Colored Anchored Visibility Representations in 2D and 3D space

Carla Binucci, Emilio Di Giacomo, Seok-Hee Hong, Giuseppe Liotta, Henk Meijer, Vera Sacristán, Stephen Wismath

Università degli Studi di Perugia, Italy
University of Sydney, Australia
University College Roosevelt
Universitat Politècnica de Catalunya, Spain,
University of Lethbridge, Canada

Abstract

In a visibility representation of a graph $G$, the vertices are represented by non-overlapping geometric objects, while the edges are represented as segments that only intersect the geometric objects associated with their end-vertices. Given a set $P$ of $n$ points, an Anchored Visibility Representation of a graph $G$ with $n$ vertices is a visibility representation such that for each vertex $v$ of $G$, the geometric object representing $v$ contains a point of $P$. We prove positive and negative results about the existence of anchored visibility representations under various models, both in 2D and in 3D space. We consider the case when the mapping between the vertices and the points is not given and the case when it is only partially given.

1. Introduction

A visibility representation (VR) of a graph $G$ maps the vertices of $G$ to non-overlapping geometric objects and the edges of $G$ to segments, called visibilities, that only intersect the geometric objects associated with their end-vertices. Various models of visibility representations have been studied in the plane using different types of objects to represent the vertices and different rules to represent the edges. Some examples are: Bar Visibility Representations (BVRs) [29, 45, 50, 51, 54], where the vertices are horizontal segments and the
edges are vertical segments (see Fig. 1(a)), *Rectangle Visibility Representations* (RVRs) [8, 11, 19, 21, 38, 48], which use axis-aligned rectangles and segments to represent vertices and edges, respectively, and *Orthopolytope Visibility Representations* [23, 41, 42], which generalize RVRs by using general orthogonal polygons.

Visibility representations in the three-dimensional space have also been considered. One of the first 3D models is the so-called *Z-parallel Visibility Representations* (ZPRs) [2, 12, 47], where vertices are represented by axis-aligned rectangles parallel to the $xy$-plane and edges are segments parallel to the $z$-axis (see Fig. 1(b)). Fekete and Meijer [33] considered the *Box Visibility Representations* where vertices are 3D boxes and visibilities are parallel to one of the three axes. Recently, 2.5D *box visibility representations* have been proposed [3, 4]; in this model vertices are 3D boxes whose bottom faces lie in the plane $z = 0$ and visibilities are parallel to the $x$- and $y$-axis. A similar 2D variant where vertices are horizontal bars whose left end points all have the same $x$-coordinate have been studied by Cobos et al. [17] and by Felsner and Massow [35].

We remark that each visibility model can be studied in two variants; in the *strong* variant (see, e.g., [19, 20, 21, 33, 38, 50, 54]) two vertices are adjacent if and only if the corresponding objects are visible (i.e. they can be connected by a visibility segment); in the *weak* variant (see, e.g., [8, 13, 23, 32, 41, 46]) visibilities between objects representing non-adjacent vertices may exist.

In this paper we study weak visibility representations with additional constraints on the “positions” of the vertices. More precisely, given a set $P$ of $n$ points with distinct coordinates along one of the directions parallel to the visibilities, an *Anchored Visibility Representation* (AVR) of a graph $G$ with $n$ vertices is a VR such that for each vertex $v$ of $G$, the geometric object representing $v$ contains a point of $P$. In particular, we consider *Anchored Bar Visibility Representations* (ABVRs) and *Anchored Z-parallel Visibility Representations* (AZPRs) (see Fig. 2). AVRs can be studied in different variants depending on whether the mapping between the vertices and the points is given or not. It is also possible that this mapping is only partially specified. We capture all these
variants within a unique framework, described in terms of colors. Given a graph
whose vertices are colored with \( k \) colors and a set of points also colored with
\( k \) colors, a \( k \)-\textit{colored AVR} of \( G \) on \( P \) is an AVR such that each point \( p \in P \)
belongs to a geometric object representing a vertex with the same color as \( p \)
(see Fig. 2 for an example with three colors). With this framework if \( k = n \)
we have a complete mapping; if \( k = 1 \) there is no mapping; for any value of
\( k \) between 1 and \( n \) we have a partial mapping. A similar framework based on
colors has been used in the study of point-set embedding, where one wants to
compute a 2D polyline drawing of a graph such that the vertices are represented
by the points of a given point set [5, 25, 26, 27, 37]. We also remark that the
problem of computing drawings with constraints on vertex positions is a clas-
sical subject in Graph Drawing (see, e.g., [9, 10, 39, 40, 43]). In particular,
Chaplick et al. [14, 16] have studied the problem of deciding whether a given
graph \( G = (V, E) \) admits a BVR when the bars representing a subset \( V' \subset V \)
are given as a part of the input. They prove that the problem is \( NP \)-complete
in general [14, 16] and it is polynomially-time solvable if \( V' = V \) [14]. The BVRs
studied in this paper, where the bars representing the vertices are not fixed but
are constrained to include the given points, can be considered as a relaxation of
those considered by Chaplick et al. An even more relaxed version where only
the \( y \)-coordinate of each bar is given has also been considered [14].

The contributions of this paper are the following:

- We first study AVRs in 2D space and prove that every 1-colored sub-
  Hamiltonian graph (i.e., a subgraph of a planar Hamiltonian graph) admits
  a 1-colored ABVR on every set of points in the plane. We show that the
  converse is also true if the points are collinear. As a consequence, deciding
  whether a graph admits a 1-colored ABVR on a given set of points is \( NP \)-
  complete. We also show that an ABVR always exists for every planar
  graph if all the points have distinct \( x \)- and \( y \)-coordinates.

- Concerning 1-colored AVRs in 3D we first show that every 1-colored graph
  with page number 4 admits a 1-colored AZPR on every given set of points.
This extends the previous results on 1-colored sub-Hamiltonian graphs since these are exactly the graphs that have page number 2. We then show that $K_n$ has a 1-colored ZPR on every set of points if and only if it admits a ZPR. This, together with known results about ZPRs of $K_n$, implies that $K_n$ admits an AZPR on any set of points if $n \leq 22$ while it does not admit an AZPR if $n \geq 51$. We finally prove that every 1-colored graph that is 3-connected 1-planar or that has geometric thickness 2 admits a 1-colored AZPR on any given point set $P$.

- Still in 3D, we consider the 2-colored version of the problem and prove that every properly 2-colored tree $T$ admits a 2-colored AZPR on any given 2-colored point set $P$.

The rest of the paper is organized as follows. Preliminary definitions are given in Section 2. The results about AVRs in the plane are presented in Section 3, while those in 3D are in Section 4. In particular, Subsection 4.1 is about 1-colored AZPRs, and Subsection 4.2 contains results about 2-colored trees. Conclusions and open problems can be found in Section 5.

2. Preliminaries

A Bar Visibility Representation (BVR) of a graph $G$ is a 2D visibility representation where the vertices of $G$ are mapped to horizontal segments, called bars, while visibilities are vertical segments. A Z-parallel Visibility Representation (ZPR) is a 3D visibility representation where vertices are mapped to axis-aligned rectangles belonging to planes parallel to the $xy$-plane, while visibilities are parallel to the $z$-axis.

In the rest of the paper, we will often transform BVRs into ZPRs; to keep the direction of the visibilities consistent between BVRs and ZPRs, we assume that a BVR is realized in the $yz$-plane with visibilities parallel to the $z$-axis. See, e.g., Fig. 3(c). Given a BVR (respectively a ZPR) $\Gamma$, the (partial) order of the bars (respectively of the rectangles), along the vertical direction is called the $z$-ordering of $\Gamma$. Throughout the paper we adopt the so-called weak visibility model where visibilities between bar/rectangles representing non-adjacent vertices may exist.

Let $G$ be a graph with $n$ vertices and let $P$ be a set of $n$ points in $\mathbb{R}^2$ or $\mathbb{R}^3$ with distinct $z$-coordinates (recall that in 2D we use the $yz$-plane). For each type of visibility representation defined above (BVR and ZPR), we define a constrained version in which for each vertex $v$ of $G$, the object representing $v$ contains a point of $P$. We will refer to these constrained versions as Anchored Bar Visibility Representations (ABVRs) and Anchored Z-parallel Visibility Representations (AZPRs). We require that the points in $P$ have different $z$-coordinates because bars or rectangles with the same $z$-coordinate cannot be connected by a visibility segment (recall that visibilities are parallel to the $z$-axis). This avoids instances that trivially do not admit anchored visibility.
representations. For example, the complete graph $K_n$ does not admit such a
representation if at least two points have the same $z$-coordinate.

Let $G = (V, E)$ be a graph with $n$ vertices. A $k$-coloring of $G$ is a partition
$\{V_1, \ldots, V_k\}$ of $V$ where integers $\{1, \ldots, k\}$ are called colors. A graph $G$ with
a $k$-coloring is called a $k$-colored graph. A graph is properly $k$-colored if it is a
$k$-colored graph and no two vertices of the same color are adjacent. Let $P$ be
a set of $n$ points in $\mathbb{R}^2$ or $\mathbb{R}^3$ with distinct $z$-coordinates. A $k$-coloring of $P$
is a partition $\{P_1, \ldots, P_k\}$ of $P$. A set of points $P$ with a $k$-coloring is called
a $k$-colored set. A $k$-colored set $P$ is compatible with a $k$-colored graph $G$ if
$|V_i| = |P_i|$ for every $i$. Let $G$ be a $k$-colored graph and let $P$ be a $k$-colored set
of points compatible with $G$ ($k \geq 1$). A $k$-colored ABVR of $G$ on $P$ is an ABVR
of $G$ on $P$ such that for each vertex $v$ of $G$, the bar representing $v$ contains a
point of $P$ with the same color as $v$. Analogous definitions hold for $k$-colored
AZPRs. The assumption that the points of $P$ have distinct $z$-coordinates avoids
straightforward negative instances where adjacent vertices of $G$ are forced to be
mapped to points with the same $z$-coordinate.

A $k$-colored sequence $\lambda$ is a sequence of (possibly repeated) colors $c_1, \ldots, c_n$
such that $c_j \in \{1, 2, \ldots, k\}$ for every $j \in \{1, 2, \ldots, n\}$. Let $G$ be a $k$-colored
graph and let $\lambda$ be a $k$-colored sequence; $\lambda$ is compatible with $G$ if the number
of elements in $\lambda$ colored $i$ is equal to $|V_i|$, for every $i = 1, 2, \ldots, k$. A total
order $\rho$ of the vertices of $G$ is consistent with $\lambda$ if the sequence of the colors
defined by $\rho$ coincides with $\lambda$. Given a $k$-colored point set $P$, we denote by
$\lambda(P)$ the sequence of colors of the points of $P$ according to their order along
the $z$-direction (this order is a total order because the points of $P$ have distinct
$z$-coordinates). Finally, given a set of $n$ points $p_1, p_2, \ldots, p_n$ we denote by $x(p_i)$,
y($p_i$), and $z(p_i)$ the $x$-, $y$-, and $z$-coordinate of point $p_i \in P$ and by $x_m$ and
$z_M$ the values $\min_{i=1}^n \{x(p_i)\}$ and $\max_{i=1}^n \{x(p_i)\}$, respectively. We analogously
define $y_m$, $y_M$, $z_m$, and $z_M$.

3. Anchored Visibility Representations in 2D

A book embedding of a graph $G = (V, E)$ consists of a total order $\rho$ of $V$
and a partition of $E$ into $k$ disjoint sets, called pages, such that no two edges
in the same page cross; that is, there are no two edges $(u_1, v_1)$ and $(u_2, v_2)$ in
the same page with $u_1 <_{\rho} u_2 <_{\rho} v_1 <_{\rho} v_2$ (see Fig 3(a)). The minimum $k$ for
which a graph $G$ admits a book embedding with $k$ pages is the page number
of $G$. A graph has page number one if and only if it is outerplanar [7]. This
also means that the graph induced by each page of a $k$-page book embedding
is an outerplanar graph. A graph is Hamiltonian if it has a simple cycle that
contains all its vertices. A graph is sub-Hamiltonian if it is a subgraph of a
planar Hamiltonian graph. A graph has page number two if and only if it is
sub-Hamiltonian [7]. A semi-bar visibility representation of a planar graph $G$
is a BVR of $G$ such that the left endpoints of all the bars representing the vertices
of $G$ belong to a vertical line. Cobos et al. [17] proved that a graph has a
semi-bar visibility representation if and only if it is outerplanar.
Theorem 1. Let $G$ be a 1-colored sub-Hamiltonian graph and let $P$ be a 1-colored point set in $\mathbb{R}^2$; $G$ has a 1-colored ABVR on $P$.

Proof. Since $G$ is sub-Hamiltonian, it admits a book embedding $\gamma$ with two pages [7]. Let $p_1, p_2, \ldots, p_n$ be the points of $P$ in the order as they appear along the $z$-axis and let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ according to the total order $\rho$ of $\gamma$. The bar $b_i$ representing $v_i$ is drawn as a segment parallel to the $y$-axis with $z$-coordinate $z(p_i)$, with minimum $y$-coordinate less than $y_m$ and maximum $y$-coordinate greater than $y_M$. This guarantees that $b_i$ contains the point $p_i$ (refer to Fig. 3 for an illustration). The amount of the extension of each $b_i$ in the half-planes $y < y_m$ and $y > y_M$ is chosen to realize the visibilities that represent the edges. In other words, we realize in each of the half-planes $y < y_m$ and $y > y_M$ two semi-bar visibility representations, one for each page of $\gamma$. Such semi-bar visibility representations exist by the result of Cobos et al. [17]. For completeness, we give a detailed description of the construction, which will also be useful to extend the result in the 3D scenario.

Denote by $b_i^-$ (resp. $b_i^+$) the length of the portion of $b_i$ that lies in the half-plane $y < y_m$ (resp. $y > y_M$). For each edge $(v_i, v_j)$, the span of $(v_i, v_j)$ in $\gamma$ is $|j - i|$. The length $b_i^-$ (resp. $b_i^+$) is chosen equal to the maximum span of an edge incident to $v_i$ in the first (resp. second) page. With this choice every pair of adjacent vertices $v_i$ and $v_j$ are visible. Suppose, as a contradiction, that $v_i$ and $v_j$, with $i < j$, are not visible, i.e., there exists a bar $b_k$ with $i < k < j$ such that $b_i^-, b_j^- < b_k^-$ and $b_i^+, b_j^+ < b_k^+$. Without loss of generality assume that $(v_i, v_j)$ is in the first page in $\gamma$. This implies that $b_i^-, b_j^+ \geq |j - i|$. On the other hand, $b_k^- > b_i^-, b_j^-$ implies that there is an edge $(v_k, v_h)$ in the first page with $|h - k| > |j - i|$, but this implies that $v_i < v_k < v_j < v_h$ or $v_h < v_i < v_k < v_j$, which is impossible because $(v_i, v_j)$ and $(v_k, v_h)$ are in the same page in $\gamma$. \square

If the points are aligned in the $z$-direction the converse of Theorem 1 holds.
Theorem 2. Let $G$ be a 1-colored planar graph and let $P$ be a 1-colored point set in $\mathbb{R}^2$ such that each point in $P$ has the same $y$-coordinate; $G$ admits a 1-colored ABVR on $P$ if and only if $G$ is sub-Hamiltonian.

Proof. The proof of sufficiency follows from Theorem 1. Consider now the necessity. Let $\Gamma$ be an ABVR of $G$ on $P$ and let $b_1, b_2, \ldots, b_n$ be the bars of $\Gamma$ according to the $z$-ordering of $\Gamma$. Since all points have the same $y$-coordinate $\bar{y}$, every $b_i$ sees $b_{i+1}$ along the line $y = \bar{y}$. Thus, we can add the visibilities (if not already present) between $b_i$ and $b_{i+1}$ ($i = 1, 2, \ldots, n$). Moreover, we can add a visibility between $b_1$ and $b_n$. To this aim, let $y_{min}$ be the minimum $y$-coordinate of a bar in $\Gamma$; we extend $b_1$ and $b_n$ in the half-plane $y < y_{min}$ and add a visibility between them. The resulting representation is an ABVR of a planar Hamiltonian supergraph $G'$ of $G$ and therefore $G$ is sub-Hamiltonian.

We remark that sub-Hamiltonian graphs include various sub-families of planar graphs, such as 2-trees (and therefore series-parallel graphs, outerplanar graphs and trees) [44], Halin graphs [18], 4-connected planar graphs [52], and planar graphs with maximum vertex degree 4 [6]. On the other hand, testing a graph for sub-Hamiltonicity is $\mathcal{NP}$-complete [53], and therefore a consequence of Theorem 2 is that testing whether a given planar graph admits an ABVR on a given set of collinear points is $\mathcal{NP}$-complete.

Corollary 1. The problem of deciding whether a 1-colored planar graph admits an ABVR on a given 1-colored point set is $\mathcal{NP}$-complete.

In contrast to the $\mathcal{NP}$-hardness result about vertically aligned points, if the points of $P$ are not vertically nor horizontally aligned, then an ABVR exists for every planar graph $G$. The next theorem is a consequence of a paper by Felsner [34] about floorplans (see also [15] for a similar proof). A generic floorplan is a partition of a rectangle into a finite set of interiorly disjoint rectangles that have no point where four rectangles meet. Two floorplans $F$ and $F'$ are weakly equivalent if there exist bijections $\phi_H$ between the horizontal segments and $\phi_V$ between the vertical segments, such that a horizontal (resp. vertical) segment $s$ has an endpoint on a vertical (resp. horizontal) segment $t$ in $F$ if and only if $\phi_H(s)$ (resp. $\phi_V(s)$) has an endpoint on $\phi_V(t)$ (resp. $\phi_H(t)$) in $F'$. A set $P$ of points in $\mathbb{R}^2$ is generic if no two points from $P$ have the same $x$- or $y$-coordinate. The following theorem has been proved by Felsner [34].

Theorem 3. [34] If $P$ is a generic set of $k$ points in a rectangle $R$ and $F$ is a generic floorplan with $n > k$ segments and $S$ is a prescribed subset of the segments of $F$ having size $k$, then there exists a generic floorplan $F'$ that is weakly equivalent to $F$ and such that every segment of $F'$ that corresponds to a segment of $S$ contains exactly one point of $P$ and no point is contained in two segments.

Theorem 4. Let $G$ be a 1-colored planar graph and let $P$ be a generic 1-colored point set in $\mathbb{R}^2$; $G$ admits a 1-colored ABVR on $P$. 7
Figure 4: (a) A generic point set P and a BVR Γ of a planar graph G; (b) A floorplan F obtained from Γ; (c) A floorplan F' weakly equivalent to F and covering P; (d) An ABVR of G on P.

Proof. Let Γ be a BVR of G. We now construct a generic floorplan starting from Γ (see Fig. 4(a)). Let b_b and b_t be the bottommost and the topmost bars in Γ; we extends b_b and b_t so that their left endpoints and their right endpoints can be connected by two vertical segments s_l and s_r. The four segments b_b, b_t, s_l and s_r are the boundary of a rectangle. Extending every bar of Γ until its endpoints touch a vertical segment, we obtain a floorplan F (see Fig. 4(b)). Let S be the set of horizontal segments of F. By Theorem 3, there exists a floorplan F' that is weakly equivalent to F and such that every point of P belongs to exactly one horizontal segment (see Fig. 4(c)). Since F' and F are weakly equivalent the adjacencies between vertical and horizontal segments are the same. We shorten the horizontal segments so to remove the adjacencies that are in F but not in Γ and we also remove the two segments s_l and s_r. We thus obtain a new BVR Γ' of G (see Fig. 4(d)). Moreover, since no point of P belongs to two segments of F', every point of P is an internal point of a horizontal segment; this means that we can shorten the horizontal segments in such a way that they still contain the points of P. This implies that Γ' is in fact an ABVR of G on P.

We conclude this section with a result about k-colored ABVR, for 2 ≤ k ≤ n, that is an immediate consequence of Theorem 2. By Theorem 2 every non-sub-Hamiltonian graph G does not admit an ABVR on a given set P of vertically aligned points. This implies that any k-colored graph G' that has G as a monochromatic subgraph does not admit an ABVR on any set of points P' that contains P as a monochromatic subset (with the color of the vertices of G only occurring in P).

Corollary 2. For every 1 ≤ k ≤ n there exists a k-colored planar graph G and a k-colored point set P such that G does not admit a k-colored ABVR on P.

4. Anchored Visibility Representations in 3D

In this section we study anchored visibility representations in 3D, in particular AZPRs, and present results about the 1-colored version (Subsection 4.1)
and the 2-colored version of the problem (Subsection 4.2). We start with two
lemmas that will be useful in both subsections.

**Lemma 1.** Let $G$ be a graph with $n$ vertices and let $P$ be a set of $n$
points in $\mathbb{R}^3$ with distinct $z$-coordinates. If $G$ has a ZPR $\Gamma$
such that each rectangle representing a vertex has nonempty intersection with the $z$-axis, then $G$ admits
an AZPR on $P$ whose $z$-ordering is the same as $\Gamma$.

**Proof.** We show that $\Gamma$ can be modified to an AZPR $\Gamma'$ on $P$ with the same
$z$-ordering as $\Gamma$. Let $p_1, p_2, \ldots, p_n$ be the sequence of the points of $P$ ordered
by increasing $z$-coordinate; let $r_1, r_2, \ldots, r_n$ be the sequence of the rectangles
ordered according to the $z$-ordering of $\Gamma$. First, we translate the rectangles
$r_1, r_2, \ldots, r_n$ so that $r_i$ has $z$-coordinate $z(p_i)$. Since the order of the rectangles
in the $z$-direction is not changed, the visibilities of $\Gamma$ are preserved. Denote by
$x'(r_i)$ and $y'(r_i)$ the maximum $x$- and $y$-coordinate of $r_i$, respectively and by
$x''(r_i)$ and $y''(r_i)$ the minimum $x$- and $y$-coordinate of $r_i$, respectively. Note
that $(x'(r_i), y'(r_i), z(p_i))$ and $(x''(r_i), y''(r_i), z(p_i))$ are two opposite corners of
$r_i$.

In order to obtain an AZPR of $G$ on $P$ we extend each rectangle $r_i$ in such a
way that the coordinates of its two opposite corners become $(x_m + x'(r_i), y_m +
y''(r_i), z(p_i))$ and $(x_m + x'(r_i), y_M + y'(r_i), z(p_i))$, respectively. Moreover, each
visibility segment $s_j$ whose $x$- and $y$-coordinate are $x(s_j)$ and $y(s_j)$, respectively,
is translated as follows. If $x(s_j) \geq 0$, then $s_j$ is translated so that its $x$-coordinate
is $x_M + x(s_j)$, while if $x(s_j) < 0$, then $s_j$ is translated so that its $x$-coordinate
is $x_m + x(s_j)$. Analogously, if $y(s_j) \geq 0$, then $s_j$ is translated so that its $y$-
coordinate is $y_M + y(s_j)$, while if $y(s_j) < 0$, then $s_j$ is translated so that its $y$-
coordinate is $y_m + y(s_j)$. See Fig. 5 for an illustration. Let $\Gamma'$ be the resulting
representation. We denote by $r'_i$ the rectangle of $\Gamma'$ obtained by extending the
rectangle $r_i$ of $\Gamma$; analogously, we denote by $s'_j$ the segment of $\Gamma'$ obtained by
translating the visibility segment $s_j$ of $\Gamma$.

We now prove that $\Gamma'$ is a valid AZPR of $G$ on $P$. Clearly, point $p_i$ belongs to
$r'_i$ because it is contained in the rectangle with opposite corners $(x_M, y_M, z(p_i))$
and $(x_m, y_m, z(p_i))$, which is contained in $r'_i$. Also, each segment $s'_j$ of $\Gamma'$ is
a valid visibility segment. Namely, assume that $x(s_j) \geq 0$ and $y(s_j) \geq 0$ (the
other cases are analogous). The coordinates of $s'_j$ are $x_M + x(s_j)$ and $y_M + y(s_j)$.
If $s'_j$ is not a valid visibility segment in $\Gamma'$, then there exists a rectangle $r'_k$ that
intersect $s'_j$ at an interior point $(x_M + x(s_j), y_M + y(s_j), z(p_k))$. This implies
that $x_M + x'(r_k) \geq x_M + x(s_j)$ and $y_M + y'(r_k) \geq y_M + y(s_j)$, i.e., $x'(r_k) \geq x(s_j)$
and $y'(r_k) \geq y(s_j)$. But this means that $s_j$ intersects $r_k$ in $\Gamma$, contradicting the
fact that $s_j$ is a valid visibility segment in $\Gamma$. □

The next lemma explains how to transform a BVR into a ZPR with the
additional properties that it is completely contained in the region of space with
$x \geq 0$ and $y \geq 0$ and such that all rectangles representing vertices have a corner
on the $z$-axis. Any such ZPR will be called cornered ZPR.

**Lemma 2.** Let $G$ be a graph that has a BVR $\Gamma$ where vertices have distinct
$z$-coordinates; $G$ admits a cornered ZPR whose $z$-ordering is the same as $\Gamma$. 9
Figure 5: Illustration for the proof of Lemma 1. (a) The projection of a rectangle \( r_i \) on the \( xy \)-plane. (b) The projection on the \( xy \)-plane of the bounding box of the point set \( P \). (c) The projection of the rectangle \( r'_i \) on the \( xy \)-plane.

**Proof.** By possibly translating it, we can assume that \( \Gamma \) is contained in the first quadrant of the \( yz \)-plane (see Fig. 6). Let \( e_i \) be an edge of \( G \), we denote by \( s_i \) the visibility segment representing \( e_i \) in \( \Gamma \). We enumerate the visibility segments from right to left and we assign to the segments integer numbers that increase from right to left. More precisely, we assign to each visibility segment \( s_i \) a number \( n(s_i) \in \mathbb{N}^+ \) so that \( n(s_i) < n(s_j) \) if there exists a \( y \)-parallel straight line that intersects both \( s_i \) and \( s_j \), and \( s_i \) is to the right of \( s_j \). We assign to each bar \( b_i \) a number \( n(b_i) \in \mathbb{N}^+ \) equal to the maximum number of a visibility segment incident to \( b_i \).

We now extend each bar \( b_i \) of \( \Gamma \) so that it touches the \( z \)-axis. Let \( \Gamma' \) be the resulting representation. We denote by \( b'_i \) the bar obtained by extending \( b_i \) and we set \( n(b'_i) = n(b_i) \). Observe that the visibility segments of \( \Gamma' \) are the same as those of \( \Gamma \). In \( \Gamma' \) bars can intersect the visibility segments. However, if a bar \( b'_i \) intersects a visibility segment \( s_j \), then \( n(b'_i) < n(s_j) \). Namely, since \( b_i \) did not intersect \( s_j \) before the extension, every point of \( b_i \) has a \( y \)-coordinate larger than the \( y \)-coordinate of \( s_j \); hence any visibility segment \( s_k \) incident to \( b_i \) has a \( y \)-coordinate greater than the \( y \)-coordinate of \( s_j \) and therefore \( n(s_k) < n(s_j) \). Since \( n(b_i) \) is equal to the maximum number of a visibility segment incident to \( b_i \) and \( n(b'_i) = n(b_i) \), we have \( n(b'_i) < n(s_j) \).

In order to construct a ZPR with the desired properties, we transform the bars representing the vertices into rectangles by extending them in the positive \( x \)-direction. In particular, a bar \( b'_i \) is transformed into a rectangle \( r_i \) with a side coincident with \( b'_i \) and whose dimension in the \( x \)-direction is equal to \( n(b'_i) \). We also translate the visibility segments in the \( x \)-direction. A visibility segment \( s_i \) is moved so that its \( x \)-coordinate is \( n(s_i) \). Denote by \( \Gamma'' \) the resulting representation. By construction, all the rectangles of \( \Gamma'' \) are in the region of
Figure 6: (a) A BVR $\Gamma$ where each visibility $s_i$ (each bar $b_i$) is associated with a number in black (gray) according to the partial order $\prec$. (b) The drawing $\Gamma'$ obtained from $\Gamma$ by extending each bar so that it touches the $z$-axis. (c) A cornered ZPR $\Gamma''$ of $\Gamma'$.

space with $x \geq 0$ and $y \geq 0$ and have a corner on the $z$-axis. Furthermore, no rectangle $r_i$ intersects a visibility segment $s_j$. Namely, if $b'_i$ (i.e. the bar that has been extended to create $r_i$) and $s_j$ did not intersect each other in $\Gamma'$, then they do not intersect in $\Gamma''$. If $b'_i$ and $s_j$ intersected in $\Gamma'$, then $n(b'_i) < n(s_j)$, and it follows that in $\Gamma''$ $s_j$ has a $x$-coordinate $n(s_j)$ while the maximum $x$-coordinate of $r_i$ is $n(b'_i)$, which implies that $r_i$ and $s_j$ do not intersect.

4.1. 1-colored AZPRs

We start with a theorem that is the 3D counterpart of Theorem 1 and that can be proven similarly.

**Theorem 5.** Let $G$ be a 1-colored graph with page number four and let $P$ be a 1-colored point set in $\mathbb{R}^3$; $G$ admits a 1-colored AZPR on $P$.

**Proof.** Let $p_1, p_2, \ldots, p_n$ be the points of $P$ in the order they appear along the $z$-axis and let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ according to the total order $\rho$ of a given 4-page book embedding of $G$. Vertex $v_i$ will be represented by a rectangle $r_i$ parallel to the $xy$-plane whose $z$-coordinate is $z(p_i)$ and such that its minimum $x$-coordinate (respectively $y$-coordinate) is less than $x_m$ (respectively $y_m$) and its maximum $x$-coordinate (respectively $y$-coordinate) is greater than $x_M$ (respectively $y_M$). This guarantees that $r_i$ contains the point $p_i$. The visibilities to represent the edges in each page are realized by choosing the amount of the extension of each $r_i$ in the half-planes $x < x_m$, $x > x_M$, $y < y_m$ and $y > y_M$, analogously to what done in the proof of Theorem 1. □

In Section 3 we showed (Theorem 2) that if a graph admits an ABVR on a set of collinear points, then it is sub-Hamiltonian and therefore has page
number two. In other words, when restricted to collinear points the converse of
Theorem 1 holds. We now show that this is not the case for Theorem 5. Indeed,
using results from [12] and [47] we can prove that $K_n$, which has page number
$\lceil \frac{n}{2} \rceil$, admits an AZPR on any given set of points (even if collinear or coplanar)
for $n \leq 22$.

**Theorem 6.** The complete graph $K_n$ admits a 1-colored AZPR on any given
set of 1-colored points if and only if it admits a ZPR.

**Proof.** The only-if direction is trivial. Suppose then that $K_n$ admits a ZPR $\Gamma$;
as observed by Bose et al. [12] if a ZPR of $K_n$ exists, then all rectangles intersect
a line parallel to the visibilities. By possibly translating $\Gamma$ we can assume that
this line is the $z$-axis and therefore by Lemma 1 $K_n$ admits an AZPR on any
given set of points. \hfill $\Box$

Bose et al. [12] construct a ZPR of $K_{22}$ and prove that a ZPR cannot exist
for $K_n$ with $n \geq 56$. Afterwards, Štola [47] lowered the upper bound to 51.
Hence we have the following.

**Corollary 3.** Let $P$ be a set of 1-colored points in $\mathbb{R}^3$. If $n \leq 22$, then $K_n$
admits a 1-colored AZPR on $P$; if $n \geq 51$ a 1-colored AZPR of $K_n$ on $P$ does
not exist.

In the rest of this section we will describe two families of graphs that admit a
1-colored AZPR on every set of points, namely the 3-connected 1-planar graphs
and the graphs with geometric thickness two. A graph is 1-planar if it has
a drawing where each edge is crossed at most once; a graph has geometric
thickness two if it has a straight-line drawing whose edges can be partitioned
into two sets and no two edges in the same set cross. This latter family includes
the RAC graphs [28] (i.e. graphs that have a straight-line drawing where each
edge crossing forms a right angle) and the graphs with maximum vertex-degree
4 [30]. Both the results are proved using a common approach based on the fact
that a graph $G$ belonging to the families above can be decomposed into two
planar graphs. The idea is to combine two cornered ZPRs of the two planar
graphs whose union is $G$ to create an AZPR of $G$ on a given set of points $P$.
The next lemma explains how to achieve this, provided that the $z$-orderings of
the two cornered ZPRs is the same.

**Lemma 3.** Let $G$ be a 1-colored graph that is the union of two planar graphs
$G_1$ and $G_2$ with the same vertex set and let $P$ be a 1-colored point set in $\mathbb{R}^3$. If
$G_1$ and $G_2$ admit two BVRs whose $z$-ordering is the same, then $G$ admits an
AZPR on $P$.

**Proof.** Consider two BVRs of $G_1$ and $G_2$ with the same $z$-ordering and denote
by $\rho$ this ordering. By Lemma 2, $G_1$ and $G_2$ admit two cornered ZPRs $\Gamma_1$ and
$\Gamma_2$, respectively, whose $z$-ordering is $\rho$. We now explain how to combine $\Gamma_1$
and $\Gamma_2$ to obtain a ZPR of $G$ such that all rectangles representing vertices have
nonempty intersection with the z-axis. By Lemma 1 this implies that \( G \) has an AZPR on \( P \).

Let \( r^1_i \) and \( r^2_i \) be the rectangles representing a vertex \( v_i \) in \( \Gamma_1 \) and \( \Gamma_2 \), respectively. We first translate the rectangles of \( \Gamma_2 \) so that \( r^1_i \) has the same z-coordinate \( z_i \) of \( r^2_i \), for every \( i = 1, \ldots, n \). Since the z-ordering of \( \Gamma_1 \) and \( \Gamma_2 \) is the same, this translation does not change the z-ordering of \( \Gamma_2 \). Next, we rotate \( \Gamma_2 \) by 180° around the z-axis. In this way, \( \Gamma_2 \) is completely contained in the region \( x \leq 0 \) and \( y \leq 0 \) and all its rectangles have a corner on the z-axis. Rectangles \( r^1_i \) and \( r^2_i \) (after the rotation) only share the point \((0, 0, z_i)\). Let \((x^1_i, y^1_i, z_i)\) be the corner of \( r^1_i \) opposite to \((0, 0, z_i)\) and let \((x^2_i, y^2_i, z_i)\) be the corner of \( r^2_i \) opposite to \((0, 0, z_i)\). Vertex \( v_i \) is represented in \( \Gamma \) by a rectangle whose opposite corners are \((x^1_i, y^1_i, z_i)\) and \((x^2_i, y^2_i, z_i)\) (see Fig. 7). \( \Gamma \) is a ZPR of \( G \) because the visibilities of \( \Gamma_1 \) and \( \Gamma_2 \) have not been destroyed: those of \( \Gamma_1 \) still exist in the region \( x \geq 0 \) and \( y \geq 0 \), while those of \( \Gamma_2 \) have been moved to the region \( x \leq 0 \) and \( y \leq 0 \) by the rotation. \( \square \)

Let \( \Gamma \) be a drawing of a graph \( G \). \( \Gamma \) has thickness \( k \geq 1 \) if the edges of \( \Gamma \) can be colored with \( k \) colors so that no two edges of the same color cross in \( \Gamma \). Let \( G \) be a directed graph; an upward planar drawing of \( G \) is a planar drawing where the edges are monotonically increasing in the vertical direction (the z-direction in our case).

**Lemma 4.** Let \( G \) be a graph that admits a drawing with thickness two that can be oriented to become an upward drawing; \( G \) admits an AZPR on any given set of points \( P \) in \( \mathbb{R}^3 \).

**Proof.** Let \( \Gamma \) be a drawing of \( G \) with thickness two that can be oriented to become an upward drawing \( \Gamma^p \). Since \( \Gamma \) has thickness two, the edges of \( \Gamma^p \) can be partitioned to obtain two upward planar drawings \( \Gamma_1 \) and \( \Gamma_2 \) with the same...
vertex set. The graph $G_i$ represented by $\vec{\Gamma}_i$ ($i = 1, 2$) admits a BVR whose $z$-ordering coincides with the vertical order $\rho$ of the vertices in $\Gamma$. Namely, $\vec{\Gamma}_i$ can be augmented to an upward planar drawing $\vec{\Gamma}'_i$ of a supergraph $G'_i$ of $G_i$ that is a planar st-graph (i.e., a planar digraph with a single source $s$ and a single sink $t$, embedded so that $s$ and $t$ are on the external face) [22, Chapter 6]; the order $\rho$ is an st-numbering of $G'_i$ (i.e., a total order $v_1, v_2, \ldots, v_n$ of the vertices of $G'_i$ such that $s = v_1$, $t = v_n$, and each vertex $v_j$ other than $s$ and $t$ is adjacent to at least two vertices $v_h$ and $v_k$ such that $h < j < k$). It is known that it is possible to compute a BVR of a graph whose $z$-ordering is a given st-numbering [49, 54]. Thus, it is possible to compute a BVR of $G'_i$ whose $z$-ordering is $\rho$. Since $G_i$ is a spanning subgraph of $G'_i$, the computed BVR contains a BVR of $G_i$ with $z$-ordering $\rho$. In other words $G$ is the union of two planar graphs that admit two BVRs whose $z$-ordering is the same and therefore, by Lemma 3, $G$ admits an AZPR on any given set of points $P$ in $\mathbb{R}^3$. \hfill \Box

We are now ready to prove the following.

**Theorem 7.** Let $G$ be an $n$-vertex 1-colored graph and let $P$ be a 1-colored point set of size $n$ in $\mathbb{R}^3$. If $G$ is 3-connected 1-planar or has geometric thickness two, then $G$ admits a 1-colored AZPR on $P$.

**Proof.** By Lemma 4 it is sufficient to prove that $G$ admits a drawing with thickness two that can be oriented to become upward.

For the case of 3-connected 1-planar graphs, Alam et al. [1] proved that every 3-connected 1-planar graph $G = (V, E)$ admits a 1-planar drawing $\Gamma$ where all edges are straight-line except one edge that has one bend. Since each edge crosses at most one other edge, the edges of $\Gamma$ can be partitioned into two sets $E_1$ and $E_2$ such that the edges in each set do not cross. By possibly rotating $\Gamma$ we can guarantee that all vertices have distinct $z$-coordinates and that the edge with one bend is monotone in the vertical direction. By orienting each edge from the vertex with lower $z$-coordinate to the vertex with higher $z$-coordinate we obtain the desired upward drawing.

In the case where $G = (V, E)$ has geometric thickness two, it admits a straight-line drawing $\Gamma$ such that $E$ can be partitioned into two sets $E_1$ and $E_2$ each containing non-intersecting edges. By possibly rotating $\Gamma$ we can guarantee that each vertex of $\Gamma$ has a distinct $z$-coordinate. Also in this case we obtain the desired upward drawing by orienting each edge of $\Gamma$ from the vertex with lower $z$-coordinate to the vertex with higher $z$-coordinate. \hfill \Box

### 4.2. 2-colored AZPRs

In this section we study 2-colored AZPRs of properly 2-colored trees. The idea is to first compute a BVR whose $z$-ordering is consistent with $\lambda(P)$ and then to use Lemmas 1 and 2 to obtain an AZPR on $P$.

Let $T$ be a properly 2-colored tree and let $\lambda$ be a 2-colored sequence compatible with $T$. We construct a BVR of $T$ whose $z$-ordering is consistent with $\lambda$. To this aim we first define a mapping of the vertices of $T$ to the elements of
The color of the root is equal to the first element of $\lambda$. The labels associated with the vertices are determined by the mapping. Root $T$ at any vertex whose color is equal to the first element of $\lambda$ and arbitrarily order the children of each node from left to right. We visit the vertices of $T$ level by level starting from the root and at each level we visit the vertices from left to right. The current vertex $v$ is mapped to the first element of $\lambda$ with the same color as $v$ that has not yet been used. The resulting ordering of the vertices $\rho$ is consistent with $\lambda$ by construction and its first element is the root of $T$ (see Fig. 8).

We now explain how to use the defined mapping to construct a BVR $\Gamma$ of $T$. First of all, we assign the $z$-coordinates to the vertices according to the ordering $\rho = \langle v_1, v_2, \ldots, v_n \rangle$. More precisely, we assign to vertex $v_j$ the $z$-coordinate $z(v_j) = j$. This implies that the $z$-ordering of $\Gamma$ will be $\rho$ and therefore consistent with $\lambda$. The children of $v_j$ whose $z$-coordinates are less than $z(v_j)$ are called backward children, the others are called forward children. Observe that the grandchildren of a vertex $v_j$ (which have the same color as $v_j$ because $T$ is properly 2-colored) have a $z$-coordinate larger than that of $v_j$; this property will be used to guarantee that there will be no crossings between bars and visibilities.

In order to actually construct $\Gamma$, we consider the vertices level by level (starting from the root) and from left to right within each level. At each step we draw a set of vertices having the same parent, which has already been drawn because it is on the previous level (clearly the root is the first vertex to be drawn). We call a region of plane delimited by two straight lines parallel to the $z$-axis a strip and we say that a bar $b$ crosses a strip $\sigma$ if $b$ intersects $\sigma$ and both endpoints of $b$ are outside $\sigma$. For example, in Fig. 9(a) the bar $b_k$ crosses the strip

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{A properly 2-colored tree $T$ and a 2-colored sequence $\lambda$ compatible with $T$.}
\end{figure}
The following invariant holds. For each bar \( b_j \) whose children have not yet been drawn, there exists a strip \( \sigma(b_j) \) such that:

1. **(P1)** The endpoint of \( b_j \) with minimum \( y \)-coordinate lies in the interior of \( \sigma(b_j) \), while the other endpoint lies outside. Furthermore, the visibility between \( b_j \) and its parent is outside \( \sigma(b_j) \). See also Fig. 9(a).

2. **(P2)** If \( \sigma(b_j) \) is crossed by a bar \( b_i \), then \( z(v_i) < z(v_j) \).

3. **(P3)** Let \( b_k \) be the bar with the maximum \( z \)-coordinate among those that cross \( \sigma(b_j) \). If \( v_k \) is not the parent of \( v_j \), then the children of \( v_j \) are forward children.

Intuitively, the strip \( \sigma(b_j) \) is a sort of a “tunnel” where the visibility between \( b_j \) and its children is guaranteed.

The root of \( T \) is drawn as a bar of arbitrary length not touching the \( z \)-axis. Clearly the invariant holds. Let \( v_j \) be the parent of the vertices to be drawn at the generic step. We process the children of \( v_j \) from left to right. If some backward child exists, then, by property **P3**, either \( \sigma(b_j) \) is not crossed by any other bar or the bar crossing it with the maximum \( z \)-coordinate represents the parent \( v_k \) of \( v_j \). Since the children of \( v_j \) have the same color as \( v_k \) (because the tree is properly colored), the backward children of \( v_j \) have a \( z \)-coordinate greater than \( z(v_k) \) (because they appear after \( v_k \) in \( \rho \)). Thus, the bars representing backward children can be drawn inside \( \sigma(b_j) \) so that they are visible from \( b_j \) (refer also to Fig. 9(b)). More precisely, let \( v_{j_1}, \ldots, v_{j_\alpha} \) be the backward children of \( v_j \) ordered according to \( \rho \) (thus \( z(v_{j_\alpha}) < z(v_{j_{\alpha+1}}) \)). The bars \( b_{j_1}, \ldots, b_{j_\alpha} \) are drawn inside \( \sigma(b_j) \) in such a way that \( y''(b_{j_i}) < y''(b_{j_{i+1}}) < y''(b_{j_1}) < y''(b_{j_\alpha}) < y'(b_{j_i}) \), where \( y''(b_{j_i}) \) and \( y'(b_{j_i}) \) represent the minimum and maximum \( y \)-coordinate of \( b_{j_i} \), respectively.

Since each bar \( b_j \) has a maximum \( y \)-coordinate larger than the maximum \( y \)-coordinate of the bars between it and \( b_j \), \( b_j \) can be connected with a visibility to \( b_j \). The strip \( \sigma(b_{j_i}) \) of \( b_{j_i} \) is determined as follows. Let \( \ell_j \), for \( i = 1, 2, \ldots, \alpha - 1 \), be a \( z \)-parallel line having a \( y \)-coordinate between \( y''(b_{j_i}) \) and \( y''(b_{j_{i+1}}) \), and let \( \ell_{j_\alpha} \) be a \( z \)-parallel line between \( y''(b_{j_\alpha}) \) and \( y''(b_{j_1}) \). The strip \( \sigma(b_{j_i}) \) with \( 1 < i < \alpha \) is contained in the region of the \( yz \)-plane delimited by the two lines \( \ell_{j_i} \) and \( \ell_{j_{i-1}} \). Further, \( \sigma(b_{j_1}) \) is contained in the region between \( \ell_{j_1} \) and a \( z \)-parallel line having a \( y \)-coordinate between the minimum \( y \)-coordinate of \( \sigma(b_{j_i}) \) and \( y''(b_{j_1}) \).

We now explain how to draw the forward children (if necessary). Refer to Fig. 9(c) for an illustration. Let \( v_{k_1}, v_{k_2}, \ldots, v_{k_\beta} \) be the forward children of \( v_j \) ordered according to \( \rho \) (thus \( z(v_{k_1}) < z(v_{k_{\beta+1}}) \)). Since these vertices have \( z \)-coordinate larger than \( z(b_j) \), by property **P2**, the bars representing forward children can be drawn inside \( \sigma(b_j) \) so that they are visible from \( b_j \). More precisely, the bars \( b_{k_1}, b_{k_2}, \ldots, b_{k_\beta} \) are drawn inside \( \sigma(b_j) \) in such a way that \( y''(b_{j_i}) < y''(b_{k_1}) < y'(b_{k_1}) < y''(b_{k_{\beta+1}}) < y'(b_{k_{\beta+1}}) \). With this construction each
Figure 9: The generic step of the construction of a BVR $\Gamma$ of a tree $T$. (a) Vertex $v_j$ and its parent $v_k$ have been already drawn; the strip $\sigma(v_j)$ is shown in gray; (b) the backward children $v_{j1}, v_{j2}, v_{j3}$ are drawn inside $\sigma(b_j)$ and their strips (dark grey) are defined; (c) the forward children $v_{k1}$ and $v_{k2}$ of $v_j$ are drawn inside $\sigma(b_j)$ and their strips (dark grey) are defined; (d) The strips of the children of $v_j$ satisfy the properties $P1$–$P3$. 
bar \( b_{ki} \) can be connected with a visibility to \( b_j \). The strip \( \sigma(b_{ki}) \) is contained in the region of the \( yz \)-plane delimited by two \( z \)-parallel lines, the first having a \( y \)-coordinate between \( y'(b_{ki-1}) \) and \( y''(b_{ki}) \) (if \( i = 1 \), between \( y''(b_j) \) and \( y''(b_{ki}) \)) and the second having a \( y \)-coordinate between \( y''(b_{ki}) \) and the \( y \)-coordinate of the visibility between \( b_{ki} \) and \( b_j \).

Fig. 10 shows a BVR of the tree \( T \) of Fig. 8 computed by the described algorithm. The next Lemma proves the correctness of the algorithm.

**Lemma 5.** Let \( T \) be a properly 2-colored tree, and let \( \lambda \) be a 2-colored sequence compatible with \( T \); \( T \) admits a BVR whose \( z \)-ordering is consistent with \( \lambda \).

**Proof.** Compute a BVR \( \Gamma \) of \( T \) by using the described algorithm. We prove that, during the construction, the strips defined for the newly drawn bars satisfy properties \( \textbf{P1}–\textbf{P3} \) (see also Fig. 9(d)). Let \( b_{ji} \) be a bar representing a backward child of \( b_j \). The strip \( \sigma(b_{ji}) \) satisfies \( \textbf{P1} \) by construction. For \( \textbf{P2} \), we observe that \( \sigma(b_{ji}) \) is completely contained inside \( \sigma(b_j) \) and the only bars that intersect \( \sigma(b_j) \) and have a \( z \)-coordinate larger than the \( z \)-coordinate of \( b_{ji} \) are those representing the backward children \( b_{jl} \) of \( b_j \) with \( l > i \) and the forward children of \( b_j \). By construction all these bars have minimum \( y \)-coordinate larger than
\[ \ell_{j_i} \text{ and therefore they do not cross } \sigma(b_{j_i}). \] For property \(P_3\), we observe that since \(b_{j_i}\) is a backward child, it does not have backward children. Namely, its children have the same color as its parent \(v_{j}\) and therefore they have a \(z\)-coordinate larger than \(z(v_{j})\) (because they appear after \(v_{j}\) in \(\rho\)). It follows that property \(P_3\) holds.

Let \(b_{k_i}\) be a bar representing a forward child of \(b_{j_i}\). Also in this case, the strip \(\sigma(b_{k_i})\) satisfies \(P_1\) by construction. For \(P_2\), \(\sigma(b_{k_i})\) is completely contained in \(\sigma(b_{j_i})\) and the only bars that intersect \(\sigma(b_{j_i})\) and have a \(z\)-coordinate larger than \(y'(b_{k_i})\) and therefore they do not cross \(\sigma(b_{k_i})\). For property \(P_3\), \(b_{k_i}\) can have backward children. On the other hand, by construction, the bar with the largest \(z\)-coordinate that crosses \(\sigma(b_{k_i})\) is \(b_{j_i}\) (the bars representing the forward children \(b_{k_l}\) of \(b_{j_i}\) with \(l > i\) do not cross \(\sigma(b_{k_i})\)). Thus property \(P_3\) holds also in this case.

We have the following theorem.

**Theorem 8.** Let \(T\) be a properly 2-colored tree and let \(P\) be a 2-colored point set in \(\mathbb{R}^3\) compatible with \(G\); \(G\) admits a 2-colored AZPR on \(P\).

**Proof.** By Lemma 5, \(T\) admits a BVR whose \(z\)-ordering is consistent with \(\lambda(P)\). By Lemma 2 \(T\) admits a cornered ZPR whose \(z\)-ordering is consistent with \(\lambda(P)\). Finally, by Lemma 1 \(T\) admits an AZPR on \(P\) whose \(z\)-ordering is consistent with \(\lambda(P)\), i.e., a 2-colored AZPR on \(P\).

5. Conclusions and Open Problems

In this paper we introduced and studied colored anchored visibility representations in 2D and in 3D space. We used a framework based on colors to describe different variants concerning how the mapping of the vertices to the points is specified. In 2D we have proved that a 1-colored ABVR always exists for sub-Hamiltonian graphs with no restriction on the point set and that only sub-Hamiltonian graphs admit an ABVR on set of vertically aligned points. This implies that the problem of deciding whether a planar graph admits an ABVR is \(NP\)-complete. If we restrict the set of points to be generic (i.e., all the points have distinct \(x\)- and \(y\)-coordinates) then a 1-colored ABVR exists for every planar graph. The case when not all points are vertically aligned but not all have distinct \(y\)-coordinate remains open.

As for the version with more than one color, we have used the results above to show that for every \(k > 1\) there exists a \(k\)-colored planar graph that does not admit a \(k\)-colored ABVR on every set of \(k\)-colored points in the plane. A question arising from Theorem 1 and Corollary 2 is whether for \(k > 1\) all \(k\)-colored sub-Hamiltonian graphs admit a \(k\)-colored ABVR on any given \(k\)-colored set of points.

The result proving the existence of ABVRs of sub-Hamiltonian graphs has been extended in 3D to prove the existence of AZPRs of graphs with page
number four. We have also shown that an AZPR of $K_n$ exists when $n \leq 22$ and does not exist for $n \geq 51$. These results derive from analogous results about ZPRs because, as stated by Theorem 6, $K_n$ has an AZPR if and only if it has a ZPR. Hence, the longstanding open problem of investigating whether $K_n$ for $22 < n < 51$ admits a ZPR or not, is of interest also for AZPRs.

We have also proven the existence of an AZPR on any set of given points for specific families of graphs both in the 1-colored case (3-connected 1-planar graphs and thickness-two graphs) and in the 2-colored case (properly 2-colored trees). It would be interesting to prove analogous results for other families of graphs. In particular, can we extend our results to general 1-colored 1-planar graphs and to general 2-colored trees? What about more than two colors?

Concerning the last question, we give a preliminary result for the case when the number of colors is equal to the number of vertices. A $z$-assignment of $G = (V, E)$ is a one-to-one mapping $\phi : V \rightarrow \{1, 2, \ldots, |V|\}$. $G$ is unlabeled level planar (ULP) if for any given $z$-assignment $\phi$, it admits a planar straight-line drawing with $z(v) = \phi(v)$ for every $v \in V$ [24, 31, 36].

Theorem 9. Let $G$ be an $n$-colored $n$-vertex graph that is the union of two ULP graphs with the same vertex set and let $P$ be an $n$-colored point set in $\mathbb{R}^3$; $G$ admits an $n$-colored AZPR on $P$.

Proof. Since both $G$ and $P$ are $n$-colored, $\lambda(P)$ defines a total order and therefore a $z$-assignment of $G$. Let $G_1$ and $G_2$ be the two ULP graphs whose union is $G$. Since each $G_i$ ($i = 1, 2$) is ULP then it admits a planar straight-line drawing $\Gamma_i$ such that $z(v) = \phi(v)$ for every $v \in V$. By orienting each edge of both $\Gamma_1$ and $\Gamma_2$ from the end-vertex with lower $z$-coordinate to the end-vertex with higher $z$-coordinate we obtain two upward planar drawings of $G_1$ and $G_2$ with the same order $\rho$ of the vertices in the vertical direction. Moreover, $\rho$ is consistent with $\lambda(P)$. By Lemma 3, $G$ has an AZPR $\Gamma$ on $P$ whose $z$-ordering is $\rho$. Since $\rho$ is consistent with $\lambda(P)$, $\Gamma$ is an $n$-colored AZPR of $G$ on $P$. □

References


[44] S. Rengarajan and C. E. Veni Madhavan. Stack and queue number of 2-
trees. In D.-Z. Du and M. Li, editors, Computing and Combinatorics, pages


[46] T. C. Shermer. On rectangle visibility graphs III. external visibility and

I. G. Tollis and M. Patrignani, editors, Graph Drawing 2008, volume 5417

[48] I. Streinu and S. Whitesides. Rectangle visibility graphs: Characterization,
construction, and compaction. In H. Alt and M. Habib, editors, STACS


[53] A. Wigderson. The complexity of the hamiltonian circuit problem for max-
imal planar graphs. Technical Report 298, Princeton University, EECS
Department, 1982.

editor, Proceedings of the First Annual Symposium on Computational Ge-
ometry, Baltimore, Maryland, USA, June 5-7, 1985, pages 147–152. ACM,
1985.