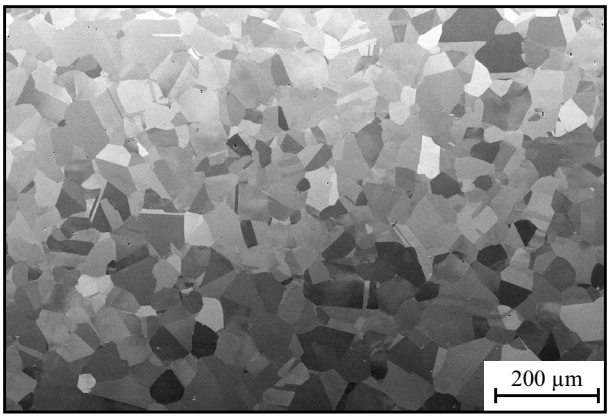

Figure 1: Microstructure in steel. Result of a scanning electron microscopy after [12].

2 THE STOCHASTIC MATERIAL POINT BEHAVIOR

An example for a typical microstructure is presented in Figure 1. Here, steel is investigated by scanning electron microscopy showing very nicely the random areas of different gray level. Each area with constant gray level possesses a constant orientation of the crystallographic lattice and is referred to as grain. Due to orientation, segregation, local defects and others, see also the introduction, the material properties are subjected to stochastic fluctuations even in the elastic regime. Since the same "chaotic" picture is present for different sampling points in a construction part, the local spatial behavior is stochastic even in a homogenized way, i.e. in terms of effective elastic constants and stresses for the entire microstructural domain.

To model this stochastic behavior in a numerically efficient and physically very reasonable manner, basically two homogenized and effective quantities have to be specified

1. a stochastic measure for the elastic constants denoted by $\bar{\mathbb{E}} = \bar{\mathbb{E}}(\boldsymbol{\xi})$ and

2. a stochastic measure for the strains denoted by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\xi})$

Combining these two measures results in a homogenized and effective but also *stochastic* measure for the stress

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\xi}) = \bar{\mathbb{E}}(\boldsymbol{\xi}) : \boldsymbol{\varepsilon}(\boldsymbol{\xi}) \tag{4}$$

A subsequent calculation of the expectation and variance of the elastic constants and the stress is then quite feasible.

The previously defined goal is achieved in two steps. In a first step, a stochastic series expansion as described in the previous section is employed to the elastic constants. A

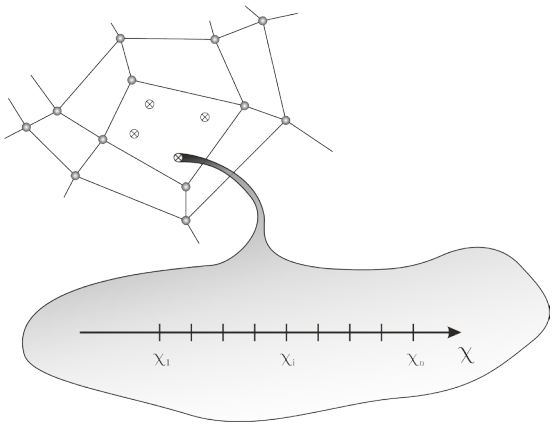

Figure 2: Material point in a finite element setting which is expanded by the microstructural coordinate χ to account for the discretized domains in a real microstructure as shown in 1. Figure 2 after [7].

relaxation of an associated energy yields the desired effective but stochastic measure for the elastic constants. In a second step, a stochastic series expansion is also employed to the strains. Relaxation of the modified energy yield the unknown coefficients of the series expansion for the strains and thus the desired stochastic measure for the effective strains.

2.1 First step: stochastic measure for the effective elastic constants

The material point is defined as the ensemble of grains (or more generally: domains) as exemplary showed in Figure 1. To merge the local behavior in each domain to an effective behavior at the material point level, we introduce a "microstructural coordinate" χ on the domain level, see Figure 2. Here, the domains are presented in a discretized way using the index i. For each discretized microstructural coordinate χ_i – referring to one discrete domain in the real material – varying stochastic elastic constants are present yielding to varying stochastic strains in each domain χ_i.

The elastic constants are expressed in terms of a stochastic series for each domain by

$$\mathbb{E}_i = \mathbb{E}(\chi_i, \xi) = \mathbb{E}_{0,i} + \sum_{k=1}^{k_{\max}} \xi_k \, \mathbb{E}_{k,i} \tag{5}$$

with the assumptions on ξ and $\mathbb{E}_{k,i}$ as outlined in Section 1. For the strains in each domain, we do not make any assumption at this stage nor do we approximate them by a series expansion but leave them completely general. Then, the Helmholtz free energy of each domain is given by

$$\Psi(\chi_i, \xi) = \Psi_i = \frac{1}{2} \boldsymbol{\varepsilon}_i : \mathbb{E}_i : \boldsymbol{\varepsilon}_i. \tag{6}$$

which allows for formulating the associated relaxation problem as

$$\Psi = \inf_{\boldsymbol{\varepsilon}_i} \left\{ \frac{1}{n} \sum_{i=1}^{n} \Psi_i \quad \middle| \quad \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon} \right\}. \tag{7}$$

The parameter n accounts for the number of domains in each material point ensemble. The unknown strains in each domain are computed by the minimization problem

$$\mathcal{L} = \frac{1}{2n} \sum_{i=1}^{n} \boldsymbol{\varepsilon}_i : \left(\mathbb{I} + \sum_{k=1}^{k_{\max}} \xi_k \, \mathbb{E}_{k,i} \right) : \boldsymbol{\varepsilon}_i + \boldsymbol{\sigma} : \left[\boldsymbol{\varepsilon} - \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\varepsilon}_i \right] \rightarrow \min_{\boldsymbol{\varepsilon}_i, \boldsymbol{\sigma}} \tag{8}$$

in which for simplicity we rescaled the elastic constants such that $\mathbb{E}_{0,i} = \mathbb{I}$. For more details we refer to [7]. The Lagrange parameter $\boldsymbol{\sigma}$ accounts to the constraint that the mean of all strains in the individual grains equals the (given) homogenized strain of the material point ensemble $\boldsymbol{\varepsilon}$. This Lagrange parameter is indeed the stress measure we are looking for. Solving the Lagrange equations yields the homogenized energy $\Psi = 1/n \sum_{i=1}^{n} \Psi_i = 1/2 \boldsymbol{\varepsilon} : \bar{\mathbb{E}} : \boldsymbol{\varepsilon}$ with the effective, *stochastic* elastic constant

$$\bar{\mathbb{E}} = \bar{\mathbb{E}}(\boldsymbol{\xi}) = \left[\frac{1}{n} \sum_{i=1}^{n} \left[\mathbb{I} + \sum_{k=1}^{k_{\max}} \xi_k \mathbb{E}_{k,i} \right]^{-1} \right]^{-1} \tag{9}$$

which is the harmonic mean of the individual elastic constants in the respective domains. The expectation value for the harmonic mean is highly inaccessible. Thus, we approximate it with a Taylor series of order two. This yields

$$\bar{\mathbb{E}} \approx \mathbb{I} + \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{k_{\max}} \xi_k \mathbb{E}_{k,i} - \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=1}^{k_{\max}} \xi_k \mathbb{E}_{k,i} \right)^2 + \frac{1}{n^2} \left(\sum_{i=1}^{n} \sum_{k=1}^{k_{\max}} \xi_k \mathbb{E}_{k,i} \right)^2 . \tag{10}$$

2.2 Second step: stochastic measure for the effective strains

The purpose of this second step is the search of an appropriate stochastic measure of the effective strains, i.e. we are seeking for a formulation for $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\xi})$. To this end, we employ the same stochastic series expansion to the strains in each domain $\boldsymbol{\varepsilon}_i$ which we also used for the elastic constants. This means

$$\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}(\chi_i, \xi) = \boldsymbol{\varepsilon}_{0,i} + \sum_{k=1}^{k_{\max}} \xi_k \, \boldsymbol{\varepsilon}_{k,i} \tag{11}$$

with the unknown expectation values in each domain $\boldsymbol{\varepsilon}_{0,i}$ and series coefficients $\boldsymbol{\varepsilon}_{k,i}$. To compute them, we employ a second relaxation approach for the homogenized energy, more precisely

$$\Psi^{\mathrm{E}} = \inf_{\boldsymbol{\varepsilon}_{0,i}, \boldsymbol{\varepsilon}_{k,i}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \Psi_i^{\mathrm{E}} \;\middle|\; \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\varepsilon}_{0,i} = \boldsymbol{\varepsilon}_0 \right\} \tag{12}$$

The superscript refers to the expectation value of the respective quantities. This procedure is similar to the derivation of the stochastic finite elements and is required to find the (deterministic) series coefficients. More details are given in [7].

The expected homogenized Helmholtz free energy is given by

$$\Psi^{\mathrm{E}} = \frac{1}{n}\sum_{i=1}^{n}\Psi_i^{\mathrm{E}} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\sum_{k=0}^{k_{\max}}\boldsymbol{\varepsilon}_{k,i} : \mathbb{I} : \boldsymbol{\varepsilon}_{k,i} + \boldsymbol{\varepsilon}_{0,i} : \sum_{k=1}^{k_{\max}}\mathbb{E}_{k,i} : \boldsymbol{\varepsilon}_{k,i}\right), \tag{13}$$

see [7]. Again, the elastic constants are rescaled to yield $\mathbb{E}_{i,0} = \mathbb{I}$. The associated minimization problem reads

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\varepsilon}_{0,i}} = \mathbf{0} = \frac{1}{n}\left(\mathbb{I} : \boldsymbol{\varepsilon}_{0,i} + \sum_{k=1}^{k_{\max}}\mathbb{E}_{k,i} : \boldsymbol{\varepsilon}_{k,i}\right) - \frac{1}{n}\hat{\boldsymbol{\sigma}} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\varepsilon}_{k,i}} = \mathbf{0} = \frac{1}{n}\left(\mathbb{I} : \boldsymbol{\varepsilon}_{k,i} + \boldsymbol{\varepsilon}_{0,i} : \mathbb{E}_{k,i}\right) \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{\boldsymbol{\sigma}}} = \mathbf{0} = \boldsymbol{\varepsilon}_0 - \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\varepsilon}_{0,i}. \tag{16}$$

with a new Lagrange parameter $\hat{\boldsymbol{\sigma}}$ which, however, is very closely related to $\boldsymbol{\sigma}$. Solving the minimization conditions for the unknown coefficients in the series expansion results finally in the desired formulation for the effective stochastic strain of the homogenized microstructure as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\xi}) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\varepsilon}_i = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{I} - \sum_{k=1}^{k_{\max}}\xi_k\mathbb{E}_{k,i}\right) : \boldsymbol{\varepsilon}_0. \tag{17}$$

2.3 Result: stochastic measure of the stress

Combing the results of the previous two subsections result in the desired stochastic measure for the stress of the homogenized material point as

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}(\boldsymbol{\xi}) = \bar{\mathbb{E}}(\boldsymbol{\xi}) : \boldsymbol{\varepsilon}(\boldsymbol{\xi}) \\ &= \left[\mathbb{I} + \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{k_{\max}}\xi_k\mathbb{E}_{k,i} - \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{k_{\max}}\xi_k\mathbb{E}_{k,i}\right)^2 + \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{k=1}^{k_{\max}}\xi_k\mathbb{E}_{k,i}\right)^2\right] \\ &\quad : \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{I} - \sum_{k=1}^{k_{\max}}\xi_k\mathbb{E}_{k,i}\right) : \boldsymbol{\varepsilon}_0\end{aligned} \tag{18}$$

The stochastic information in terms of expectation and variance can now be calculated for both the elastic constants and the strains. For the elastic constants they read

$$\mathrm{E}[\bar{\mathbb{E}}] = \mathbb{E}_{0,1} - \mathbb{E}_{0,1}^{1/2} : \mathbb{V} : \mathbb{E}_{0,1}^{1/2} + \frac{1}{n}\mathbb{E}_{0,1}^{1/2} : \mathbb{C} : \mathbb{E}_{0,1}^{1/2} \tag{19}$$

and

$$\mathrm{Var}(\mathbb{E}_{0,1}^{-1/2} : \bar{\mathbb{E}} : \mathbb{E}_{0,1}^{-1/2}) = \frac{1}{n}\mathbb{C} + \frac{1}{n}\mathbb{C}^{(2)} + o\left(\frac{1}{n}\right), \tag{20}$$

respectively, where here we give the general formula without assuming $\mathbb{E}_{0,1} = \mathbb{I}$ anymore.

For the stress, they are calculated as

$$\mathrm{E}[\boldsymbol{\sigma}] = \mathbb{E}_{0,1}^{1/2} : (\mathbb{I} - \mathbb{V}) : \mathbb{E}_{0,1}^{1/2} : \boldsymbol{\varepsilon}_0. \tag{21}$$

and

$$\mathrm{Var}(\mathbb{E}_{0,1}^{-1/2} : \boldsymbol{\sigma} : \mathbb{E}_{0,1}^{-1/2}) = \frac{1}{n}\boldsymbol{\varepsilon}_0 : \hat{\mathbb{C}}^{(2)} : \boldsymbol{\varepsilon}_0 + \frac{1}{n}\boldsymbol{\varepsilon}_0 : \hat{\mathbb{C}}_{\mathbb{V}} : \boldsymbol{\varepsilon}_0 + o\left(\frac{1}{n}\right). \tag{22}$$

Note that here the variance is calculated as a fourth order tensor as in (3). The following abbreviations have been used

$$\mathbb{V} = \mathrm{E}[\mathbb{X}_1 : \mathbb{X}_1], \tag{23}$$

$$\mathbb{C} = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{E}[\mathbb{X}_i : \mathbb{X}_j], \tag{24}$$

$$\hat{\mathbb{C}} = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{E}[\mathbb{X}_i \cdot \mathbb{X}_j], \tag{25}$$

$$\hat{\mathbb{C}}_{\mathbb{V}} = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{E}[\mathbb{X}_i : \mathbb{V} \cdot \mathbb{V} : \mathbb{X}_j] \tag{26}$$

and

$$\mathbb{C}^{(2)} = \frac{1}{n}\sum_{i,j=1}^{n} (\mathrm{E}[\mathbb{X}_i^2 : \mathbb{X}_j^2] - \mathbb{V} : \mathbb{V}) \tag{27}$$

$$\hat{\mathbb{C}}^{(2)} = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{E}[\mathbb{X}_i^2 \cdot \mathbb{X}_j^2] - \mathbb{V} \cdot \mathbb{V}) \tag{28}$$

with

$$\mathbb{X}_i = \mathbb{E}_{0,1}^{-1/2} : (\mathbb{E}_i - \mathbb{E}_{0,1}) : \mathbb{E}_{0,1}^{-1/2}. \tag{29}$$

Details can be found in [7]. We emphasize two important consequences of our approach: firstly, the variances and covariances as in (23) to (28) can be estimated from data. Once they are known, one does not need to calculate the coefficients $\mathbb{E}_{k,i}$ in the expansion of the random field. Secondly, this formulation of the results is independent of the number $k_{\max}$ of terms in the expansion. The expansion (1) is important for our derivation of the results, but $k_{\max}$ may be supposed to be very high, thereby allowing for a more general distribution of the random field. Together with the first point, that there is no need to compute the coefficients in the expansion, this implies the results have a much higher precision.

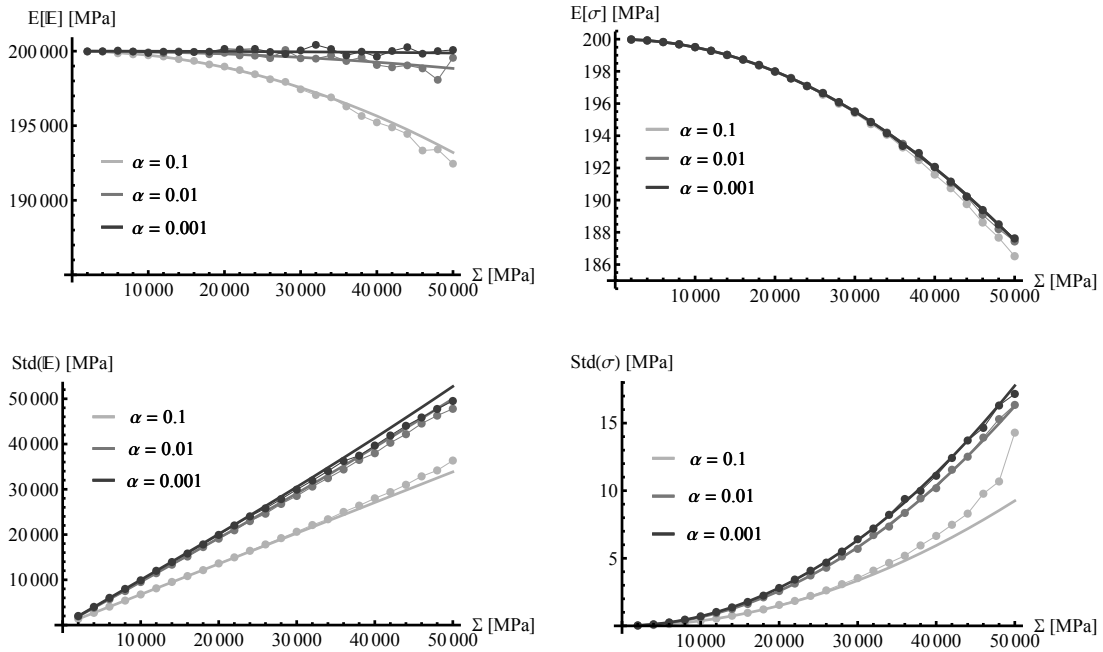

Figure 3: Mean of the stochastic simulation for 20,000 random variables (dots) vs. the analytical values proposed by our model (lines) for Case 1 and varying α. The number of domains is $n = 30$. After [7], Figure 6.

3 NUMERICAL RESULTS

To illustrate our results collected in Section 2.3, we discuss two numerical examples for the case of a one-dimensional stochastic field, which are recalled from [7].

We model two different distributions of $\mathbb{E}_i$, a Gaussian with square-exponential covariance, and a finitely dependent linear combination of uniformly varying random variables.

Case 1: The ξ_k are independent standard Gaussian and the covariance is given by $C(\chi_i, \chi_j) = \Sigma^2 \exp(-\alpha|i-j|^2)$. This is a two-parameter family with variance Σ and α corresponding to the strength of the correlations.

Case 2: The ξ_k are independent and uniform distributed on $[-\sqrt{3}, \sqrt{3}]$. Each $\mathbb{E}_i$ is a homogeneous linear combination of three ξ_k, such that $\mathbb{E}_i$ and $\mathbb{E}_j$ are independent whenever $|i-j| \geq 2$ and $\mathrm{Var}(\mathbb{E}_i) = \Sigma^2$.

In both cases we use an expectation value of $\mathbb{E}_0 = 200'000$ [MPa] and a strain of $\varepsilon_0 = 1 \times 10^{-4}$ [-] and let the standard deviation Σ vary between 2,000 and 50,000. For Case 1, we additionally vary $\alpha = \{0.001, 0.01, 0.1\}$ and we let $n = 30$. The expectation $\mathrm{E}[\bar{\mathbb{E}}]$ and $\mathrm{E}[\sigma]$ and the standard deviation $\mathrm{Std}(\bar{\mathbb{E}}) = \sqrt{\mathrm{Var}(\bar{\mathbb{E}})}$ and $\mathrm{Std}(\sigma) = \sqrt{\mathrm{Var}(\sigma)}$ are calculated for the effective Young's modulus and the stress, respectively, evaluated according to the formulas above. These analytic results are compared with the empirical estimated from a Monte Carlo simulation with 20,000 iterations in Figures 3 and 4.

Several conclusions can be drawn from our results:

1. Our analytical formulas show excellent agreement with the numerical simulations;

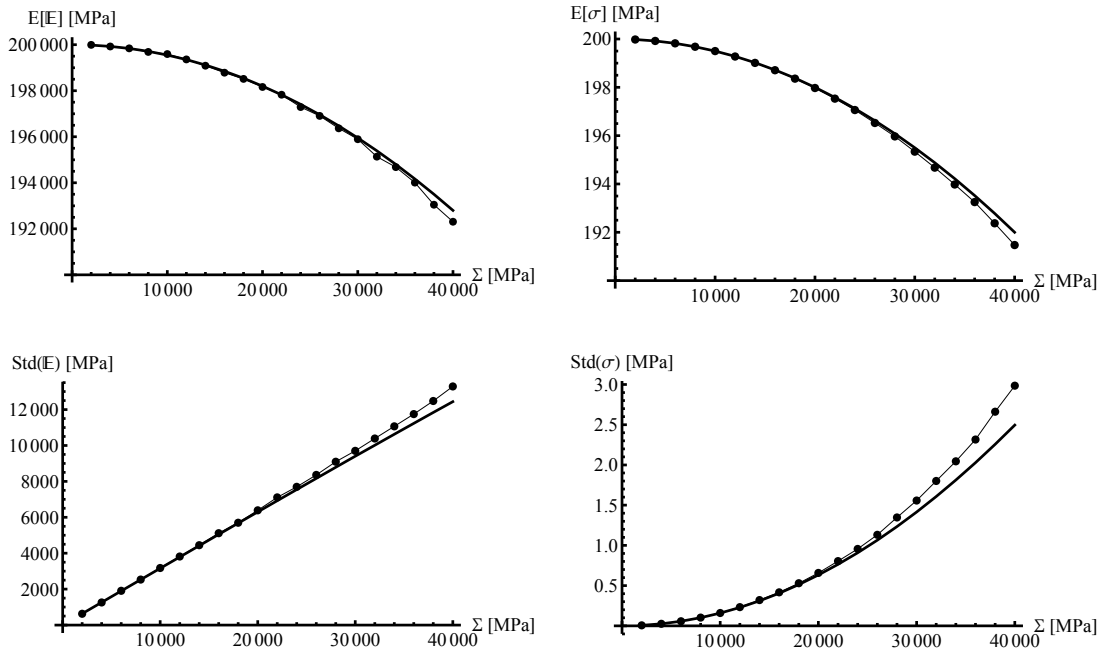

Figure 4: Mean of the stochastic simulation for 20,000 random variables (dots) vs. the analytical values proposed by our model (lines) for Case 2. The number of domains is $n = 30$. After [7], Figure 8.

in particular for small values of the variance.

2. The Gaussian distribution in Case 1, which a priori does not satisfy the ellipticity constraint, also performs well in the numerical comparison.

3. The stochastic stress **cannot** be calculated simply by $\mathbb{E}\,\varepsilon_0$; it decreases quite strongly for all model parameters (n, Σ, α); our equation captures this aspect correctly.

4. The standard deviation of the stress is large for high standard deviations Σ of the Young's modulus (close to 10%); it also **cannot** be concluded solely from the standard deviation Σ without our equations.

We emphasize that our equations can be evaluated basically without any computational effort. In contrast, 20,000 stochastic simulations are necessary in order to receive the same behavior just by averaging the realizations of the stochastic behavior. The effect is even more pronounced for a finite element simulation. Furthermore, the presented model is also much faster than a stochastic finite element framework due to the increased number of nodal unknowns in the latter method. The excellent agreement between simulation and evaluation of our equations, which captures the averaged stochastic behavior very well, proves that our assumptions are very reasonable and even the broken Taylor series produces only negligible errors.

4 OUTLOOK

It is possible to extend the method to the modeling of inelastic materials. We give here a very brief outlook on how this yields a formula for the expected stress measure and refer to the forthcoming paper [8] for the derivation. The strains can be decomposed into the elastic and inelastic parts as $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_i^{\mathrm{e}} + \boldsymbol{\varepsilon}_i^{\mathrm{p}}$, and applying the relaxation method in Sections 2.1 and 2.2 to the elastic parts results in

$$\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}} = \left[\mathbb{I} - \sum_{k=1}^{k_{\max}} \xi_k \mathbb{E}_0^{-1} : \mathbb{E}_{k,i}\right] : (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_0^{\mathrm{p}}) \tag{30}$$

and $\boldsymbol{\sigma} = \bar{\mathbb{E}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{p}})$ with $\bar{\mathbb{E}}$ as in (10). It remains to obtain a formula for the inelastic strains, which can be done by employing the Hamilton principle in its form for absent gradients of $\boldsymbol{\varepsilon}_{\mathrm{p}}$ reading

$$\mathcal{L} = \dot{\Psi}^{\mathrm{E}} + \Delta^{\mathrm{E}} + \mathrm{cons} \to \underset{\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{p}}, \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}}{\mathrm{stat}}, \tag{31}$$

where $\Delta = r\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ is a dissipation function homogeneous of order two. This results in the differential equation

$$\dot{\boldsymbol{\varepsilon}}_0^{\mathrm{p}} = r^{-1}\mathrm{dev}\left[\mathbb{E}_{0,1}^{1/2} : (\mathbb{I} - \mathbb{V}) : \mathbb{E}_{0,1}^{1/2} : (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_0^{\mathrm{p}})\right] \tag{32}$$

for $\boldsymbol{\varepsilon}_0^{\mathrm{p}}$. Solving this equation allows then to compute the expected stress as in formula (21). Figure 5 shows a comparison of this analytic formula with the estimation obtained from a Monte-Carlo simulation.

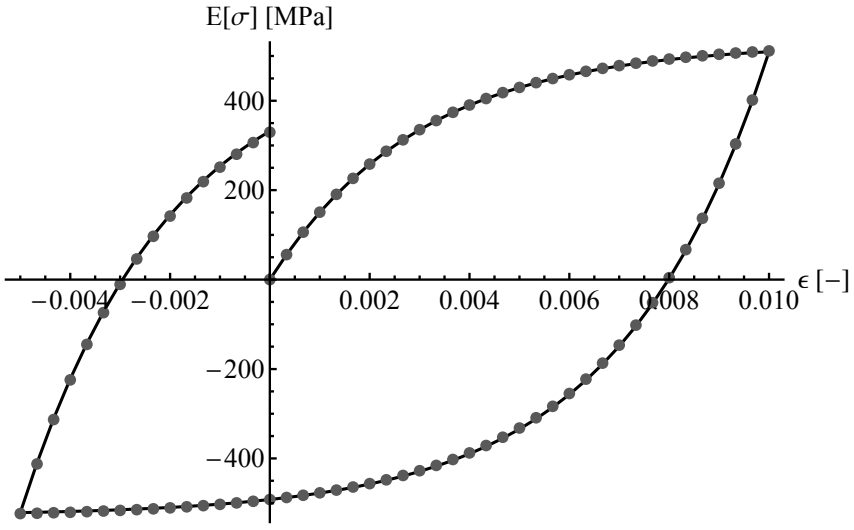

Figure 5: Mean of the stochastic simulation for 2,000 random variables (dots) vs. the analytical values proposed by our model (lines) in the viscous material for Case 1. The number of domains is $n = 30$.

5 CONCLUSIONS

The inclusion of stochastic information in modeling of materials is highly appreciated since construction parts are increasingly being designed at the edge of the sustainability. The stochastic fluctuations of material properties have a strong impact on the respective material behavior during operation. In this contribution, we propose a novel approach that is based on a stochastic field at the material point which is physically motivated. In a first step, we applied a stochastic expansion to the elastic constants and performed a homogenization over the material point. This results in a stress measure, which includes a stochastic effective elastic constant that depends on the harmonic mean of the elastic constants in each domain. In a second step, we employed the same stochastic expansion as for the elastic constants for the strains at the microlevel. The relaxation of the expected Helmholtz free energy yielded then the stochastic coefficients of the strain expansion. Double contraction of the stochastic elastic constants and the stochastic strains gives the appropriate stress measure. Using these formulas for the stress measure, we were able to calculate the expectation and variance. For this computation, only the knowledge of covariances of the elastic constants needs to be given so that the stress as well as its expectation and variance can be calculated in a closed form. For an implementation into a finite element routine, only "modified" elastic constants have to be used.

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