On a Conjecture Concerning Positive Semi-definiteness

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Abstract

In [7] a conjecture relating the positive definiteness of a similarity with its transitivity with respect to the Łukasiewicz t-norm is made. In its current form, the conjecture is not true but from a modified version interesting consequences can be derived.

Keywords: Positive definite matrix, similarity, transitivity.

1 Introduction

In the paper [7] published in this journal an interesting Conjecture is presented concerning the positive definiteness of some similarities very much related to Fuzzy Logic [11] and especially to the theory of indistinguishability operators [8]. This Conjecture is not true in its current form as will be stated in the next section but in Section 3 a reformulation leading to interesting consequences is stated and proved.

Let us recall the definition of similarity and the conjecture presented in [7].

Definition 1.1. *[7] Let E be a finite set and let $P(E)$ be its power set. A similarity is a mapping s from $P(E) \times P(E)$ into $\mathbb{R}^+$ such that*

a) $s(X, Y) = s(Y, X)$ for all $X, Y \in P(E)$

b) $s(X,Y) \leq s(X,X)$ *for all* $X, Y \in P(E)$.

A similarity s gives rise to a matrix $S = (s(A_i, A_j))$ that is called a similarity matrix in [7].

Conjecture 1.2. *[7] Let* $s : P(E) \times P(E) \to \mathbb{R}^+$ *be a similarity such that* $s(X,X) = k$ *for all* $X \in P(E)$ *and* $s(X,Y) + s(Y,Z) \leq s(X,Z) + k$ *for all* $X, Y, Z \in P(E)$. *Then the corresponding similarity matrix* S *is positive semi-definite.*

2 Counterexample and Comments

First of all let us notice that A is a positive semi-definite matrix if and only if $p \cdot A$ is positive semi-definite for all $p > 0$. So that dividing the matrix S by k in 1.2 we can assume that $k = 1$ (i.e., it is reflexive) and that s is valued in $[0,1]$. Then the condition of Conjecture 1.2 can be rewritten as

$$\max(s(X,Y) + s(Y,Z) - 1, 0) \leq s(X,Z).$$

Definition 2.1. *[5] The operation* $Ł : [0,1] \times [0,1] \to [0,1]$ *defined for all* $x, y \in [0,1]$ *by*

$$Ł(x,y) = \max(x + y - 1, 0)$$

is called the Łukasiewicz t-norm.

Definition 2.2. *Given a set* X, *a similarity* $s : X \times X \to [0,1]$ *is Ł-transitive if for all* $x, y, z \in X$,

$$Ł(s(x,y), s(y,z)) \leq s(x,z).$$

A generalization of Conjecture 1.2 to finite sets of any cardinality is then:

Conjecture 2.3. *If a reflexive similarity* $s : X \times X \to [0,1]$ *on a finite set* X *is Ł-transitive, then its corresponding similarity matrix* S *is positive semi-definite.*

The next counterexample shows that the conjecture fails for sets of cardinality greater than or equal to 5.

Counterexample 2.4. *The similarity with matrix*

$$S = \begin{pmatrix} 1 & 0.4 & 0.6 & 0.2 & 0.8 \\ 0.4 & 1 & 0.8 & 0.4 & 0.6 \\ 0.6 & 0.8 & 1 & 0.6 & 0.4 \\ 0.2 & 0.4 & 0.6 & 1 & 0.4 \\ 0.8 & 0.6 & 0.4 & 0.4 & 1 \end{pmatrix}$$

is reflexive and Ł-transitive but its determinant is -0.03584 *and one of its eigenvalues is* -0.0512922301693901.

The reason for this comes from the following results.

Definition 2.5. *If a metric space* (S, d) *is isometrically embeddable in an Euclidean space, we will say that* d *is Euclidean.*

Proposition 2.6. *[9] Let* (S, d), $S = \{x_0, x_1, ..., x_n\}$, *be a finite metric space of* $n+1$ *points. Then* d *is Euclidean if and only if the matrix* A *with entries* $x_{ij} = d_{0i}^2 + d_{0j}^2 - d_{ij}^2$, $i, j = 1, ..., n$ *where* d_{ij} *stands for* $d(x_i, x_j)$ *is positive semi-definite.*

We can send x_0 to the origin of coordinates and in the case that the matrix A is reflexive, we have that

$$d(x_i, x_j) = \sqrt{2}\sqrt{1 - x_{ij}} \text{ for } i, j = 1, ..., n.$$

From this, the next result follows (see also [4]).

Corollary 2.7. *Let* s *be a reflexive similarity on a finite set* $X = \{x_1, ...x_n\}$ *with positive semi-definite matrix* $S = (x_{ij})_{i,j=1,..n}$ *where* x_{ij} *stands for* $s(x_i, x_j)$. *Then* $d : X \times X \to [0, 1]$ *defined for all* $x_i, x_j \in X$ *by* $d(x_i, x_j) = \sqrt{1 - x_{ij}}$ *is a metric and* X *is isometrically embeddable in an Euclidean space.*

It is clear that if a distance d is Euclidean, then $k \cdot d$, $k > 0$ is also Euclidean. Hence, in order to consider euclidianity of distances we can assume that they are valued in $[0, 1]$.

The next proposition provides a relationship between distances and Ł-transitive reflexive similarities.

Proposition 2.8. *[3, 8] Let* $s : X \times X \to [0, 1]$ *be a reflexive similarity on a set* X. s *is Ł-transitive if and only if* $1 - s$ *is a pseudometric on* X.

Hence, every distance can be written in the form $1 - s$ where s is a reflexive and Ł-transitive similarity. Therefore, if the conjecture were true, this would say that the square root of any distance would be Euclidean, a fact that contradicts the results in [2].

Indeed, in [2] the authors study the values c for which, given a set X of cardinality n and a distance d on X, the power of d to c (d^c) is Euclidean. In particular they prove that for a set of cardinality 6, the greatest value c_6 of c is $\frac{1}{2} \log_2 \frac{3}{2} \sim 0.2924$ which is smaller than $\frac{1}{2}$. Of course, as the cardinality n of the set increases, the corresponding greatest value c_n decreases.

Thanks to a result by Blumenthal [1], $c_4 = \frac{1}{2}$ and the conjecture is true for sets of cardinality smaller than or equal to 4.

3 A Reformulation

In this section we will modify the hypothesis of Conjecture 1.2 in order to obtain a valid result with interesting consequences. For this, we need to recall the definition of continuous Archimedean t-norm [5] and a couple of considerations regarding [2].

Definition 3.1. *A continuous Archimedean t-norm T is an operation $T : [0,1] \times [0,1] \to [0,1]$ such that there exists a continuous decreasing mapping $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ and such that for all $x, y \in [0,1]$*

$$T(x,y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ is the pseudoinverse of t defined for all $x \in [0,1]$ by

$$t^{[-1]}(x) = \begin{cases} t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

t is called an additive generator of T.

Definition 3.2. *[8] Given a set X and a continuous Archimedean t-norm T, a similarity $s : X \times X \to [0,1]$ is T-transitive if for all $x, y, z \in X$,*

$$T(s(x,y), s(y,z)) \leq s(x,z).$$

The next result relates T-transitive similarities with distances.

Proposition 3.3. *[8] Let X be a set, T a continuous Archimedean t-norm and t an additive generator of T. $s : X \times X \to [0,1]$ a reflexive and T-transitive similarity on X if and only if $t \circ s$ is a pseudodistance on X.*

The next family of continuous Archimedean t-norms (Yager's family) will be useful.

Example 3.4. *[5] The Yager's family of continuous Archimedean t-norms $(T_\lambda)_{\lambda \in (0,\infty)}$ is defined for all $x, y \in [0,1]$ by*

$$T_\lambda(x,y) = \max((1-(1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}, 0).$$

t_λ defined by $t_\lambda(x) = (1-x)^\lambda$ for all $x \in [0,1]$ is an additive generator of T_λ.

N.B.

- If $\lambda > \mu$, then $T_\lambda(x,y) \geq T_\mu(x,y)$ for all $x, y \in [0,1]$.
- If $\lambda = 1$, then we recover the Łukasiewicz t-norm and $t_1(x) = 1-x$ is an additive generator.
- $\lim_{\lambda\to\infty} T_\lambda(x,y) = \min(x,y)$ for all $x, y \in [0,1]$.

Conjecture 1.2 is not true in its curent forma but now we can state and prove an alternative result.

Proposition 3.5. *Let n be a positive integer and c_n the greatest value satisfying that for every distance d on any finite set of cardinality n, d^{c_n} is an Euclidean distance. Then a reflexive similarity $s : X \times X \to [0,1]$ on a set X of cardinality n is $T_{\frac{1}{2c_n}}$-transitive if and only if its matrix S is positive semi-definite.*

Proof. If s is $T_{\frac{1}{2c_n}}$-transitive, then, thanks to Proposition 3.3, $(1-s)^{\frac{1}{2c_n}}$ is a pseudodistance and by Corollary 2.7 $(1-s)^{\frac{1}{2c_n}\cdot c_n} = (1-s)^{\frac{1}{2}}$ is Euclidean. Hence S is positive semi-definite. □

c_n is not known except for very few values (for $n = 2,3,4,6$, the corresponding c_n are $c_2 = \infty$, $c_3 = 1$, $c_4 = \frac{1}{2}$, $c_6 = \frac{1}{2}\log_2\frac{3}{2} \sim 0.2924$ [2]) but in [2] a lower bound k_n for c_n is given. Namely, $k_n = \frac{1}{2n}\log_2 \mathrm{e} \sim \frac{0.7213}{n}$. Therefore we have the following result

Proposition 3.6. *If a reflexive similarity $s : X \times X \to [0, 1]$ on a set X of cardinality n is $T_{\frac{n}{\log_2 e}}$-transitive, then its matrix S is positive semi-definite.*

In [2] it is conjectured that the value of c_n is

$$c_n = \begin{cases} \frac{1}{2}\log_2(\frac{n}{n-2}) & \text{if } n \text{ is even} \\ \frac{1}{2}\log_2(\frac{n^2-1}{n^2-2n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

From this we can conjecture the following.

Conjecture 3.7.

- *A reflexive similarity $s : X \times X \to [0, 1]$ on a set X of even cardinality n is $T_{\frac{1}{\log_2(\frac{n}{n-2})}}$-transitive if and only if its matrix S is positive semi-definite.*
- *A reflexive similarity $s : X \times X \to [0, 1]$ on a set X of odd cardinality n is $T_{\frac{1}{\log_2(\frac{n^2-1}{n^2-2n-1})}}$-transitive if and only if its matrix S is positive semi-definite.*

We end this note by showing that Propositions 3.5 and 3.6 provide alternative proofs of two important well known facts.

- Since $\min(x, y) \geq T_\lambda(x, y)$ for all $\lambda \in (0, \infty)$ and $x, y \in [0, 1]$, every min-transitive and reflexive similarity on a finite set is also T_λ-transitive for all $\lambda \in (0, \infty)$. From Proposition 3.5 it follows the next result (see [6] for an alternative proof).

  Proposition 3.8. *Every reflexive and* min*-transitive similarity on a finite set has a positive semi-definite matrix.*

- It is well known that s is a reflexive and min-transitive similarity on a set X if and only if $1 - s$ is a pseudoultrametric [8]. By the last proposition, its matrix S is positive semi-definite and therefore $\sqrt{1 - s}$ is Euclidean. Since the power of pseudoultrametrics are also pseudoultrametrics we obtain a new proof of this well-known result ([10]).

  Proposition 3.9. *Every ultrametric on a finite set is Euclidean.*

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