A New Lower Bound on the Maximum Number of Plane Graphs using Production Matrices

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Abstract

We use the concept of production matrices to show that there exist sets of n points in the plane that admit $\Omega(42.11^n)$ crossing-free geometric graphs. This improves the previously best known bound of $\Omega(41.18^n)$ by Aichholzer et al. (2007).

1 Introduction

A *geometric graph* on a set S of n labeled points in the Euclidean plane is a graph with vertex set S where each edge is represented by a straight line segment between the corresponding points. In this work, we are interested in the number of *crossing-free* geometric graphs on a set of n points, i.e., geometric graphs in which all segments are interior-disjoint, also referred as *plane graphs*. It is easy to see that, for any n points, this number is at least exponential in n. In 1982, Ajtai et al. [6] showed that the upper bound on this number is also exponential. Currently, it is known that any set of n points admits not more than $O(187.53^n)$ crossing-free graphs [19]. While the number of crossing-free graphs is minimized if the point set is in convex position [3], not much is known about sets maximizing this number. The best known example by now is the so-called *double-zig-zag chain* [3], with $\Omega(41.18^n)$ crossing-free graphs. As usual, such lower-bound constructions rely on describing a family of point sets with convenient structural properties. In this paper, we improve this bound by showing that another well-known family of point sets, a generalization of the double-zig-zag chain, admits $\Omega(42.11^n)$ crossing-free graphs. This generalization has also been used for similar bounds on triangulations [10] and, recently, on crossing-free perfect matchings [7], but the number of general crossing-free graphs on this configuration was not known. The method that allows us to analyze these point sets is the use of *production matrices*, a technique that we consider interesting on its own.

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This method works by implicitly arranging the graphs in a *generating tree*, describing a rule to produce a graph from one on fewer points. We consider a partition of the set of graphs on $i \leq n$ points into n parts according to their degree at an arbitrarily defined *root vertex*, and represent the cardinality of each part in a vector $\vec{v}^{\,i}$. The first element of $\vec{v}^{\,i}$ is the number of graphs with the root vertex having degree 0, the second one that of graphs with root vertex with degree 1, and so on. We then devise how to generate graphs on $i + c$ points (for some small positive number c) with a new root vertex, from the graphs counted in $\vec{v}^{\,i}$, and again give the cardinalities of their parts in a vector $\vec{v}^{\,i+c}$. In the production matrix approach, the relation between $\vec{v}^{\,i}$ and $\vec{v}^{\,i+c}$ is encoded in an $n \times n$ *production matrix* $A \in \mathbb{N}_0^{n \times n}$ such that $\vec{v}^{\,i+c} = A\vec{v}^{\,i}$. In this way, we obtain the number of graphs on n vertices in $\vec{v}^{\,n} = A^j \vec{v}^{\,n_0}$ from the graphs on a constant number n_0 of vertices, with $j = (n - n_0)/c$. Deriving a production matrix in general is a difficult task, but in this paper we show how that can be done for point sets with a particular structure.

In this paper, we focus on obtaining an asymptotic lower bound on the number of crossing-free graphs. To that end, we obtain the corresponding production matrix A, and apply the Perron–Frobenius theorem to obtain a lower bound on the elements of A^j when j tends to infinity, by approximating the largest eigenvalue of the matrix. This gives us a lower bound on the number of crossing-free graphs on such a point set.

For points in convex position, generating trees have been described for triangulations [16], spanning trees [12], and very recently for a few other crossing-free graphs [13]. They are also the basis of the ECO method [8]. The term *production matrix* was introduced in [9], although the equivalent term *AGT matrix* [17] is sometimes used. In a recent paper together with Seara [14], we already studied characteristic polynomials of production matrices for various classes of geometric graphs, which can give rise to new relations between well-known combinatorial objects. Asinowski and Rote [7] use similar matrices to bound the number of crossing-free perfect matchings of point sets; in fact, the classes of point sets they consider are the same as ours.

Indeed, the work by Asinowski and Rote [7] is closely related to ours, both in the classes of point sets considered, as well as in the counting methods. In their paper, they use various methods for counting crossing-free perfect matchings on particular point sets; in Section 5 of their work, they obtain a sequence of infinite vectors whose elements are the number of matchings on n vertices, partitioned by the number of unmatched points (i.e., points that still have to be matched). They devise an infinite band matrix A to obtain this sequence of vectors. Inspired by the Perron–Frobenius theorem (which we will use for our fixed-size matrices), they show that the growth rate for perfect matchings is equal to the column sum of A after stabilization. This way, they also show that the bound is tight for that class of point sets. While our production matrices can also be considered infinite, we will eventually consider constant-size matrices (and thus only obtain a subset of all possible crossing-free graphs), in order to obtain a lower bound using the Perron–Frobenius theorem.

For many classes of crossing-free graphs, it is known that their number is minimal when the points are in convex position [2, 3]. A remarkable exception are triangulations, where it is conjectured that so-called *double circles* are the minimizing point set class [5]; the current best bound [1] is, however, far from the number of triangulations of the double circle. Much less seems to be known about point sets that maximize the number of graphs. See the online list by Sheffer [20] for current bounds on these numbers for various graph classes.

Outline. We begin in Section 2 by introducing the production matrix technique with an example, i.e., counting the number of plane graphs on points in convex position. In the following section we define the family of point sets used to obtain our improved lower bound,

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Table 1: Matrix for plane graphs in convex position, for $n = 6$.

the generalized double zig-zag chain. In Section 4 we provide production matrices to count sub-graphs in its different parts, together with an additional technique that allows to improve the lower bound even further. Using this, in Section 5, we argue that bounds on the Perron roots of the matrices give us a lower bound on the number of crossing-free graphs, leading to our main result.

2 Warm-up for points in convex position

Our point sets will be described as sequences $p_1, \ldots, p_n$, where each p_i ($1 \le i \le n$) is a vertex. Consider any $(i+1)$-vertex graph G drawn on vertices p_1 to p_{i+1}. We can associate G to a graph G' with i vertices, by replacing every edge $p_j p_{i+1}$ by the edge $p_j p_i$ for all $1 \le j \le i$ (disregarding duplicates and loops). The graph G' that we obtain is called the *parent* of G. Our sets will be such that G' is crossing-free. In the other direction, we can select some edges incident to p_i in G' and replace them by edges incident to p_{i+1} in a way that G' is the parent of the new graph $\tilde{G}$, and such that $\tilde{G}$ is crossing-free. We say that G' *produces* $\tilde{G}$, and that the edges incident to p_{i+1} that are mapped to edges of G' are *inherited* from G'. The degree of p_i in G' determines how many graphs can be produced from it.

In our construction, p_i will be the root vertex, and the vector $\vec{v}^{\,i}$ will contain the number of graphs with root vertex p_i of degree j, for $0 \le j \le n$. The relation between a graph on n vertices and those that can be produced from it define the implicit generating tree. For our purposes, we do not need the tree explicitly, but are only interested in counting how many graphs can be produced from another one.

We will introduce the production matrix approach via obtaining the (known) lower bound for the number of graphs on points in convex position. The base of this lower bound is the largest eigenvalue of the matrix shown in Table 1.

We have n points $p_1, \ldots, p_n$ in convex position, indexed from 1 to n in, say, clockwise order along the convex hull boundary. Note that we can always add an edge of the convex hull to any graph without introducing any crossings. Hence, we may only count plane graphs without edges on the convex hull boundary and then multiply their number by 2^n, accounting for all possibilities of adding such edges. (Recall that we are considering labeled graphs.)

Each graph on $i+1$ vertices is mapped to its unique parent graph on i vertices that is obtained by identifying p_{i+1} with p_i, and possibly deleting the edge $p_{i-1}p_i$ that is now on the convex hull boundary and stems from $p_{i-1}p_{i+1}$ (see Figure 1 (left)). Note that apart from this edge, there is only one other possibility for the number of edges to be less in the parent graph than in the original one. This happens when there are both the edges $p_k p_i$ and $p_k p_{i+1}$, which are mapped to the same edge (which is shown in Figure 1 (right)). Note that, since the graph is crossing-free, there is no other pair of such edges $p_{k'} p_i$ and $p_{k'} p_{i+1}$.

Let us now translate this relation to a production matrix. Suppose we are given the vector $\vec{v}^{\,i}$ that contains the number of graphs partitioned by their degree at vertex p_i. For example,

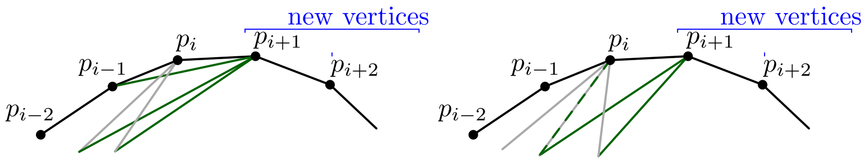

Figure 1: Part of a convex chain. Left: Vertex p_{i+1} has degree k and the graph is obtained from one where p_i has degree $k-1$. This requires the presence of edge $p_{i-1}p_{i+1}$; p_{i+1} inherits all edges incident to p_i. Right: Vertex p_{i+1} has degree k and the graph is obtained from one where p_i has degree at least k. In this case, p_{i+1} inherits k edges, and the last inherited edge may be duplicated and remain incident to p_i.

we can start with $\vec{v}^{\,3} = (1, 0, 0, \dots)^T$. Now we want to obtain the vector $\vec{v}^{\,i+1} = C\vec{v}^{\,i}$ by finding an appropriate matrix C. The jth row of C is thus used to produce the number of graphs where p_{i+1} has degree $j-1$. Next we derive the shape of the different rows of C.

First row. The plane graphs where p_{i+1} has degree 0 are equal to all the graphs counted in $\vec{v}^{\,i+1}$. This gives a first row of 1s in the matrix C.

Second row. If p_{i+1} has degree 1, there are two possibilities to obtain that degree. If the degree of p_i is 0, we can add the edge $p_{i-1}p_{i+1}$, and we get a one in the first column of the second row. Otherwise, p_i has degree at least 1, and p_{i+1} can inherit one edge from p_i. Moreover, there is the option of keeping (a copy of) the inherited edge incident to p_i without creating any crossing. In total, for each graph in which p_i has degree at least one, that gives two ways for making p_{i+2} have degree 1. Thus, the rest of the row is made of 2s.

Other rows. The following rows are analogous, shifted by one column every time: There are two ways for p_{i+1} to have degree k. Either k edges are inherited from p_{i+1}, for which the minimum degree for p_i needs to be k; since we can always choose to keep the last inherited edge incident to p_{i+1}, we get 2 options every time (cf. Figure 1 (left)). Otherwise, p_i needs to have exactly $k-1$ edges, which are inherited by p_{i+1}; by adding the edge $p_{i-1}p_{i+1}$, the degree of p_{i+1} becomes k (see Figure 1 (right)). This results in matrix C in Table 1.

For n vertices, we need that the size of C is at least n, and then we can obtain $\vec{v}^{\,n} = C^{n-3}\vec{v}^{\,3}$. We will see in Section 5 how to get a lower bound on the entries of $\vec{v}^{\,n}$ from a lower bound on the largest eigenvalue of a constant-size version of C.

The main contribution of our paper will be the generalization of this approach to point sets with a more complicated structure.

As we will see, while this captures the basic idea of our proofs, we will actually have to use more involved constructions, in which we add a constant number of points at once and add edges, some inherited, and some not, in a well-defined, local way. To make this more precise we need to define our construction, which we do next.

3 Generalized double zig-zag chains

In this section, we describe the classes of point sets we will investigate and provide an outline of our general counting approach.

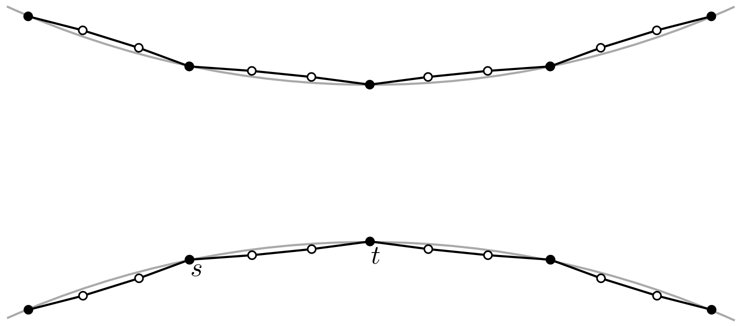

Figure 2: A generalized double-zig-zag chain Z_2. The arcs for the construction are gray, the solid edges are not crossed by any segment between two points of the set. The points placed on the arcs are black, while the $k = 2$ points in interior of the pockets are white. Any two consecutive black point, such as s and t, together with the interior points between them, form one pocket.

3.1 The generalized double-zig-zag chain

The construction that we will analyze, and will allow us to improve on the existing lower bound on the number of crossing-free graphs, is the *generalized double-zig-zag chain*, illustrated in Figure 2. It is a family of point sets parameterized by two values n and k, for n the total number of points, and k a parameter that defines the *pocket size*. Next we make this more precise.

Let Z_k be a set of $n = 2z$ points with $z \equiv 1 \pmod{(k+1)}$, for k a parameter, that is arranged in the following way. Consider two x-monotone circular arcs facing each other as in Fig. 2, such that each point on one arc can *see* each point on the other arc (where two points can see each other if the interior of the line segment connecting them does not intersect one of the arcs). On each arc, we place $\lceil z/(k+1) \rceil$ points (shown black in the figure). Consider the segment between two consecutive such points s and t on the lower arc. We now place a "flat" circular arc between s and t with circle center above the arc, and place k points on it (shown white in the figure); here, flat means that moving the center of the arc up (and thus the k points on it) does not change the set of crossing-free graphs drawable on Z_k. We call the group formed by s, t, and the k points in between them a *pocket* (of *size* k). We place k such points between each pair of consecutive points of the lower arc (obtaining the *lower chain*), and also in an analogous way on the upper arc (resulting in the *upper chain*). The example in Figure 2 shows Z_2, where each pocket has size two (i.e, $k = 2$). It will be sometimes useful to refer not only to the k points in the pocket, but also to the two points defining it (i.e., the k points together with s and t above). For this we will use the term *cup*. Thus, we will refer to either a pocket of size k, or to the corresponding cup of size $k + 2$, which we will denote a $(k + 2)$-cup.

We label the points along the lower arc, including pockets, from left to right, $p_1, \ldots, p_z$, and those on the upper arc $q_1, \ldots, q_z$.

Double chains (i.e., $k = 0$) have provided extremal configurations for various settings in the investigation of geometric graphs [11]. The generalization to the double zig-zag chain (i.e., $k = 1$) was devised in [3, 4] to obtain improved bounds on the number of crossing-free graphs and triangulations. (Each of the chains can be considered as a *double circle*, a point configuration that is conjectured to have the fewest number of triangulations [5].)

Generalizations to larger pocket sizes allowed for improving the bound for triangulations [10] and perfect matchings [7].

3.2 Counting strategy

First we observe that the segment between any two consecutive points $p_i p_{i+1}$ is not crossed by any other segment between two points of the set, and thus can co-exist with any other edge in a crossing-free graph. For this reason, these edges will be disregarded first in our counting, and will be considered in the end by multiplying by a factor of 2^n.

Therefore, in the next section we will split the counting into two parts. On the one hand, we will count the graphs with edges below the path $(p_1, \ldots, p_z)$ (and, symmetrically, those above the path $(q_1, \ldots, q_z)$) called the *outer part*. On the other hand, we will count the edges that connect vertices of the two paths, which are in the *inner part*.

Our counting will be on Z_k for $2 \leq k \leq 6$.

4 Counting for the outer and inner parts

In order to count the number of crossing-free graphs in the outer and inner parts of Z_k, we will derive production matrices for them. We begin with the outer part, for which we will present a matrix that counts the exact number of graphs. For the case of the inner part, we first present a matrix for Z_2 that will give an almost exact lower bound. Already for Z_3, this method seems not to give a good estimate. Therefore, we derive our values for the inner part by estimating polynomial coefficients. There are n edges that separate the two parts (connecting consecutive vertices of the chains and the two chains on the convex hull boundary). As the number of possibilities to add (or not) such edges to a plane graph is 2^n, we will not consider these edges here, and will add the 2^n term directly in the end.

4.1 Outer part

In this section we deduce matrices to count the number of plane graphs with edges below the path $(p_1, \ldots, p_z)$, as in Figure 3. Recall that a chain is composed of a series of pockets; each pocket of size k forms a cup or reflex chain on $k+2$ vertices. The first and last vertices are convex, while the k middle ones are reflex. The first (say, with smallest index) reflex vertex is called the *leading* vertex of the chain. All other vertices we call *regular*.

We will present a matrix to count the number of plane graphs in the outer part after adding one whole pocket. This matrix will be the product of several matrices, one related to each new vertex of the pocket. For instance, for $k = 2$, we will have one related to each of $p_{i+1}, p_{i+2}, p_{i+3}$ (recall that p_i coincides with the last vertex of the previous pocket).

4.1.1 Matrix for regular vertices

For simplicity, we present the following for $k = 2$, but it works in the same way for larger sizes. Consider a regular vertex like p_{i+2} (refer to Figure 3). Our goal is to find a matrix R such that $\vec{v}^{\,i+2} = R\vec{v}^{\,i+1}$. Assume that the vector $\vec{v}^{\,i+1}$, containing the number of plane graphs for each possible degree of p_{i+1}, is known.

First row. The plane graphs where p_{i+2} has degree 0 are equal to all the graphs counted in $\vec{v}^{\,i+1}$. This gives a first row of 1s in the matrix R.

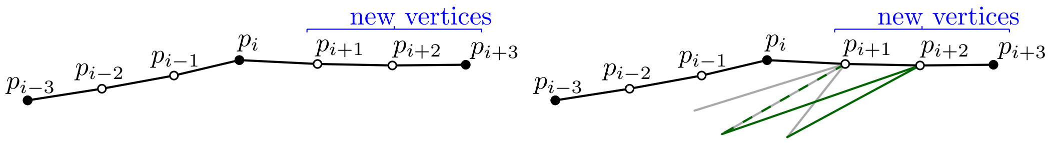

Figure 3: Left: Part of an almost convex chain with two interior vertices (i.e., $k = 2$). Vertices p_{i-2} and p_{i+1} are leading vertices. The other vertices are regular. Right: Since p_{i+2} is a regular vertex, any edge incident to p_{i+2} present in a plane graph can be obtained by inheriting an edge from the previous vertex p_{i+1}. The example shows p_{i+2} inheriting two edges from p_{i+1}. The last inherited edge (dashed) may also be kept at p_{i+1} without influencing the degree of p_{i+2}.

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Table 2: Matrices for computing the outer part, for $n = 6$.

Second row. If p_{i+2} has degree 1, it needs to inherit one edge from p_{i+1}. If the degree of p_{i+1} is 0, this is not possible, thus we get a zero in the first column of the second row. As soon as p_{i+1} has degree at least 1, p_{i+2} can inherit one edge from p_{i+1}. Moreover, there is the option of keeping (a copy of) the inherited edge incident to p_{i+1} without creating any crossing. In total, for each graph in which p_{i+1} has degree at least one, that gives two ways for making p_{i+2} have degree 1. Thus the rest of the row is made of 2s.

Other rows. The following rows are analogous, shifted by one column every time: in order for p_{i+2} to have degree k, k edges need to be inherited from p_{i+1}, thus the minimum degree for p_{i+1} is k. Since we can always choose to keep the last inherited edge incident to p_{i+1}, we get 2 options every time.

This results in matrix R in Table 2. Exactly the same matrix applies to p_{i+3}, and to all other regular vertices when $k > 2$.

4.1.2 Matrix for leading vertices

Leading vertices like p_{i+1} in Figure 3 require a different approach, as there are edges incident to p_{i+1} that cannot be obtained by inheriting from p_i (i.e., edges $p_{i+1}p_{i-1}$, $p_{i+1}p_{i-2}$, $p_{i+1}p_{i-3}$, as p_ip_{i-1}, p_ip_{i-2}, p_ip_{i-3} are not in the outer part).

In general, for pockets of size k, we partition the graphs depending on which edges connect p_{i+1} to $p_{i-k-1}, \dots, p_{i-1}$.

- **No edges from p_{i+1} to any of $p_{i-k-1}, \dots, p_{i-1}$.** All these graphs can be produced by inheriting edges from p_i like for regular vertices, i.e., by applying matrix R; see Figure 4, left.
- **First warm-up: One edge from p_{i+1} to p_{i-1} (but none to $p_{i-k-1}, \dots, p_{i-2}$).** Such graphs can be produced by inheriting edges from p_{i-1}. The vector for p_{i-1} can

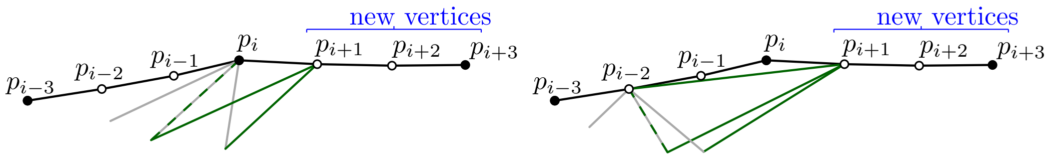

Figure 4: Computing leading vertices on Z_2. Left: When edges $p_{i+1}p_{i-1}$, $p_{i+1}p_{i-2}$ and $p_{i+1}p_{i-3}$ are not included, p_{i+1} can inherit edges from p_i. The example shows p_{i+1} inheriting two edges from p_i. The last inherited edge (dashed) may be kept without influencing the degree of p_{i+1}. Right: We distinguish cases on which of the edges $p_{i+1}p_{i-1}$, $p_{i+1}p_{i-2}$ or $p_{i+1}p_{i-3}$ are included. In the example $p_{i+1}p_{i-2}$ is included, and p_{i+1} inherits two edges from p_{i-2}. The dashed edge can be optionally kept. Note that in this case, p_{i+1} cannot inherit any edge from p_i.

be obtained by applying R^{-1} to the one for p_i. However, inheriting cannot be done the same way as from p_i, as the edge $p_{i+1}p_{i-1}$ increases the degree of p_{i+1} by one. This increase can be captured by shifting the entries of matrix R vertically one row (i.e., we obtain no graphs with degree 0, as they have been counted in the previous case, we get exactly one graph of degree one for every parent graph, etc.). This shift of R is obtained by multiplying it with the matrix S in Table 2. The number of graphs with exactly the edge $p_{i+1}p_{i-1}$ added is thus obtained by multiplying with $SRR^{-1} = S$.[1]

- **Second warm-up: An edge from p_{i+1} to p_{i-2}.** We can apply the same reasoning as before, inheriting from p_{i-2}. The corresponding vector is obtained by multiplying with R^{-2}. We have the edge $p_{i-2}p_{i+1}$, and count both graphs that do and do not contain the edge $p_{i-1}p_{i+1}$. For the graphs not containing this edge, we apply the shift matrix S once (see Figure 4, left), and for the ones containing it, we have to apply it twice (as the degree of p_{i+1} is increased by two). The graphs in this case are thus obtained by multiplying the vector with $SRR^{-2} + S^2RR^{-2} = (S + S^2)R^{-1}$.

- **In general, an edge from p_{i+1} to p_{i-m}.** The reader may by now already have realized the pattern to follow for counting graphs with an edge $p_{i+1}p_{i-m}$. We obtain the vector at p_{i-m} by applying R^{-m}. Then, edges are inherited by applying R, but we have to shift R once to account for the degree increase by the edge $p_{i-1}p_{i+1}$ at p_{i+1}. Then we have to consider the different possibilities for edges between p_{i+1} and $p_{i-m+1}, p_{i-m+2}, \ldots$; in general, when adding $l \leq m$ edges between p_{i+1} and those vertices, we have to add $p_{i-m}p_{i+1}$, and have $m-1$ possibilities for the remaining $l-1$ edges. We get

$$\sum_{l=1}^{m} \binom{m-1}{l-1} S^l R^{1-m}$$

whenever p_{i-m} is the vertex with the smallest index on that pocket with an edge to p_{i+1}.

When summing over all m, the sum of the matrices is the matrix giving the number of graphs when adding the leading vertex p_{i+1}. Note that p_i cannot share an edge in the outer part with any of the vertices on that pocket, which also includes $p_{i-(k+1)}$.

[1] Observe that inheriting from p_i and adding the edge $p_{i-1}p_{i+1}$ are exactly the "operations" that we used for points in convex position in Section 2. Indeed, $C = R + S$. While this line of arguments (using the inverse of R) is slightly more involved, it generalizes nicely, as we will see in the next two items.

$$Q = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 10 & 6 & 0 & 0 & 0 & 0 \\ 5 & 10 & 6 & 0 & 0 & 0 \\ 1 & 5 & 10 & 6 & 0 & 0 \\ 0 & 1 & 5 & 10 & 6 & 0 \\ 0 & 0 & 1 & 5 & 10 & 6 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Table 3: Matrices for computing the inner part, for $n = 6$.

$$\vec{v}^{\,i+1} = \left(R + \sum_{m=1}^{k+1} \sum_{l=1}^{m} \binom{m-1}{l-1} S^l R^{1-m} \right) \vec{v}^{\,i} \tag{1}$$

4.1.3 Putting things together

The final production matrix for the outer part is obtained by combining matrices R and S accordingly. For each of the regular vertices it is enough to multiply the previous vector by R. For the leading vertex we use the expression in (1). Thus the final combined matrix for the outer part, for pockets of size k, is

$$R^k \left(R + \sum_{m=1}^{k+1} \sum_{l=1}^{m} \binom{m-1}{l-1} S^l R^{1-m} \right) . \tag{2}$$

4.2 Inner part

The number of graphs on the inner part can be bounded similar to [3]. However, we also show how to obtain a lower bound using production matrices, based on two additional matrices, Q and F, shown in Table 3.

4.2.1 Using a production matrix for Z_2

The main difference with the outer part is that inheriting edges from a point on one chain to a point on the other chain does not work here. One way of coping with this is to only count some of the graphs; it will turn out that this is sufficient to get a good bound for Z_2.

We add three points on the lower and the upper chain in alternation, and keep track of the numbers at root vertex q_j (that is always on the upper chain, and may not be the last added point). The degree of root vertex q_j will be the number of edges incident to vertices of the other chain. See Figure 5. However, when adding pockets on the lower chain, we only add edges from these points to q_j. Note that by doing this we do not count possible graphs in which the added pocket connects to a vertex q_l with $l < j$. Thus in our calculations we will be missing some graphs, and we will get only a lower bound, instead of an exact count, as we had for the outer part.

We add a pocket $(p_i, p_{i+1}, p_{i+2}, p_{i+3})$ to the chain. We analyze several cases depending on the degree of q_j (considering only connections to $p_{i+1}, p_{i+2}, p_{i+3}$), which lies between zero and three.

q_j has degree zero. If the degree of q_j is zero, there are six ways to add edges among the four points of the pocket without creating crossings: we have three possibilities by adding either $p_i p_{i+2}$ or $p_{i+1} p_{i+3}$ or none, and for each of these we may or may not add the edge $p_i p_{i+3}$.

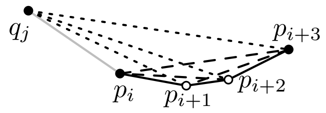

Figure 5: We compute the production matrix when adding three points to the right to the lower chain, forming a new pocket. The production matrix produces the degree vector at q_j with respect to $p_{i+1}, p_{i+2}, p_{i+3}$. The dotted edges influence the degree, and the dashed edges determine the multiplicity.

q_j has degree one. For having one edge between q_j and one of p_{i+1}, p_{i+2}, or p_{i+3}, we in total have ten possibilities; six when adding q_jp_{i+3} as in the previous case, and for each of q_jp_{i+1} and q_jp_{i+2} we have two possibilities by adding or omitting the edges $p_{i+1}p_{i+3}$ and p_ip_{i+2}, respectively.

q_j has degree two. For two such edges from q_j, there are five possibilities; one for adding both q_jp_{i+1} and q_jp_{i+2}, and the other four for adding one of them and q_jp_{i+3} (again, one of p_ip_{i+2} and $p_{i+1}p_{i+3}$ is not crossed by these edges and may or may not be added).

q_j has degree three. Finally, for three edges there is clearly only one possibility.

Since in our analysis the previous four are the only degrees that q_j can have in the relevant subgraph (i.e., the possible increment of the degree by adding the pocket), the values above give us the non-zero column entries of the first column of the matrix, which repeats with vertical shifts in the other columns. The result is matrix Q in Table 3. Recall that we do not count the possibilities of adding an edge to some $q_l, l < j$.

For the upper chain $(q_1, \ldots, q_z)$, we can apply matrices similar to CCR (in Tables 1 and 3). Again, we measure the degree in the number of edges going to the other chain. For the first convex (w.r.t. the interior) vertex q_{i+1} of a pocket, we inherit the edges of the reflex vertex q_i in the usual way, using matrix R. For the second convex vertex q_{i+2}, we have the same options, but in addition may add the edge to q_i; note that this does not increase the number of edges from q_{i+2} to the other chain. This is captured in matrix F in Table 3.

Now consider the last vertex q_{i+3} of a pocket. Note that q_{i+3} only inherits edges that go from q_{i+2} to the other chain. Potentially we also can add edges from q_{i+3} to q_i and q_{i+1}.

We can therefore define that, for the case in which q_{i+3} inherits all the edges, if there is an edge q_iq_{i+2}, we add the edge q_iq_{i+3}, and if there is no such edge, we add the edge $q_{i+1}q_{i+3}$. These graphs are produced by F. So we did not count the graphs that had an edge q_iq_{i+3} and not the edge q_iq_{i+2}, and for those, we can inherit the edges from q_i and double the number, as we have the option of adding $q_{i+1}q_{i+3}$ (we did not produce the latter so far, as the edge q_iq_{i+3} is not inherited from q_{i+2} using matrix F). Thus, we apply $2R$ to count these graphs.

We obtain that the matrix $(FFR + 2R)Q$ gives a lower bound on the number of plane graphs for the inner part when adding first three points at the bottom chain and then three points at the top chain. We conclude that a lower bound on the number of plane graphs in the inner part is given by $((FFR + 2R)Q)^{n/6}$.

Observe that this method does not count graphs in which there are edges from a pocket on the upper chain to one of a lower chain that is further to the right. We will show in Section 5 that, despite this, this approach for Z_2 already gives a better lower bound on the number of crossing-free graphs. However, it seems that the number of graphs that we do not count with this approach gets more significant when generalizing it for Z_3. For that reason,

Figure 6: Left: A tapped 4-cup. The brown thick edge hides the inner two vertices from the vertices of the other chain. Right: One vertex of a 4-cup is tapped.

in the following we use a more conventional way of counting the graphs of the inner part, which will allow us to improve the lower bound even further for larger values of k.

4.2.2 Using an estimation of binomial coefficients

This approach is a generalization of the one presented in [4]. Consider one of the two chains of Z_k, which has $z = \frac{n}{2}$ vertices. Then the number of pockets on the chain is $\frac{z}{k+1}$, forming convex cups of $k+2$ elements. The main idea of the approach is to count the possibilities of putting edges between vertices of a pocket; we say that such an edge is *tapping* the vertices behind them. For the remaining edges, the set of un-tapped vertices behaves like a double chain, for which the number of plane graphs is known. The number of vertices tapped will depend on parameters, $\alpha, \beta, \dots$, which we will then optimize.

We provide the main idea for small pocket sizes. For Z_2, the number of caps of a chain is $\frac{z}{3}$. To construct a plane graph, we first choose $\alpha \frac{z}{3} = \alpha \frac{n}{6}$ 4-cups from the chain and tap them. Then there are three ways to get a plane graph in the interior of a tapped 4-cup. This gives us a factor of

$$\binom{\frac{n}{6}}{\alpha \frac{n}{6}} 3^{\alpha n/6} \quad .$$

Of the remaining $\frac{n}{6} - \alpha \frac{n}{6}$ 4-cups, we choose $\beta \frac{n}{6}$ 4-cups and tap one point. There are two ways to choose the tapped point for each of these 4-cups. This gives us a factor of

$$\binom{(1-\alpha)\frac{n}{6}}{\beta \frac{n}{6}} 2^{\frac{\beta n}{6}} \quad .$$

To get the asymptotic behavior of these factors, we estimate a binomial coefficient as

$$\binom{n}{\lambda n} \approx 2^{H(\lambda) n} \quad .$$

where $H(x)$ is the entropy function

$$H(x) = -x \log_2(x) - (1-x) \log_2(1-x) \quad .$$

This also implies

$$\binom{\mu n}{\lambda n} \approx 2^{H(\frac{\lambda}{\mu}) \mu n} \quad .$$

We get

$$2^{H(\alpha) n/6} 2^{\log(3) \alpha n/6} \cdot 2^{H(\beta/(1-\alpha))(1-\alpha) n/6} 2^{\beta n/6} \quad . \tag{3}$$

Then, the number of points of the chain which are not tapped is

$$z' = \frac{n}{2} - 2\alpha \frac{n}{6} - \beta \frac{n}{6} = \frac{n}{6}(3 - 2\alpha - \beta) \quad .$$

The number of graphs having only edges between the chains is thus the same as for the double chain. A double chain of $2z'$ vertices has $\Omega\left(\left(\frac{10+7\sqrt{2}}{3+2\sqrt{2}}\right)^{2z'}\right)$ such plane graphs [11]. By multiplying this with (3), we get the number of plane graphs for our two parameters. Numerically maximizing over them gives us a lower bound on the number of plane graphs in the interior part of the two chains of

$$\Omega(2^{\frac{6.19683}{3}n}) = \Omega(4.18611^n) \ .$$

Let us now consider Z_3. Let α, β, and γ be the fraction of cups in which three, two, and one vertices are tapped, respectively. For three tapped vertices, we have 11 different ways of plane graphs in the tapped part (see Figure 7), seven for tapping two vertices (see Figure 8), and three for tapping one. For the first part, we get

$$\binom{\frac{n}{8}}{\alpha\frac{n}{8}} 11^{\alpha n/8} \ ,$$

for the next one

$$\binom{(1-\alpha)\frac{n}{8}}{\beta\frac{n}{8}} 7^{\beta n/8} \ ,$$

and for the third one

$$\binom{(1-\alpha-\beta)\frac{n}{8}}{\gamma\frac{n}{8}} 3^{\gamma n/8} \ .$$

We get

$$2^{(H(\alpha)+\log(11)\alpha+H(\beta/(1-\alpha))(1-\alpha)+\log(7)\beta+H(\gamma/(1-\alpha-\beta))(1-\alpha-\beta)+\log(3)\gamma)n/8} \ . \tag{4}$$

The number of points of the chain that are not tapped is

$$z' = \frac{n}{2} - 3\alpha\frac{n}{8} - 2\beta\frac{n}{8} - \gamma\frac{n}{8} = \frac{n}{8}\left(4 - 3\alpha - 2\beta - \gamma\right) \ .$$

Again, we can consider the remaining vertices as being the ones of a double chain and count the graphs in the inner part. The overall number is the product with (4). The maximum is obtained for $\alpha \approx 0.100302$, $\beta \approx 0.217924$, and $\gamma \approx 0.318874$. We obtain $\Omega(4.398895942833997^n)$ graphs for the inner part.

In the same way, we obtain bounds for Z_k with larger k. The tedious part of this computation is to obtain the number of possibilities for tapping a certain number of vertices. The numbers are summarized in Table 4. Note that the number of plane graphs of a k-cup, without its boundary edges, is given by the formula for the number of plane graphs on k points in convex position, divided by 2^k. We sketch the counting for the possibilities for the tappings of Z_5 in Figure 9, as this pocket size gives our best bound.

5 A lower bound using the largest eigenvalue

In order to use the production matrices devised to obtain bounds on the number of crossing-free graphs, we need to bound the elements of the matrix powers as n tends to infinity. This asymptotic information is given by the largest eigenvalue of the production matrix, which is what we analyze next.

Z_2		Z_3		Z_4		Z_5		Z_6	
t:	p:	t:	p:	t:	p:	t:	p:	t:	p:
1	2	1	3	1	4	1	5	1	6
2	3	2	7	2	12	2	18	2	25
		3	11	3	28	3	52	3	84
				4	45	4	121	4	237
						5	197	5	550
								6	903
4.18^n		4.39^n		4.55^n		4.67^n		4.77^n	
41.77^n		42.01^n		42.09^n		42.11^n		42.08^n	

Table 4: Number of possibilities p to tap t vertices in different generalized double zig-zag chains. The last-but-one line contains the bound for the graphs in the inner part (numbers rounded down). The bottom line contains the obtained overall bounds.

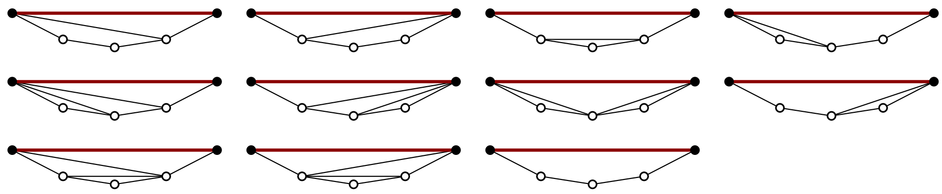

Figure 7: Eleven different graphs for a 5-cup with all three vertices tapped.

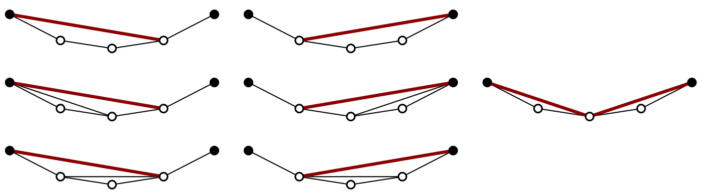

Figure 8: Seven different graphs for a 5-cup with two tapped vertices.

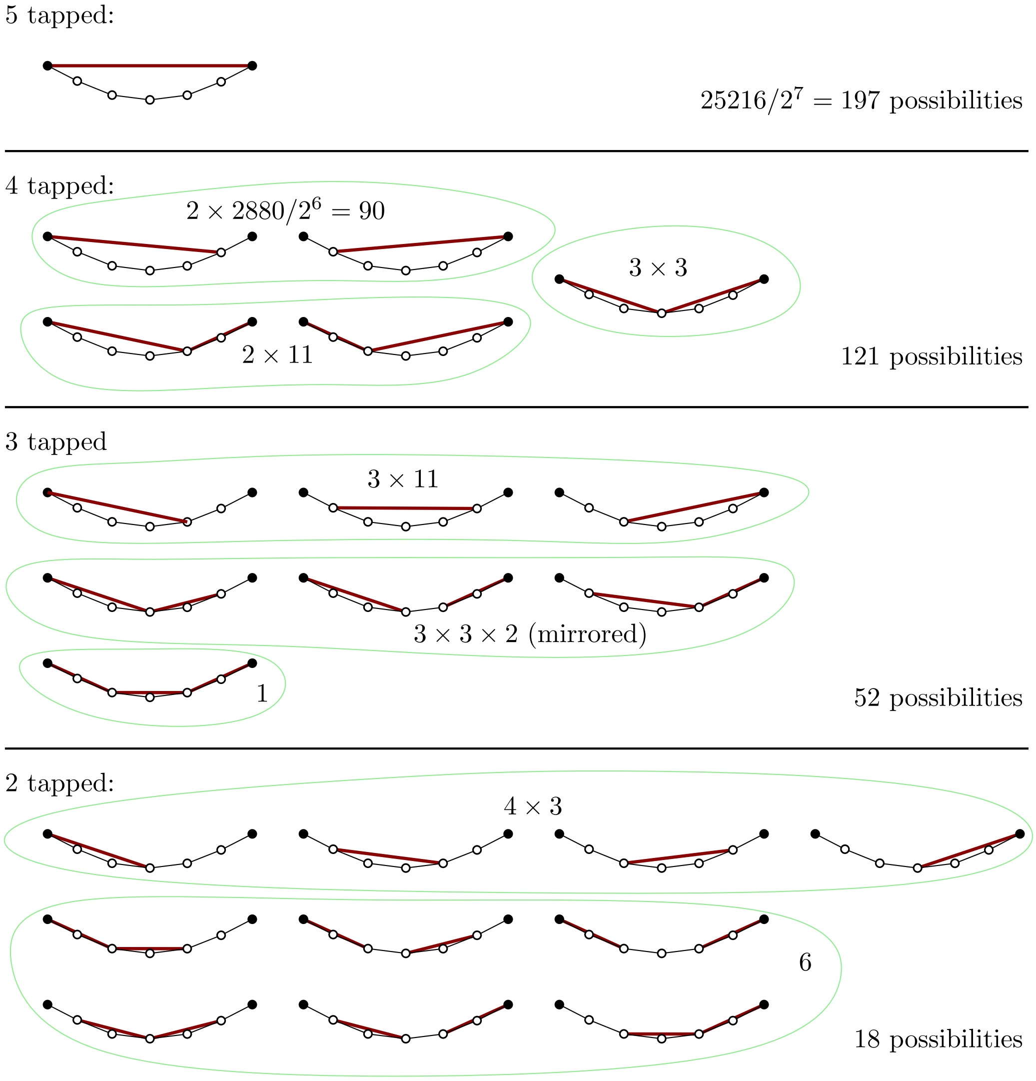

Figure 9: Different possibilities for tapping pockets of size 5, i.e. 7-cups. We distinguish the different types of paths that tap the vertices, and multiply it with the number of graphs in the convex sub-parts. The numbers for the convex sub-parts can be obtained from the known number of crossing-free graphs on small point sets in convex position. See, e.g, [21, s.v. A054726].

All our production matrices are non-negative. The zero entries are exactly those below a sub-diagonal. Thus, they are irreducible and primitive (Frobenius' test for primitivity holds, cf. [18, p. 678][2]). Let A be a production matrix of fixed size $m \times m$. We know therefore that

$$\lim_{n\to\infty} \left(\frac{A}{r}\right)^n = \frac{\vec{p}\vec{q}^T}{\vec{q}^T\vec{p}} > 0 \ ,$$

where $\vec{p}$ and $\vec{q}$ are the Perron vectors of A and A^T, respectively, and r is the Perron root (i.e., largest eigenvalue) of A [18, p. 674]. As these values are constant and each entry of A^n is in $\Theta(r^n)$, this provides a means of obtaining the asymptotic number of elements constructed by the production matrix: multiplying the initial degree vector with A^i gives the degree vector for $ci < m$ points. However, there is one caveat. The exponent n tends to infinity, and thus we cannot use this to argue about matrices of size n. The matrix size must be fixed. However, for obtaining lower bounds, we can take the nth power of a $(m \times m)$ production matrix for some constant m to obtain a lower bound on the number of graphs on n vertices. In the first iteration where we add a point larger than the size of the matrix, we do not count some graphs with high degree at the last point. These are also not taken into account in the following iterations, where we also produce graphs of smaller degree at the last point. Still, the degree vector gives a lower bound on the number of graphs. Hence, we can consider the Perron root r of a constant-size production matrix, and know that the number of graphs on n vertices in that class is in $\Omega(r^n)$.

For Z_2, the largest eigenvalue is at least 124.22239555, when taking the constant-size production matrix large enough.[3] For the inner part, the largest eigenvalue of the matrix $(FFR + 2R)Q$ is at least 5380.90657056. Accounting for the 2^n ways to add edges along the chains, we get $\Omega((\sqrt[3]{124.22239555} \cdot \sqrt[6]{5380.90657056} \cdot 2)^n) = \Omega(41.773981586^n)$ crossing-free graphs (eigenvalues computed using *Mathematica 11.3* with $m = 1024$).

The best bound we were able to find is the one for Z_5. The largest eigenvalue for the matrix obtained from (2) is 8296.0181565661924828. For the inner part, we obtain $\Omega(4.67964430624674625069^n)$ (as we add six vertices for each pocket). This results in $\Omega((\sqrt[6]{8303.6171640967198428} \cdot 4.67964430624674625069 \cdot 2)^n) = \Omega(42.116673256039055102^n)$ graphs. This leads to the main result in this paper.

Theorem 1. *There exist sets of n points with $\Omega(42.116673256039055102^n)$ crossing-free graphs.*

6 Note: mixing pocket sizes

Suppose we have a point set that consists of two chains like a generalized double-zig-zag chain, but where the pockets have different sizes. It is interesting to observe that the order of the pockets does not matter. For the inner part, this follows from the counting in Section 4.2.2; the number only depends on the number and variants of tappings.

[2] A non-negative matrix is primitive if, for some natural number n, the matrix A^n is positive. In our case, the existence of such a number n follows from the fact that every graph produces graphs of any degree at the root vertex after sufficiently many iterations.

[3] After the presentation of this work at the *European Workshop on Computational Geometry 2018*, Günter Rote (personal communication) applied an extension of a method that was first used in [7, Theorem 12] for a geometric counting problem similar to ours: non-crossing perfect matchings in repetitively structured point sets. He derived a polynomial system for characterizing the largest eigenvalue. The numerical solution of the system gives $x \approx 124.225396744416$. By trying to find a polynomial that fits this value, x has experimentally been established to be a root of the polynomial $x^3 - 125x^2 + 96x + 28$.

For the outer part, we can use a similar argument. The two outer parts of such chains correspond to so-called *almost convex polygons* (in which the points are connected by the polygon boundary from left to right), that were previously considered by Hurtado and Noy [15]. They discovered a statement for triangulations that is analogous to the following.

Lemma 2. *Consider an almost convex polygon P with two adjacent pockets $A = (p_1, \ldots, p_k)$ and $B = (p_k, \ldots p_l)$ with convex vertices p_1, p_k and p_l. Then the almost convex polygon P' in which the pockets A and B are swapped has the same number of plane graphs as P.*

Proof. If the two pockets have the same size (i.e., $l = 2k - 1$), then there is nothing to prove. We map the set of plane graphs on P to the set of plane graphs on P', as before disregarding edges on the boundary.

Let $p'_1, \ldots p'_l$ be the points on the boundary of P', and note that p_{l-k+1} is a convex vertex of P'. If a graph does not contain an edge between the two pockets, then we can map each edge ab in P to the corresponding edge $a'b'$ in P', which gives a bijection between these graphs. For a plane graph G with an edge between these two pockets, let $p_i p_j$ be the edge such that i is minimal and j is maximal (i.e., the edge covers all other such edges from the "interior"). We map this graph to a plane graph G' on P' in the following way. The edge $p_i p_j$ is mapped to the edge $p'_{l-i+1} p'_{l-j+1}$. In both polygons, we have now a chain of $l - j + i + 1$ vertices; for edges with only one endpoint on the pockets, we map that endpoint to the corresponding endpoint of the chain. All other edges ab are mapped to $a'b'$. The region bounded by the pockets and $p_i p_j$ is the mirror image of the region bounded by the pockets and $p'_{l-i+1} p'_{l-j+1}$ in P', which also defines a mapping for the edges inside these regions. Hence, the number of plane graphs in P and P' is the same. □

To count the number of plane graphs with different pocket sizes, we merely have to multiply a constant number of matrices for different k, and we get the matrix for a longer chain that is a combination of pockets of different sizes. We were experimenting with such combinations of pockets, but this did not lead to improved bounds.

7 Conclusions

We slightly improved the previously best lower bound on the maximum number of crossing-free geometric graphs on n points using production matrices. Applying production matrices to families of well-structured point sets appears to be a conceptually simple way of obtaining bounds for important families of graphs. It is interesting that with this technique it is also possible to obtain bounds when using a mix of different pocket sizes. While we could not find combinations that improve the presented bound in this way, our search was not exhaustive, and we cannot rule out that such an approach could allow to improve the lower bound even further.

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