A micro Lie theory for
state estimation in robotics

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Abstract—A Lie group is an old mathematical abstract object dating back to the XIX century, when mathematician Sophus Lie laid the foundations of the theory of continuous transformation groups. As it often happens, its usage has spread over diverse areas of science and technology many years later. In robotics, we are recently experiencing an important trend in its usage, at least in the fields of estimation, and particularly in motion estimation for navigation.

Yet for a vast majority of roboticians, Lie groups are highly abstract constructions and therefore difficult to understand and to use. This may be due to the fact that most of the literature on Lie theory is written by and for mathematicians and physicists, who might be more used than us, perhaps by their academic formation, to the deep abstractions this theory deals with.

In estimation for robotics, it is often not necessary to exploit the full capacity of the theory, and therefore an effort of selection of materials is required. In this paper, we will walk through the most basic principles of the Lie theory, with the aim of conveying clear and useful ideas, and leave a significant corpus of the Lie theory behind. Even with this mutilation, the material included here has proven to be extremely useful in modern estimation algorithms for robotics, especially in the fields of SLAM, visual odometry, and the like.

We provide also a vast reference of formulas for the major groups used in robotics, including most jacobian matrices and the way to easily manipulate them.

I. INTRODUCTION

There has been a remarkable effort in the last years in the robotics community to formulate estimation problems properly. This is motivated by an increasing demand for precision, consistency and stability of the solutions. Indeed, a proper modeling of the states and measurements, the functions relating them, and their uncertainties, is crucial to achieve these goals. This has led to designs involving what has been known as 'manifolds', which in this context are no less than the smooth topologic surfaces of the Lie groups where the state representations evolve. Relying on the Lie theory (LT) we are able to construct a rigorous calculus corpus to handle uncertainties, derivatives and integrals with precision and ease. Typically, these works have focused on the well known manifolds of rotation SO(3) and rigid motion SE(3).

Lie theory is by no means simple. To grasp a minimum idea of what LT can be, we may consider the following two references. On one hand, Howe's *"Very basic Lie theory"* [1] comprises more than 600 pages. On another hand, the more modern Stillwell's *"Naive Lie theory"* [2] comprises more than 200 pages. With such precedents labeled as 'very basic' and 'naive', the aim of this paper at merely 14 pages is to simplify Lie theory much more (thus our adjective 'micro' in the title). This we do in two forms. First, we select a very
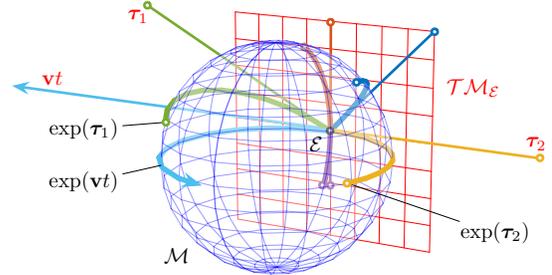

Figure 1. Representation of the relation between the Lie group and the Lie algebra. The Lie algebra (red plane) is the tangent space to the Lie group's manifold (here represented as a blue sphere) at the identity $\mathcal{E}$. Through the exponential map, each straight path $\mathbf{v}t$ through the origin on the Lie algebra produces a path $\exp(\mathbf{v}t)$ over the manifold which runs along the respective geodesic. Conversely, each element of the group has an equivalent in the Lie algebra. This relation is so profound that (nearly) all operations in the group, which is curved and nonlinear, have an exact equivalent in the Lie algebra, which is a linear vector space. Though the sphere in $\mathbb{R}^3$ is not a Lie group (we just use it as a representation that can be drawn on paper), that in $\mathbb{R}^4$ is, and describes the group of unit quaternions —see Fig. 4.

restrictive subset of material from the LT. This subset is so small that it merely explores the potential of LT. However, it appears very useful for uncertainty management in the kind of estimation problems we deal with in robotics (*e.g.* inertial pre-integration, odometry and SLAM, visual servoing, and the like), thus enabling elegant and rigorous designs of optimal optimizers. Second, we explain it in a didactical way, with plenty of redundancy so as to reduce the entry gap to LT even more, which we believe is still needed. That is, we insist on the efforts in this direction of, to name a paradigmatic title, Stillwell's [2], and provide yet a more simplified version. The main text body is generic, though we try to keep the abstraction level to a minimum. Inserted examples serve as grounding base for the general concepts when applied to known groups (rotation and motion matrices, quaternions, etc.). Also, plenty of figures with very verbose captions re-explain the same concepts once again. We put special attention to the computation of Jacobians, which are essential for most optimal optimizers and the source of much trouble when designing new algorithms. And finally, several appendices contain ample reference for the most relevant details of the most commonly used groups in robotics.

When being introduced to Lie groups for the first time, it is important to try to regard them from different points of view. The topological viewpoint, see Fig. 1, involves the shape of the manifold and conveys powerful intuitions of its relation to the tangent space and the exponential map. The algebraic viewpoint involves the group operations and their concrete

realization, allowing the exploitation of algebraic properties to develop closed-form formulas or to simplify them. The geometrical viewpoint, particularly useful in robotics, associates group elements to the position, velocity, orientation, and/or other modifications of bodies or reference frames. The origin frame may be identified with the group's identity, and any other point on the manifold represents a certain 'local' frame. By resorting to these analogies, many mathematical abstractions of the LT can be brought closer to intuitive notions in vector spaces, geometry, kinematics and other more classical fields.

Yet the most important simplification is in terms of scope. The following passage from Howe [1] may serve us to illustrate what we leave behind: "*The essential phenomenon of Lie theory is that one may associate in a natural way to a Lie group $\mathcal{G}$ its Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is first of all a vector space and secondly is endowed with a bilinear nonassociative product called the Lie bracket [...]. Amazingly, the group $\mathcal{G}$ is almost completely determined by $\mathfrak{g}$ and its Lie bracket. Thus for many purposes one can replace $\mathcal{G}$ with $\mathfrak{g}$. Since $\mathcal{G}$ is a complicated nonlinear object and $\mathfrak{g}$ is just a vector space, it is usually vastly simpler to work with $\mathfrak{g}$. [...] This is one source of the power of Lie theory.*" In [2], Stillwell even speaks of "the miracle of Lie theory". In this work we will effectively relegate the Lie algebra to a second plane in favor of its equivalent vector space $\mathbb{R}^n$, and will not introduce the Lie bracket at all. Therefore, the connection between the Lie group and its Lie algebra will not be made here as profound as it should. Our position is that, given the target application areas that we foresee, this material is often not necessary. Moreover, if included, then we would fail in the objective of being clear and useful, because the reader would have to go into mathematical concepts that, by their abstraction or subtleness, are unnecessarily complicated.

Our effort is in line with other recent works on the subject [3], [4], which have also identified this need of bringing the LT closer to the robotician. Our approach aims at appearing familiar to the target audience of this paper: an audience that is skilled in state estimation (Kalman filtering, graph-based optimization, and the like), but not yet familiar with the theoretical corpus of the Lie theory. We have for this taken some initiatives with regard to notation, especially in the definition of the derivative, bringing it close to the vectorial counterparts, thus making the chain rule clearly visible. As said, we opted to practically avoid the material proper to the Lie algebra, and prefer instead to work on its isomorphic tangent vector space $\mathbb{R}^n$, which is where we ultimately represent uncertainty or (small) state increments. All these steps are undertaken with absolutely no loss in precision or exactness, and we believe they make the understanding of the LT and the manipulation of its tools easier.

II. A MICRO LIE THEORY

A. The Lie group

The Lie group encompasses the concepts of *group* and *smooth manifold* in a unique body: a Lie group $\mathcal{G}$ is a smooth manifold whose elements satisfy the group axioms. We briefly present these two concepts before joining them together.
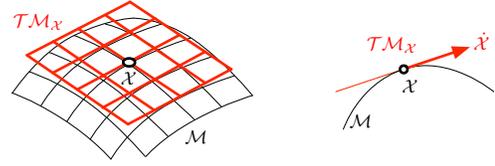

Figure 2. A manifold $\mathcal{M}$ and the vector space $\mathcal{TM} \simeq \mathbb{R}^2$ tangent at the point $\mathcal{X}$, and a convenient side-cut. The velocity element, $\dot{\mathcal{X}} = \partial \mathcal{X}/\partial t$, does not belong to the manifold $\mathcal{M}$ but to the tangent space of $\mathcal{M}$ at $\mathcal{X}$.

On one hand, a differentiable or *smooth manifold* is a topological space that locally resembles linear space. The reader should be able to visualize the idea of manifold (Fig. 2): it is like a curved, smooth (hyper)-surface, with no edges or spikes, embedded in a space of higher dimension. In robotics, we say that our state vector evolves on this surface, that is, the manifold describes or is defined by the constraints imposed on the state. For example, vectors with the unit norm constraint define a spherical manifold of radius one. The smoothness of the manifold implies the existence of a unique tangent space at each point. This space is a linear or vector space on which we are allowed to do calculus.

On the other hand, a *group* $(\mathcal{G}, \circ)$ is a set, $\mathcal{G}$, with a composition operation, $\circ$, that satisfies the following axioms,

$$\text{Closure under '}\circ\text{'} : \quad \mathcal{X} \circ \mathcal{Y} \in \mathcal{G} \tag{1}$$

$$\text{Identity } \mathcal{E} : \quad \mathcal{E} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{E} = \mathcal{X} \tag{2}$$

$$\text{Inverse } \mathcal{X}^{-1} : \quad \mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{E} \tag{3}$$

$$\text{Associativity} : \quad (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z}) . \tag{4}$$

for elements $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{G}$.

In a *Lie group*, the manifold looks the same at every point (like *e.g.* in the surface of a sphere, see Exs. 1 and 2), and therefore all tangent spaces at any point are alike. The group structure imposes that the composition of elements of the manifold remains on the manifold, and that each element has an inverse also in the manifold. A special one of these elements is the identity, and thus a special one of the tangent spaces is the tangent at the identity, which we call the Lie algebra of the Lie group. Lie groups join the local properties of smooth manifolds, allowing us to do calculus, with the global properties of groups, enabling nonlinear composition of distant objects.

In this work, for simplicity and as it has been common in robotics works, we will oftentimes refer to Lie groups as just 'manifolds'.

B. The tangent spaces and the Lie algebra

Given $\mathcal{X}$ a point on a Lie group $\mathcal{M}$, its velocity $\dot{\mathcal{X}} = \partial \mathcal{X}/\partial t$ belongs to the space tangent to $\mathcal{M}$ at $\mathcal{X}$ (Fig. 2), which we note $\mathcal{TM}_{\mathcal{X}}$. The smoothness of the manifold, *i.e.*, the absence of edges or spikes, implies that the tangent space at each point is unique. The tangent space has the same structure at every tangent point.

1) The Lie algebra $\mathfrak{m}$*:* The tangent space at the identity, $\mathcal{TM}_{\mathcal{E}}$, is called the *Lie algebra* of $\mathcal{M}$, and noted $\mathfrak{m}$,

$$\text{Lie algebra} : \qquad \mathfrak{m} \triangleq \mathcal{TM}_{\mathcal{E}} . \tag{5}$$

Table I
TYPICAL LIE GROUPS USED IN 2D AND 3D MOTION, INCLUDING THE TRIVIAL $\mathbb{R}^n$

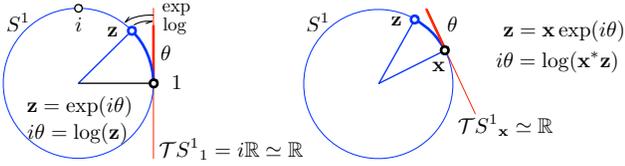| Lie group $\mathcal{M}, \circ$		size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$	$\boldsymbol{\tau} \in \mathbb{R}^m$	$\text{Exp}(\boldsymbol{\tau})$	Action
n-D vector	$\mathbb{R}^n, +$	n	n	$\mathbf{v} \in \mathbb{R}^n$	None	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} = \exp(\mathbf{v})$	$\mathbf{v} + \mathbf{x}$
circle	$S^1, \cdot$	2	1	$\mathbf{z} \in \mathbb{C}$	$\mathbf{z}^* \mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$	$\mathbf{z} = \exp(i\theta)$	$\mathbf{z}\,\mathbf{x}$
Rotation	$SO(2), \cdot$	4	1	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$	$\mathbf{R} = \exp([\theta]_\times)$	$\mathbf{R}\,\mathbf{x}$
Rigid motion	$SE(2), \cdot$	9	3	$\mathbf{M} = \left[\begin{smallmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{smallmatrix}\right]$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\left[\begin{smallmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{se}(2)$	$\left[\begin{smallmatrix} \boldsymbol{\rho} \\ \theta \end{smallmatrix}\right] \in \mathbb{R}^3$	$\exp\left(\left[\begin{smallmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 1 \end{smallmatrix}\right]\right)$	$\mathbf{R}\,\mathbf{x} + \mathbf{t}$
3-sphere	$S^3, \cdot$	4	3	$\mathbf{q} \in \mathbb{H}$	$\mathbf{q}^* \mathbf{q} = 1$	$\mathbf{u}\theta/2 \in \mathbb{H}_p$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{q} = \exp(\mathbf{u}\theta/2)$	$\mathbf{q}\,\mathbf{x}\,\mathbf{q}^*$
Rotation	$SO(3), \cdot$	9	3	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\boldsymbol{\theta}]_\times \in \mathfrak{so}(3)$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{R} = \exp([\boldsymbol{\theta}]_\times)$	$\mathbf{R}\,\mathbf{x}$
Rigid motion	$SE(3), \cdot$	16	6	$\mathbf{M} = \left[\begin{smallmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{smallmatrix}\right]$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\left[\begin{smallmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{se}(3)$	$\left[\begin{smallmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{smallmatrix}\right] \in \mathbb{R}^6$	$\exp\left(\left[\begin{smallmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{smallmatrix}\right]\right)$	$\mathbf{R}\,\mathbf{x} + \mathbf{t}$

Figure 3. The S^1 manifold is a unit circle (blue) in the plane $\mathbb{C}$, where the unit complex numbers $\mathbf{z}^*\mathbf{z} = 1$ live. The Lie algebra $\mathfrak{s}^1 = \mathcal{T}S^1_\mathcal{E}$ is the line of imaginary numbers $i\mathbb{R}$ (red), and any tangent space $\mathcal{T}S^1$ is isomorphic to the line $\mathbb{R}$ (red). Tangent vectors (red segment) wrap the manifold creating the arc of circle (blue arc). Mappings exp and log (arrows) map (wrap and unwrap) elements of $\mathbb{R}$ to/from elements of S^1 (blue arc). Increments between unit complex numbers are expressed in the tangent space via composition and the exponential map (and we will define special operators for this). See the text for explanations.
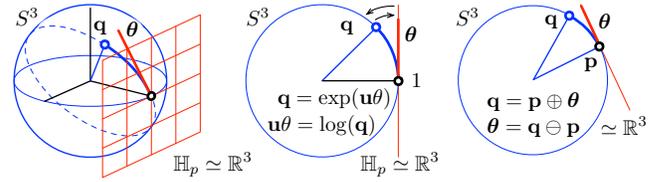

Figure 4. The S^3 manifold is a unit 3-sphere (blue) in the 4-space of quaternions $\mathbb{H}$, where the unit quaternions $\mathbf{q}^*\mathbf{q} = 1$ live. The Lie algebra is the space of pure imaginary quaternions $ix + jy + kz \in \mathbb{H}_p$, isomorphic to the hyperplane $\mathbb{R}^3$ (red grid), and any other tangent space $\mathcal{T}S^3$ is also isomorphic to $\mathbb{R}^3$. Tangent vectors (red segment) wrap the manifold over the great arc or *geodesic* (dashed). The centre and right figures show a side-cut through this geodesic (notice how it ressembles S^1 in Fig. 3). Mappings exp and log (arrows) map (wrap and unwrap) elements of $\mathbb{H}_p$ to/from elements of S^3 (blue arc). Increments between quaternions are expressed in the tangent space via the operators $\oplus, \ominus$ (see text).

Example 1: The unit complex numbers group S^1

Our first example of Lie group, which is the easiest to visualize, is the group of unit complex numbers under complex multiplication (Fig. 3). Unit complex numbers take the form $\mathbf{z} = \cos\theta + i\sin\theta$.

– *Action:* Vectors $\mathbf{x} = x + iy$ rotate in the plane by an angle θ, through complex multiplication, $\mathbf{x}' = \mathbf{z}\,\mathbf{x}$.

– *Group facts:* The product of unit complex numbers is a unit complex number, the identity is 1, and the inverse is the conjugate.

– *Manifold facts:* The unit norm constraint defines the unit circle in the complex plane (which can be viewed as the 1-sphere, and hence the name S^1). This is a 1-DoF curve in 2-dimensional space. Unit complex numbers evolve with time on this circle. The group (the circle) ressembles the linear space (the tangent line) locally, but not globally.

Example 2: The unit quaternions group S^3

A second example of Lie group, which is also relatively easy to visualize, is the group of unit quaternions under quaternion multiplication (Fig. 4). Unit quaternions take the form $\mathbf{q} = \cos(\theta/2) + \mathbf{u}\sin(\theta/2)$, with $\mathbf{u} = iu_x + ju_y + ku_z$ a unitary axis and θ a rotation angle.

– *Action:* Vectors $\mathbf{x} = ix + jy + kz$ rotate in 3D space by an angle θ around the unit axis $\mathbf{u}$ through the double quaternion product $\mathbf{x}' = \mathbf{q}\,\mathbf{x}\,\mathbf{q}^*$.

– *Group facts:* The product of unit quaternions is a unit quaternion, the identity is 1, and the inverse is the conjugate.

– *Manifold facts:* The unit norm constraint defines the 3-sphere S^3, a spherical 3-dimensional surface or *manifold* in 4-dimensional space. Unit quaternions evolve with time on this surface. The group (the sphere) ressembles the linear space (the tangent hyperplane $\mathbb{R}^3 \subset \mathbb{R}^4$) locally, but not globally.

Every Lie group has an associated Lie algebra. In the present work, the way we exploit this relation is through the following facts [4] (see Figs. 1 and 5):

- The Lie algebra $\mathfrak{m}$ is a vector space. As such, its elements can be *identified* with vectors in $\mathbb{R}^m$, whose dimension m matches the number of degrees of freedom of $\mathcal{M}$
- The *exponential map* $\exp : \mathfrak{m} \to \mathcal{M}$ exactly converts elements of the Lie algebra into elements of the group. The log map is its inverse.
- Vectors of the tangent space at $\mathcal{X}$ can be transformed

to the tangent space at the identity $\mathcal{E}$ (the Lie algebra) through a linear transform. This transform is called the *adjoint*.

2) The vector space $\mathbb{R}^m$: The elements $\boldsymbol{\tau}^\wedge$ of the Lie algebra have non-trivial structures (skew-symmetric matrices, imaginary numbers, pure quaternions, see Table I) but the key aspect for us is that they can be expressed as linear
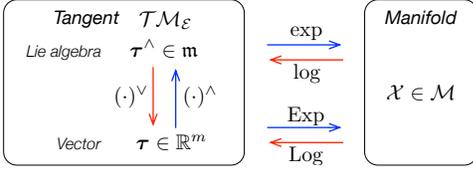
Figure 5. Mappings between manifold $\mathcal{M}$ and the representations of its tangent space at the origin $\mathcal{TM_E}$ (Lie algebra $\mathfrak{m}$ and Cartesian $\mathbb{R}^m$).

combinations of some base elements E_i, where E_i are called the *generators* of $\mathfrak{m}$ (they are the derivatives of $\mathcal{X}$ around the origin in the i-th direction). We may pass from $\mathfrak{m}$ to $\mathbb{R}^m$ and vice versa through two mutually inverse linear maps or isomorphisms, commonly called *hat* and *vee*,

$$\text{Hat}: \quad \mathbb{R}^m \to \mathfrak{m}; \qquad \boldsymbol{\tau} \mapsto \boldsymbol{\tau}^\wedge = \sum_{i=1}^m \tau_i E_i \tag{6}$$

$$\text{Vee}: \quad \mathfrak{m} \to \mathbb{R}^m; \quad \boldsymbol{\tau}^\wedge \mapsto (\boldsymbol{\tau}^\wedge)^\vee = \boldsymbol{\tau} = \sum_{i=1}^m \tau_i \mathbf{e}_i, \tag{7}$$

with $\mathbf{e}_i$ the vectors of the base of $\mathbb{R}^m$ (we have $\mathbf{e}_i^\wedge = E_i$). This means that $\mathfrak{m}$ is isomorphic to the vector space $\mathbb{R}^m$ — one writes $\mathfrak{m} \simeq \mathbb{R}^m$, or $\boldsymbol{\tau}^\wedge \simeq \boldsymbol{\tau}$. Vectors $\boldsymbol{\tau} \in \mathbb{R}^m$ are handier for our purposes than their isomorphs $\boldsymbol{\tau}^\wedge \in \mathfrak{m}$, since they can be stacked in larger state vectors, and more importantly, manipulated with linear algebra using matrix operators. In this work, we enforce this preference of $\mathbb{R}^m$ over $\mathfrak{m}$, to the point that most of the operators and objects that we define (specifically: the adjoint, the Jacobians, the perturbations and their covariances matrices, as we will see soon) are on $\mathbb{R}^m$.

C. The group action

Importantly, Lie groups come with the power to transform elements of other sets, producing *e.g.* rotations, translations, scalings, and combinations of them. These are extensively used in robotics, both in 2D and 3D.

Given a Lie group $\mathcal{M}$ and a set $\mathcal{V}$, we note $\mathcal{X} \cdot v$ the *action* of $\mathcal{X} \in \mathcal{M}$ on $v \in \mathcal{V}$,

$$\cdot : \quad \mathcal{M} \times \mathcal{V} \to \mathcal{V}; \; (\mathcal{X}, v) \mapsto \mathcal{X} \cdot v. \tag{8}$$

For $\cdot$ to be a group action, it must satisfy the axioms,

$$\text{Identity}: \qquad \mathcal{E} \cdot v = v \tag{9}$$
$$\text{Compatibility}: \qquad (\mathcal{X} \circ \mathcal{Y}) \cdot v = \mathcal{X} \cdot (\mathcal{Y} \cdot v). \tag{10}$$

Common examples are the groups of rotation matrices $SO(n)$, the group of unit quaternions, and the groups of rigid motion $SE(n)$. Their respective actions on vectors satisfy

$$
\begin{aligned}
SO(n) &: \text{rotation matrix} & \mathbf{R} \cdot \mathbf{x} &\triangleq \mathbf{R}\mathbf{x} \\
SE(n) &: \text{Euclidean matrix} & \mathbf{H} \cdot \mathbf{x} &\triangleq \mathbf{R}\mathbf{x} + \mathbf{t} \\
S^1 &: \text{unit complex} & \mathbf{z} \cdot \mathbf{x} &\triangleq \mathbf{z}\,\mathbf{x} \\
S^3 &: \text{unit quaternion} & \mathbf{q} \cdot \mathbf{x} &\triangleq \mathbf{q}\,\mathbf{x}\,\mathbf{q}^*
\end{aligned}
$$

See Table I for a more detailed exposition, and the appendices.

Example 3: The rotation group $SO(3)$, its Lie algebra $\mathfrak{so}(3)$, and the vector space $\mathbb{R}^3$

In the rotation group $SO(3)$, of 3×3 rotation matrices $\mathbf{R}$, we have the orthogonality condition $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$. The tangent space may be found by taking the time derivative of this constraint, that is $\mathbf{R}^\top \dot{\mathbf{R}} + \dot{\mathbf{R}}^\top \mathbf{R} = 0$, which we rearrange as

$$\mathbf{R}^\top \dot{\mathbf{R}} = -(\mathbf{R}^\top \dot{\mathbf{R}})^\top.$$

This expression reveals that $\mathbf{R}^\top \dot{\mathbf{R}}$ is a skew-symmetric matrix (the negative of its transpose). Skew-symmetric matrices are often noted $[\boldsymbol{\omega}]_\times$ and have the form

$$[\boldsymbol{\omega}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

This gives $\mathbf{R}^\top \dot{\mathbf{R}} = [\boldsymbol{\omega}]_\times$. When $\mathbf{R} = \mathbf{I}$ we have

$$\dot{\mathbf{R}} = [\boldsymbol{\omega}]_\times,$$

that is, $[\boldsymbol{\omega}]_\times$ is in the Lie algebra of $SO(3)$, which we name $\mathfrak{so}(3)$. Since $[\boldsymbol{\omega}]_\times \in \mathfrak{so}(3)$ has 3 DoF, the dimension of $SO(3)$ is $m = 3$. The Lie algebra is a vector space whose elements can be decomposed into

$$[\boldsymbol{\omega}]_\times = \omega_x \mathbf{E}_x + \omega_y \mathbf{E}_y + \omega_z \mathbf{E}_z$$

with $\mathbf{E}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{E}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathbf{E}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the generators of $\mathfrak{so}(3)$, and where $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ is the vector of angular velocities. The linear relation above allows us to identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ — we write $\mathfrak{so}(3) \simeq \mathbb{R}^3$. We pass from $\mathfrak{so}(3)$ to $\mathbb{R}^3$ and viceversa using the linear operators *hat* and *vee*,

$$\text{Hat}: \qquad \mathbb{R}^3 \to \mathfrak{so}(3); \qquad \boldsymbol{\omega} \mapsto \boldsymbol{\omega}^\wedge = [\boldsymbol{\omega}]_\times$$
$$\text{Vee}: \qquad \mathfrak{so}(3) \to \mathbb{R}^3; \qquad [\boldsymbol{\omega}]_\times \mapsto [\boldsymbol{\omega}]_\times^\vee = \boldsymbol{\omega}.$$

The group composition (1) may be viewed as an action of the group on itself, $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. We can consider also the group action on elements of its own Lie algebra,

$$\cdot : \quad \mathcal{M} \times \mathfrak{m} \to \mathfrak{m}; \quad (\mathcal{X}, \boldsymbol{\tau}^\wedge) \mapsto \mathcal{X} \cdot \boldsymbol{\tau}^\wedge \cdot \mathcal{X}^{-1}, \tag{11}$$

which as we will see constitutes the definition of the adjoint.

D. Maps and operations

We sketch the maps relating $\mathcal{M}$, $\mathfrak{m}$ and $\mathbb{R}^m$ through the following mutually inverse commutative diagrams (see also Fig. 5),

$$
\begin{array}{ccc}
\mathcal{TM_E} = \mathfrak{m} \xrightarrow{\;\exp\;} \mathcal{M} & \qquad & \mathfrak{m} \xleftarrow{\;\log\;} \mathcal{M} \\
{}^\wedge\uparrow \quad \swarrow_{\text{Exp}} & & {}^\vee\downarrow \quad \swarrow_{\text{Log}} \\
\mathbb{R}^m & & \mathbb{R}^m
\end{array}
\tag{12}
$$

where hat $(\cdot)^\wedge$ and vee $(\cdot)^\vee$ are the linear invertible maps or *isomorphisms* (6–7), $\exp(\cdot)$ and $\log(\cdot)$ map the Lie algebra to/from the manifold, and $\text{Exp}(\cdot)$ and $\text{Log}(\cdot)$ are shortcuts to map directly the vector space $\mathbb{R}^m$ to/from $\mathcal{M}$.

1) The exponential map: The exponential map $\exp()$ allows us to exactly transfer elements of the Lie algebra to the group (Fig. 1). Intuitively, $\exp()$ wraps the tangent element over the manifold following the great arc or *geodesic* (as when wrapping a string around a ball, Figs. 1, 3 and 4). The inverse map is the $\log()$, *i.e.*, the unwrapping operation. $\exp()$ arises naturally by considering the time-derivatives of $\mathcal{X} \in \mathcal{M}$ over the manifold, as follows. For multiplicative groups, deriving the generalized group constraint $\mathcal{X}^{-1}\mathcal{X} = \mathcal{E}$ (3) yields

$$\mathcal{X}^{-1}\dot{\mathcal{X}} = -\dot{\mathcal{X}}^{-1}\mathcal{X} \ .$$

This equation has converted the global group constraint into a constraint on the tangent space. As such, it reveals the structure of the tangent space, and thus the Lie algebra (see Ex. 3 for an example). If we define $\mathbf{v}^{\wedge} \triangleq \mathcal{X}^{-1}\dot{\mathcal{X}}$ we can write

$$\dot{\mathcal{X}} = \mathcal{X}\mathbf{v}^{\wedge} \qquad \in \mathcal{TM}_{\mathcal{X}} \ . \tag{13}$$

For $\mathbf{v}$ constant, this is an ordinary differential equation (ODE) whose solution is

$$\mathcal{X}(t) = \mathcal{X}(0)\exp(\mathbf{v}^{\wedge}t) \ . \tag{14}$$

Since $\mathcal{X}(t)$ and $\mathcal{X}(0)$ are elements of the group, then $\exp(\mathbf{v}^{\wedge}t) = \mathcal{X}(0)^{-1}\mathcal{X}(t)$ must be in the group too, and so $\exp(\boldsymbol{\tau}^{\wedge}t)$ maps elements $\mathbf{v}^{\wedge}t$ of the Lie algebra to the group. This is known as the *exponential map*. The element $\mathbf{v}^{\wedge}t$ is indeed in the Lie algebra for all t, since $\mathbf{v}$ is constant and, at $\mathcal{X}(0) = \mathcal{E}$, (13) reduces to $\mathbf{v}^{\wedge} = \dot{\mathcal{X}} \in \mathcal{TM}_{\mathcal{E}} \triangleq \mathfrak{m}$.

In order to provide a more generic definition of the exponential map, let us define the tangent increment $\boldsymbol{\tau} \triangleq \mathbf{v}t \in \mathbb{R}^m$ as velocity per time, so that we have $\boldsymbol{\tau}^{\wedge} = \mathbf{v}^{\wedge}t \in \mathfrak{m}$ a point in the Lie algebra. The exponential map, and its inverse the logarithmic map, can be now written as,

$$\exp : \quad \mathfrak{m} \to \mathcal{M} \quad ; \quad \boldsymbol{\tau}^{\wedge} \mapsto \mathcal{X} = \exp(\boldsymbol{\tau}^{\wedge}) \tag{15}$$

$$\log : \quad \mathcal{M} \to \mathfrak{m} \quad ; \quad \mathcal{X} \mapsto \boldsymbol{\tau}^{\wedge} = \log(\mathcal{X}) \ . \tag{16}$$

Closed forms of the exponential in multiplicative groups are obtained by writing the absolutely convergent Taylor series,

$$\exp(\boldsymbol{\tau}^{\wedge}) = \mathcal{E} + \boldsymbol{\tau}^{\wedge} + \tfrac{1}{2}\boldsymbol{\tau}^{\wedge 2} + \tfrac{1}{3!}\boldsymbol{\tau}^{\wedge 3} + \cdots \ , \tag{17}$$

and taking advantage of the algebraic properties of the powers of $\boldsymbol{\tau}^{\wedge}$ (see Ex. 4 and 5 for developments of the exponential map in $SO(3)$ and S^3). These are then inverted to find the logarithmic map. Key properties of the exponential map are

$$\exp((t+s)\boldsymbol{\tau}^{\wedge}) = \exp(t\boldsymbol{\tau}^{\wedge})\exp(s\boldsymbol{\tau}^{\wedge}) \tag{18}$$

$$\exp(t\boldsymbol{\tau}^{\wedge}) = \exp(\boldsymbol{\tau}^{\wedge})^t \tag{19}$$

$$\exp(-\boldsymbol{\tau}^{\wedge}) = \exp(\boldsymbol{\tau}^{\wedge})^{-1} \tag{20}$$

$$\exp(\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}) = \mathcal{X}\exp(\boldsymbol{\tau}^{\wedge})\mathcal{X}^{-1} \ , \tag{21}$$

where (21) can be checked by expanding the Taylor series.

2) The capitalized exponential map: The capitalized Exp and Log maps are convenient shortcuts to map vector elements $\boldsymbol{\tau} \in \mathbb{R}^m \ (\simeq \mathcal{TM}_{\mathcal{E}})$ directly with elements $\mathcal{X} \in \mathcal{M}$. We have,

$$\text{Exp} : \quad \mathbb{R}^m \to \mathcal{M} \quad ; \quad \boldsymbol{\tau} \mapsto \mathcal{X} = \text{Exp}(\boldsymbol{\tau}) \tag{22}$$

$$\text{Log} : \quad \mathcal{M} \to \mathbb{R}^m \quad ; \quad \mathcal{X} \mapsto \boldsymbol{\tau} = \text{Log}(\mathcal{X}) \ . \tag{23}$$

Example 4: The exponential map of $SO(3)$

We have seen in Ex. 3 that $\dot{\mathbf{R}} = \mathbf{R}[\boldsymbol{\omega}]_{\times} \in \mathfrak{so}(3)$. For $\boldsymbol{\omega}$ constant, this is an ordinary differential equation (ODE), whose solution is $\mathbf{R}(t) = \mathbf{R}_0\exp([\boldsymbol{\omega}]_{\times}t)$. At the origin $\mathbf{R}_0 = \mathbf{I}$ we have the exponential map,

$$\mathbf{R}(t) = \exp([\boldsymbol{\omega}]_{\times}t) \qquad \in SO(3) \ .$$

We now define the vector $\boldsymbol{\theta} \triangleq \mathbf{u}\theta \triangleq \boldsymbol{\omega}t \in \mathbb{R}^3$ as the integrated rotation in angle-axis form, with angle θ and unit axis $\mathbf{u}$. Thus $[\boldsymbol{\theta}]_{\times} \in \mathfrak{so}(3)$ is the total rotation expressed in the Lie algebra. We substitute it above and write the exponential as a power series,

$$\mathbf{R} = \exp([\boldsymbol{\theta}]_{\times}) = \sum_k \frac{\theta^k}{k!}([\mathbf{u}]_{\times})^k \ .$$

In order to find a closed-form expression, we write down a few powers of $[\mathbf{u}]_{\times}$,

$$\begin{aligned}
[\mathbf{u}]_{\times}^0 &= \mathbf{I}, & [\mathbf{u}]_{\times}^1 &= [\mathbf{u}]_{\times} \ , \\
[\mathbf{u}]_{\times}^2 &= [\mathbf{u}]_{\times}^2 = \mathbf{u}\mathbf{u}^{\top} - \mathbf{I}, & [\mathbf{u}]_{\times}^3 &= -[\mathbf{u}]_{\times} \ , \\
[\mathbf{u}]_{\times}^4 &= -[\mathbf{u}]_{\times}^2 \ , & &\cdots
\end{aligned}$$

and realize that all can be expressed as multiples of $\mathbf{I}$, $[\mathbf{u}]_{\times}$ or $[\mathbf{u}]_{\times}^2$. We thus rewrite the series as,

$$\begin{aligned}
\mathbf{R} = \mathbf{I} &+ [\mathbf{u}]_{\times}\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5 - \cdots\right) \\
&+ [\mathbf{u}]_{\times}^2\left(\tfrac{1}{2}\theta^2 - \tfrac{1}{4!}\theta^4 + \tfrac{1}{6!}\theta^6 - \cdots\right) \ ,
\end{aligned}$$

where we identify the series of $\sin\theta$ and $\cos\theta$, yielding the closed form,

$$\mathbf{R} = \exp([\mathbf{u}\theta]_{\times}) = \mathbf{I} + [\mathbf{u}]_{\times}\sin\theta + [\mathbf{u}]_{\times}^2(1-\cos\theta) \ .$$

This expression is the well known Rodrigues rotation formula. It can be used as the capitalized exponential just by doing $\mathbf{R} = \text{Exp}(\mathbf{u}\theta) = \exp([\mathbf{u}\theta]_{\times})$.

Clearly from (12),

$$\mathcal{X} = \text{Exp}(\boldsymbol{\tau}) \triangleq \exp(\boldsymbol{\tau}^{\wedge}) \tag{24}$$

$$\boldsymbol{\tau} = \text{Log}(\mathcal{X}) \triangleq \log(\mathcal{X})^{\vee} \ . \tag{25}$$

See the Appendices for details on the implementation of these maps for different manifolds.

3) Plus and minus: Plus and minus allow us to introduce increments between elements of a (curved) manifold, and express them in its (flat) tangent vector space. Denoted by $\oplus$ and $\ominus$, they combine one Exp/Log operation with one composition. Because of the non-commutativity of the composition, they are defined in right- and left- versions depending on the order of the operands. The right operators are (see Fig. 4-*right*),

$$\text{right-}\oplus : \quad \mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} \triangleq \mathcal{X} \circ \text{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \in \mathcal{M} \tag{26}$$

$$\text{right-}\ominus : \quad {}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \triangleq \text{Log}(\mathcal{X}^{-1}\circ\mathcal{Y}) \in \mathcal{TM}_{\mathcal{X}} \ , \tag{27}$$

Example 5: The unit quaternions group S^3 (cont.)

In the group S^3 (recall Ex. 2 and see *e.g.* [5]), the time derivative of the unit norm condition $\mathbf{q}^*\mathbf{q} = 1$ yields

$$\mathbf{q}^*\dot{\mathbf{q}} = -(\mathbf{q}^*\dot{\mathbf{q}})^* .$$

This reveals that $\mathbf{q}^*\dot{\mathbf{q}}$ is a pure quaternion (its real part is zero). Pure quaternions $\mathbf{u}v \in \mathbb{H}_p$ have the form

$$\mathbf{u}v = (iu_x + ju_y + ku_z)v = iv_x + jv_y + kv_z,$$

where $\mathbf{u} \triangleq iu_x + ju_y + ku_z$ is pure and unitary, v is the norm, and i, j, k are the generators of $\mathfrak{s}^3 = \mathbb{H}_p$. Rewriting the condition above we have,

$$\dot{\mathbf{q}} = \mathbf{q}\,\mathbf{u}\,v \qquad \in \mathcal{T}S^3_{\mathbf{q}_0},$$

which integrates to $\mathbf{q} = \mathbf{q}_0 \exp(\mathbf{u}vt)$. Letting $\mathbf{q}_0 = 1$ and defining $\boldsymbol{\phi} \triangleq \mathbf{u}\phi \triangleq \mathbf{u}vt$ we get the exponential map,

$$\mathbf{q} = \exp(\mathbf{u}\phi) \triangleq \sum \frac{\phi^k}{k!}\mathbf{u}^k \qquad \in S^3 .$$

The powers of $\mathbf{u}$ follow the pattern $1, \mathbf{u}, -1, -\mathbf{u}, 1, \cdots$. We group the terms in 1 and $\mathbf{u}$ and identify the series of $\cos\phi$ and $\sin\phi$. We get the closed form,

$$\mathbf{q} = \exp(\mathbf{u}\phi) = \cos(\phi) + \mathbf{u}\sin(\phi) ,$$

which is a beautiful extension of the Euler formula, $\exp(i\phi) = \cos\phi + i\sin\phi$. The elements of the Lie algebra $\mathbf{u}\phi \in \mathfrak{s}^3$ can be identified with the rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$ trough the mappings *hat* and *vee*,

$$\text{Hat}: \qquad \mathbb{R}^3 \to \mathfrak{s}^3; \qquad \boldsymbol{\theta} \mapsto \boldsymbol{\theta}^\wedge = 2\boldsymbol{\phi}$$
$$\text{Vee}: \qquad \mathfrak{s}^3 \to \mathbb{R}^3; \qquad \boldsymbol{\phi} \mapsto \boldsymbol{\phi}^\vee = \boldsymbol{\theta}/2 ,$$

where the factor 2 accounts for the double effect of the quaternion in the rotation action, $\mathbf{x}' = \mathbf{q}\,\mathbf{x}\,\mathbf{q}^*$. With this choice of Hat and Vee, the quaternion exponential

$$\mathbf{q} = \text{Exp}(\mathbf{u}\theta) = \cos(\theta/2) + \mathbf{u}\sin(\theta/2)$$

is equivalent to the rotation matrix $\mathbf{R} = \text{Exp}(\mathbf{u}\theta)$.

Figure 6. Two paths, $\mathcal{X} \circ {}^{\mathcal{X}}\delta$ and ${}^{\mathcal{E}}\delta \circ \mathcal{X}$, join the origin $\mathcal{E}$ with the point $\mathcal{Y}$. They both compose the element $\mathcal{X}$ with increments or 'deltas' expressed either in the local frame, ${}^{\mathcal{X}}\delta$, or in the origin, ${}^{\mathcal{E}}\delta$. Due to non-commutativity, the elements ${}^{\mathcal{X}}\delta$ and ${}^{\mathcal{E}}\delta$ are not equal. Their associated tangent vectors ${}^{\mathcal{X}}\boldsymbol{\tau} = \text{Log}({}^{\mathcal{X}}\delta)$ and ${}^{\mathcal{E}}\boldsymbol{\tau} = \text{Log}({}^{\mathcal{E}}\delta)$ are therefore unequal too. They are related by a linear transform ${}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau}$. $\mathbf{Ad}_{\mathcal{X}}$ is the adjoint of $\mathcal{M}$ at $\mathcal{X}$.

the right- forms of $\oplus$ and $\ominus$ by default.

4) The adjoint, and the adjoint matrix: If we identify $\mathcal{Y}$ in (26, 28), we arrive at ${}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}$, which determines a relation between the local and global tangent elements (Fig. 6). This is developed with (26–29) and (21) as

$$\text{Exp}({}^{\mathcal{E}}\boldsymbol{\tau})\mathcal{X} = \mathcal{X}\,\text{Exp}({}^{\mathcal{X}}\boldsymbol{\tau})$$
$$\exp({}^{\mathcal{E}}\boldsymbol{\tau}^\wedge) = \mathcal{X}\exp({}^{\mathcal{X}}\boldsymbol{\tau}^\wedge)\mathcal{X}^{-1} = \exp(\mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1})$$
$${}^{\mathcal{E}}\boldsymbol{\tau}^\wedge = \mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}$$

We thus define the *adjoint* of $\mathcal{M}$ at $\mathcal{X}$, noted $\text{Ad}_{\mathcal{X}}$, to be

$$\text{Ad}_{\mathcal{X}} : \mathfrak{m} \to \mathfrak{m}; \quad \boldsymbol{\tau}^\wedge \mapsto \text{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^\wedge) \triangleq \mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1} , \quad (30)$$

so that ${}^{\mathcal{E}}\boldsymbol{\tau}^\wedge = \text{Ad}_{\mathcal{X}}({}^{\mathcal{X}}\boldsymbol{\tau}^\wedge)$. The adjoint has two interesting (and easy to prove) properties,

$$\text{Linear}: \quad \text{Ad}_{\mathcal{X}}(a\boldsymbol{\tau} + b\boldsymbol{\sigma}) = \text{Ad}_{\mathcal{X}}(a\boldsymbol{\tau}) + \text{Ad}_{\mathcal{X}}(b\boldsymbol{\sigma})$$
$$\text{Homomorphism}: \quad \text{Ad}_{\mathcal{X}}(\text{Ad}_{\mathcal{Y}}(\boldsymbol{\tau})) = \text{Ad}_{\mathcal{X}\mathcal{Y}}(\boldsymbol{\tau}) .$$

Since it is linear, we can find an equivalent matrix operator $\mathbf{Ad}_{\mathcal{X}}$ that maps the isomorphic tangent vectors,

$$\mathbf{Ad}_{\mathcal{X}} : \mathbb{R}^m \to \mathbb{R}^m; \quad {}^{\mathcal{X}}\boldsymbol{\tau} \mapsto {}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}{}^{\mathcal{X}}\boldsymbol{\tau} , \quad (31)$$

which we call the *adjoint matrix*. This can be computed by applying $^\vee$ to (30), thus writing

$$\mathbf{Ad}_{\mathcal{X}}\,\boldsymbol{\tau} = (\mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1})^\vee , \quad (32)$$

then developing the right hand side to identify the adjoint matrix (see Ex. 6 and the appendices). Additional properties of the adjoint matrix are,

$$\mathcal{X} \oplus \boldsymbol{\tau} = (\mathbf{Ad}_{\mathcal{X}}\,\boldsymbol{\tau}) \oplus \mathcal{X} \quad (33)$$
$$\mathbf{Ad}_{\mathcal{X}^{-1}} = \mathbf{Ad}_{\mathcal{X}}^{-1} \quad (34)$$
$$\mathbf{Ad}_{\mathcal{X}\mathcal{Y}} = \mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}} . \quad (35)$$

We will use the adjoint matrix often as a way to linearly transform vectors of the tangent space at $\mathcal{X}$ onto vectors of the tangent space at the origin, with ${}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}{}^{\mathcal{X}}\boldsymbol{\tau}$. In this work, the adjoint matrix will be referred to as simply the adjoint.

and the left operators,

$$\text{left-}\oplus: \quad \mathcal{Y} = {}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X} \triangleq \text{Exp}({}^{\mathcal{E}}\boldsymbol{\tau}) \circ \mathcal{X} \in \mathcal{M} \quad (28)$$
$$\text{left-}\ominus: \quad {}^{\mathcal{E}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \triangleq \text{Log}(\mathcal{Y} \circ \mathcal{X}^{-1}) \in \mathcal{T}\mathcal{M}_{\mathcal{E}} . \quad (29)$$

Because in (26) $\text{Exp}({}^{\mathcal{X}}\boldsymbol{\tau})$ appears at the right hand side of the composition, ${}^{\mathcal{X}}\boldsymbol{\tau}$ belongs to the tangent space at $\mathcal{X}$ (see (27)): we say by convention[1] that ${}^{\mathcal{X}}\boldsymbol{\tau}$ is expressed in the *local* frame at $\mathcal{X}$ — and we note it with a left superscript. Conversely, in (28) $\text{Exp}({}^{\mathcal{E}}\boldsymbol{\tau})$ is on the left and we have ${}^{\mathcal{E}}\boldsymbol{\tau} \in \mathcal{T}\mathcal{M}_{\mathcal{E}}$: we say that ${}^{\mathcal{E}}\boldsymbol{\tau}$ is expressed in the *global* frame. Notice that while left- and right- $\oplus$ are distinguished by the operands order, the notation $\ominus$ in (27) and (29) is ambiguous. In this work we express perturbations locally by default and therefore we use
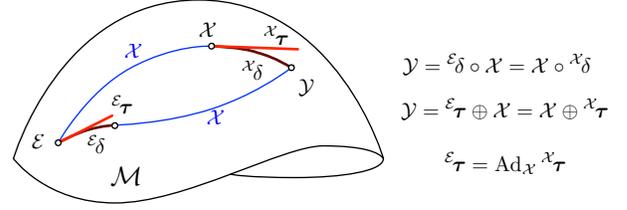
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[1] The convention sticks to that of frame transformation, *e.g.* ${}^G\mathbf{x} = \mathbf{R}\,{}^L\mathbf{x}$, where the matrix $\mathbf{R} \in SO(3)$ transforms local vectors into global. Notice that this convention is not shared by all authors, and for example [6] uses the opposite, ${}^L\mathbf{x} = \mathbf{R}\,{}^G\mathbf{x}$.

Example 6: The adjoint matrix of $SE(3)$

The $SE(3)$ group of rigid body motions (see App. D) has group, Lie algebra and vector elements,

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} , \quad \boldsymbol{\tau}^\wedge = \begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} , \quad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} .$$

The adjoint matrix is identified by developing (32) as

$$\begin{aligned}
\mathbf{Ad_M}\,\boldsymbol{\tau} &= (\mathbf{M}\boldsymbol{\tau}^\wedge\mathbf{M}^{-1})^\vee = \cdots = \\
&= \left(\begin{bmatrix} \mathbf{R}\,[\boldsymbol{\theta}]_\times\,\mathbf{R}^\top & \mathbf{R}\,[\boldsymbol{\theta}]_\times\,\mathbf{R}^\top\mathbf{t} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)^\vee \\
&= \left(\begin{bmatrix} [\mathbf{R}\boldsymbol{\theta}]_\times & -[\mathbf{t}]_\times\,\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)^\vee \\
&= \begin{bmatrix} -[\mathbf{t}]_\times\,\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\boldsymbol{\rho} \\ \mathbf{R}\boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -[\mathbf{t}]_\times\,\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}
\end{aligned}$$

where we used $[\mathbf{R}\boldsymbol{\theta}]_\times = \mathbf{R}\,[\boldsymbol{\theta}]_\times\,\mathbf{R}^\top$ and $[\mathbf{a}]_\times\,\mathbf{b} = -[\mathbf{b}]_\times\,\mathbf{a}$. So the adjoint is identified as

$$\mathbf{Ad_M} = \begin{bmatrix} \mathbf{R} & -[\mathbf{t}]_\times\,\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \quad \in \mathbb{R}^{6\times 6} .$$
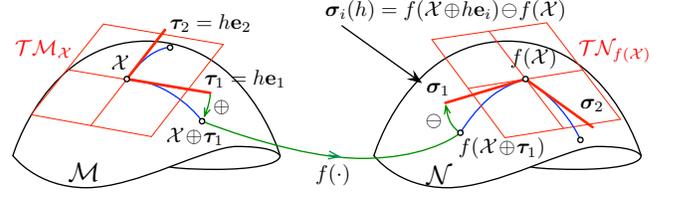

Figure 7. Right Jacobian of a function $f : \mathcal{M} \to \mathcal{N}$. The perturbation vectors in the canonical directions, $\boldsymbol{\tau}_i = h\mathbf{e}_i \in \mathcal{T}\mathcal{M}_{\mathcal{X}}$, are propagated to perturbation vectors $\boldsymbol{\sigma}_i \in \mathcal{T}\mathcal{N}_{f(\mathcal{X})}$ through the processes of plus, apply $f()$, and minus (arrows), obtaining $\boldsymbol{\sigma}_i(h) = f(\mathcal{X} \oplus h\mathbf{e}_i) \ominus f(\mathcal{X})$. The Jacobian columns are $\mathbf{j}_i = \lim_{h\to 0} \boldsymbol{\sigma}_i(h)/h$. We write the full Jacobian simply as $\mathbf{J} = \partial f / \partial\mathcal{X} = \lim_{\boldsymbol{\tau}\to 0} \sigma(\boldsymbol{\tau})/\boldsymbol{\tau}$.

E. Derivatives on Lie groups

Among the different ways to define derivatives in the context of Lie groups, we concentrate in those in the form of Jacobian matrices mapping vector tangent spaces. This is sufficient here, since in these spaces uncertainties and increments can be properly and easily defined. Using these Jacobians, the formulas for uncertainty management in Lie groups will largely resemble those in vector spaces.

The Jacobians described hereafter fulfill the chain rule, so that we can easily compute any derivative from the partial derivative blocks of *inversion*, *composition*, *exponentiation* and *action*. See Section III-A for details and proofs.

1) Reminder: Jacobians on vector spaces: For a multivariate function $f : \mathbb{R}^m \to \mathbb{R}^n$, the Jacobian matrix is defined as the $n \times m$ matrix stacking all partial derivatives,

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \in \mathbb{R}^{n\times m} . \quad (36)$$

It is handy to define this matrix in the following form. Let us partition $\mathbf{J} = [\mathbf{j}_1 \cdots \mathbf{j}_m]$, and let $\mathbf{j}_i = [\frac{\partial f_1}{\partial x_i} \cdots \frac{\partial f_n}{\partial x_i}]^\top$ be its i-th column vector. This column vector responds to

$$\mathbf{j}_i = \frac{\partial f(\mathbf{x})}{\partial x_i} \triangleq \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h} \in \mathbb{R}^n , \quad (37)$$

where $\mathbf{e}_i$ is the i-th vector of the natural basis of $\mathbb{R}^m$. Regarding the numerator, notice that the vector

$$\mathbf{v}_i(h) = f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x}) \quad \in \mathbb{R}^n \quad (38)$$

is the variation of $f(\mathbf{x})$ when $\mathbf{x}$ is perturbed in the direction of $\mathbf{e}_i$, and that the respective Jacobian column is just $\mathbf{j}_i = \partial\mathbf{v}_i(h)/\partial h = \lim_{h\to 0} \mathbf{v}_i(h)/h$. In this work, for the sake of

convenience, we introduce the compact form,

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \lim_{\mathbf{h}\to 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \in \mathbb{R}^{n\times m} , \quad (39)$$

with $\mathbf{h} \in \mathbb{R}^m$, which aglutinates all columns (37) to form the definition of (36). We remark that (39) is just a notation convenience, since division by the vector $\mathbf{h}$ is undefined and proper computation requires (37). However, this form may be used to identify Jacobians by developing the numerator into a form linear in $\mathbf{h}$, and identifying the left hand side as the Jacobian, that is,

$$\lim_{\mathbf{h}\to 0} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} = \cdots = \lim_{\mathbf{h}\to 0} \frac{\mathbf{J}\mathbf{h}}{\mathbf{h}} \triangleq \frac{\partial \mathbf{J}\mathbf{h}}{\partial \mathbf{h}} = \mathbf{J}. \quad (40)$$

Notice finally that for small values of $\mathbf{h}$ we have the linear approximation,

$$f(\mathbf{x} + \mathbf{h}) \xrightarrow{\mathbf{h}\to 0} f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\mathbf{h} . \quad (41)$$

2) Right Jacobians on Lie goups: Inspired by the standard derivative definition above, we can now use our right- $\oplus$ and $\ominus$ operators to define Jacobians of functions $f : \mathcal{M} \to \mathcal{N}$ acting on manifolds (see Fig. 7), obtaining a form akin to the above,

$$\begin{aligned}
\frac{^\mathcal{X}\partial f(\mathcal{X})}{\partial \mathcal{X}} &\triangleq \lim_{\boldsymbol{\tau}\to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \quad \in \mathbb{R}^{n\times m} \quad (42) \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \mathrm{Exp}(\boldsymbol{\tau}))\right)}{\boldsymbol{\tau}} \\
&= \frac{\partial \mathrm{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \mathrm{Exp}(\boldsymbol{\tau}))\right)}{\partial \boldsymbol{\tau}} \Bigg|_{\boldsymbol{\tau}=0} ,
\end{aligned}$$

where (42) mimics (39), and the other forms are alternative ways to look at it.[2] We call this the *right Jacobian* of f. Crucially, and thanks to the way right- $\oplus$ and $\ominus$ operate, variations in $\mathcal{X}$ and $f(\mathcal{X})$ are now expressed as vectors in the local tangent spaces, *i.e.*, tangent respectively at $\mathcal{X} \in \mathcal{M}$ and $f(\mathcal{X}) \in \mathcal{N}$. This derivative is then a proper Jacobian matrix $\mathbb{R}^{n\times m}$ linearly mapping the *local* tangent spaces $\mathcal{T}\mathcal{M}_{\mathcal{X}} \to \mathcal{T}\mathcal{N}_{f(\mathcal{X})}$ (and we mark the derivative with a local $\mathcal{X}$ superscript). Just as in vector spaces, the columns of this

---

[2] The notation $\frac{\partial f}{\partial \mathcal{X}}$ is chosen in front of other alternatives in order to make the chain rule readable, *i.e.*, $\frac{\partial \mathcal{Z}}{\partial \mathcal{X}} = \frac{\partial \mathcal{Z}}{\partial \mathcal{Y}} \frac{\partial \mathcal{Y}}{\partial \mathcal{X}}$.
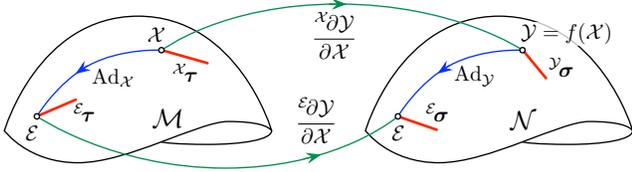
Figure 8. Linear maps between all tangent spaces involved in a function $\mathcal{Y} = f(\mathcal{X})$, from $\mathcal{M}$ to $\mathcal{N}$. The linear maps $^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\mathcal{X}}\,^{\mathcal{X}}\boldsymbol{\tau}$, $^{\mathcal{E}}\boldsymbol{\sigma} = \mathbf{Ad}_{\mathcal{Y}}\,^{\mathcal{Y}}\boldsymbol{\sigma}$, $^{\mathcal{E}}\boldsymbol{\sigma} = \frac{^{\mathcal{E}}\partial \mathcal{Y}}{^{\mathcal{X}}\partial \mathcal{X}}\,^{\mathcal{E}}\boldsymbol{\tau}$, and $^{\mathcal{Y}}\boldsymbol{\sigma} = \frac{^{\mathcal{X}}\partial \mathcal{Y}}{\partial \mathcal{X}}\,^{\mathcal{X}}\boldsymbol{\tau}$, form a loop that leads to (47).

matrix correspond to directional derivatives. That is, the vector (see Fig. 7 again, and compare the folowing to (38))

$$\boldsymbol{\sigma}_i(h) = f(\mathcal{X} \oplus h\mathbf{e}_i) \ominus f(\mathcal{X}) \quad \in \mathbb{R}^n \tag{43}$$

is the variation of $f(\mathcal{X})$ when $\mathcal{X}$ varies in the direction of $\mathbf{e}_i$, and the respective Jacobian column is $\mathbf{j}_i = \partial \boldsymbol{\sigma}_i(h)/\partial h$.

As before, we use (42) to effectively find Jacobians by resorting to the same mechanism (40). For example, for a 3D rotation $f : SO(3) \to \mathbb{R}^3$; $f(\mathbf{R}) = \mathbf{Rp}$, we have $\mathcal{M} = SO(3)$ and $\mathcal{N} = \mathbb{R}^3$ and so,

$$
\begin{aligned}
\frac{^{\mathbf{R}}\partial \mathbf{Rp}}{\partial \mathbf{R}} &= \lim_{\boldsymbol{\theta} \to 0} \frac{(\mathbf{R} \oplus \boldsymbol{\theta})\mathbf{p} \ominus \mathbf{Rp}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}\,\mathrm{Exp}(\boldsymbol{\theta})\mathbf{p} - \mathbf{Rp}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}(\mathbf{I} + [\boldsymbol{\theta}]_\times)\mathbf{p} - \mathbf{Rp}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}\,[\boldsymbol{\theta}]_\times\,\mathbf{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \to 0} \frac{-\mathbf{R}\,[\mathbf{p}]_\times\,\boldsymbol{\theta}}{\boldsymbol{\theta}} = -\mathbf{R}\,[\mathbf{p}]_\times \quad \in \mathbb{R}^{3 \times 3} .
\end{aligned}
$$

Many examples of this can be observed in Section III and in the appendices. Remark that whenever the function f passes from one manifold to another, the plus and minus operators in (42) must be selected appropriately: $\oplus$ for the domain $\mathcal{M}$, and $\ominus$ for the codomain or image $\mathcal{N}$.

For small values of $\boldsymbol{\tau}$, the following approximation holds,

$$f(\mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}) \xrightarrow[{}^{\mathcal{X}}\boldsymbol{\tau} \to 0]{} f(\mathcal{X}) \oplus \frac{^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau} \quad \in \mathcal{N} . \tag{44}$$

3) Left Jacobians on Lie groups: Derivatives can also be defined from the left- plus and minus operators, leading to,

$$
\begin{aligned}
\frac{^{\mathcal{E}}\partial f(\mathcal{X})}{\partial \mathcal{X}} &\triangleq \lim_{\boldsymbol{\tau} \to 0} \frac{f(\boldsymbol{\tau} \oplus \mathcal{X}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n \times m} \tag{45} \\
&= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}(f(\mathrm{Exp}(\boldsymbol{\tau}) \circ \mathcal{X}) \circ f(\mathcal{X})^{-1})}{\boldsymbol{\tau}} ,
\end{aligned}
$$

which we call the *left Jacobian* of f. Notice that now $\boldsymbol{\tau} \in \mathcal{TM}_{\mathcal{E}}$, and the numerator belongs to $\mathcal{TN}_{\mathcal{E}}$, thus the left Jacobian is a $n \times m$ matrix mapping the *global* tangent spaces, $\mathcal{TM}_{\mathcal{E}} \to \mathcal{TN}_{\mathcal{E}}$, which are the Lie algebras of $\mathcal{M}$ and $\mathcal{N}$ (and we mark the derivative with a global or origin $\mathcal{E}$ superscript). For small values of $\boldsymbol{\tau}$ the following holds,

$$f(^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X}) \xrightarrow[^{\mathcal{E}}\boldsymbol{\tau} \to 0]{} \frac{^{\mathcal{E}}\partial f(\mathcal{X})}{\partial \mathcal{X}}\,{}^{\mathcal{E}}\boldsymbol{\tau} \oplus f(\mathcal{X}) \quad \in \mathcal{N} . \tag{46}$$

We can show from (33, 44, 46) (see Fig. 8) that left and right derivatives are related by the adjoints of $\mathcal{M}$ and $\mathcal{N}$,

$$\frac{^{\mathcal{E}}\partial f(\mathcal{X})}{\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}} . \tag{47}$$
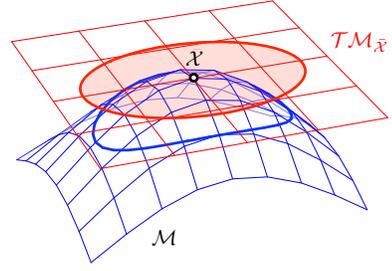

Figure 9. Uncertainty around a point $\bar{\mathcal{X}} \in \mathcal{M}$ is properly expressed as a covariance on the vector space tangent at the point (red). Using $\oplus$ (52), the probability ellipses in the tangent space are wrapped over the manifold (blue), thus illustrating the probability concentration region on the group.

4) Crossed right–left Jacobians: One can also define derivatives using right-plus but left-minus, or vice versa. These are useful in some practical cases, since they map local to global tangents or vice versa. To keep it short, we will just relate them to the other derivatives through the adjoints,

$$\frac{^{\mathcal{E}}\partial \mathcal{Y}}{^{\mathcal{X}}\partial \mathcal{X}} = \frac{^{\mathcal{E}}\partial \mathcal{Y}}{^{\mathcal{E}}\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{\mathcal{Y}} \frac{^{\mathcal{Y}}\partial \mathcal{Y}}{^{\mathcal{X}}\partial \mathcal{X}} \tag{48}$$

$$\frac{^{\mathcal{Y}}\partial \mathcal{Y}}{^{\mathcal{E}}\partial \mathcal{X}} = \frac{^{\mathcal{Y}}\partial \mathcal{Y}}{^{\mathcal{X}}\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}}^{-1} = \mathbf{Ad}_{\mathcal{Y}}^{-1} \frac{^{\mathcal{E}}\partial \mathcal{Y}}{^{\mathcal{E}}\partial \mathcal{X}} \tag{49}$$

where $\mathcal{Y} = f(\mathcal{X})$. Now, the sub- and super-scripts indicate the reference frames where the differentials are expressed. Respective small-tau approximations read,

$$f(\mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}) \xrightarrow[^{\mathcal{X}}\boldsymbol{\tau} \to 0]{} \frac{^{\mathcal{E}}\partial f(\mathcal{X})}{^{\mathcal{X}}\partial \mathcal{X}}\,{}^{\mathcal{X}}\boldsymbol{\tau} \oplus f(\mathcal{X}) \tag{50}$$

$$f(^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X}) \xrightarrow[^{\mathcal{E}}\boldsymbol{\tau} \to 0]{} f(\mathcal{X}) \oplus \frac{^{f(\mathcal{X})}\partial f(\mathcal{X})}{^{\mathcal{E}}\partial \mathcal{X}}\,{}^{\mathcal{E}}\boldsymbol{\tau} \tag{51}$$

F. Uncertainty in manifolds, covariance propagation

We define local perturbations $\boldsymbol{\tau}$ around a point $\bar{\mathcal{X}} \in \mathcal{M}$ in the tangent vector space $\mathcal{TM}_{\bar{\mathcal{X}}}$, using right- $\oplus$ and $\ominus$,

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \boldsymbol{\tau} , \qquad \boldsymbol{\tau} = \mathcal{X} \ominus \bar{\mathcal{X}} \in \mathcal{TM}_{\bar{\mathcal{X}}} . \tag{52}$$

Covariances matrices can be properly defined on this tangent space at $\bar{\mathcal{X}}$ through the standard expectation operator $\mathbb{E}[\cdot]$,

$$\boldsymbol{\Sigma}_{\mathcal{X}} \triangleq \mathbb{E}[\boldsymbol{\tau}\boldsymbol{\tau}^\top] = \mathbb{E}[(\mathcal{X} \ominus \bar{\mathcal{X}})(\mathcal{X} \ominus \bar{\mathcal{X}})^\top] \in \mathbb{R}^{n \times n} , \tag{53}$$

allowing us to define Gaussian variables on manifolds, $\mathcal{X} \sim \mathcal{N}(\bar{\mathcal{X}}, \boldsymbol{\Sigma}_{\mathcal{X}})$, see Fig. 9. Notice that although we write $\boldsymbol{\Sigma}_{\mathcal{X}}$, the covariance is rather that of the tangent perturbation $\boldsymbol{\tau}$. Since the dimension m of $\mathcal{TM}$ matches the degrees of freedom of $\mathcal{M}$, these covariances are well defined.[3]

Perturbations can also be expressed in the global reference, that is, in the tangent space at the origin $\mathcal{TM}_{\mathcal{E}}$, using left- $\oplus$ and $\ominus$,

$$\mathcal{X} = \boldsymbol{\tau} \oplus \bar{\mathcal{X}} , \qquad \boldsymbol{\tau} = \mathcal{X} \ominus \bar{\mathcal{X}} \in \mathcal{TM}_{\mathcal{E}} . \tag{54}$$

This allows global specification of covariance matrices using left-minus in (53). For example, a 3D orientation that is known up to rotations in the horizontal plane can be associated to a

---

[3]A naive definition $\boldsymbol{\Sigma}_{\mathcal{X}} \triangleq \mathbb{E}[(\mathcal{X} - \bar{\mathcal{X}})(\mathcal{X} - \bar{\mathcal{X}})^\top]$ is always ill-defined if $\mathrm{size}(\mathcal{X}) > \dim(\mathcal{M})$, which is the case for most non-trivial manifolds.
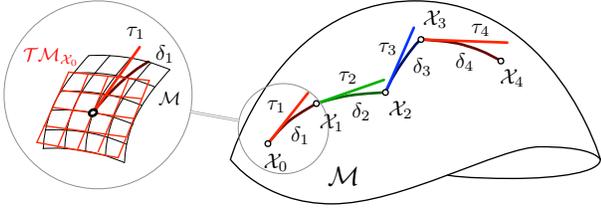
Figure 10. Motion integration on a manifold. Each motion data produces a step $\boldsymbol{\tau}_k \in \mathcal{T}\mathcal{M}_{\mathcal{X}_{k-1}}$, which is wrapped to a local motion increment or 'delta' $\delta_k = \mathrm{Exp}(\boldsymbol{\tau}_k) \in \mathcal{M}$, and then composed with $\mathcal{X}_{k-1}$ to yield $\mathcal{X}_k = \mathcal{X}_{k-1} \circ \mathrm{Exp}(\boldsymbol{\tau}_k) = \mathcal{X}_{k-1} \circ \delta_k \in \mathcal{M}$.

covariance $^{\mathcal{E}}\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_\phi^2, \sigma_\theta^2, \infty)$ only if $^{\mathcal{E}}\boldsymbol{\Sigma}$ is specified in the global reference.

Notice finally that there is no consensus on this last point. While [7] and us use local perturbations $\mathcal{X} = \bar{\mathcal{X}} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}$, [4], [8] use perturbations around the origin, $\mathcal{X} = {}^{\mathcal{E}}\boldsymbol{\tau} \oplus \bar{\mathcal{X}}$, yielding global covariance specifications. Since global and local perturbations are related by the adjoint (31), their covariances can be transformed according to

$$^{\mathcal{E}}\boldsymbol{\Sigma}_\mathcal{X} = \mathbf{Ad}_\mathcal{X} \, {}^{\mathcal{X}}\boldsymbol{\Sigma}_\mathcal{X} \, \mathbf{Ad}_\mathcal{X}^\top \ . \tag{55}$$

Covariance propagation through a function $f : \mathcal{M} \to \mathcal{N}; \mathcal{X} \mapsto \mathcal{Y} = f(\mathcal{X})$ just requires the linearization (44) with Jacobian matrices (42) to yield the familiar formula,

$$\boldsymbol{\Sigma}_\mathcal{Y} \approx \frac{\partial f}{\partial \mathcal{X}} \boldsymbol{\Sigma}_\mathcal{X} \frac{\partial f}{\partial \mathcal{X}}^\top \ \in \mathbb{R}^{m \times m} \ . \tag{56}$$

G. Discrete integration on manifolds

The exponential map $\mathcal{X}(t) = \mathcal{X}_0 \circ \mathrm{Exp}(\mathbf{v}t)$ performs the continuous-time integral of constant velocities $\mathbf{v} \in \mathcal{T}\mathcal{M}_{\mathcal{X}_0}$ onto the manifold. Non-constant velocities $\mathbf{v}(t)$ are typically handled by segmenting them into piecewise constant bits $\mathbf{v}_k \in \mathcal{T}\mathcal{M}_{\mathcal{X}_{k-1}}$, of (short) duration δt_k, and writing the discrete integral

$$\mathcal{X}_k = \mathcal{X}_0 \circ \mathrm{Exp}(\mathbf{v}_1 \delta t_1) \circ \mathrm{Exp}(\mathbf{v}_1 \delta t_2) \circ \cdots \circ \mathrm{Exp}(\mathbf{v}_k \delta t_k) \ .$$

Equivalently (Fig. 10), we can define $\boldsymbol{\tau}_k = \mathbf{v}_k \delta t_k$ and use $\oplus$ to construct the integral as a "sum" of (small) discrete tangent steps $\boldsymbol{\tau}_k \in \mathcal{T}\mathcal{M}_{\mathcal{X}_{k-1}}$, i.e., $\mathcal{X}_k \triangleq \mathcal{X}_0 \oplus \boldsymbol{\tau}_1 \oplus \cdots \oplus \boldsymbol{\tau}_k$. We write all these variants in recursive form,

$$\mathcal{X}_k = \mathcal{X}_{k-1} \oplus \boldsymbol{\tau}_k = \mathcal{X}_{k-1} \circ \mathrm{Exp}(\boldsymbol{\tau}_k) = \mathcal{X}_{k-1} \circ \mathrm{Exp}(\mathbf{v}_k \delta t_k) \ . \tag{57}$$

Common examples are the integration of 3D angular rates $\boldsymbol{\omega}$ into the rotation matrix, $\mathbf{R}_k = \mathbf{R}_{k-1} \mathrm{Exp}(\boldsymbol{\omega}_k \delta t)$, or into the quaternion, $\mathbf{q}_k = \mathbf{q}_{k-1} \mathrm{Exp}(\boldsymbol{\omega}_k \delta t)$.

III. DERIVATION RULES ON MANIFOLDS

For all the typical manifolds $\mathcal{M}$ that we use, we can determine closed forms for the elementary derivatives of *inversion*, *composition*, *exponentiation* and *action*. Moreover, some of these forms can be related to the adjoint $\mathbf{Ad}_\mathcal{X}$, which becomes a central block of the derivation process. Other forms for Log, $\oplus$ and $\ominus$ can be easily derived from them. Once these forms or 'blocks' are found, all other derivatives follow by the

chain rule. Except for the so called *left Jacobian*, which we also present below, all derivatives developed here are right-derivatives, *i.e.*, defined by (42). By following the hints here, the interested reader should find no particular difficulties in developing the left-derivatives. For the reader not willing to do this effort, equation (47) can be used to this end, since

$$\frac{^{\mathcal{E}}\partial f(\mathcal{X})}{\partial \mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}} \mathbf{Ad}_\mathcal{X}^{-1} \ . \tag{58}$$

We use the notations $\mathbf{J}_\mathcal{X}^{f(\mathcal{X})} \triangleq \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}}$ and $\mathbf{J}_\mathcal{X}^\mathcal{Y} \triangleq \frac{\partial \mathcal{Y}}{\partial \mathcal{X}}$.

A. The chain rule

For $\mathcal{Y} = f(\mathcal{X})$ and $\mathcal{Z} = g(\mathcal{Y})$ we have $\mathcal{Z} = g(f(\mathcal{X}))$. The chain rule simply states,

$$\frac{\partial \mathcal{Z}}{\partial \mathcal{X}} = \frac{\partial \mathcal{Z}}{\partial \mathcal{Y}} \frac{\partial \mathcal{Y}}{\partial \mathcal{X}} \qquad \text{or} \qquad \mathbf{J}_\mathcal{X}^\mathcal{Z} = \mathbf{J}_\mathcal{Y}^\mathcal{Z} \mathbf{J}_\mathcal{X}^\mathcal{Y} \ . \tag{59}$$

We prove it here for the right Jacobian using (44) thrice,

$$g(f(\mathcal{X})) \oplus \mathbf{J}_\mathcal{X}^\mathcal{Z} \boldsymbol{\tau} \leftarrow g(f(\mathcal{X} \oplus \boldsymbol{\tau})) \rightarrow g(f(\mathcal{X}) \ \oplus \mathbf{J}_\mathcal{X}^\mathcal{Y} \boldsymbol{\tau})$$
$$\rightarrow g(f(\mathcal{X})) \oplus \mathbf{J}_\mathcal{Y}^\mathcal{Z} \mathbf{J}_\mathcal{X}^\mathcal{Y} \boldsymbol{\tau}$$

with the arrows indicating limit as $\boldsymbol{\tau} \to 0$, and so $\mathbf{J}_\mathcal{X}^\mathcal{Z} = \mathbf{J}_\mathcal{Y}^\mathcal{Z} \mathbf{J}_\mathcal{X}^\mathcal{Y}$. The proof for the left and crossed Jacobians is akin, using respectively (46, 50, 51). Notice that when mixing right, left and crossed Jacobians, we need to chain also the reference frames, as in *e.g.*

$$\frac{^z\partial\mathcal{Z}}{^\varepsilon\partial\mathcal{X}} = \frac{^z\partial\mathcal{Z}}{^y\partial\mathcal{Y}}\frac{^y\partial\mathcal{Y}}{^\varepsilon\partial\mathcal{X}} = \frac{^z\partial\mathcal{Z}}{^\varepsilon\partial\mathcal{Y}}\frac{^\varepsilon\partial\mathcal{Y}}{^\varepsilon\partial\mathcal{X}} \tag{60}$$

$$\frac{^\varepsilon\partial\mathcal{Z}}{^x\partial\mathcal{X}} = \frac{^\varepsilon\partial\mathcal{Z}}{^y\partial\mathcal{Y}}\frac{^y\partial\mathcal{Y}}{^x\partial\mathcal{X}} = \frac{^\varepsilon\partial\mathcal{Z}}{^\varepsilon\partial\mathcal{Y}}\frac{^\varepsilon\partial\mathcal{Y}}{^x\partial\mathcal{X}} \ , \tag{61}$$

where the two identities of (60) are proven below,

$$g(f(^\mathcal{E}\boldsymbol{\tau} \oplus \mathcal{X})) \xrightarrow[\varepsilon\boldsymbol{\tau}\to 0]{(51)} g(f(\mathcal{X})) \oplus \mathbf{J}_{\varepsilon\mathcal{X}}^{z\mathcal{Z}} {}^\varepsilon\boldsymbol{\tau} \ ;$$

$$g(f(^\mathcal{E}\boldsymbol{\tau} \oplus \mathcal{X})) \xrightarrow[\varepsilon\boldsymbol{\tau}\to 0]{(51)} g(f(\mathcal{X}) \ \oplus \mathbf{J}_{\varepsilon\mathcal{X}}^{y\mathcal{Y}} {}^\varepsilon\boldsymbol{\tau}) \rightarrow$$
$$\xrightarrow[\varepsilon\boldsymbol{\tau}\to 0]{(44)} g(f(\mathcal{X})) \oplus \mathbf{J}_{y\mathcal{Y}}^{z\mathcal{Z}} \mathbf{J}_{\varepsilon\mathcal{X}}^{y\mathcal{Y}} {}^\varepsilon\boldsymbol{\tau} \ , \quad Q.E.D. \ ,$$

$$g(f(^\mathcal{E}\boldsymbol{\tau} \oplus \mathcal{X})) \xrightarrow[\varepsilon\boldsymbol{\tau}\to 0]{(46)} g(\mathbf{J}_{\varepsilon\mathcal{X}}^{\varepsilon\mathcal{Y}} {}^\varepsilon\boldsymbol{\tau} \oplus f(\mathcal{X})) \rightarrow$$
$$\xrightarrow[\varepsilon\boldsymbol{\tau}\to 0]{(51)} g(f(\mathcal{X})) \oplus \mathbf{J}_{\varepsilon\mathcal{Y}}^{z\mathcal{Z}} \mathbf{J}_{\varepsilon\mathcal{X}}^{\varepsilon\mathcal{Y}} {}^\varepsilon\boldsymbol{\tau} \ , \quad Q.E.D. \ .$$

B. Elementary derivative blocks

1) Jacobian of Inverse: We define with (42)

$$\mathbf{J}_\mathcal{X}^{\mathcal{X}^{-1}} \triangleq \frac{^\mathcal{X}\partial \mathcal{X}^{-1}}{\partial \mathcal{X}} \in \mathbb{R}^{m \times m} \ . \tag{62}$$

This can be determined from the adjoint using (21) and (32),

$$\mathbf{J}_\mathcal{X}^{\mathcal{X}^{-1}} = \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Log}((\mathcal{X}^{-1})^{-1}(\mathcal{X}\,\mathrm{Exp}(\boldsymbol{\tau}))^{-1})}{\boldsymbol{\tau}}$$
$$= \lim_{\boldsymbol{\tau}\to 0} \frac{\mathrm{Log}(\mathcal{X}\,\mathrm{Exp}(-\boldsymbol{\tau})\mathcal{X}^{-1})}{\boldsymbol{\tau}}$$
$$= \lim_{\boldsymbol{\tau}\to 0} \frac{(\mathcal{X}(-\boldsymbol{\tau})^\wedge \mathcal{X}^{-1})^\vee}{\boldsymbol{\tau}} = -\mathbf{Ad}_\mathcal{X} \ . \tag{63}$$

2) Jacobians of Composition: We define with (42)

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X} \circ \mathcal{Y}} \triangleq \frac{^{\mathcal{X}} \partial \mathcal{X} \circ \mathcal{Y}}{\partial \mathcal{X}} \qquad \in \mathbb{R}^{m \times m} \qquad (64)$$

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X} \circ \mathcal{Y}} \triangleq \frac{^{\mathcal{Y}} \partial \mathcal{X} \circ \mathcal{Y}}{\partial \mathcal{Y}} \qquad \in \mathbb{R}^{m \times m} \; , \qquad (65)$$

and using (32) as above and (34),

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X} \circ \mathcal{Y}} = \mathbf{Ad}_{\mathcal{Y}}^{-1} \qquad (66)$$

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X} \circ \mathcal{Y}} = \mathbf{I} \qquad (67)$$

3) Jacobians of $\mathcal{M}$: We define the right Jacobian of $\mathcal{M}$ as the right Jacobian of $\mathcal{X} = \mathrm{Exp}(\boldsymbol{\tau})$, *i.e.*, for $\boldsymbol{\tau} \in \mathbb{R}^m$,

$$\mathbf{J}_r(\boldsymbol{\tau}) \triangleq \frac{^{\boldsymbol{\tau}} \partial \mathrm{Exp}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} \in \mathbb{R}^{m \times m} \; , \qquad (68)$$

which is defined with (42). The right Jacobian maps variations of the argument $\boldsymbol{\tau}$ into variations in the *local* tangent space at $\mathrm{Exp}(\boldsymbol{\tau})$. From (42) it is easy to prove that, for small $\delta\boldsymbol{\tau}$, the following approximations hold,

$$\mathrm{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) \approx \mathrm{Exp}(\boldsymbol{\tau}) \mathrm{Exp}(\mathbf{J}_r(\boldsymbol{\tau})\delta\boldsymbol{\tau}) \qquad (69)$$

$$\mathrm{Exp}(\boldsymbol{\tau}) \mathrm{Exp}(\delta\boldsymbol{\tau}) \approx \mathrm{Exp}(\boldsymbol{\tau} + \mathbf{J}_r^{-1}(\boldsymbol{\tau})\, \delta\boldsymbol{\tau}) \qquad (70)$$

$$\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau}) \mathrm{Exp}(\delta\boldsymbol{\tau})) \approx \boldsymbol{\tau} + \mathbf{J}_r^{-1}(\boldsymbol{\tau})\, \delta\boldsymbol{\tau} \; . \qquad (71)$$

Complementarily, the left Jacobian $\mathbf{J}_l$ is defined by,

$$\mathbf{J}_l(\boldsymbol{\tau}) \triangleq \frac{^{\mathcal{E}} \partial \mathrm{Exp}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} \in \mathbb{R}^{m \times m} \; , \qquad (72)$$

using the left derivative (45), leading to the approximations

$$\mathrm{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) \approx \mathrm{Exp}(\mathbf{J}_l(\boldsymbol{\tau})\delta\boldsymbol{\tau}) \mathrm{Exp}(\boldsymbol{\tau}) \qquad (73)$$

$$\mathrm{Exp}(\delta\boldsymbol{\tau}) \mathrm{Exp}(\boldsymbol{\tau}) \approx \mathrm{Exp}(\boldsymbol{\tau} + \mathbf{J}_l^{-1}(\boldsymbol{\tau})\, \delta\boldsymbol{\tau}) \qquad (74)$$

$$\mathrm{Log}(\mathrm{Exp}(\delta\boldsymbol{\tau}) \mathrm{Exp}(\boldsymbol{\tau})) \approx \boldsymbol{\tau} + \mathbf{J}_l^{-1}(\boldsymbol{\tau})\, \delta\boldsymbol{\tau} \; . \qquad (75)$$

The left Jacobian maps variations of the argument $\boldsymbol{\tau}$ into variations in the *global* tangent space or Lie algebra. From (69, 73) we can relate left- and right- Jacobians with the adjoint,

$$\mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})} = \mathbf{J}_l(\boldsymbol{\tau})\, \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \; . \qquad (76)$$

Also, the chain rule allows us to develop

$$\begin{aligned} \mathbf{J}_r(-\boldsymbol{\tau}) \triangleq \mathbf{J}_{-\boldsymbol{\tau}}^{\mathrm{Exp}(-\boldsymbol{\tau})} &= \mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(-\boldsymbol{\tau})} \mathbf{J}_{-\boldsymbol{\tau}}^{\boldsymbol{\tau}} = \mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(\boldsymbol{\tau})^{-1}}(-\mathbf{I}) \\ &= -\mathbf{J}_{\mathrm{Exp}(\boldsymbol{\tau})}^{\mathrm{Exp}(\boldsymbol{\tau})^{-1}} \mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(\boldsymbol{\tau})} = \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})} \mathbf{J}_r(\boldsymbol{\tau}) \\ &= \mathbf{J}_l(\boldsymbol{\tau}) \; . \end{aligned} \qquad (77)$$

Closed forms of $\mathbf{J}_r$, $\mathbf{J}_r^{-1}$, $\mathbf{J}_l$ and $\mathbf{J}_l^{-1}$ exist for the typical manifolds in use. See the appendices for reference.

4) Jacobians of the group action: For $\mathcal{X} \in \mathcal{M}$ and $v \in \mathcal{V}$, we define with (42)

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X} \cdot v} \triangleq \frac{^{\mathcal{X}} \partial \mathcal{X} \cdot v}{\partial \mathcal{X}} \qquad (78)$$

$$\mathbf{J}_v^{\mathcal{X} \cdot v} \triangleq \frac{^{v} \partial \mathcal{X} \cdot v}{\partial v} \; . \qquad (79)$$

Since group actions depend on the set $\mathcal{V}$, these expressions cannot be generalized. See the appendices for reference.

C. Useful, but deduced, derivative blocks

1) Jacobian of Log: For $\boldsymbol{\tau} = \mathrm{Log}(\mathcal{X})$, and from (71),

$$\mathbf{J}_{\mathcal{X}}^{\mathrm{Log}(\mathcal{X})} = \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \; . \qquad (80)$$

2) Jacobians of $\oplus$ and $\ominus$: We have

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{X} \oplus \boldsymbol{\tau}} = \mathbf{J}_{\mathcal{X}}^{\mathcal{X} \circ (\mathrm{Exp}(\boldsymbol{\tau}))} \qquad = \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}^{-1} \qquad (81)$$

$$\mathbf{J}_{\boldsymbol{\tau}}^{\mathcal{X} \oplus \boldsymbol{\tau}} = \mathbf{J}_{\mathrm{Exp}(\boldsymbol{\tau})}^{\mathcal{X} \circ (\mathrm{Exp}(\boldsymbol{\tau}))} \mathbf{J}_{\boldsymbol{\tau}}^{\mathrm{Exp}(\boldsymbol{\tau})} = \mathbf{J}_r(\boldsymbol{\tau}) \qquad (82)$$

and given $\mathcal{Z} = \mathcal{X}^{-1} \circ \mathcal{Y}$ and $\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} = \mathrm{Log}(\mathcal{Z})$,

$$\mathbf{J}_{\mathcal{X}}^{\mathcal{Y} \ominus \mathcal{X}} = \mathbf{J}_{\mathcal{Z}}^{\mathrm{Log}(\mathcal{Z})} \mathbf{J}_{\mathcal{X}^{-1}}^{\mathcal{Z}} \mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = -\mathbf{J}_l^{-1}(\boldsymbol{\tau}) \qquad (83)$$

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{Y} \ominus \mathcal{X}} = \mathbf{J}_{\mathcal{Z}}^{\mathrm{Log}(\mathcal{Z})} \mathbf{J}_{\mathcal{Y}}^{\mathcal{Z}} \qquad = \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \; . \qquad (84)$$

where the former is proven here

$$\begin{aligned} \mathbf{J}_{\mathcal{X}}^{\mathcal{Y} \ominus \mathcal{X}} &= \mathbf{J}_{(\mathcal{X}^{-1} \circ \mathcal{Y})}^{\mathrm{Log}(\mathcal{X}^{-1} \circ \mathcal{Y})} \mathbf{J}_{\mathcal{X}^{-1}}^{(\mathcal{X}^{-1} \circ \mathcal{Y})} \mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} \\ (80, 66, 63) &= \quad \mathbf{J}_r^{-1}(\boldsymbol{\tau}) \quad \mathbf{Ad}_{\mathcal{Y}}^{-1} \quad (-\mathbf{Ad}_{\mathcal{X}}) \\ (34, 35) &= -\mathbf{J}_r^{-1}(\boldsymbol{\tau}) \mathbf{Ad}_{\mathcal{Y}^{-1} \mathcal{X}} \\ &= -\mathbf{J}_r^{-1}(\boldsymbol{\tau}) \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\tau})}^{-1} \\ (76) &= -\mathbf{J}_l^{-1}(\boldsymbol{\tau}) \; . \end{aligned}$$

APPENDIX A
THE 2D ROTATION GROUPS S^1 AND $SO(2)$

The Lie group S^1 is the group of unit complex numbers under complex product. Its topology is the unit circle, or the unit 1-sphere, and therefore the name S^1. The group, Lie algebra and vector elements have the form,

$$\mathbf{z} = \cos\theta + i\sin\theta, \qquad \tau^{\wedge} = i\theta, \qquad \tau = \theta \; . \qquad (85)$$

Inversion and composition are achieved by conjugation $\mathbf{z}^{-1} = \mathbf{z}^*$, and product $\mathbf{z}_a \circ \mathbf{z}_b = \mathbf{z}_a \mathbf{z}_b$.

The group $SO(2)$ is the group of special orthogonal matrices in the plane, or rotation matrices, under matrix multiplication. Group, Lie algebra and vector elements have the form,

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \tau^{\wedge} = [\theta]_{\times} \triangleq \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}, \quad \tau = \theta \; . \qquad (86)$$

Inversion and composition are achieved by transposition $\mathbf{R}^{-1} = \mathbf{R}^{\top}$, and product $\mathbf{R}_a \circ \mathbf{R}_b = \mathbf{R}_a \mathbf{R}_b$.

Both groups rotate 2-vectors, and they have isomorphic tangent spaces. We thus study them together.

A. Exp and Log maps

Exp and Log maps may be defined for complex numbers of S^1 and rotation matrices of $SO(2)$. For S^1 we have,

$$\mathbf{z} = \mathrm{Exp}(\theta) = \cos\theta + i\sin\theta \qquad \in \mathbb{C} \qquad (87)$$

$$\theta = \mathrm{Log}(\mathbf{z}) = \arctan(\mathrm{Im}(\mathbf{z}), \mathrm{Re}(\mathbf{z})) \qquad \in \mathbb{R} \; , \qquad (88)$$

where (87) is the Euler formula, whereas for $SO(2)$,

$$\mathbf{R} = \mathrm{Exp}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \in \mathbb{R}^{2 \times 2} \qquad (89)$$

$$\theta = \mathrm{Log}(\mathbf{R}) = \arctan(r_{21}, r_{11}) \qquad \in \mathbb{R} \; . \qquad (90)$$

B. Inverse, composition

Since our defined derivatives map tangent vector spaces, and these spaces coincide for the planar rotation manifolds of S^1 and $SO(2)$, *i.e.*, $\theta = \mathrm{Log}(\mathbf{z}) = \mathrm{Log}(\mathbf{R})$, it follows that the Jacobians are independent of the representation used ($\mathbf{z}$ or $\mathbf{R}$). We thus consider generic 2D rotation elements, and note them with the sans-serif font R.

We have

$$\mathsf{R}(\theta)^{-1} = \mathsf{R}(-\theta) \tag{91}$$
$$\mathsf{Q} \circ \mathsf{R} = \mathsf{R} \circ \mathsf{Q} \;, \tag{92}$$

since planar rotations are commutative. It follows that,

$$\mathrm{Exp}(\theta_1 + \theta_2) = \mathrm{Exp}(\theta_1) \circ \mathrm{Exp}(\theta_2) \tag{93}$$
$$\mathrm{Log}(\mathsf{Q} \circ \mathsf{R}) = \mathrm{Log}(\mathsf{Q}) + \mathrm{Log}(\mathsf{R}) \tag{94}$$
$$\mathsf{Q} \ominus \mathsf{R} = \theta_Q - \theta_R \;. \tag{95}$$

C. Derivative blocks

1) Adjoint and other trivial Jacobians: From (42), Section III-B and the properties above, the following scalar derivative blocks become trivial,

$$\mathbf{Ad}_{\mathsf{R}} = 1 \qquad\qquad \in \mathbb{R} \tag{96}$$
$$\mathbf{J}_{\mathsf{R}}^{\mathsf{R}^{-1}} = -1 \qquad\qquad \in \mathbb{R} \tag{97}$$
$$\mathbf{J}_{\mathsf{Q}}^{\mathsf{Q} \circ \mathsf{R}} = \mathbf{J}_{\mathsf{R}}^{\mathsf{Q} \circ \mathsf{R}} = 1 \qquad\qquad \in \mathbb{R} \tag{98}$$
$$\mathbf{J}_r(\theta) = \mathbf{J}_l(\theta) = 1 \qquad\qquad \in \mathbb{R} \tag{99}$$
$$\mathbf{J}_{\mathsf{R}}^{\mathsf{R} \oplus \theta} = \mathbf{J}_{\theta}^{\mathsf{R} \oplus \theta} = 1 \qquad\qquad \in \mathbb{R} \tag{100}$$
$$\mathbf{J}_{\mathsf{Q}}^{\mathsf{Q} \ominus \mathsf{R}} = -\mathbf{J}_{\mathsf{R}}^{\mathsf{Q} \ominus \mathsf{R}} = 1 \qquad\qquad \in \mathbb{R} \tag{101}$$

2) Rotation action: For the action $\mathsf{R} \cdot \mathbf{v}$ we have,

$$\begin{aligned}
\mathbf{J}_{\mathsf{R}}^{\mathsf{R} \cdot \mathbf{v}} &= \lim_{\theta \to 0} \frac{\mathbf{R}\,\mathrm{Exp}(\theta)\mathbf{v} - \mathbf{R}\mathbf{v}}{\theta} \\
&= \lim_{\theta \to 0} \frac{\mathbf{R}(\mathbf{I} + [\theta]_\times)\mathbf{v} - \mathbf{R}\mathbf{v}}{\theta} \\
&= \lim_{\theta \to 0} \frac{\mathbf{R}\,[\theta]_\times \mathbf{v}}{\theta} = \mathbf{R}\,[1]_\times \mathbf{v} \qquad \in \mathbb{R}^{2 \times 1} \tag{102}
\end{aligned}$$

and

$$\mathbf{J}_{\mathbf{v}}^{\mathsf{R} \cdot \mathbf{v}} = \frac{\partial \mathbf{R}\mathbf{v}}{\partial \mathbf{v}} = \mathbf{R} \qquad\qquad \in \mathbb{R}^{2 \times 2} \;. \tag{103}$$

APPENDIX B
THE 3D ROTATION GROUPS S^3 AND $SO(3)$

The Lie group S^3 is the group of unit quaternions under the quaternion multiplication. Its topology is the unit 3-sphere in $\mathbb{R}^4$, and therefore its name S^3. Quaternions may be represented by either of these equivalent forms,

$$\begin{aligned}
\mathbf{q} = w + ix + jy + kz = w + \mathbf{v} &\in \mathbb{H} \\
= \begin{bmatrix} w & x & y & z \end{bmatrix}^\top = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} &\in \mathbb{H} \;, \tag{104}
\end{aligned}$$

where $w, x, y, z \in \mathbb{R}$, and i, j, k are three unit imaginary numbers such that $i^2 = j^2 = k^2 = ijk = -1$. The scalar w is known as the scalar or real part, and $\mathbf{v} \in \mathbb{H}_p$ as the vector or imaginary part. We note $\mathbb{H}_p$ the set of pure quaternions, *i.e.*, of null scalar part, with dimension 3. Inversion and composition

are achieved by conjugation $\mathbf{q}^{-1} = \mathbf{q}^*$, where $\mathbf{q}^* \triangleq w - \mathbf{v}$ is the conjugate, and product $\mathbf{q}_a \circ \mathbf{q}_b = \mathbf{q}_a \, \mathbf{q}_b$.

The group $SO(3)$ is the group of special orthogonal matrices in 3D space, or rotation matrices, under matrix multiplication. Inversion and composition are achieved with transposition and product as in all groups $SO(n)$.

Both groups rotate 3-vectors. They have isomorphic tangent spaces whose elements are identifiable with rotation vectors in $\mathbb{R}^3$, so we study them together. It is in this space $\mathbb{R}^3$ where we define the vectors of rotation rate $\boldsymbol{\omega} \triangleq \mathbf{u}\omega$, angle-axis $\boldsymbol{\theta} \triangleq \mathbf{u}\theta$, and all perturbations and uncertainties.

The quaternion manifold S^3 is a double cover of $SO(3)$, *i.e.*, $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation $\mathbf{R}$. The first cover corresponds to quaternions with positive real part $w > 0$. These groups can be considered isomorphic up to the first cover. For an in-depth reference consult [5].

A. Exp and Log maps

The Exp and Log maps may be defined for quaternions of S^3 and rotation matrices of $SO(3)$. For quaternions $\mathbf{q} = (w, \mathbf{v}) \in \mathbb{H}$ we have (see Ex. 5),

$$\mathbf{q} = \mathrm{Exp}(\theta\mathbf{u}) \triangleq \cos(\theta/2) + \mathbf{u}\sin(\theta/2) \qquad \in \mathbb{H} \tag{105}$$
$$\theta\mathbf{u} = \mathrm{Log}(\mathbf{q}) \triangleq 2\,\mathbf{v}\frac{\arctan(\|\mathbf{v}\|, w)}{\|\mathbf{v}\|} \qquad \in \mathbb{R}^3 \;, \tag{106}$$

We can avoid eventual problems due to the double cover of $\mathbf{q}$ by ensuring that its scalar part w is positive before doing the Log. If it is not, we can substitute $\mathbf{q}$ by $-\mathbf{q}$ before the Log.

For rotation matrices we have (see Ex. 4),

$$\mathbf{R} = \mathrm{Exp}(\theta\mathbf{u}) \triangleq \mathbf{I} + \sin\theta\,[\mathbf{u}]_\times + (1 - \cos\theta)\,[\mathbf{u}]_\times^2 \quad \in \mathbb{R}^{3\times3} \tag{107}$$

$$\theta\mathbf{u} = \mathrm{Log}(\mathbf{R}) \triangleq \frac{\theta(\mathbf{R} - \mathbf{R}^\top)^\vee}{2\sin\theta} \qquad \in \mathbb{R}^3 \;, \tag{108}$$

with $\theta = \cos^{-1}\left(\frac{\mathrm{trace}(\mathbf{R})-1}{2}\right)$.

B. Rotation action

Given the expressions above for the quaternion and the rotation matrix, the rotation action of quaternions on 3-vectors is performed by the double product,

$$\mathbf{x}' = \mathbf{q}\,\mathbf{x}\,\mathbf{q}^* \tag{109}$$

while rotation matrices use a single matrix product,

$$\mathbf{x}' = \mathbf{R}\mathbf{x} \;, \tag{110}$$

Both correspond to a right-hand rotation of θ rad around the axis $\mathbf{u}$. Identifying in them $\mathbf{x}$ and $\mathbf{x}'$ yields the identity

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & w^2 - x^2 + y^2 - z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & w^2 - x^2 - y^2 + z^2 \end{bmatrix} \tag{111}$$

C. Elementary derivative blocks

Since our defined derivatives map tangent vector spaces, and these spaces coincide for the 3D rotation manifolds of S^3 and $SO(3)$, *i.e.*, $\boldsymbol{\theta} = \mathrm{Log}(\mathbf{q}) = \mathrm{Log}(\mathbf{R})$, it follows that the Jacobians are independent of the representation used ($\mathbf{q}$ or $\mathbf{R}$). We thus consider generic 3D rotation elements, and note them with the sans-serif font R.

1) Adjoint: We have from (32)

$$\mathbf{Ad}_\mathsf{R}\boldsymbol{\tau} = (\mathbf{R}\,[\boldsymbol{\tau}]_\times\,\mathbf{R}^\top)^\vee = ([(\mathbf{R}\boldsymbol{\tau})]_\times)^\vee = \mathbf{R}\boldsymbol{\tau}$$

therefore

$$\mathbf{Ad}_\mathsf{R} = \mathbf{R} \;, \tag{112}$$

which means, just to clarify it once again, that $\mathbf{Ad}_\mathbf{q} = \mathbf{R}(\mathbf{q})$, see (111), and $\mathbf{Ad}_\mathbf{R} = \mathbf{R}$.

2) Inversion, composition: We have from Section III-B,

$$\mathbf{J}_\mathsf{R}^{\mathsf{R}^{-1}} = -\mathbf{Ad}_\mathsf{R} = -\mathbf{R} \tag{113}$$

$$\mathbf{J}_\mathsf{Q}^{\mathsf{QR}} = \mathbf{Ad}_\mathsf{R}^{-1} = \mathbf{R}^\top \tag{114}$$

$$\mathbf{J}_\mathsf{R}^{\mathsf{QR}} = \mathbf{I} \;. \tag{115}$$

3) Right and left Jacobians: They admit the closed forms [9, pag. 40],

$$\mathbf{J}_r(\boldsymbol{\theta}) = \mathbf{I} - \frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times + \frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2 \tag{116}$$

$$\mathbf{J}_r^{-1}(\boldsymbol{\theta}) = \mathbf{I} + \frac{1}{2}\,[\boldsymbol{\theta}]_\times + \left(\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta}\right)[\boldsymbol{\theta}]_\times^2 \tag{117}$$

$$\mathbf{J}_l(\boldsymbol{\theta}) = \mathbf{I} + \frac{1-\cos\theta}{\theta^2}\,[\boldsymbol{\theta}]_\times + \frac{\theta-\sin\theta}{\theta^3}\,[\boldsymbol{\theta}]_\times^2 \tag{118}$$

$$\mathbf{J}_l^{-1}(\boldsymbol{\theta}) = \mathbf{I} - \frac{1}{2}\,[\boldsymbol{\theta}]_\times + \left(\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta}\right)[\boldsymbol{\theta}]_\times^2 \tag{119}$$

where we can observe that

$$\mathbf{J}_l = \mathbf{J}_r^\top \;, \qquad \mathbf{J}_l^{-1} = \mathbf{J}_r^{-\top} \;. \tag{120}$$

4) Right- plus and minus: We have for $\boldsymbol{\theta} = \mathsf{Q} \ominus \mathsf{R}$,

$$\mathbf{J}_\mathsf{R}^{\mathsf{R}\oplus\boldsymbol{\theta}} = \mathbf{R}(\boldsymbol{\theta})^\top \qquad \mathbf{J}_{\boldsymbol{\theta}}^{\mathsf{R}\oplus\boldsymbol{\theta}} = \mathbf{J}_r(\boldsymbol{\theta}) \tag{121}$$

$$\mathbf{J}_\mathsf{Q}^{\mathsf{Q}\ominus\mathsf{R}} = \mathbf{J}_r^{-1}(\boldsymbol{\theta}) \qquad \mathbf{J}_\mathsf{R}^{\mathsf{Q}\ominus\mathsf{R}} = \mathbf{J}_l^{-1}(\boldsymbol{\theta}) \tag{122}$$

5) Rotation action: We have

$$\mathbf{J}_\mathsf{R}^{\mathsf{R}\cdot\mathbf{v}} \triangleq \lim_{\theta\to0} \frac{(\mathbf{R}\oplus\theta)\mathbf{v} - \mathbf{R}\mathbf{v}}{\theta} =$$

$$\lim_{\theta\to0} \frac{\mathbf{R}\,\mathrm{Exp}(\theta)\mathbf{v} - \mathbf{R}\mathbf{v}}{\theta} = \lim_{\theta\to0} \frac{\mathbf{R}(\mathbf{I}+[\theta]_\times)\mathbf{v} - \mathbf{R}\mathbf{v}}{\theta}$$

$$= \lim_{\theta\to0} \frac{\mathbf{R}\,[\theta]_\times\,\mathbf{v}}{\theta} = \lim_{\theta\to0} \frac{-\mathbf{R}\,[\mathbf{v}]_\times\,\theta}{\theta} = -\mathbf{R}\,[\mathbf{v}]_\times \tag{123}$$

where we used the properties $\mathrm{Exp}(\theta) \approx \mathbf{I}+[\theta]_\times$ and $[\mathbf{a}]_\times\,\mathbf{b} = -[\mathbf{b}]_\times\,\mathbf{a}$. The second Jacobian yields,

$$\mathbf{J}_\mathbf{v}^{\mathsf{R}\cdot\mathbf{v}} \triangleq \lim_{\partial\mathbf{v}\to0} \frac{\mathbf{R}(\mathbf{v}+\partial\mathbf{v}) - \mathbf{R}\mathbf{v}}{\partial\mathbf{v}} = \mathbf{R} \;. \tag{124}$$

APPENDIX C
THE 2D RIGID MOTION GROUP $SE(2)$

We write elements of the planar rigid motion group $SE(2)$ as

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(2) \;,$$

with $\mathbf{R} \in SO(2)$ a rotation and $\mathbf{t} \in \mathbb{R}^2$ a translation. The Lie algebra and vector tangents are formed by elements of the type

$$\boldsymbol{\tau}^\wedge = \begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathfrak{se}(2) \;, \qquad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3 \;.$$

A. Inverse, composition

Inversion and composition are performed respectively with matrix inversion and product,

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{125}$$

$$\mathbf{H}_a\,\mathbf{H}_b = \begin{bmatrix} \mathbf{R}_a\mathbf{R}_b & \mathbf{t}_a + \mathbf{R}_a\mathbf{t}_b \\ \mathbf{0} & 1 \end{bmatrix} \;. \tag{126}$$

B. Exp and Log maps

Exp and Log are implemented via exponential maps directly from the scalar tangent space $\mathbb{R}^3 \simeq \mathfrak{se}(2) = \mathcal{T}SE(2)$ — see [4] for the derivation,

$$\mathbf{H} = \mathrm{Exp}(\boldsymbol{\tau}) \triangleq \begin{bmatrix} \mathrm{Exp}(\theta) & \mathbf{V}(\theta)\,\boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{127}$$

$$\boldsymbol{\tau} = \mathrm{Log}(\mathbf{H}) \triangleq \begin{bmatrix} \mathbf{V}^{-1}(\theta)\,\mathbf{p} \\ \mathrm{Log}(\mathbf{R}) \end{bmatrix} \;. \tag{128}$$

with

$$\mathbf{V}(\theta) = \frac{1}{\theta} \begin{bmatrix} \sin(\theta) & -(1-\cos(\theta)) \\ 1-\cos(\theta) & \sin(\theta) \end{bmatrix} \;. \tag{129}$$

C. Derivative blocks

1) Adjoint: The adjoint is easily found from (32) using the fact that planar rotations commute,

$$\mathbf{Ad}_\mathbf{H}\boldsymbol{\tau} = (\mathbf{H}\boldsymbol{\tau}^\wedge\mathbf{H}^{-1})^\vee = \begin{bmatrix} \mathbf{R}\boldsymbol{\rho} - [\theta]_\times\,\mathbf{t} \\ \theta \end{bmatrix} = \mathbf{Ad}_\mathbf{H} \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \;,$$

leading to

$$\mathbf{Ad}_\mathbf{H} = \begin{bmatrix} \mathbf{R} & -[1]_\times\,\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \;. \tag{130}$$

All derivatives follow from (42).

2) Inversion, composition: We have from Section III-B,

$$\mathbf{J}_\mathbf{H}^{\mathbf{H}^{-1}} = -\mathbf{Ad}_\mathbf{H} = \begin{bmatrix} -\mathbf{R} & [1]_\times\,\mathbf{t} \\ \mathbf{0} & -1 \end{bmatrix} \tag{131}$$

$$\mathbf{J}_{\mathbf{H}_a}^{\mathbf{H}_a\mathbf{H}_b} = \mathbf{Ad}_{\mathbf{H}_b}^{-1} = \begin{bmatrix} \mathbf{R}_b^\top & \mathbf{R}_b^\top\,[1]_\times\,\mathbf{t}_b \\ \mathbf{0} & 1 \end{bmatrix} \tag{132}$$

$$\mathbf{J}_{\mathbf{H}_b}^{\mathbf{H}_a\mathbf{H}_b} = \mathbf{I} \;. \tag{133}$$

3) Right and left Jacobians: We have from [9, pag. 36],

$$\mathbf{J}_r = \begin{bmatrix} \sin\theta/\theta & (1-\cos\theta)/\theta & (\theta v_1 - v_2 + v_2\cos\theta - v_1\sin\theta)/\theta^2 \\ (1-\cos\theta)/\theta & \sin\theta/\theta & (v_1 + \theta v_2 - v_1\cos\theta - v_2\sin\theta)/\theta^2 \\ 0 & 0 & 1 \end{bmatrix} \tag{134}$$

$$\mathbf{J}_l = \begin{bmatrix} \sin\theta/\theta & (\cos\theta-1)/\theta & (\theta v_1 + v_2 - v_2\cos\theta - v_1\sin\theta)/\theta^2 \\ (1-\cos\theta)/\theta & \sin\theta/\theta & (-v_1 + \theta v_2 + v_1\cos\theta - v_2\sin\theta)/\theta^2 \\ 0 & 0 & 1 \end{bmatrix} \;. \tag{135}$$

4) Rigid motion action: We have the action on points $\mathbf{p}$,

$$\mathbf{H} \cdot \mathbf{p} \triangleq \mathbf{t} + \mathbf{R}\mathbf{p} \;, \tag{136}$$

therefore from (127),

$$\mathbf{J}_\mathbf{H}^{\mathbf{H}\cdot\mathbf{P}} = \lim_{\boldsymbol{\tau}\to0} \frac{\mathbf{H}\,\mathrm{Exp}(\boldsymbol{\tau})\cdot\mathbf{p} - \mathbf{H}\cdot\mathbf{p}}{\boldsymbol{\tau}} = \begin{bmatrix} \mathbf{R} & \mathbf{R}\,[1]_\times\,\mathbf{p} \end{bmatrix} \tag{137}$$

$$\mathbf{J}_\mathbf{p}^{\mathbf{H}\cdot\mathbf{P}} = \mathbf{R} \;. \tag{138}$$

APPENDIX D
THE 3D RIGID MOTION GROUP $SE(3)$

We write elements of the 3D rigid motion group $SE(3)$ as

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3) \;,$$

with $\mathbf{R} \in SO(3)$ a rotation matrix and $\mathbf{t} \in \mathbb{R}^3$ a translation vector. The Lie algebra and vector tangents are formed by elements of the type

$$\boldsymbol{\tau}^\wedge = \begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathfrak{se}(3) \;, \qquad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6 \;.$$

A. Inverse, composition

Inversion and composition are performed respectively with matrix inversion and product,

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{139}$$

$$\mathbf{H}_a \mathbf{H}_b = \begin{bmatrix} \mathbf{R}_a \mathbf{R}_b & \mathbf{t}_a + \mathbf{R}_a \mathbf{t}_b \\ \mathbf{0} & 1 \end{bmatrix} \;. \tag{140}$$

B. Exp and Log maps

Exp and Log are implemented via exponential maps directly from the vector tangent space $\mathbb{R}^6 \simeq \mathfrak{se}(3) = \mathcal{T}SE(3)$ — see [4] for the derivation,

$$\mathbf{H} = \mathrm{Exp}(\boldsymbol{\tau}) \triangleq \begin{bmatrix} \mathrm{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{141}$$

$$\boldsymbol{\tau} = \mathrm{Log}(\mathbf{H}) \triangleq \begin{bmatrix} \mathbf{V}^{-1}(\boldsymbol{\theta})\mathbf{p} \\ \mathrm{Log}(\mathbf{R}) \end{bmatrix} \;. \tag{142}$$

with

$$\mathbf{V}(\boldsymbol{\theta}) = \mathbf{I} + \frac{1-\cos\theta}{\theta^2} [\boldsymbol{\theta}]_\times + \frac{\theta - \sin\theta}{\theta^3} [\boldsymbol{\theta}]_\times^2 \;. \tag{143}$$

C. Derivative blocks

1) Adjoint: We have,

$$\mathbf{Ad}_\mathbf{H} \boldsymbol{\tau} = (\mathbf{H}\boldsymbol{\tau}^\wedge \mathbf{H}^{-1})^\vee = \begin{bmatrix} \mathbf{R}\boldsymbol{\rho} + [\mathbf{t}]_\times \mathbf{R}\boldsymbol{\theta} \\ \mathbf{R}\boldsymbol{\theta} \end{bmatrix} = \mathbf{Ad}_\mathbf{H} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}$$

therefore,

$$\mathbf{Ad}_\mathbf{H} = \begin{bmatrix} \mathbf{R} & [\mathbf{t}]_\times \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{6\times 6} \;. \tag{144}$$

2) Inversion, composition: We have from Section III-B,

$$\mathbf{J}_\mathbf{H}^{\mathbf{H}^{-1}} = - \begin{bmatrix} \mathbf{R} & [\mathbf{t}]_\times \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} \tag{145}$$

$$\mathbf{J}_{\mathbf{H}_a}^{\mathbf{H}_a \mathbf{H}_b} = \begin{bmatrix} \mathbf{R}_b & [\mathbf{t}_b]_\times \mathbf{R}_b \\ 0 & \mathbf{R}_b \end{bmatrix}^{-1} \tag{146}$$

$$\mathbf{J}_{\mathbf{H}_b}^{\mathbf{H}_a \mathbf{H}_b} = \mathbf{I}_6 \;. \tag{147}$$

3) Right and left Jacobians: A closed form of the left Jacobian is given by Barfoot in [8],

$$\mathbf{J}_l(\boldsymbol{\rho}, \boldsymbol{\theta}) = \begin{bmatrix} \mathbf{J}_l^{-1}(\boldsymbol{\theta}) & -\mathbf{J}_l^{-1}(\boldsymbol{\theta})\mathbf{Q}(\boldsymbol{\rho}, \boldsymbol{\theta})\mathbf{J}_l^{-1}(\boldsymbol{\theta}) \\ \mathbf{0} & \mathbf{J}_l^{-1}(\boldsymbol{\theta}) \end{bmatrix} \tag{148}$$

where $\mathbf{J}_l^{-1}(\boldsymbol{\theta})$ is the inverse left Jacobian of $SO(3)$, see (119), and

$$
\begin{aligned}
\mathbf{Q}(\boldsymbol{\rho}, \boldsymbol{\theta}) =& \frac{1}{2}\boldsymbol{\rho}_\times + \frac{\theta - \sin\theta}{\theta^3}(\boldsymbol{\theta}_\times \boldsymbol{\rho}_\times + \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times + \boldsymbol{\theta}_\times \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times) \\
& - \frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4}(\boldsymbol{\theta}_\times^2 \boldsymbol{\rho}_\times + \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times^2 - 3\boldsymbol{\theta}_\times \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times) \\
& - \frac{1}{2}\left(\frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4} - 3\frac{\theta - \sin\theta - \frac{\theta^3}{6}}{\theta^5} \right) \\
& \times (\boldsymbol{\theta}_\times \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times^2 + \boldsymbol{\theta}_\times^2 \boldsymbol{\rho}_\times \boldsymbol{\theta}_\times) \;.
\end{aligned}
\tag{149}
$$

The right Jacobian is obtained using (77), that is, changing the signs of all $\boldsymbol{\rho}$, θ and $\boldsymbol{\theta}$ above.

4) Rigid motion action: We have the action on points $\mathbf{p}$,

$$\mathbf{H} \cdot \mathbf{p} \triangleq \mathbf{t} + \mathbf{R}\mathbf{p} \;, \tag{150}$$

therefore from (141),

$$\mathbf{J}_\mathbf{H}^{\mathbf{H}\cdot\mathbf{p}} = \lim_{\boldsymbol{\tau} \to 0} \frac{\mathbf{H}\,\mathrm{Exp}(\boldsymbol{\tau}) \cdot \mathbf{p} - \mathbf{H} \cdot \mathbf{p}}{\boldsymbol{\tau}} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\,[\mathbf{p}]_\times \end{bmatrix} \tag{151}$$

$$\mathbf{J}_\mathbf{p}^{\mathbf{H}\cdot\mathbf{p}} = \mathbf{R} \;. \tag{152}$$

APPENDIX E
THE TRANSLATION GROUPS $(\mathbb{R}^n, +)$ AND $T(n)$

The group $(\mathbb{R}^n, +)$ is the group of vectors under addition, and can be regarded as a translation group. We deem it *trivial* in the sense that the group elements, the Lie algebra, and the tangent vectors are all the same, so $\mathbf{t} = \mathbf{t}^\wedge = \mathrm{Exp}(\mathbf{t})$. Its equivalent matrix group (under multiplication) is the translation group $T(n)$, whose group, Lie algebra and tangent vector elements are,

$$\mathbf{T} \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in T(n), \quad \mathbf{t}^\wedge \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{t} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathfrak{t}(n), \quad \mathbf{t} \in \mathbb{R}^n \;.$$

Equivalence is easily verified by observing that $\mathbf{T}(\mathbf{0}) = \mathbf{I}$, $\mathbf{T}(-\mathbf{t}) = \mathbf{T}(\mathbf{t})^{-1}$, and that the composition

$$\mathbf{T}_1 \mathbf{T}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{t}_1 + \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix} \;,$$

effectively adds the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ together. Since $T(n)$ is a subgroup of $SE(n)$ with $\mathbf{R} = \mathbf{I}$, we can easily determine its exponential map from generalizations of $(127, 141)$,

$$\mathrm{Exp}: \quad \mathbb{R}^n \to T(n) \quad \mathbf{T} = \mathrm{Exp}(\mathbf{t}) = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \;. \tag{153}$$

This serves as immediate proof for the equivalent exponential of the $(\mathbb{R}^n, +)$ group, which is the identity,

$$\mathrm{Exp}: \quad \mathbb{R}^n \to \mathbb{R}^n \quad \mathbf{t} = \mathrm{Exp}(\mathbf{t}) \;. \tag{154}$$

We have also in $\mathbb{R}^n$,

$$\mathbf{t}_1 \oplus \mathbf{t}_2 = \mathbf{t}_1 + \mathbf{t}_2 \tag{155}$$

$$\mathbf{t}_2 \ominus \mathbf{t}_1 = \mathbf{t}_2 - \mathbf{t}_1 \tag{156}$$

A. Derivative blocks

We express translations in $T(n)$ and $\mathbb{R}^n$ as S and T. The groups are commutative, and so the Jacobians are trivial (compare them with those of $SO(2)$ in Section A-C1).

$$\mathbf{Ad}_{\mathsf{T}} = \mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (157)$$

$$\mathbf{J}_{\mathsf{T}}^{\mathsf{T}^{-1}} = -\mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (158)$$

$$\mathbf{J}_{\mathsf{T}}^{\mathsf{T}\,\mathsf{S}} = \mathbf{J}_{\mathsf{S}}^{\mathsf{T}\,\mathsf{S}} = \mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (159)$$

$$\mathbf{J}_r = \mathbf{J}_l = \mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (160)$$

$$\mathbf{J}_{\mathsf{T}}^{\mathsf{T}\oplus\mathbf{v}} = \mathbf{J}_{\mathbf{v}}^{\mathsf{T}\oplus\mathbf{v}} = \mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (161)$$

$$\mathbf{J}_{\mathsf{S}}^{\mathsf{S}\ominus\mathsf{T}} = -\mathbf{J}_{\mathsf{T}}^{\mathsf{S}\ominus\mathsf{T}} = \mathbf{I} \qquad \in \mathbb{R}^{n \times n} \qquad (162)$$

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