SOURCES FROM 16th CENTURY FOR THE TEACHING AND LEARNING OF MATHEMATICS[1]

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ABSTRACT

The use of original texts from History of Mathematics is a way to introduce the sources on which mathematical knowledge is based into the classroom. Choosing historical texts carefully can help students to develop their mathematical reasoning skills and to realize the humanistic aspects of mathematical knowledge through the understanding of the formation process of mathematical thinking. In this paper we present two problems drawn from four mathematical treatises from the 15th and 16th centuries. For the first one, we present three similar wordings drawn from the *Arithmetica* (1484) by Pietro Borghi, the *Coss* (1525) by Christoff Rudolff, and the *Libro Primero* (1552) by Marco Aurel, which concern the hiring of a worker. The second one is a situation that was posed by one student to another in the last book of the *Arithmetica practica y speculative* (1562) by Juan Pérez de Moya. Our aim is to analyse some aspects of the development of algebraic thinking through the mathematical activity of problem solving, implemented in the classroom. The analysis of the different mathematical procedures for solving these historical situations can encourage students to devise their own methods when faced with an unknown problematic situation.

1 Introduction

The history of mathematics shows how mathematics has frequently been used to solve problems concerning human activity, as well as for helping to understand the world that surrounds us. The use of original texts from the history of mathematics is a way to introduce in the classroom the sources on which mathematical knowledge is based.[2]

In this paper we present two historical problems drawn from four mathematical treatises from the 15th and 16th centuries. We have chosen three versions of the first one, drawn from the *Arithmetica* (1484) by Pietro Borghi, the *Coss* (1525) by Christoff Rudolff, and the *Libro Primero* (1552) by Marco Aurel, which deal with the question of hiring a worker. The second one involves a situation that was posed by one student to another in the last book of the *Arithmetica practica y speculative* (1562) by Juan Pérez de Moya. This book is written as a dialogue between students, which allows the author to present different points of view about the usefulness of mathematics and the reasons for the importance of acquiring such knowledge.

Our aim is to analyse some aspects of the development of algebraic thinking through the mathematical activity of solving historical problems, implemented in the classroom (Radford, 2006; Filloy *et al.*, 2008; Broker & Windsor, 2010). The analysis of the different mathematical procedures for solving these historical situations can encourage students to devise their own methods when faced with an unknown problematic situation.

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[2] All this knowledge will undoubtedly enrich the mathematical background and training of teachers. Historians of mathematics have pursued various research lines at an international level that investigate how to use knowledge of the history of mathematics effectively in the classroom in order to improve the teaching and learning of mathematics. For more, see, Barbin, 2000, pp.63-66; Jahnke, 1996; Fauvel & Maanen, 2000; Demattè, 2006; Massa Esteve *et al.*, 2011.

2 History of Mathematics in the Construction of Mathematic Knowledge of Students

The knowledge of the history of mathematics can assist in the enrichment of the construction of students' mathematic knowledge in two ways. First, the use of the history of mathematics in the mathematics classroom can provide students with a conception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science (see Massa-Esteve, 2003; Romero Vallhonesta *et al.*, 2015) and second, the use of the history of mathematics in the mathematics classroom can provide students with a further relevant feature of mathematics – that it can be understood as a cultural activity (Furinghetti, 1997; Katz, 1997b). History reveals that societies develop as a result of the scientific activity undertaken by successive generations, and that mathematics is a fundamental part of this process. Mathematics can be presented as an intellectual activity for solving problems in each period. Furthermore, the history of mathematics as an implicit and explicit didactic resource can provide tools to enable students to understand mathematical concepts better.[3]

First of all, history can be employed explicitly in the content of compulsory research work undertaken by students in Catalonia in their second year of Baccalaureate (17-year-olds). History situates students in a more general context, since problems are addressed within a global framework of mathematics and within the overall field of science. These research projects not only show the historical evolution of an idea or a concept, but also involve mathematical research enabling students to become familiar with mathematical reasoning from other periods and cultures, as well as in other contexts.[4] The mathematical work that can be undertaken in each of these research fields is highly diverse and range from very simple problems to more complicated proofs.

Secondly, the holding of workshops, centenaries and conferences provides further types of activities in which history can be used explicitly to achieve a more comprehensive learning experience for students. For instance, the workshop devoted to the study of the life and work of René Descartes (1596-1650), held in 1996 at the INS Carles Riba (a high school in Barcelona), afforded students additional education from a mathematical, philosophical, physical and historical perspective.

Finally, the explicit use of significant original sources in the classroom is an activity that can provide students with more valuable means for a better understanding of mathematical concepts, that is the case presented below (Massa-Esteve, 2012; Romero-Vallhonesta & Massa-Esteve, 2015).[5]

[3]The history of mathematics as an implicit resource can be employed by teachers in the design phase by choosing contexts, by preparing activities (problems and auxiliary sources) and also by drawing up the teaching syllabus for a concept or an idea.

[4] The list of titles of research works dealing with the history of mathematics proposed by mathematics teachers is quite extensive, but as examples we may quote the following: Pythagoras and Music; The Golden Mean; On Fermat's Theorem; Pascal's Arithmetic Triangle as a Tool for Resolution; Perspective and its History in the Work of Leonardo da Vinci, Luca Pacioli and Albert Dürer; Women and Science; On Incommensurability: A Mathematical and Philosophical Problem, etc.

[5] Historical texts can be used throughout the different steps in the teaching and learning process; to introduce a mathematical concept; to explore it more deeply; to explain the differences between two contexts; to motivate study of a particular type of problem or to clarify a process of reasoning

3 Catalonian Curriculum 2015. Problem Solving and the History of Mathematics

The history of mathematics has enjoyed an official place in the Catalonian curriculum of mathematics for secondary schools (ESO) since 2007. In the academic year 2007-2008, the Catalan Government Department of Education introduced some compulsory elements of the history of science into the curriculum for secondary education. Specifically, the new mathematics curriculum for secondary schools in Catalonia, published in June 2007, contains notions of the historical genesis of relevant mathematical subjects within the syllabus.[6]

In Decree 187/2015, of August 25th, on the regulation of Catalonian compulsory secondary education, the basic skills in the different areas that students must achieve at this stage are established. These skills are associated with the standards of each area. Therefore, since the last curriculum decree in 2015, the history of mathematics forms part of the contents of the mathematical concepts to be taught.

Also in the last curriculum, problem-solving has played an important role as one of the main methods for teaching and learning mathematics, which goes beyond finding simple solutions or just knowing the basic facts and formulas (Bednarz, Kieran, & Lee, 1996). The best known strategies for solving problems can be found in the work of Pólya (1957), which include, for example, trying special values or cases, working backwards, guessing and checking. By learning how to tackle problem-solving, students acquire confidence in a variety of situations that can help them to use mathematical approaches to solving real-life problems. Problem-solving is a complex mathematical activity, which contributes to the development of algebraic thinking.

The problem-solving standard of mathematics consists of four basic skills:

1. Translate a problem into mathematical language or to a mathematical representation using variables, symbols, diagrams or appropriate models.
2. Use concepts, tools and mathematical strategies for solving problems.
3. Maintain a research attitude towards a problem by testing different strategies.
4. Generate questions of a mathematical nature and raise issues.

In the explanation given in the decree on the problem-solving standard, it is stated that a problem is a proposal to confront an unknown situation that is posed through a set of data within a context, and which initially has no obvious solution and requires reflection, decision-making and strategies. The decree also emphasizes the need to distinguish a problem from an exercise.

From this explanation, it can be inferred that while for some students a proposal may represent a problem because they do not know the algorithm by which it could be solved almost immediately, for others it is the simple application of a technique.

[6] The list of these historical contexts includes: The origins of the numeration system; the introduction of zero and the systems of positional numeration; geometry in ancient civilizations (Egypt, Babylonia); initial approaches to the number π (Egypt, China and Greece); Pythagoras' theorem in Euclid's *Elements* and in China; the origins of symbolic algebra (Arab world, Renaissance); the relationship between geometry and algebra and the introduction of Cartesian coordinates; the geometric resolution of equations (Greece, India, Arab World); the use of geometry to measure the distance Earth - Sun and Earth - Moon (Greece).

4 Setting the scene

We outline a brief historical journey through the development of algebraic equations for setting the scene (Katz, 1997a; Bashmakova & Smirnova, 2000; Massa-Esteve, 2005). While it is possible to deduce an algorithm for solving equations of second degree from Babylonian tablets (1800 BC), Arabian mathematicians were those who took the decisive step in the development of algebra. The mathematician, astronomer and member of the House of Wisdom in Baghdad, Mohamed Ben-Musa al-Khwarizmi (850 AD), is regarded as the creator of the rules of algebra fully rhetorical. In his work *Hisâb al-jabrwal-muqqabala* (813 and 830), he classified equalities (now called equations) up to the second degree according to six different types, as well as explaining rhetorically the method for solving them. It was Leonardo de Pisa, son of Bonacci (1180-1250), better known as Fibonacci, who disseminated, also rhetorically, all this knowledge in the West. Many of the problems addressed in the algebra of the Arabs are to be found in Fibonacci'swork *Liber abaci* (1202), as well as methods for calculating with Indian numerals. The sketchiest period in the development of algebraic equations corresponds to the 13th and 14th centuries, when commercial mathematics flourished with the *Mercantile Arithmetic*, works that are still being explored and analyzed. The knowledge of these mercantile arithmetic and Arab algebra were collected in a work by Luca Pacioli (1447-1517) entitled *Summa de Arithmetica, Geometria, Proportioni & Proportionalità* (1494), which was widely known at that time and very influent in Spanish algebra's authors quoted in the practical activity.[7]

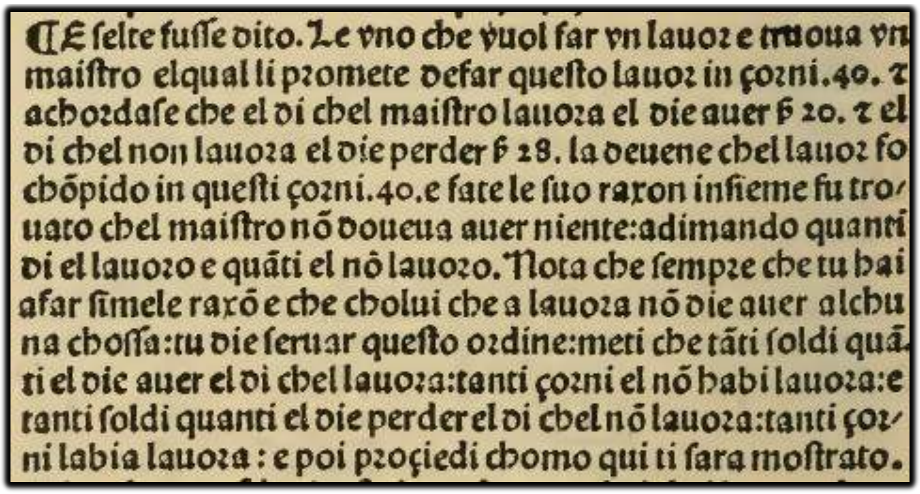
4.1 Practical activity in the classroom

What we now propose are two historical problems from 16th century's sources, period that began the development of resolution of algebraic equations rhetorically. Several versions exist of the first problem, which consists of a problem that can be solved immediately by solving simultaneous equations.

From this problem we present three versions to students; the first by Pietro Borghi, the second by Christoff Rudolff and the third by Marco Aurel. After solving the problems, we discussed with the students the solutions given by the authors themselves, and also explained the characteristics of the mathematics of that time. In this paper we focus on the resolution of the problems.

The first version, from the text by Borghi (Figure 4.1), that contains the wording of the problem and some considerations about the resolution, and the resolution itself (Figure 4.2):

[7] Later, Girolamo Cardano (1501-1576) and Rafael Bombelli (1526-1573), among other algebraists of the *Cinquecento*, also contributed to the solution of cubic and quartic equations with syncopated algebra through their respective works *Artis Magnae sive de Regulis Algebraicis* (1545) and *Algebra* (1572), respectively.



E selte fusse dito. Le uno che vuol far un lauor e truoua un maistro elqual li promete defar questo lauor in çorni.40. e achordase che el di chel maistro lauora el die auer ß 20. e el di chel non lauora el die perder ß 28. la deuene chel lauor fo chōpido in questi çorni.40.e fate le suo raxon insieme fu trouato chel maistro nō doueua auer niente:adimando quanti di el lauoro e quāti el nō lauoro. Nota che sempre che tu hai afar simele raxō e che cholui che a lauora nō die auer alchuna chossa:tu die seruar questo ordine:meti che tāti soldi quāti el die auer el di chel lauora:tanti çorni el nō habi lauora:e tanti soldi quanti el die perder el di chel nō lauora:tanti çorni el labia lauora : e poi proçiedi chomo qui ti fara mostrato.

Figure 4.1: [Borghi, 1484, 111v]

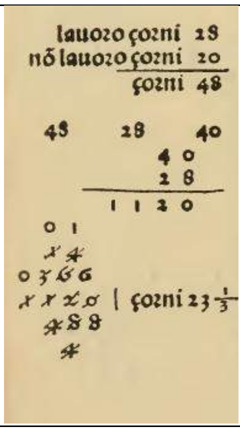
The resolution by Borghi	In current notation
	20 (days not worked) + 28 (days worked)= 48 Let us suppose that in every 48 days, the worker works for 20 days and doesnot work for 28. Therefore, the profit for the worker will be: 28·20 – 20·28 = 0. This would be the solution if the contract was for 48 days, but it was for 40 days. Solving this rule of three: 48 → 28 we obtain 40→ x $x = 23\frac{1}{3}$

Figure 4.2: Theresolution of theproblemabouthiring a worker, by Borghi

We have adapted the situation to a modern context to make it more familiar to the students[8] but maintaining the same quantities:

A girl is looking for a part time job to help in houses doing odd jobs (installing wall sockets, hanging pictures, painting and decorating, etc.) and a woman hires her for 40 days. The woman knows that the girl is not very reliable and the conditions she imposes on the girl are the following: for every day worked, she will receive 20 euros, but if she doesn't show up, she has to pay 28 euros to the owner. After 40 days, the woman doesn't owe anything to the girl. How many days has the girl worked?

The other problem is taken from a book on arithmetic in which various questions are formulated in the form of a dialogue. The first part consists of a dialogue between two students: Sophornio and Antimacho. Sophronio argues for the importance of knowing arithmetic by posing some questions to Antimacho, who at the beginning is doubtful about

[8] Una noia busca feina per fer petits arranjaments a les cases (posar endolls, penjar quadres, pintar alguntros de paret deteriorat…) i una senyora n'hi dóna perquè hi vagi una estona els matins durant 40 dies. La senyoras' assabenta que la noia no ésgaire formal i li proposa pagar-li 20 euros per cada dia treballat i, si algundia no va a treballar, haurà de donar 28 euros a la senyora. Passats els 40 dies, la senyora no li deu res a la noia. Quants dies va treballar i quants dies no va treballar?

this importance, but who in the end has to agree with Sophronio. What we are concerned with here relates to proportionality,which is one of the key concepts of mathematics at all stages of education.

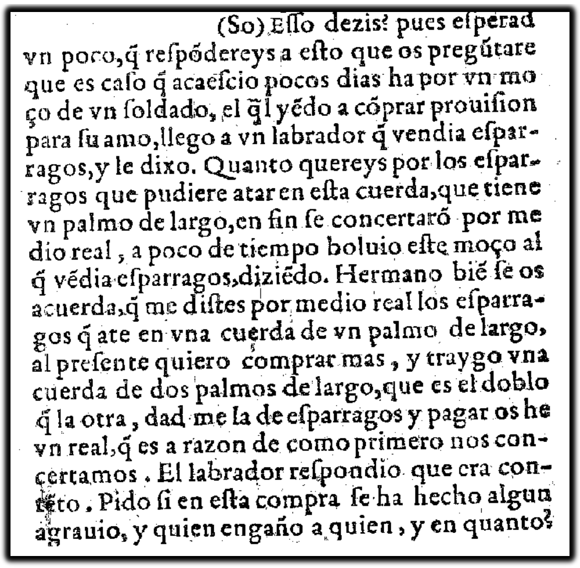

(So) Eſſo dezis? pues eſperad vn poco, q̃ reſpõdereys a eſto que os pregũtare que es caſo q̃ acaeſcio pocos dias ha por vn moço de vn ſoldado, el q̃l yẽdo a cõprar prouiſion para ſu amo, llego a vn labrador q̃ vendia eſparragos, y le dixo. Quanto quereys por los eſparragos que pudiere atar en eſta cuerda, que tiene vn palmo de largo, en fin ſe concertarõ por medio real, a poco de tiempo boluio eſte moço al q̃ vẽdia eſparragos, diziẽdo. Hermano biẽ ſe os acuerda, q̃ me diſtes por medio real los eſparragos q̃ ate en vna cuerda de vn palmo de largo, al preſente quiero comprar mas, y traygo vna cuerda de dos palmos de largo, que es el doblo q̃ la otra, dad me la de eſparragos y pagar os he vn real, q̃ es a razon de como primero nos concertamos. El labrador reſpondio que era contento. Pido ſi en eſta compra ſe ha hecho algun agrauio, y quien engaño a quien, y en quanto?

Figure 4.3: [Pérez de Moya, 1562, 701]

The problem as posed to the students (not adapted in this case, but translated into Catalan[9] from the original 16th century Spanish, with little changes in the two final questions) is as follows:

A soldier went to the market to buy asparagus from a farmer and asked about the price of a bunch of asparagus that could be tied with a string measuring the span of his palm? It was agreed that the soldier would pay half a farthing to the farmer. After a few days, the soldier returned to the same farmer and told him that he wanted to buy a bunch of asparagus that could be tied with a string measuring 2 spans of his palm. For this he offered to pay the farmer double, 1 farthing.
Is it fair for the soldier to pay the farmer 1 farthing?
Is this correct? If you think that this is not right, how much should be paid and why?[10]

In this case, the solution is given as a rhetorical reasoning. The question is posed by Sophronio to Antimacho. Antimacho believes that there is no doubt that the soldier has to pay one farthing for the asparagus tied by a string measuring 1 handspan. Sophronio asks him to take one string and another of double its length and to check himself the quantity of asparagus that can be bound by in each string, in order to see that the quantity of asparagus in the second string will be four times the quantity in the first. Therefore, the

[9] Un soldat va anar al mercat i va anar a comprar a un pagès que venia espàrrecs i li va dir: quant em demaneu pels espàrrecs que puc encerclar amb aquesta corda que té una longitud d'1pam? Van acordar que el soldat pagaria mig ral al pagès. Al cap d'uns dies, el soldatva tornar a anar al mateix pagès i li va recordar que feia uns dies li havia cobrat mig ral pels espàrrecs que es podien encerclar en una corda d'1 pam i que ara en voliamés: totsels que es podienencerclaramb una corda de 2 pams, que és el doble de l'altra. Per tant, posa-m'hi els espàrrecs que hi caben, i jo et pago el doble, 1 ral.
Ésjust el que li volia pagar el soldat? Justifica-ho.
Si creus que no ésjust, digues quant li hauria de pagar i justifica-ho.
[10] We have changed the last question from the original text, in order to be more understandable for the students.

amount of money that should be paid to the farmer is 2 farthings.

These problems were put to 28 3rd-grade students of ESO[11] and to 29 1st-grade students of ESO[12] at the Eugeni Xammar Secondary School [13] in l'Ametlladel Vallès. In secondary schools in Catalonia, students start their secondary education at age twelve and they can continue until they are eighteen, although compulsory education finishes in 4th of ESO when students are sixteen. The 3rd-grade students had already been taught how to solve simultaneous equations, while the 1st-grade students had not.

In the presentation of the activity to students, they were told the following:

1. This activity is part of a research project.
2. It is important that you note down any possible ideas, strategies or reflections about the resolution of the problems, even if you have doubts about the accuracy.

The students were given this advice because, as they were not used to solving problems in which they were not required to apply known techniques, we wished to avoid the risk that many of them might not attempt to solve the problems for fear of an incorrect answer.

In the case of the 3rd-grade ESO students, the only correct answers were given by students who solved the problem by using simultaneous equations. None of the students who tried to resolve the problem using a more imaginative procedure managed to arrive at the solution, as shown in the following table:

Resolution of the first problem by 3rd-grade ESO students				
Using simultaneous equations		Without using equations		Unanswered
correct	incorrect	correct	incorrect	
7	4	0	13	4

The following image shows a typical correct resolution by a 3rd-grade ESO student:

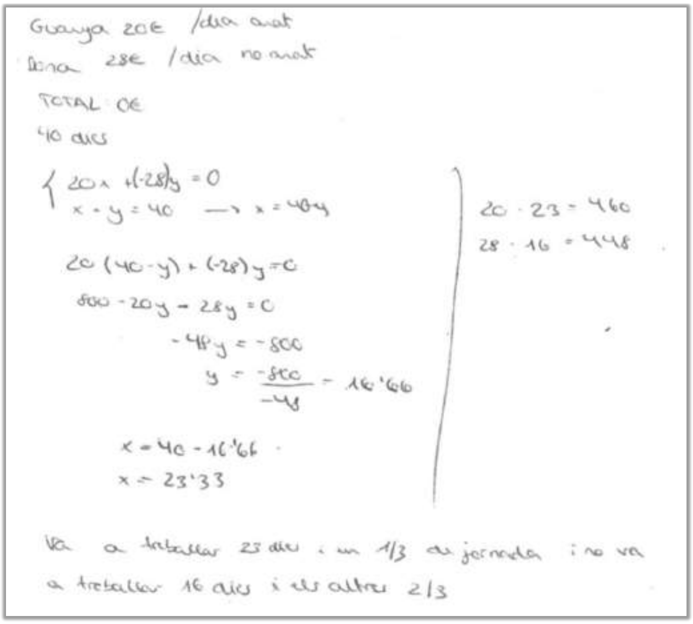


Figure 4.4: Resolution by a 3rd-grade ESO student.

[11]14-year-old students.
[12]12-year-old students.
[13] We express our gratitude to the mathematics teaching stafffor their collaboration and effort in implementing this activity.

The types of solutions given by the 1st-grade students were more varied. Most of them attempted to find the solution by looking for common multiples of 20 and 28, with the condition that the numbers by which we have to multiply 20 and 28, added together, gave 40. In this case, the way to solve the problem is more interesting. In the case of the 3rd-grade students who solved the problem by solving simultaneous equations, on arriving at the solution they took it for granted that it was correct. However, some of the 1st-grade students thought that a non-integer solution did not make sense, and since they were unable to find two entire quantities according to the conditions of the wording, they said that the problem had no solution. In these cases, if the procedure adopted was correct, we considered the resolution to be correct.

Resolution of the first problem by 1st-grade ESO students			
Resolve the problem correctly	Use a good method but do not arrive at the solution	Try to solve the problem with an unclear method and do not arrive at any conclusion	Unanswered
8 (6 of whom say that the problem has no solution)	10	5	6

In this regard, we present two interesting resolutions. In the first one, the student says that the circled numbers in the table represent the days on which neither the woman nor the girl are indebted to each other. The sum of these days should be 40, but in none of the cases is this the result. Therefore, the problem has no solution. However, the student probably found it strange that the problem has no solution, and adds: if we read the question carefully, it is stated that the woman does not owe the girl anything, but nothing is said about the possible debt owed by the girl to the woman. We could assume that the girl has not worked at all, and in this case, she would have to pay 1120 euros to the woman, who does not owe the girl anything.

In fact, as the student remarks about the statement of the problem, nothing is said about the debt that the girl owes to the woman, but according to the solution provided by Borghi, we see that the author supposes that the girl owes the woman nothing.

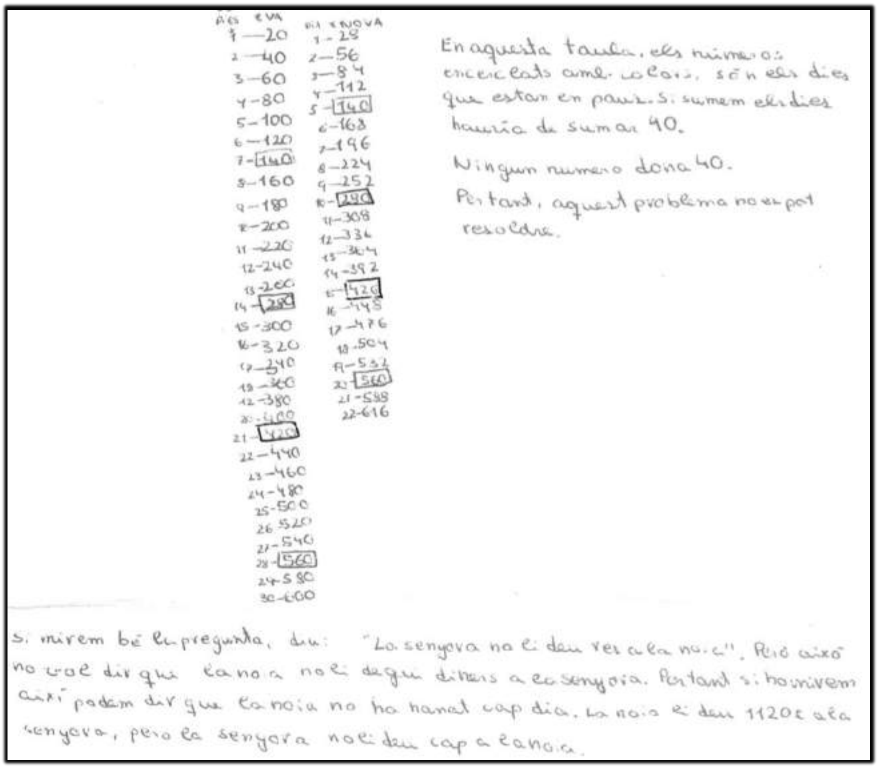


Figure 4.5: Resolution by a 1st-grade ESO student.

In the second case, the student considers that the ratio between the days worked and those not worked must be 7/5, which is the ratio between 28 and 20. For the balance to be 0, then for every set of 12 days the girl would work on 7 days and on 5 days she would fail to show up. As the number of days worked cannot be a decimal (as the student supposes), the total number of days should be a multiple of 12 (7+5). The multiple of 12 nearest to 40 is 36 (12·3). Therefore, the girl goes to work on 7·3=21 days and fails to show up on 5·3=15 days, which leaves 4 days unaccounted for. On considering what may have happened on these days, she believes that since the two parties reached no agreement on what might happen if the girl went to work but did not do anything, then neither would the woman pay the girl nor would the girl owe anything to the woman.

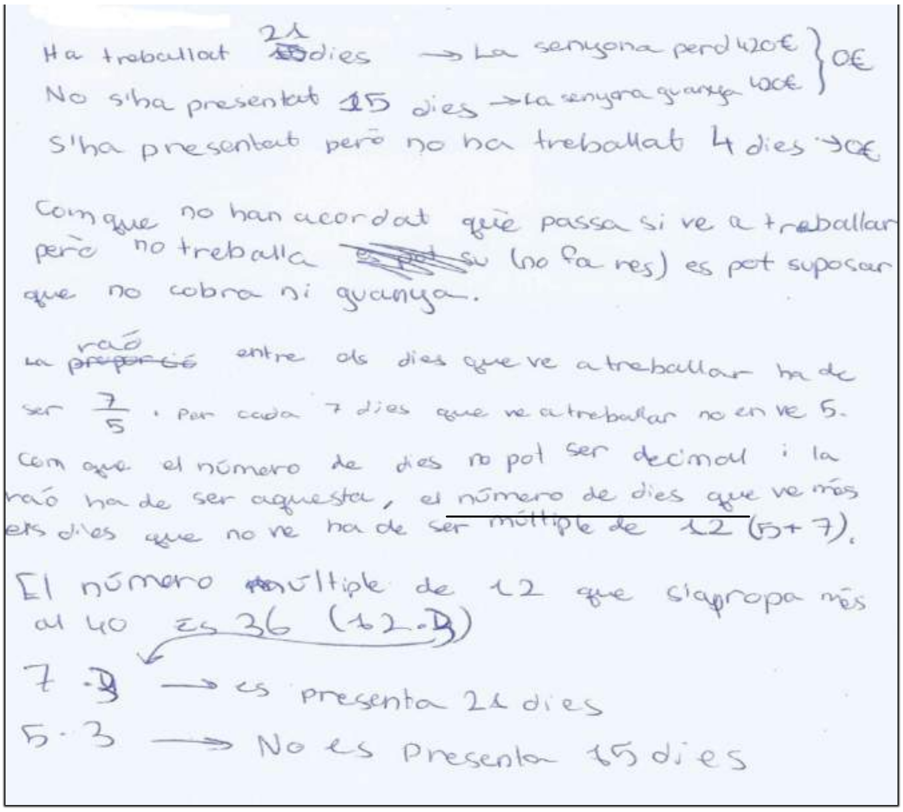


Figure 4.6: The resolution of a 1st-grade ESO student.

It is interesting to remark that only the students who solved the problem using their own procedures gave any consideration to the authenticity of the solution. None of those who solved the problem by using simultaneous equations add any comments to the result.

Resolution of the second problem by 3rd-grade ESO students		
State that it is not fair, but fails to justify or do it incorrectly	State that it is fair	Unanswered
9	16	3

None of the students gave the correct solution to the question and none of them used formulas for the perimeter of a circumference and the area of the circle.

In the case of the 1st-grade ESO students:

Resolution of the second problem by 1st-grade ESO students			
Give a good solution	State that it is not fair, but fail to justify or do it incorrectly	State that it is fair	Unanswered
1	2	9	17

For the second problem, we present the only correct answer:

In this case, since the student probably did not know the formulas for the area of a circle and the perimeter of a circumference, he drew a square with each side measuring 1 cm (the area of thissquare being easier to calculate than that of a circle) and another square with each side measuring 2 cm. He realizes that the area of the second is four times the area of the first and concludes that it is not fair to pay 1 farthing to the farmer.

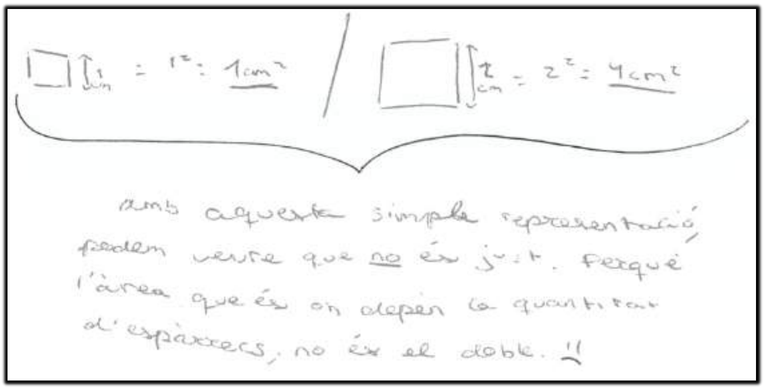


Figure 4.7: Solution by a 1st-grade ESO student.

We might ask ourselves why there were many students who tried to solve the first problem by guessing and checking, whilethe only imaginative solution for the second problem is that shown in Figure 4.7. One of the reasons could be that in the first case it is easy to guess and check using pen and paper, while in the second case it would be more practical to use objects like pieces of thread, and pencils for the asparagus,and the students are less familiar with this method.

5 Practical activity for pre-service teachers

The problems in the case under study were also posed to pre-service teachers, who were divided in six groups of five or six. They were asked to think about how they might solve these problems without using equations and also to explain in detail the four steps of the Pólya's approach to problem-solving:

- Understanding the problem
- Devising a plan
- Carrying out the plan
- Looking back

In the case of the *lazy worker,* for the first step, all the groups recommended identification of the main facts in the wording of the problem and then trying to solve it with some specific values in order to determine what the outcome would be.

Five of the six groups drew a table, two of whom regarded it as part of devising a plan and three of whom considered it as part of carrying the plan out. For this problem, they all thought that a table was a good device for visualizing all the possibilities.

None of the pre-service teachers thought about a non-integer quantity as a solution. On arriving at the conclusion that there was no solution, they proposed a re-reading of the statement of the problem and realized that no reference was made about the debt owed by the girl to the woman, and then put forward different solutions. Except for minor differences, their reasoning is similar to that followed by the student whose resolution is shown in Figure 4.5.

One of the groups analyzed the problem in a way similar to that in the work of the student shown in Figure 4.6; that is, by considering sets of 12 days and concluding that there is no solution. This group also re-read the wording and remarked that nothing is statedabout the debt owed by the girl to the woman and said that is not possible to obtain the right result, but only an approximation.

In the case of the problem of the asparagus, all the groups came to the conclusion that the problem for the students would be that they would fail to take into account that the proportionality between areas and perimeters is not the same.

One of the groups proposed a drawing with GeoGebra (Figure 5.1), for the conception of the plan, as follows:

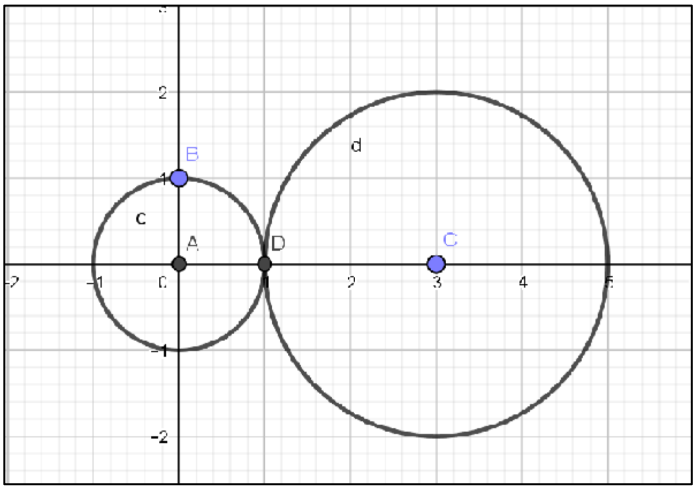


Figure 5.1: Devising a plan by pre-service teachers.

They believed that even though the relationship between the areas of these two circles might not be clear to the students, it seemed obvious that the area of the biggest circle was greater than double the area of the smaller one.

Another group proposed that students use their hand-spans or use threads or similar objects in order to experiment.

The solution given by one student, as shown above in Figure 4.7, was also taken into account by one of the groups (Figure 5.2) with the following representation:

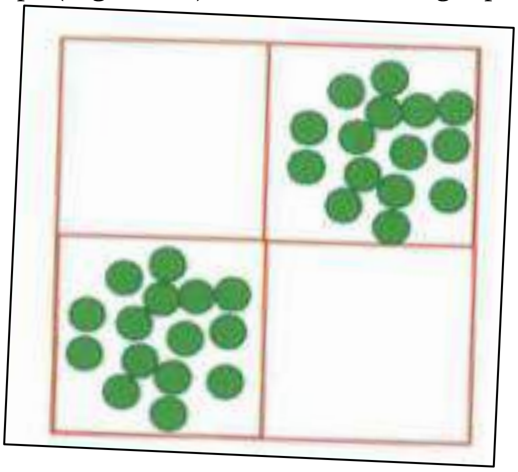


Figure 5.2: Solution by pre-service teachers.

This clearly shows that the soldier's proposal is not fair for the farmer.

All the groups found the proposal interesting, since it admits different levels of resolution and also enables teachers to follow the reasoning of the students.They also agreed that it should be considered when equations and some algorithms are introduced.

6 Concluding Remarks

The activities based on the analysis of historical texts using original sources as a point of departure contribute to improving students' overall education and provides them with additional knowledge of the social and scientific context of the periods involved.

These activities can be designed to allow for different levels of development and, in some cases, the distribution of tasks among students according to their skills.

We presented to students four texts written in Italian, German and Spanish, three of them about a *lazy worker* and the other related to proportionality. In the case of the problem of the *lazy worker,* which is a typical problem to be solved by simultaneous equations, we have seen different approaches, the most interesting of which were carried out by students who had not been taught about simultaneous equations. These students proved to be more creative, because they were unfamiliar with the standard method of resolution.

When students are told how to solve simultaneous equations and are then required to apply the method to solving a problem, the situation that initially arises could constitute a difficulty for them, since they may find it hard to translate the situation into mathematical language. However, for some students this translation may occur to them almost immediately, and the situation is not considered a problem following the explanation in the curriculum.

Early introduction of the simultaneous equations may restrict students in their exploration of methods that might otherwise be employed. In the problem of the asparagus, the student who gave the best solution was unaware of the relationship of the proportion between the lengths and the areas. He did not remember the formula for calculating the area of the circle and drew two squares, one of which was double the perimeter that the other one. He calculated the areas of these squares and deduced that the relationship between the areas of the circles, on which the number of asparagus depends, should be the same.

When teachers pose problems to students, it is important that they take into account the different backgrounds of the pupils and adapt their approach accordingly, depending on the outcomes desired.

Reading ancient texts enables students to acquire an understanding of mathematics, not as a final product but as a science that has developed on the basis of trying to answer the questions that humanity has been asking about the world throughout history.

Finally, we would like to stress the importance of using different approaches to the curriculum, for a better understanding of some concepts. In this paper, by looking at the solutions of students, using historical problems, we are aware of the difficulties that some of them have, and the strategies of others to cope with non-standard situations.

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