AUTOMORPHIC SL_2-PERIODS AND THE SUBCONVEXITY PROBLEM FOR $\mathrm{GL}_2 \times \mathrm{GL}_3$

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Abstract. We prove a new (conditional) result towards the subconvexity problem for certain automorphic L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_3$. This follows from the computation of new SL_2-period integrals associated with newforms f and g of even weight and odd squarefree level. The same computations also lead to a central value formula for degree 6 complex L-series of the form $L(f \otimes \mathrm{Ad}(g), s)$, extending previous work in [PdVP19].

1. Introduction

Let $f \in S_{2k}(N_f)$ and $g \in S_{\ell+1}(N_g)$ be two normalized newforms of weight $2k$ and $\ell + 1$, and level $\Gamma_0(N_f)$ and $\Gamma_0(N_g)$, respectively. We assume throughout that $\ell \geq k \geq 1$ are both odd integers, and that the levels N_f and N_g are both squarefree and odd. We set $\ell - k = 2m$, with $m \geq 0$. We emphasize that we consider level structure of Γ_0-type, hence both f and g have trivial Nebentype character.

Associated with f and g, one has a degree 6 complex L-series

$$L(f \otimes \mathrm{Ad}(g), s),$$

which is the Artin L-series corresponding to the tensor product $V(f) \otimes \mathrm{Ad}(V(g))$ of the (compatible system of p-adic) Galois representation(s) attached to f and the adjoint of the one attached to g. This L-series admits a representation as an Euler product

$$L(f \otimes \mathrm{Ad}(g), s) = \prod_p L_p(f \otimes \mathrm{Ad}(g), s),$$

where p varies over all rational primes. For example, if p is a rational prime not dividing $N_f N_g$, and $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ are the Satake parameters of f and g at p, respectively, so that

$$1 - a_f(p)X + p^{2k-1}X^2 = (1 - p^{k-1/2}\alpha_p X)(1 - p^{k-1/2}\alpha_p^{-1}X),$$
$$1 - a_g(p)X + p^{\ell}X^2 = (1 - p^{\ell/2}\beta_p X)(1 - p^{\ell/2}\beta_p^{-1}X),$$

then one has

$$L_p(f \otimes \mathrm{Ad}(g), s) = \det(\mathbf{1}_6 - A_p \otimes B_p \cdot p^{-s-\ell})^{-1},$$

where we put

$$A_p = p^{k-1/2}\begin{pmatrix} \alpha_p & 0 \\ 0 & \alpha_p^{-1} \end{pmatrix}, \qquad B_p = p^{\ell}\begin{pmatrix} \beta_p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta_p^{-2} \end{pmatrix}.$$

The above Euler product converges absolutely for $\mathrm{Re}(s) \gg 0$, and the completed L-series

$$\Lambda(f \otimes \mathrm{Ad}(g), s) = L_\infty(f \otimes \mathrm{Ad}(g), s) \prod_p L_p(f \otimes \mathrm{Ad}(g), s),$$

where $L_\infty(f \otimes \mathrm{Ad}(g), s) := \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+\ell)\Gamma_{\mathbb{C}}(s+\ell-2k+1)$, $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, has analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(f \otimes \mathrm{Ad}(g), 2k - s) = \Lambda(f \otimes \mathrm{Ad}(g), s),$$

with center of symmetry at $s = k$. In our previous paper [PdVP19], we proved an explicit central value formula for $\Lambda(f \otimes \mathrm{Ad}(g), k)$ under certain hypotheses, extending a previous formula of Ichino [Ich05]. Such formula was obtained by making explicit a decomposition formula due to Qiu [Qiu14] for a certain automorphic SL_2-period, and in classical terms it involves a half-integral weight modular form $h \in S^+_{k+1/2}(N_f)$ in Shimura–Shintani correspondence with f and its Saito–Kurokawa lift. The purpose of this note is two-fold: on one hand, we generalize the central value formula in [PdVP19], and on the other hand, we make some progress towards the subconvexity problem for automorphic L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_3$. For both goals we need new computations of local SL_2-periods, and for the second one we also use recent work of Nelson [Nel19].

To be more precise, let π and τ be the automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ (actually, of $\mathrm{PGL}_2(\mathbb{A})$) associated with f and g, respectively. In addition, let ψ denote the standard additive character of $\mathbb{A}/\mathbb{Q}$, $\omega_{\bar{\psi}}$ be the Weil representation of the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ on the space $\mathcal{S}(\mathbb{A})$ of Bruhat–Schwartz functions (on the one dimensional quadratic space with bilinear form $(x, y) = xy/2$) with respect to $\bar{\psi} = \psi^{-1}$, and

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$\tilde{\pi} \in \mathrm{Wald}_{\bar{\psi}}(\pi)$ be an automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ belonging to the *Waldspurger packet* of π with respect to $\bar{\psi}$. Associated with the triple $\tilde{\pi}, \tau, \omega_{\bar{\psi}}$, Qiu defines a natural automorphic SL_2-period functional

$$\mathcal{Q} : \tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega_{\bar{\psi}} \otimes \omega_{\bar{\psi}} \longrightarrow \mathbb{C}$$

and studies its main features. Most importantly, he shows that when $\mathcal{Q}$ is not identically zero, then it decomposes as a product of local SL_2-periods

$$\mathcal{I}_v : \tilde{\pi}_v \otimes \tilde{\pi}_v \otimes \tau_v \otimes \tau_v \otimes \omega_{\bar{\psi},v} \otimes \omega_{\bar{\psi},v} \longrightarrow \mathbb{C}$$

up to certain L-values, including the central value $\Lambda(f \otimes \mathrm{Ad}(g), k)$. The non-vanishing of $\mathcal{Q}$ is well-understood, and it is equivalent to the central value $\Lambda(f \otimes \mathrm{Ad}(g), k)$ being non-zero together with some local conditions on the choice of $\tilde{\pi}$ in $\mathrm{Wald}_{\bar{\psi}}(\pi)$. With this, the strategy followed in [PdVP19] consists in finding a *test vector* on which $\mathcal{Q}$ does not vanish, and then evaluating both the global period $\mathcal{Q}$ and the local periods $\mathcal{I}_v$ at such vector. From Qiu's decomposition formula, one can then isolate the desired central value $\Lambda(f \otimes \mathrm{Ad}(g), k)$.

The assumptions made in [PdVP19], mainly that $N_g = N_f$ and $\ell = k$, simplified the still involved computations of the local periods $\mathcal{I}_v$, as well as the evaluation of the global period itself. Both of these assumptions can be relaxed, leading to the following result:

Theorem 1.1. *Suppose that N_f, N_g are both odd and squarefree, and that $N_g \mid N_f$. Suppose that $\ell \geq k \geq 1$ are both odd, and set $\ell - k = 2m$. If the Atkin–Lehner eigenvalue of f at p is $+1$ at all primes p dividing $M_g := N_f/N_g$, then there exists a half-integral weight modular form $h \in S^+_{k+1/2}(N_f)$ in Shimura–Shintani correspondence with f such that*

$$\Lambda(f \otimes \mathrm{Ad}(g), k) = 2^{6m+k+1-\nu(M_g)} C_0(f,g) C_\infty(f,g) \cdot \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle \breve{F}_{|\mathcal{H}\times\mathcal{H}}, g \times V_{M_g} g\rangle|^2}{\langle g, g\rangle^2},$$

where $\breve{F} \in S^{nh}_{\ell+1}(\Gamma_0^{(2)}(N_f))$ is a nearly holomorphic Siegel form closely related to the Saito–Kurokawa lift of h (cf. Proposition 7.9), $\nu(M_g)$ denotes the number of primes dividing M_g, and $C_0(f,g)$ and $C_\infty(f,g)$ are non-zero rational constants that depend on the levels and weights of f and g, respectively (cf. Theorem 4.1 for their explicit value).

When $N_g = N_f$ and $\ell = k$, one has $m = 0$ and $C_\infty(f,g) = 1$, $\breve{F} = \mathrm{SK}(h) \in S_{k+1}(\Gamma_0^{(2)}(N_f))$ is the Saito–Kurokawa lift of h, and the above formula recovers [PdVP19, Theorem 1.1] (assuming in loc. cit. that g has trivial Nebentype character, see Remark 1.2 in op. cit.). If in addition $N_g = N_f = 1$, then it recovers the original formula of Ichino [Ich05]. We also point out that the case $N_g = N_f$ and $\ell \geq k$ has been considered in [Che19], by extending Ichino's approach instead of using Qiu's strategy via SL_2-periods. In the above statement, the Petersson products $\langle f, f\rangle$, $\langle g, g\rangle$ are defined as usual, namely

$$\langle f, f\rangle := \mu_{N_f}^{-1} \int_{\Gamma_0(N_f)\backslash\mathcal{H}} f(z)\overline{f(z)} y^{2k-2} dz, \qquad \langle g, g\rangle := \mu_{N_g}^{-1} \int_{\Gamma_0(N_g)\backslash\mathcal{H}} g(z)\overline{g(z)} y^{\ell-1} dz,$$

where $z = x + \sqrt{-1}y$ and $\mu_t = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(t)]$ for $t \in \mathbb{Z}_{\geq 1}$. For the half-integral weight modular form h we similarly have

$$\langle h, h\rangle := \mu_{4N_f}^{-1} \int_{\Gamma_0(4N_f)\backslash\mathcal{H}} h(z)\overline{h(z)} y^{k-3/2} dz.$$

Finally, the Petersson product $\langle \breve{F}_{|\mathcal{H}\times\mathcal{H}}, g \times V_{M_g} g\rangle$ is defined by (notice that $N_f = \mathrm{lcm}(N_f, N_g)$ because of our assumption that $N_g \mid N_f$)

$$\langle \breve{F}_{|\mathcal{H}\times\mathcal{H}}, g \times V_{M_g} g\rangle := \mu_{N_f}^{-2} \int_{\Gamma_0(N_f)\backslash\mathcal{H}} \int_{\Gamma_0(N_f)\backslash\mathcal{H}} \breve{F}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right) \overline{g(z_1) g(M_g z_2)} y_1^{\ell-1} y_2^{\ell-2} dz_1 dz_2.$$

As we have already pointed out above, the above theorem is an extension of the main result of [PdVP19]. The proof requires to extend both the global and local computations involved in our strategy of making explicit Qiu's decomposition formula. Special attention in this paper is deserved to the local side, because the new computations of local SL_2-periods $\mathcal{I}_v$ at a specific test vector done in this note, together with those already carried out in [PdVP19], allow us to derive new advances in the subconvexity problem for $\mathrm{GL}_2 \times \mathrm{GL}_3$ by using recent work of Nelson [Nel19]. This is the most interesting novelty of this paper, and also the main motivation that led us to write this note. Namely, in Section 8 we address the *subconvexity problem* for automorphic L-functions[1]

(1) $$L(\pi \otimes \mathrm{ad}(\tau), s), \qquad \pi \text{ on } \mathrm{PGL}_2 \text{ fixed}, \ \tau \text{ on } \mathrm{GL}_2 \text{ varying}.$$

[1]In analogy with classical L-series, we follow the convention that automorphic L-functions $L(\Pi, s)$ refer always to the finite part of the L-function, omitting the Γ-factors at the archimedean place. When including such factors, we will write $\Lambda(\Pi, s)$.

This problem consists in establishing a *subconvex bound* for $L(\pi \otimes \mathrm{ad}(\tau), 1/2)$ when π is a fixed automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ and τ varies in a family $\mathcal{G}$ of automorphic representations of GL_2, i.e. proving the existence of constants $c = c(\mathcal{G}) \geq 0$ and $\delta = \delta(\mathcal{G}) > 0$ such that

$$|L(\pi \otimes \mathrm{ad}(\tau), 1/2)| \leq cC(\pi \otimes \mathrm{ad}(\tau))^{1/4-\delta} \tag{2}$$

for all $\tau \in \mathcal{G}$, where $C(\pi \otimes \mathrm{ad}(\tau)) \in \mathbb{R}_{\geq 1}$ denotes the analytic conductor of $\pi \otimes \mathrm{ad}(\tau)$. The inequality analogous to (2) with $\delta = 0$ is the so-called *convex bound*, and can be obtained by using the Phragmen–Lindelöf principle. Therefore, establishing a subconvex bound requires to break this barrier and improve the convex bound. Interest in subconvexity problems as the above one relies on their relation to fundamental arithmetic equidistribution questions. In the case of (1), it has applications towards the limiting mass distribution of automorphic forms, also known as the 'arithmetic quantum unique ergodicity' (see [Sar95], [IS00], [HS10], [NPS14], [Sar11]).

Our contribution to the subconvexity problem in (1), under some assumptions on the family $\mathcal{G}$, follows from the observation that our computations of local SL_2-periods provide the required bounds in recent work of Nelson [Nel19] concerning this subconvexity problem. And it is important to remark that the local SL_2-periods computed in [PdVP19] alone would not have been enough to improve Nelson's result. Let us illustrate in this introduction an easy but relevant example in which we can push Nelson's result one step further in the above subconvexity problem, referring the reader to Section 8 for a more detailed and general statement.

In line with our notation above, fix at the outset an odd integer $\ell \geq 1$, and let q traverse an infinite increasing sequence $\mathfrak{Q}$ of (odd) prime numbers. For each prime $q \in \mathfrak{Q}$, choose a newform $g \in S_{\ell+1}^{new}(q)$ of weight $\ell + 1$ and level $\Gamma_0(q)$, and let $\mathcal{G}$ be the infinite collection of all the automorphic representations $\tau = \tau(g)$ of $\mathrm{PGL}_2(\mathbb{A})$ associated with the newforms g as q varies. We assume the following hypothesis on the family $\mathcal{G}$, which is the existence of a subconvex bound for $L(\tau \otimes \tau \otimes \chi, 1/2)$ in the τ-aspect with polynomial dependence upon the Hecke character χ:

Hypothesis: there exist absolute constants $c_0, A_0 \geq 0$, $\delta_0 > 0$ such that for all $\tau \in \mathcal{G}$ and all unitary characters χ of $\mathbb{A}^\times/\mathbb{Q}^\times$, one has

$$|L(\tau \otimes \tau \otimes \chi, 1/2)| \leq c_0 C(\tau \otimes \tau \otimes \chi)^{1/4-\delta_0} C(\chi)^{A_0}.$$

The following statement is a particular instance of Theorem 8.1 in Section 8, which strengthens [Nel19, Theorem 1] in the sense that we allow f to have arbitrary (odd) squarefree level instead of level 1. Modulo the above hypothesis, the main novelty of the following result is precisely that we remove the assumption on f having trivial level.

Theorem 1.2. *With the above notation, assume that the family $\mathcal{G}$ satisfies the above hypothesis. Then, there exist absolute constants $c, A \geq 0$ and $\delta > 0$ such that*

$$L(\pi \otimes \mathrm{ad}(\tau), 1/2) \leq cC(\pi \otimes \mathrm{ad}(\tau))^{1/4-\delta} C(\pi)^A$$

for all $\tau \in \mathcal{G}$ and every automorphic representation $\pi = \pi(f)$ of $\mathrm{PGL}_2(\mathbb{A})$ associated with a newform $f \in S_{2k}^{new}(N_f)$ of weight $2k$, with $1 \leq k \leq \ell$ an odd integer, and odd squarefree level N_f.

Observe that we have omitted the absolute value on the left hand side of the inequality in the statement. This is because the L-value $L(\pi \otimes \mathrm{ad}(\tau), 1/2) = L(f \otimes \mathrm{Ad}(g), k)$ is non-negative (this can be deduced from Theorem 1.1).

The emphasized hypothesis in the above theorem is inspired by the work of Munshi in [Mun]. Via the factorization

$$L(\tau \otimes \tau \otimes \chi, 1/2) = L(\chi, 1/2) L(\mathrm{ad}(\tau) \otimes \chi, 1/2),$$

the subconvexity problem for $L(\tau \otimes \tau \otimes \chi, s)$ can be reduced to that for $L(\mathrm{ad}(\tau) \otimes \chi, s)$ (with τ varying and χ fixed). A proof for the latter is given in op. cit. when χ is trivial, and the general case is expected to follow by the same techniques. Hence, relying on this, the above theorem would be unconditional and it would lead to strong quantitative forms of the arithmetic quantum unique ergodicity conjecture in the prime level aspect.

Let us close this introduction by briefly describing the organization of the paper. Section 2 below collects general notation that is used through all the text, mainly about metaplectic groups and orthogonal groups. In Section 3, we recall some general notions on quadratic spaces and theta correspondences, with special attention to the cases needed for our purposes. In Section 4 we explain in more detail the strategy to prove Theorem 1.1, and state again the result in more precise terms (see Theorem 4.1). After that, Sections 5 and 6 are devoted to describe our choice of test vector and to compute the local periods alluded to above. Section 6 deserves special attention, since the local period computations therein (together with those in [PdVP19]) are the key ingredient for our application in Section 8 towards the subconvexity problem for automorphic L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the proof of Theorem 1.2 above (which is a particular case of the more general version in Theorem 8.1). Section 7 is devoted to complete the proof of the central value formula stated in Theorem 4.1, and can be skipped by the reader interested in the subconvexity problem considered in Theorem 1.2.

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2. Notation

Through all the paper, we write $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ for the ring of adeles over $\mathbb{Q}$. We write $\zeta(s)$ for Riemann's zeta function, admitting the usual Euler product representation $\zeta(s) = \prod_p \zeta_p(s)$ for $\mathrm{Re}(s) > 1$, where $\zeta_p(s) = (1-p^{-s})^{-1}$. We write $\zeta_{\mathbb{Q}}(s)$ for the completed Riemann zeta function given by $\zeta_{\mathbb{Q}}(s) := \Gamma_{\mathbb{R}}(s)\zeta(s)$, where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s)$ and $\Gamma(s)$ is the usual Gamma function. We will also use $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$.

If $r, M \geq 1$ are integers, and ψ is a Dirichlet character modulo M, we denote by $S_r(M,\psi)$ the (complex) space of cusp forms of weight r, level M and character ψ. When ψ is trivial, we just write $S_r(M)$ or $S_r(\Gamma_0(M))$, where $\Gamma_0(M) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is the usual Hecke congruence subgroup of level M. Similarly $S_r(\Gamma_0^{(2)}(M))$ will stand for the (complex) space of Siegel forms of degree 2 and weight r for the Hecke congruence subgroup $\Gamma_0^{(2)}(M) \subseteq \mathrm{Sp}_2(\mathbb{Z})$ of level M.

If $M \geq 1$ is an odd integer and $k \geq 0$ is an integer, we write $S_{k+1/2}(4M, (\frac{4}{\cdot}))$ for the (complex) space of cusp forms of half-integral weight $k+1/2$, level $4M$ and character $(\frac{4}{\cdot})$, in the sense of Shimura [Shi73]. We denote by $S^+_{k+1/2}(M)$ Kohnen's plus subspace in $S^+_{k+1/2}(4M, (\frac{4}{\cdot}))$ consisting of those forms h whose q-expansion has the form

$$h = \sum_{\substack{n\geq 1,\\ (-1)^k n \equiv 0,1\,(4)}} c(n)q^n.$$

We refer the reader to [Koh82] for a careful study of these spaces.

If v is a place of $\mathbb{Q}$, we write $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ for the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Q}_v)$, and similarly, we denote by $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$. We will identify $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$, resp. $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, with $\mathrm{SL}_2(\mathbb{Q}_v) \times \{\pm 1\}$, resp. $\mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$, where the product is given by the rule

$$[g_1,\epsilon_1][g_2,\epsilon_2] = [g_1g_2, \epsilon_v(g_1,g_2)\epsilon_1\epsilon_2] \qquad (\text{resp. } [g_1,\epsilon_1][g_2,\epsilon_2] = [g_1g_2, \epsilon(g_1,g_2)\epsilon_1\epsilon_2]).$$

At each place v, $\epsilon_v(g_1,g_2)$ is defined as follows. First one defines $x : \mathrm{SL}_2(\mathbb{Q}_p) \to \mathbb{Q}_p$ by

$$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0; \end{cases}$$

then, $\epsilon_v(g_1,g_2) = (x(g_1)x(g_1g_2), x(g_2)x(g_1g_2))_v$. When $g_1, g_2 \in \mathrm{SL}_2(\mathbb{A})$, we set $\epsilon(g_1,g_2) = \prod_v \epsilon_v(g_1,g_2)$. When $v = \infty$, we put $s_\infty(g) = 1$ for all $g \in \mathrm{SL}_2(\mathbb{R})$, and when $v = p$ is a finite place, we set

$$s_p\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \begin{cases} (c,d)_p & \text{if } cd \neq 0,\ \mathrm{ord}_p(c) \text{ odd}, \\ 1 & \text{otherwise}, \end{cases}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}_p)$. With this, at each place v, $\mathrm{SL}_2(\mathbb{Q}_v)$ embeds as a subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ through $g \mapsto [g, s_v(g)]$. If p is an odd prime, then this homomorphism gives a splitting of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ over the maximal compact subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$, while for $p = 2$ this is only a splitting over $\Gamma_1(4;\mathbb{Z}_2) \subset \mathrm{SL}_2(\mathbb{Z}_2)$. If p is an odd prime (resp. if $p = 2$), and G is a subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ (resp. of $\Gamma_1(4;\mathbb{Z}_2)$), then we will write $\tilde{G} \subseteq \widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$ for the image of G under the previous splitting. We will also regard $\mathrm{SL}_2(\mathbb{Q})$ as a subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ through the homomorphism $g \mapsto [g, \prod_v s_v(g)]$.

When working in $\mathrm{SL}_2(\mathbb{Q}_v)$ (or $\mathrm{SL}_2(\mathbb{A})$), we will often use the notation

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for $x \in \mathbb{Q}_v$ (or $\mathbb{A}$) and $a \in \mathbb{Q}_v^\times$ (or $\mathbb{A}^\times$).

If V is a finite-dimensional quadratic space over $\mathbb{Q}$, with bilinear form $(\ ,\ )$, and ψ is an additive character of $\mathbb{A}/\mathbb{Q}$, we equip $V(\mathbb{A})$ with the Haar measure which is self-dual with respect to ψ, unless otherwise stated. In other words, we consider the Haar measure such that $\mathcal{F}(\mathcal{F}(\phi))(x) = \phi(-x)$, where $\mathcal{F}(x) = \int_{V(\mathbb{A})} \phi(y)\psi((x,y))dy$ is the Fourier transform of ϕ. We note that the orthogonal group $\mathrm{O}(V)$ is *not* connected, and choose a measure on $\mathrm{O}(V)(\mathbb{A})$ as follows. First, we equip $\mathrm{SO}(V)(\mathbb{A})$ with the Tamagawa measure. Secondly, at each place v we extend the local measure on $\mathrm{SO}(V)(\mathbb{Q}_v)$ to the non-identity component of $\mathrm{O}(V)(\mathbb{Q}_v)$. And finally, we consider the measure dh_v on $\mathrm{O}(V)(\mathbb{Q}_v)$ to be half of this extended measure, and define $dh = \prod_v dh_v$. This is the Tamagawa measure on $\mathrm{O}(V)(\mathbb{A})$, and $[\mathrm{O}(V)] = \mathrm{O}(V)(\mathbb{Q})\backslash\mathrm{O}(V)(\mathbb{A})$ has volume 1 with respect to dh. If $\mathcal{S}(V(\mathbb{A}))$ denotes the space of Bruhat–Schwartz functions on $V(\mathbb{A})$, and ϕ_1, $\phi_2 \in \mathcal{S}(V(\mathbb{A}))$, we set $\langle\phi_1,\phi_2\rangle = \int_{V(\mathbb{A})} \phi_1(x)\overline{\phi_2(x)}dx$, where dx is the Haar measure that is self-dual with respect to ψ. If π is an irreducible cuspidal unitary representation of $G(\mathbb{A})$, and $f_1, f_2 \in \pi$, we define the pairing $\langle f_1, f_2\rangle$ to be:

i) $\int_{[\mathrm{SL}_2]} f_1(g)\overline{f_2(g)}dg$, if $G = \widetilde{\mathrm{SL}}_2$;
ii) $\int_{[\mathrm{PGL}_2]} f_1(g)\overline{f_2(g)}dg$, if $G = \mathrm{GL}_2$;
iii) $\int_{[G]} f_1(g)\overline{f_2(g)}dg$, if $G = \mathrm{SO}(V)$ or $\mathrm{O}(V)$.

3. Quadratic spaces and theta correspondences

3.1. **Quadratic spaces.** Let F be a field with $\mathrm{char}(F) \neq 2$, and V be a quadratic space over F, i.e. a finite-dimensional vector space over F equipped with a non-degenerate symmetric bilinear form $(\, , \,)$. We denote by Q the associated quadratic form on V, given by

$$Q(x) = \frac{1}{2}(x, x), \quad x \in V.$$

If $m = \dim(V)$, fixing a basis $v_1, \dots v_m$ of V and identifying V with the space of column vectors F^m, the bilinear form $(\, , \,)$ determines a matrix (that we still denote with the same letter) $Q \in \mathrm{GL}_m(F)$ by setting $Q = ((v_i, v_j))_{i,j}$. Then we have $(x, y) = {}^t xQy$ for $x, y \in V$. We define $\det(V)$ to be the image of $\det(Q)$ in $F^\times/(F^\times)^2$. The orthogonal similitude group of V is

$$\mathrm{GO}(V) = \{h \in \mathrm{GL}_m : {}^t hQh = \nu(h)Q, \, \nu(h) \in \mathbb{G}_m\},$$

and $\nu : \mathrm{GO}(V) \to \mathbb{G}_m$ is the similitude morphism (also called scale map). From the very definition, observe that $\det(h)^2 = \nu(h)^m$ for every $h \in \mathrm{GO}(V)$. If m is even, we also set

$$\mathrm{GSO}(V) = \{h \in \mathrm{GO}(V) : \det(h) = \nu(h)^{m/2}\}.$$

Finally, we let $\mathrm{O}(V) = \ker(\nu)$ denote the orthogonal group of V, and write $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}_m$ for the special orthogonal group.

3.2. **Explicit realizations in low rank.** We are particularly interested in orthogonal groups for vector spaces of dimension 3, 4 and 5. We fix in this paragraph the explicit realizations that will be used later on to describe automorphic representations for $\mathrm{SO}(V)(\mathbb{A})$ and $\mathrm{GSO}(V)(\mathbb{A})$. We keep the same choices as in [PdVP19], which follow quite closely the ones in [Ich05, Qiu14].

When $\dim(V) = 3$, there exist a unique quaternion algebra B over F and an element $a \in F^\times$ such that $(V, q) \simeq (V_B, aq_B)$, where $V_B = \{x \in B : \mathrm{Tr}_B(x) = 0\}$ is the subspace of elements in B with zero trace, and $q_B(x) = -\mathrm{Nm}_B(x)$. The group of invertible elements $B^\times$ acts on V_B by conjugation, and this action gives rise to an isomorphism

$$PB^\times \xrightarrow{\simeq} \mathrm{SO}(V_B, q_B) \simeq \mathrm{SO}(V, q).$$

When $B = \mathrm{M}_2$ is the split algebra of 2-by-2 matrices, $PB^\times = \mathrm{PGL}_2$ and the above identifies PGL_2 with the special orthogonal group of a three-dimensional quadratic space.

In dimension 4, we consider the vector space $V_4 := \mathrm{M}_2(F)$ of 2-by-2 matrices, equipped with the quadratic form $q(x) = \det(x)$. The associated non-degenerate bilinear form is $(x, y) = \mathrm{Tr}(xy^*)$, where

$$x^* = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{M}_2(F).$$

There is an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GL}_2 \times \mathrm{GL}_2 \xrightarrow{\rho} \mathrm{GSO}(V_4) \longrightarrow 1, \tag{3}$$

where $\iota(a) = (a\mathbf{1}_2, a^{-1}\mathbf{1}_2)$, $\rho(h_1, h_2)x = h_1 x h_2^*$. One has $\nu(\rho(h_1, h_2)) = \det(h_1 h_2) = \det(h_1)\det(h_2)$. In particular, when F is a number field, automorphic representations of $\mathrm{GSO}(V_4)$ can be seen as automorphic representations of $\mathrm{GL}_2 \times \mathrm{GL}_2$ through the homomorphism ρ in the above short exact sequence. Here we warn the reader that our choice for ρ in (3) agrees with the one on [Qiu14] and [GT11], but differs from the one considered in [Ich05] (or [II08]), leading to a slightly different model for $\mathrm{GSO}(V_4)$.

Finally, in dimension 5 we will describe a realization of $\mathrm{SO}(3, 2)$, the special orthogonal group of a 5-dimensional quadratic space (V, q) of Witt index 2. Although the isomorphism class of such a quadratic space depends on $\det(V)$, the group $\mathrm{SO}(V, q)$ does not. We describe a model V_5 of such a quadratic space with determinant 1 (modulo $F^{\times,2}$). Namely, start considering the 4-dimensional space F^4 of column vectors, on which $\mathrm{GSp}_2 \subset \mathrm{GL}_4$ acts on the left. Let $e_1 = {}^t(1, 0, 0, 0)$, $\dots$, $e_4 = {}^t(0, 0, 0, 1)$ denote the standard basis on F^4, and equip $\tilde{V} := \wedge^2 F^4$ with the non-degenerate symmetric bilinear form $(\, , \,)$ defined by the rule

$$x \wedge y = (x, y) \cdot (e_1 \wedge e_2 \wedge e_3 \wedge e_4), \quad \text{for all } x, y \in \tilde{V}.$$

Set $x_0 := e_1 \wedge e_3 + e_2 \wedge e_4$, and define the 5-dimensional subspace $V_5 \subset \tilde{V}$ to be the orthogonal complement of the span of x_0, i.e.

$$V_5 := \{x \in \tilde{V} : (x, x_0) = 0\}.$$

Then the homomorphism $\tilde{\rho} : \mathrm{GSp}_2 \to \mathrm{SO}(\tilde{V})$ given by $\tilde{\rho}(h) = \nu(h)^{-1} \wedge^2 (h)$ satisfies $\tilde{\rho}(h)x_0 = x_0$, and therefore induces an exact sequence

$$(4) \qquad 1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GSp}_2 \xrightarrow{\rho} \mathrm{SO}(V_5) \longrightarrow 1,$$

where $\iota(a) = a\mathbf{1}_4$ for $a \in \mathbb{G}_m$. This short exact sequence induces an isomorphism $\mathrm{PGSp}_2 \simeq \mathrm{SO}(V_5)$.

We fix an identification of V_5 with the 5-dimensional space F^5 of column vectors by

$$\sum_{i=1}^{5} x_i v_i \longmapsto {}^t(x_1, x_2, x_3, x_4, x_5),$$

where $v_1 = e_2 \wedge e_1,\ v_2 = e_1 \wedge e_4,\ v_3 = e_1 \wedge e_3 - e_2 \wedge e_4,\ v_4 = e_2 \wedge e_3,\ v_5 = e_3 \wedge e_4$. Upon this identification, we consider the non-degenerate bilinear symmetric form $(\ ,\ )$ on V defined by $(x, y) = {}^t x Q y$ for $x, y \in F^5$, where

$$Q = \begin{pmatrix} & & -1 \\ & Q_1 & \\ -1 & & \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We shall distinguish the 3-dimensional subspace $V_3 \subset V_5$ spanned by v_2, v_3, v_4, equipped with the bilinear form $(x, y) = {}^t x Q_1 y$, for $x, y \in F^3$, under the identification $V_3 = F^3$ induced by restricting the above one for $V = F^5$. Notice that $V_5 = \langle v_1 \rangle \oplus V_3 \oplus \langle -v_5 \rangle$, where v_1 and $-v_5$ are isotropic vectors with $(v_1, -v_5) = 1$, and V_3 is the orthogonal complement of $\langle v_1, -v_5 \rangle = \langle v_1, v_5 \rangle$.

Also, we shall distinguish a 4-dimensional subspace of V_5. Indeed, the subspace $\{x \in V : (x, v_3) = 0\} = \langle v_3 \rangle^{\perp} \subset V_5$ is a quadratic 4-dimensional subspace of V_5, and it can be identified with the space V_4 defined above by means of the linear map

$$\langle v_3 \rangle^{\perp} \longrightarrow V_4, \quad x_1 v_2 + x_2 v_1 + x_3 v_5 + x_4 v_4 \longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

By restricting the homomorphism ρ from the exact sequence in (3) to

$$G(\mathrm{SL}_2 \times \mathrm{SL}_2)^- := \{(h_1, h_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(h_1)\det(h_2) = 1\} \subseteq \mathrm{GL}_2 \times \mathrm{GL}_2,$$

one gets an exact sequence

$$(5) \qquad 1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} G(\mathrm{SL}_2 \times \mathrm{SL}_2)^- \xrightarrow{\rho} \mathrm{SO}(V_4) \longrightarrow 1.$$

Now notice that $G(\mathrm{SL}_2 \times \mathrm{SL}_2)^-$ is isomorphic to

$$G(\mathrm{SL}_2 \times \mathrm{SL}_2) := \{(h_1, h_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(h_1)\det(h_2)^{-1} = 1\} \subseteq \mathrm{GL}_2 \times \mathrm{GL}_2$$

through the morphism $(h_1, h_2) \mapsto (h_1, \det(h_2)^{-1} h_2)$. Composing this isomorphism with the natural embedding $G(\mathrm{SL}_2 \times \mathrm{SL}_2) \hookrightarrow \mathrm{GSp}_2$ given by

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix},$$

one gets a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota} & G(\mathrm{SL}_2 \times \mathrm{SL}_2)^- & \xrightarrow{\rho} & \mathrm{SO}(V_4) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota} & \mathrm{GSp}_2 & \xrightarrow{\rho} & \mathrm{SO}(V_5) & \longrightarrow & 1 \end{array}$$

and hence an embedding $\mathrm{SO}(V_4) \subset \mathrm{SO}(V_5)$. This embedding will be of crucial interest later on.

3.3. **Weil representations.** Let now F be a local field with $\mathrm{char}(F) \neq 2$ (for the purposes of this paper, we can think of F being $\mathbb{Q}_v$ for a rational place v), and (V, Q) be a quadratic space over F of dimension m as above. Let $\mathcal{S}(V)$ denote the space of locally constant and compactly supported complex-valued functions on V. This is usually referred to as the space of Bruhat–Schwartz functions on V. If F is archimedean, we rather consider $\mathcal{S}(V)$ to be the Fock model (which is a smaller subspace, see [YZZ13, Section 2.1.2]).

We fix a non-trivial additive character ψ of F. The Weil representation $\omega_{\psi,V}$ of $\widetilde{\mathrm{SL}}_2(F) \times \mathrm{O}(V)$ on $\mathcal{S}(V)$, which depends on the choice of the character ψ, is given by the following formulae. If $a \in F^{\times}$, $b \in F$, $h \in \mathrm{O}(V)$, and $\phi \in \mathcal{S}(V)$, then

$$\begin{aligned} \omega_{\psi,V}(h)\phi(x) &= \phi(h^{-1}x), \\ \omega_{\psi,V}\left([t(a),\epsilon]\right)\phi(x) &= \epsilon^m \chi_{\psi,V}(a)|a|^{m/2}\phi(ax) \\ \omega_{\psi,V}\left([u(b),1]\right)\phi(x) &= \psi(Q(x)b)\phi(x), \\ \omega_{\psi,V}\left([s,1]\right)\phi(x) &= \gamma(\psi,V)\int_V \phi(y)\psi((x,y))dy. \end{aligned}$$

Here, $\gamma(\psi, V)$ is the Weil index, which is an 8-th root of unity, and $\chi_{\psi,V} : F^\times \to S^1$ is a function satisfying $\chi_{\psi,V}(ab) = (a,b)_F^m \chi_{\psi,V}(a)\chi_{\psi,V}(b)$ for $a, b \in F^\times$, where $(\cdot,\cdot)_F$ denotes the Hilbert symbol. When $m = 1$ and $Q(x) = x^2$, we will simply write ω_ψ, χ_ψ, and $\gamma(\psi)$ for $\omega_{\psi,V}, \chi_{\psi,V}$, and $\gamma(\psi, V)$, respectively. In this case, the function χ_ψ can be written as

$$\chi_\psi(a) = (a,-1)_F\gamma(a,\psi) = (a,-1)_F\frac{\gamma(\psi^a)}{\gamma(\psi)},$$

where the function $\gamma(\cdot,\psi) : F^\times \to S^1$ is defined by $\gamma(a,\psi) = \gamma(\psi^a)/\gamma(\psi)$ and satisfies

$$\gamma(ab,\psi) = (a,b)_p\gamma(a,\psi)\gamma(b,\psi), \qquad \gamma(ab^2,\psi) = \gamma(a,\psi) \quad \text{for all } a,b \in F^\times.$$

Thus $\chi_\psi(ab) = (a,b)_p\chi_\psi(a)\chi_\psi(b)$ and $\chi_\psi(ab^2) = \chi_\psi(a)$ for all $a, b \in F^\times$, and furthermore $\chi_{\psi^a} = \chi_\psi \cdot \chi_a$, where χ_a stands for the quadratic character $(a,\cdot)_F$.

For the standard additive character ψ_p of $F = \mathbb{Q}_p$, with p an *odd* prime, one has $\gamma(\psi_p) = 1$ and

$$\gamma(a,\psi_p) = 1 \text{ for all } a \in \mathbb{Z}_p^\times, \qquad \gamma(p,\psi_p) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -\sqrt{-1} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This completely determines the functions $\gamma(\cdot,\psi_p)$ and χ_{ψ_p} by the above properties. One can easily deduce similar formulae for twists ψ_p^d of the standard additive character.

For a general quadratic space V, if $Q(x) = a_1x_1^2 + \cdots + a_mx_m^2$ with respect to some orthogonal basis, then

$$\gamma(\psi,V) = \prod_i \gamma(\psi^{a_i}) \quad \text{and} \quad \chi_{\psi,V} = \prod_i \chi_{\psi^{a_i}}.$$

This does not depend on the chosen basis. For example, consider the 3-dimensional quadratic space V_3 as before, with quadratic form whose matrix is Q_1. The eigenvalues of this matrix are 1, -1, and 2, thus $\gamma(\psi, V_3) = \gamma(\psi)\gamma(\psi^{-1})\gamma(\psi^2)$. If $\psi = \psi_p^a$ for some unit $a \in \mathbb{Z}_p^\times$, this yields $\gamma(\psi, V_3) = \gamma(\psi_p)^3 = 1$. Besides, we also have $\chi_{\psi,V_3} = \chi_\psi \cdot \chi_{\psi^{-1}} \cdot \chi_{\psi^2} = \chi_\psi^3 \cdot \chi_{-2}$.

When m is even, the above simplifies considerably. Indeed, if m is even the Weil representation descends to a representation of $\mathrm{SL}_2(F) \times \mathrm{O}(V)$ on $\mathcal{S}(V)$. Further, the Weil index $\gamma(\psi, V)$ is a 4-th root of unity in this case, and $\chi_{\psi,V}$ becomes the quadratic character associated with the quadratic space (V,Q). This means that

$$\chi_{\psi,V}(a) = (a, (-1)^{m/2}\det(V))_F, \quad a \in F^\times.$$

It will be useful in some settings to extend the Weil representation $\omega_{\psi,V}$. If m is even, one defines

$$R = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V)) = \{(g,h) \in \mathrm{GL}_2 \times \mathrm{GO}(V) : \det(g) = \nu(h)\},$$

and then $\omega_{\psi,V}$ extends to a representation of $R(F)$ on $\mathcal{S}(V)$ by setting

$$\omega_{\psi,V}(g,h)\phi = \omega_{\psi,V}\left(g\begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1}\end{pmatrix}, 1\right)L(h)\phi \quad \text{for } (g,h) \in R(F) \text{ and } \phi \in \mathcal{S}(V),$$

where $L(h)\phi(x) = |\nu(h)|_F^{-m/4}\phi(h^{-1}x)$ for $x \in V$.

3.4. **Theta functions and theta lifts.** Now let F be a number field (for our purposes, we can think of $F = \mathbb{Q}$), and consider a quadratic space V over F of dimension m. Fix a non-trivial additive character ψ of $\mathbb{A}_F/F$ and let $\omega = \omega_{\psi,V}$ be the Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$ on $\mathcal{S}(V(\mathbb{A}_F))$ with respect to ψ. Given $(g,h) \in \widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$ and $\phi \in \mathcal{S}(V(\mathbb{A}_F))$, let

$$\theta(g,h;\phi) := \sum_{x \in V(F)} \omega(g,h)\phi(x).$$

Then $(g,h) \mapsto \theta(g,h;\phi)$ defines an automorphic form on $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$, called a *theta function*. When m is even, this may be regarded as an automorphic form on $\mathrm{SL}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$.

Let f be a cusp form on $\mathrm{SL}_2(\mathbb{A}_F)$ if m is even, and a genuine cusp form on $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)$ if m is odd. If $\phi \in \mathcal{S}(V(\mathbb{A}_F))$, put

$$\theta(h;f,\phi) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \theta(g,h;\phi)f(g)dg, \quad h \in \mathrm{O}(V)(\mathbb{A}_F).$$

Then $\theta(f,\phi) : h \mapsto \theta(h; f, \phi)$ defines an automorphic form on $\mathrm{O}(V)(\mathbb{A}_F)$. If m is even and π is an irreducible cuspidal automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$, or if m is odd and π is an irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)$, put

$$\Theta_{\widetilde{\mathrm{SL}}_2\times \mathrm{O}(V)}(\pi) := \{\theta(f,\phi) : f \in \pi, \phi \in \mathcal{S}(V(\mathbb{A}_F))\}.$$

Then $\Theta_{\widetilde{\mathrm{SL}}_2\times \mathrm{O}(V)}(\pi)$ is an automorphic representation of $\mathrm{O}(V)(\mathbb{A}_F)$, called the *theta lift of* π. Going in the opposite direction, one defines similarly the theta lift $\theta(f',\phi)$ of a cusp form f' on $\mathrm{O}(V)(\mathbb{A}_F)$ and the theta lift $\Theta_{\mathrm{O}(V)\times\widetilde{\mathrm{SL}}_2}(\pi')$ of an irreducible cuspidal automorphic representation π' of $\mathrm{O}(V)(\mathbb{A})$.

Suppose that m is even. As we did for the Weil representation, theta lifts can also be extended. If $(g,h) \in R(\mathbb{A}_F)$ and $\phi \in \mathcal{S}(V(\mathbb{A}_F))$, one defines $\theta(g,h;\phi)$ via the same expression as above (using the extended Weil representation). Then, if f is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$ and $h \in \mathrm{GO}(V)(\mathbb{A}_F)$, choose $g' \in \mathrm{GL}_2(\mathbb{A}_F)$ with $\det(g') = \nu(h)$ and set

$$\theta(h; f, \phi) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \theta(gg', h; \phi) f(gg') dg.$$

The integral does not depend on the choice of the auxiliary element g', and $\theta(f,\phi) : h \mapsto \theta(h; f, \phi)$ defines now an automorphic form on $\mathrm{GO}(V)(\mathbb{A}_F)$. If π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, then its theta lift $\Theta_{\mathrm{GL}_2\times\mathrm{GO}(V)}(\pi)$ is formally defined exactly as before (and the same applies for $\Theta_{\mathrm{GO}(V)\times\mathrm{GL}_2}(\pi')$ if π' is an irreducible cuspidal automorphic representation of $\mathrm{GO}(V)$).

4. SL_2-periods and a central value formula

Let $f \in S_{2k}^{new}(N_f)$ and $g \in S_{\ell+1}^{new}(N_g)$ be two normalized newforms as in the Introduction. Thus $\ell \geq k \geq 1$ are odd integers, and $N_f, N_g \geq 1$ are odd squarefree integers with $N_g \mid N_f$. We let $m \in \mathbb{Z}$ be such that $\ell - k = 2m$.

Let π and τ denote the irreducible cuspidal automorphic representation associated with f and g, respectively. These are automorphic representations of $\mathrm{PGL}_2(\mathbb{A})$, although we will often regard them as automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character.

Let $\psi : \mathbb{A}/\mathbb{Q} \to S^1$ be the standard additive character of $\mathbb{A}$, and $\overline{\psi}$ be its twist by -1. Fix a fundamental discriminant $D < 0$ such that $\chi_D(p) = w_p$ for all primes $p \mid N_f$, where w_p denotes the eigenvalue of the p-th Atkin–Lehner involution acting on f, and consider the automorphic representation $\tilde{\pi} := \Theta(\pi \otimes \chi_D, \overline{\psi}^D)$. The assumptions on D guarantee that $\tilde{\pi} \neq 0$, and hence it belongs to the so-called *(global) Waldspurger packet*

$$\mathrm{Wald}_{\overline{\psi}}(\pi) = \{\text{non-zero } \Theta(\pi \otimes \chi_a, \overline{\psi}^a) : a \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2\} = \mathrm{Wald}_{\overline{\psi}^D}(\pi \otimes \chi_D).$$

Waldspurger's theory (see [Wal91]) tells us that the set $\mathrm{Wald}_{\overline{\psi}}(\pi)$ is finite. Further, these global packets can be conveniently described by means of *local* Waldspurger packets. Namely, let v be a rational place and B_v be the quaternion division algebra over $\mathbb{Q}_v$. Set $\tilde{\pi}_v^+ = \Theta(\pi_v, \overline{\psi}_v)$ and $\tilde{\pi}_v^- = \Theta(\mathrm{JL}(\pi_v), \overline{\psi}_v)$, where $\mathrm{JL}(\pi_v)$ is the Jacquet–Langlands lift of π_v to $PB_v^\times$. Then the local Waldspurger packet $\mathrm{Wald}_{\overline{\psi}_v}(\pi_v)$ is defined as the singleton $\{\tilde{\pi}_v^+\}$ if π_v is not square-integrable, and as the set $\{\tilde{\pi}_v^+, \tilde{\pi}_v^-\}$ if π_v is square-integrable. If $\epsilon = (\epsilon_v)_v$ is a collection of signs $\epsilon_v \in \{\pm 1\}$, one for each rational place, such that $\epsilon_v = +1$ whenever π_v is not square-integrable (or equivalently, for each $\epsilon \in \{\pm1\}^{|\Sigma(\pi)|}$, where $\Sigma(\pi)$ is the set of rational places where π is square-integrable), we set $\tilde{\pi}^\epsilon = \otimes \tilde{\pi}_v^{\epsilon_v}$. Then

$$\mathrm{Wald}_{\overline{\psi}}(\pi) = \{\tilde{\pi}^\epsilon : \prod_v \epsilon_v = \epsilon(1/2, \pi)\}.$$

The labelling $\pm$ of a given element in $\mathrm{Wald}_{\overline{\psi}}(\pi)$ at each place $v \in \Sigma(\pi)$ depends on the choice of the additive character. For the representation $\tilde{\pi} = \Theta(\pi \otimes \chi_D, \overline{\psi}_D)$, we have $\epsilon_\infty = -1$ and $\epsilon_p = \chi_D(p) = w_p$ for each prime $p \mid N_f$.

We let $h \in S_{k+1/2}^{+,new}(N_f)$ be any (non-zero) newform in Shimura–Shintani correspondence with f. Then the adelization of h belongs to $\tilde{\pi}$, and h is unique up to non-zero multiples. We let also $F = \mathrm{SK}(h) \in S_{k+1}(\Gamma_0^{(2)}(N_f))$ be the Saito–Kurokawa lift of h, and Π be the automorphic representation of $\mathrm{PGSp}_2(\mathbb{A})$ associated with it. The Siegel modular form F admits a nice Fourier expansion

$$F(Z) = \sum_B A_F(B) e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}, \quad Z = X + \sqrt{-1}Y \in \mathcal{H}_2,$$

where B runs over the half-integral, positive definite symmetric two-by-to matrices, and $A_F(B)$ is given in an elementary way in terms of the Fourier coefficients of h (see (25)).

For each integer $\kappa \geq 1$, consider the classical Maass differential operator (see (27) below for the precise definition)

$$\Delta_\kappa : S_\kappa^{nh}(\Gamma_0^{(2)}(N_f)) \longrightarrow S_{\kappa+2}^{nh}(\Gamma_0^{(2)}(N_f))$$

sending nearly holomorphic Siegel forms of weight κ (and level $\Gamma_0^{(2)}(N_f)$) to nearly holomorphic Siegel forms of weight $\kappa+2$ (and level $\Gamma_0^{(2)}(N_f)$). By applying $\Delta_{k+1}^m := \Delta_{\ell-1} \circ \Delta_{\ell-3} \circ \cdots \circ \Delta_{k+1}$ to F, one obtains a nearly holomorphic Siegel form

$$\Delta_{k+1}^m F \in S_{\ell+1}^{nh}(\Gamma_0^{(2)}(N_f))$$

of weight $\ell+1$ and level $\Gamma_0^{(2)}(N_f)$. By using the definition of the Maass differential operator, one shows that the Fourier expansion of $\Delta_{k+1}^m F$ is expressed as

$$\Delta_{k+1}^m F(Z) = \sum_B A_F(B) C(B,Y) e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)},$$

where $C(B,Y)$ can be written down explicitly by an induction argument (see (28)).

Theorem 4.1. *With the above notation, suppose that $w_p = 1$ for each prime dividing $M_g := N_f/N_g$. Then*

$$\Lambda(f \otimes \mathrm{Ad}(g), k) = 2^{6m+k+1-\nu(M_g)} C_0(f,g) C_\infty(f,g) \cdot \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle \breve{F}_{|\mathcal{H}\times\mathcal{H}}, g \times g\rangle|^2}{\langle g, g\rangle^2},$$

where $\breve{F} \in S_{\ell+1}^{nh}(\Gamma_0^{(2)}(N_f))$ is a Siegel modular form closely related to $\Delta_{k+1}^m F$ defined explicitly in Proposition 7.9, $\breve{F}_{|\mathcal{H}\times\mathcal{H}}$ denotes its restriction or pullback to $\mathcal{H}\times\mathcal{H} \subset \mathcal{H}_2$, and the constants $C_0(f,g)$, $C_\infty(f,g) \in \mathbb{Q}^\times$ are

$$C_0(f,g) = \frac{M_g^2 \mu_{N_g}^2}{N_f} = \frac{M_g^2}{N_f} \prod_{p|N_g} (p+1)^2,$$

$$C_\infty(f,g) = \frac{(2m)!}{m!} \frac{(k+m-1)!}{(\ell-1)!} \sum_{\substack{0\leq s\leq 2m,\\ s \text{ even}}} \prod_{\substack{0\leq j\leq s-2,\\ j \text{ even}}} \frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)}.$$

The strategy to prove the central value formula in this theorem is the same as in [PdVP19]. Indeed, let $\omega = \omega_{\overline{\psi}}$ denote the Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ acting on the space $\mathcal{S}(\mathbb{A})$ of Bruhat–Schwartz functions (for the one dimensional quadratic space endowed with bilinear form $(x,y) = 2xy$) with respect to the additive character $\overline{\psi}$ (note that $\tilde{\pi}$ belongs to $\mathrm{Wald}_{\overline{\psi}}(\pi)$). Associated with $\tilde{\pi}$, τ and ω, there is a (global) SL$_2$-period functional

$$\mathcal{Q} : \tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega \otimes \omega \longrightarrow \mathbb{C}$$

defined by associating to each choice of decomposable vectors $\mathbf{h}_1, \mathbf{h}_2 \in \tilde{\pi}$, $\mathbf{g}_1, \mathbf{g}_2 \in \tau$, $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \omega$, the product of integrals

$$\mathcal{Q}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) := \left(\int_{[\mathrm{SL}_2]} \overline{\mathbf{h}_1(g)} \mathbf{g}_1(g) \Theta_{\boldsymbol{\phi}_1}(g) dg\right) \cdot \overline{\left(\int_{[\mathrm{SL}_2]} \overline{\mathbf{h}_2(g)} \mathbf{g}_2(g) \Theta_{\boldsymbol{\phi}_2}(g) dg\right)}.$$

Let us assume for now that the global SL$_2$-period functional $\mathcal{Q}$ is non-vanishing (which is true under the assumptions of Theorem 4.1, see Proposition 4.2 and Corollary 4.3 below). Then, we know by [Qiu14, Theorem 4.5] that $\mathcal{Q}$ decomposes as a product of local SL$_2$-periods up to certain L-values. Namely, one has

$$\mathcal{Q}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = \frac{1}{4} \frac{\Lambda(\pi \otimes \mathrm{ad}(\tau), 1/2)}{\Lambda(1, \pi, \mathrm{ad})\Lambda(1, \tau, \mathrm{ad})} \prod_v \mathcal{I}_v(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2), \tag{6}$$

where for each rational place v, the local period $\mathcal{I}_v(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ is defined by integrating a product of matrix coefficients, and equals

$$\frac{L(1, \pi_v, \mathrm{ad}) L(1, \tau_v, \mathrm{ad})}{L(\pi_v \otimes \mathrm{ad}(\tau_v), 1/2)} \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \overline{\langle \tilde{\pi}(g_v) \mathbf{h}_{1,v}, \mathbf{h}_{2,v}\rangle} \langle \tau(g_v) \mathbf{g}_{1,v}, \mathbf{g}_{2,v}\rangle \langle \omega_v(g_v) \boldsymbol{\phi}_{1,v}, \boldsymbol{\phi}_{2,v}\rangle dg_v.$$

Now, the L-value $\Lambda(\pi \otimes \mathrm{ad}(\tau), 1/2)$ coincides with the central value $\Lambda(f \otimes \mathrm{Ad}(g), k)$ in the above theorem. Thus, Qiu's decomposition formula provides a way to compute this central value, by finding a *test vector* at which the global period does not vanish, and then computing both the global period and all the corresponding local periods evaluated at this test vector.

Let us elaborate a bit on formula (6) above, writing $\mathcal{Q}(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) := \mathcal{Q}(\breve{\mathbf{h}}, \breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}, \breve{\boldsymbol{\phi}})$ for each pure tensor $\breve{\mathbf{h}} \otimes \breve{\mathbf{g}} \otimes \breve{\boldsymbol{\phi}} \in \tilde{\pi} \otimes \tau \otimes \omega$, and using similar conventions with the local periods. Setting

$$\mathcal{I}_v^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) := \frac{\mathcal{I}_v(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})}{\langle \breve{\mathbf{h}}_v, \breve{\mathbf{h}}_v\rangle \langle \breve{\mathbf{g}}_v, \breve{\mathbf{g}}_v\rangle \langle \breve{\boldsymbol{\phi}}_v, \breve{\boldsymbol{\phi}}_v\rangle} = \frac{\mathcal{I}_v(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})}{||\breve{\mathbf{h}}_v||^2 ||\breve{\mathbf{g}}_v||^2 ||\breve{\boldsymbol{\phi}}_v||^2},$$

one has

$$\mathcal{I}_v^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \frac{L(1, \pi_v, \mathrm{ad}) L(1, \tau_v, \mathrm{ad})}{L(\pi_v \otimes \mathrm{ad}(\tau_v), 1/2)} \alpha_v^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}), \tag{7}$$

with

$$\alpha_v^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) := \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \frac{\overline{\langle\tilde\pi(g_v)\breve{\mathbf{h}}_v,\breve{\mathbf{h}}_v\rangle}}{||\breve{\mathbf{h}}_v||^2}\frac{\langle\tau(g_v)\breve{\mathbf{g}}_v,\breve{\mathbf{g}}_v\rangle}{||\breve{\mathbf{g}}_v||^2}\frac{\langle\omega_{\overline{\psi}_v}(g_v)\breve{\boldsymbol{\phi}}_v,\breve{\boldsymbol{\phi}}_v\rangle}{||\breve{\boldsymbol{\phi}}_v||^2}dg_v. \tag{8}$$

If $\breve{\mathbf{h}}$, $\breve{\mathbf{g}}$ and $\breve{\boldsymbol{\phi}}$ are chosen so that $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ is non-zero, then we deduce from (6) that

$$\Lambda(f\otimes\mathrm{Ad}(g),k) = \frac{4\Lambda(1,\pi,\mathrm{ad})\Lambda(1,\tau,\mathrm{ad})}{\langle\breve{\mathbf{h}},\breve{\mathbf{h}}\rangle\langle\breve{\mathbf{g}},\breve{\mathbf{g}}\rangle\langle\breve{\boldsymbol{\phi}},\breve{\boldsymbol{\phi}}\rangle}\left(\prod_v \mathcal{I}_v^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})^{-1}\right)\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}). \tag{9}$$

We will choose a suitable *test vector* $\breve{\mathbf{h}}\otimes\breve{\mathbf{g}}\otimes\breve{\boldsymbol{\phi}}\in\tilde\pi\otimes\tau\otimes\omega$ such that $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})\neq 0$, and we will compute the terms on the right hand side of the above expression to obtain the central value formula claimed in the theorem. By virtue of a comparison theorem between the global SL_2-period $\mathcal{Q}$ and a global $\mathrm{SO}(4)$-period due to Qiu (see [Qiu14, Theorem 5.4], or Section 7.3 below), the global contribution $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ is the responsible of the term $|\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}}, g\times V_{M_g}g\rangle|^2/\langle g,g\rangle^2$. Hence, the proof of Theorem 4.1 follows by making explicit the right hand side of (9).

For the sketched strategy to work, it is essential that the SL_2-period $\mathcal{Q}$ is non-vanishing. A criterion for this is proved in [Qiu14, Proposition 4.1] (see also [GG09, Theorem 7.1]):

Proposition 4.2. *The functional $\mathcal{Q}$ is non-zero if and only if the following conditions hold:*

i) $\Lambda(\pi\otimes\mathrm{ad}(\tau),1/2)\neq 0$*;*
ii) $\tilde\pi=\tilde\pi^\epsilon$ *with* $\epsilon_v=\epsilon(1/2,\pi_v\otimes\tau_v\otimes\tau_v^\vee)$*;*
iii) $\epsilon(1/2,\pi_v\otimes\tau_v\otimes\tau_v^\vee)=1$ *whenever* π_v *is not square-integrable.*

In our current setting, the non-vanishing of $\mathcal{Q}$ is equivalent to the non-vanishing of $\Lambda(f\otimes\mathrm{Ad}(g),k)$:

Corollary 4.3. *With the same assumptions as in Theorem 4.1, the functional $\mathcal{Q}$ is non-zero if and only if* $\Lambda(f\otimes\mathrm{Ad}(g),k)\neq 0$.

Proof. Condition iii) in the above proposition clearly holds, thus we may prove that ii) is satisfied under the sign assumptions made in Theorem 4.1. We only need to consider places $v\mid N_f\infty$. At $v=\infty$, we have $\epsilon_\infty=-1$ and also $\epsilon(1/2,\pi_v\otimes\tau_v\otimes\tau_v^\vee)$ because of our choice of weights with $\ell\geq k$. Let p be a prime dividing N_g. Then both π_p and τ_p are (quadratic twists of) Steinberg representations, and in this case [Pra90, Proposition 8.6] implies that $\epsilon(1/2,\pi_p\otimes\tau_p\otimes\tau_p^\vee)=\epsilon(1/2,\pi_p)=w_p$, which agrees with $\chi_D(p)=\epsilon_p$. At primes $p\mid N_f/N_g$, the representation τ_p is an unramified principal series instead, and [Pra90, Proposition 8.4] tells us that $\epsilon(1/2,\pi_p\otimes\tau_p\otimes\tau_p^\vee)=1$. Since we assume that $\chi_D(p)=w_p=1$ at such primes, we see that this coincides with ϵ_p, and hence condition ii) in the previous proposition holds. □

5. Choice of the test vector

We keep the notation and assumptions as in the previous section, and proceed now to describe our choice of *test vector*

$$\breve{\mathbf{h}}\otimes\breve{\mathbf{g}}\otimes\breve{\boldsymbol{\phi}}\in\tilde\pi\otimes\tau\otimes\omega$$

that will be used to prove Theorem 4.1 following the already explained strategy.

To begin with, let us describe the Bruhat–Schwartz function $\breve{\boldsymbol{\phi}}=\otimes_v\breve{\boldsymbol{\phi}}_v\in\mathcal{S}(\mathbb{A})$. Recall that we regard $\mathcal{S}(\mathbb{A})$ as the space of Bruhat–Schwartz functions on the one-dimensional quadratic space endowed with quadratic form $Q(x)=x^2$. Our choice $\breve{\boldsymbol{\phi}}$ is determined by its local components, which are defined as follows:

i) if $v=p$ is a prime, then we let $\breve{\boldsymbol{\phi}}_p=\mathbf{1}_{\mathbb{Z}_p}$ be the characteristic function of $\mathbb{Z}_p$ in the space $\mathcal{S}(\mathbb{Q}_p)$ of Bruhat–Schwartz functions;
ii) at the archimedean place $v=\infty$, we define $\breve{\boldsymbol{\phi}}_\infty$ by setting $\breve{\boldsymbol{\phi}}_\infty(x)=e^{-2\pi x^2}$ for all $x\in\mathbb{R}$.

Lemma 5.1. *For each rational prime p, $\breve{\boldsymbol{\phi}}_p$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z}_p)\subseteq\mathrm{SL}_2(\mathbb{Q}_p)$, and in addition $||\breve{\boldsymbol{\phi}}_p||^2=\langle\breve{\boldsymbol{\phi}}_p,\breve{\boldsymbol{\phi}}_p\rangle=1$. At the archimedean place, one has $||\breve{\boldsymbol{\phi}}_\infty||^2=2^{-1}$, hence also $||\breve{\boldsymbol{\phi}}||^2=2^{-1}$.*

Proof. The invariance assertion follows easily from the definitions. If p is a prime, then

$$||\breve{\boldsymbol{\phi}}_p||^2=\langle\breve{\boldsymbol{\phi}}_p,\breve{\boldsymbol{\phi}}_p\rangle=\int_{\mathbb{Q}_p}\breve{\boldsymbol{\phi}}_p(x)\overline{\breve{\boldsymbol{\phi}}_p(x)}dx=\mathrm{vol}(\mathbb{Z}_p)=1.$$

And besides, $||\breve{\boldsymbol{\phi}}_\infty||^2=\int_{\mathbb{R}}e^{-4\pi x^2}dx=1/2$. □

Now let us describe our choice for $\breve{\mathbf{h}}\in\tilde\pi$ and $\breve{\mathbf{g}}\in\tau$. To do so, let $\mathbf{h}=\otimes_v\mathbf{h}_v\in\tilde\pi$ and $\mathbf{g}=\otimes_v\mathbf{g}_v\in\tau$ denote the adelizations of the cuspidal forms $h\in S^+_{k+1/2}(N_f)$ and $g\in S_{\ell+1}(N_g)$, respectively. At each rational prime $p\nmid N_f$ (resp. $p\nmid N_g$), the local component $\mathbf{h}_p$ (resp. $\mathbf{g}_p$) is an unramified or spherical vector in the local representation $\tilde\pi_p$ (resp. τ_p). These are unique up to scalar multiples. If instead p is a prime dividing N_f (resp. N_g), then $\mathbf{h}_p$ (resp. $\mathbf{g}_p$) is a newvector in $\tilde\pi_p$ (resp. τ_p) fixed under the action of $\tilde\Gamma_0(p)$ (resp. $K_0(p)$).

Such local newforms are also unique up to scalar multiples. At the archimedean place, τ_∞ is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of weight $\ell+1$, and $\mathbf{g}_\infty$ is a lowest weight vector in τ_∞. Similarly, $\tilde{\pi}_\infty$ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest $\widetilde{\mathrm{SO}}(2)$-type $k+1/2$, and $\mathbf{h}_\infty$ is a lowest weight vector in $\tilde{\pi}_\infty$. Again, such lowest weight vectors are uniquely determined up to multiples. We will define $\breve{\mathbf{h}} = \otimes_v \breve{\mathbf{h}}_v$ and $\breve{\mathbf{g}} = \otimes_v \breve{\mathbf{g}}_v$ by describing their local components at each place v, according to the following cases:

(1) $v = p$ is a prime not dividing N_f;
(2) $v = p$ is a prime dividing N_g;
(3) $v = p$ is a prime dividing $M_g = N_f/N_g$;
(4) $v = \infty$ is the archimedean place.

5.1. **Primes not dividing N_f.** If p is a prime not dividing $2N_f$, then both $\tilde{\pi}_p$ and τ_p are unramified principal series representations, of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and $\mathrm{PGL}_2(\mathbb{Q}_p)$ respectively. At such primes, we choose both $\breve{\mathbf{h}}_p = \mathbf{h}_p$ and $\breve{\mathbf{g}}_p = \mathbf{g}_p$ to be an unramified (or spherical) vector in $\tilde{\pi}_p$ and τ_p, respectively. At $p=2$, we adopt the same choice as the one explained in [PdVP19, Section 9]: we let $\breve{\mathbf{g}}_2 = \mathbf{g}_2^\sharp := \tau_2(t(2)^{-1})\mathbf{g}_2$ and $\breve{\mathbf{h}}_2 = \tilde{\pi}_2(t(2))\mathbf{h}_2$, where $t(2) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2^{-1}\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}_2)$.

5.2. **Primes dividing N_g.** Let p be a prime dividing N_g. By our assumption that N_g is squarefree, τ_p is a twist of the Steinberg representation by an unramified quadratic character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. That is to say, τ_p is the unique irreducible subrepresentation of the induced representation $\pi(\chi,\chi^{-1})$. The subspace $\tau_p^{K_0} \subseteq \tau_p$ of vectors fixed by

$$K_0 = K_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p} \right\}$$

is one-dimensional, and it is generated by the newvector $\mathbf{g}_p : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ in the induced model characterized by the property that

$$\mathbf{g}_{p|\mathrm{GL}_2(\mathbb{Z}_p)} = \mathbf{1}_{K_0} - \frac{1}{p}\mathbf{1}_{K_0 w K_0},$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. We choose the p-th component of $\breve{\mathbf{g}}$ to be this newvector: $\breve{\mathbf{g}}_p = \mathbf{g}_p$.

As for $\tilde{\pi}_p$, observe first that π_p is also a twist of the Steinberg representation by an unramified quadratic character. Then, by our choice of $\tilde{\pi} = \Theta(\pi \otimes \chi_D, \overline{\psi}^D)$ in the Waldspurger packet $\mathrm{Wald}_{\overline{\psi}}(\pi) = \mathrm{Wald}_{\overline{\psi}^D}(\pi \otimes \chi_D)$, the local representation $\tilde{\pi}_p = \Theta(\pi_p \otimes \chi_D, \overline{\psi}_p^D)$ is the special representation $\tilde{\sigma}^\delta(\overline{\psi}_p^D)$ of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$, where $\delta \in \mathbb{Z}_p^\times$ is any non-quadratic residue. This representation is realized as the space of functions $\tilde{\varphi} : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \to \mathbb{C}$ such that

$$\tilde{\varphi}\left(\left[\begin{pmatrix} a & * \\ & a^{-1}\end{pmatrix}, \epsilon\right] g\right) = \epsilon \chi_{\bar{\psi}_p^D}(a)\chi_\delta(a)|a|_p^{3/2}\tilde{\varphi}(g), \tag{10}$$

where $\chi_\delta = (\cdot,\delta)_p$ is the quadratic character associated with $\delta \in \mathbb{Z}_p^\times$ and $\chi_{\bar{\psi}_p^D} = \chi_{\psi_p^{-D}} : \mathbb{Q}_p^\times \to S^1$ is as in Section 3.3. Recall that the fundamental discriminant $D \in \mathbb{Q}^\times$ has been chosen so that $D \in \mathbb{Z}_p^\times$ and $\chi_D(p) = w_p$.

If $\tilde{\Gamma}$ denotes the image of $\Gamma = \Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ into $\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$ under the canonical splitting, then the space of $\tilde{\Gamma}$-fixed vectors in $\tilde{\pi}_p$ is one-dimensional. Moreover, this space is generated by the newvector $\mathbf{h}_p : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \to \mathbb{C}$ characterized by the property that

$$\mathbf{h}_{p|\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)} = \mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)} - (p+1)\mathbf{1}_{\tilde{\Gamma}}.$$

Here, $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)}$ denotes the (genuine) function on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ sending $[g,\epsilon]$ to 0 if $g \notin \mathrm{SL}_2(\mathbb{Z}_p)$, and to $\epsilon s_p(g)$ otherwise. Similarly, $\mathbf{1}_{\tilde{\Gamma}}$ is the function on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ that sends $[g,\epsilon]$ to 0 if $g \notin \Gamma$, and to $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}([g,\epsilon])$ otherwise. We choose the p-th component of $\breve{\mathbf{h}}$ to be this newvector: $\breve{\mathbf{h}}_p = \mathbf{h}_p$.

5.3. **Primes dividing M_g.** Let now p be a prime dividing N_f but not N_g, i.e. p divides M_g. In this case, the local type of $\tilde{\pi}_p$ is as in the previous paragraph, and we continue to choose $\breve{\mathbf{h}}_p = \mathbf{h}_p$ to be the newvector as described there. Besides, τ_p is now the unramified principal representation $\pi(\chi,\chi^{-1})$ associated with an unramified character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. The representation being unitary, we have $\chi^{-1} = \overline{\chi}$. The subspace $\tau_p^{\mathrm{GL}_2(\mathbb{Z}_p)} \subseteq \tau_p$ of $\mathrm{GL}_2(\mathbb{Z}_p)$-fixed vectors is one-dimensional, and generated by the unramified (or spherical) vector $\mathbf{g}_p : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$ characterized by the property that

$$\mathbf{g}_p\left(\left(\begin{smallmatrix} a & * \\ & d\end{smallmatrix}\right) x\right) = \begin{cases} \chi(a)\overline{\chi}(d)|ad^{-1}|_p^{1/2} & \text{if } x \in \mathrm{GL}_2(\mathbb{Z}_p),\ \left(\begin{smallmatrix} a & * \\ & d\end{smallmatrix}\right) \in B(\mathbb{Q}_p) \text{ with } a,d \in \mathbb{Q}_p^\times, \\ 0 & \text{otherwise,}\end{cases} \tag{11}$$

where B denotes the Borel subgroup of GL_2 of upper-triangular matrices. In particular, notice that $\mathbf{g}_p(x) = 1$ for all $x \in \mathrm{GL}_2(\mathbb{Z}_p)$. This gives a well-defined element $\mathbf{g}_p$ by virtue of Iwasawa decomposition for $\mathrm{GL}_2(\mathbb{Z}_p)$. We define the p-th component $\breve{\mathbf{g}}_p$ of $\breve{\mathbf{g}}$ to be the old vector

$$\breve{\mathbf{g}}_p := \mathbf{V}_p\mathbf{g}_p = \tau_p(\varpi_p)\mathbf{g}_p, \quad \text{where } \varpi_p = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p). \tag{12}$$

It is elementary to check that the vector $\breve{\mathbf{g}}_p$ is now fixed by $K_0 = K_0(\mathbb{Z}_p)$ (and it is not fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$), and furthermore we can easily give an explicit description of $\breve{\mathbf{g}}_p(x)$ in terms of $\mathbf{g}_p(x)$ for $x \in \mathrm{GL}_2(\mathbb{Q}_p)$:

Lemma 5.2. *With the above notation,*

$$\breve{\mathbf{g}}_p(x) = \begin{cases} p^{1/2}\overline{\chi}(p)\mathbf{g}_p(x) & \text{if } x \in B(\mathbb{Q}_p)K_0(\mathbb{Z}_p), \\ p^{-1/2}\chi(p)\mathbf{g}_p(x) & \text{if } x \notin B(\mathbb{Q}_p)K_0(\mathbb{Z}_p). \end{cases}$$

Proof. By virtue of Iwasawa decomposition, write $x = \left(\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right)\gamma$, with $a, d \in \mathbb{Q}_p^\times$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z}_p)$. If $\gamma \in K_0$, then $\gamma\varpi_p = \varpi_p\gamma'$ for some $\gamma' \in \mathrm{GL}_2(\mathbb{Z}_p)$, and therefore

$$x\varpi_p = \begin{pmatrix} a & * \\ & d \end{pmatrix}\gamma\varpi_p = \begin{pmatrix} a & * \\ & d \end{pmatrix}\varpi_p\gamma' = \begin{pmatrix} ap^{-1} & * \\ & d \end{pmatrix}\gamma'.$$

By applying (12) and (11), one easily checks that $\breve{\mathbf{g}}_p(x) = p^{1/2}\overline{\chi}(p)\mathbf{g}_p(x)$. The case $\gamma \notin K_0$ follows similarly using again (12) and (11), and one finds $\breve{\mathbf{g}}_p(x) = p^{-1/2}\chi(p)\mathbf{g}_p(x)$. □

Corollary 5.3. *With the above notation, we have* $||\breve{\mathbf{g}}_p||^2 = 1$.

Proof. By using the decomposition $\mathrm{GL}_2(\mathbb{Z}_p) = K_0 \sqcup K_0wK_0$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the previous lemma implies that

$$||\breve{\mathbf{g}}_p||^2 = \langle \breve{\mathbf{g}}_p, \breve{\mathbf{g}}_p\rangle = \int_{\mathrm{GL}_2(\mathbb{Z}_p)} \breve{\mathbf{g}}_p(x)\overline{\breve{\mathbf{g}}_p(x)}dx = \int_{K_0} \breve{\mathbf{g}}_p(x)\overline{\breve{\mathbf{g}}_p(x)}dx + \int_{K_0wK_0} \breve{\mathbf{g}}_p(x)\overline{\breve{\mathbf{g}}_p(x)}dx =$$

$$= p\int_{K_0} \mathbf{g}_p(x)\overline{\mathbf{g}_p(x)}dx + p^{-1}\int_{K_0wK_0} \mathbf{g}_p(x)\overline{\mathbf{g}_p(x)}dx = p\mathrm{vol}(K_0) + p^{-1}\mathrm{vol}(K_0wK_0).$$

Since $\mathrm{vol}(K_0) = (p+1)^{-1}$ and $\mathrm{vol}(K_0wK_0) = p(p+1)^{-1}$, the statement follows. □

5.4. **The archimedean place.** We consider now the archimedean components of τ and $\tilde{\pi}$, and choose the corresponding local vectors $\breve{\mathbf{g}}_\infty \in \tau_\infty$ and $\breve{\mathbf{h}}_\infty \in \tilde{\pi}_\infty$. On the one hand, τ_∞ is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of weight $\ell + 1$, and we choose $\breve{\mathbf{g}}_\infty \in \tau_\infty$ to be a lowest weight vector. Similarly, π_∞ is a discrete series representation of weight $2k$, and consequently $\tilde{\pi}_\infty$ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest $\widetilde{\mathrm{SO}}(2)$-type $k + 1/2$. We choose a lowest weight vector $\mathbf{h}_\infty \in \tilde{\pi}_\infty$, and define $\breve{\mathbf{h}}_\infty$ as follows. Let $\mathfrak{gl}(2,\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$, and $\mathfrak{gl}(2,\mathbb{R})_{\mathbb{C}}$ be its complexification. Consider the weight raising element

$$V_+ := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sqrt{-1} \in \mathfrak{gl}(2,\mathbb{R})_{\mathbb{C}},$$

and normalize it setting $\tilde{V}_+ := -\frac{1}{8\pi}V_+$. Then we define $\breve{\mathbf{h}}_\infty := \tilde{V}_+^m\mathbf{h}_\infty$, where recall that $2m = \ell - k$. Thus $\breve{\mathbf{h}}_\infty$ is a weight $\ell + 1/2$ vector in $\tilde{\pi}_\infty$. The vector $\mathbf{h}_\infty$ is (up to a non-zero multiple) the archimedean component of the adelization of the modular form $h \in S^+_{k+1/2}(N_f)$, while $\breve{\mathbf{h}}_\infty$ is (up to a non-zero multiple) the archimedean component of the adelization of the nearly holomorphic modular form $\delta^m_{k+1/2}h \in S^{+,nh}_{\ell+1/2}(N_f)$, where

$$\delta_{k+1/2} : S^{nh}_{k+1/2}(N_f) \longrightarrow S^{nh}_{k+5/2}(N_f)$$

is the usual Shimura–Maass differential operator sending nearly holomorphic modular forms of weight $k + 1/2$ to nearly holomorphic modular forms of weight $k + 5/2$. It is defined as

$$\delta_{k+1/2} := \frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial\tau} + \frac{2k+1}{4y\sqrt{-1}}\right), \qquad \tau = x + \sqrt{-1}y \in \mathcal{H},$$

and we set $\delta^m_{k+1/2} := \delta_{\ell-3/2} \circ \cdots \circ \delta_{k+1/2}$.

6. Computation of local periods

Let $\breve{\mathbf{h}} \otimes \breve{\mathbf{g}} \otimes \breve{\boldsymbol{\phi}} \in \tilde{\pi} \otimes \tau \otimes \omega$ be the test vector as described in the previous section. The goal of this section is to compute the value of all the regularized local periods $\mathcal{I}^\sharp_v(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})$, for v a rational place.

For every rational prime $p \nmid M_g = N_f/N_g$, the local components $\breve{\mathbf{h}}_p$, $\breve{\mathbf{g}}_p$, and $\breve{\boldsymbol{\phi}}_p$ are the same as in [PdVP19], and therefore the computations done there still apply:

Proposition 6.1. *If p is a prime not dividing M_g, then*

$$\mathcal{I}^\sharp_p(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \begin{cases} 1 & \text{if } p \nmid N_f, \\ p^{-1} & \text{if } p \mid N_g. \end{cases}$$

Proof. The case $p \nmid 2N_f$ actually follows already from [Qiu14, Lemma 4.4], and the case $p = 2$ is proved in [PdVP19, Proposition 9.2]. The case $p \mid N_g$ is covered in [PdVP19, Proposition 7.15]. □

It only remains to perform the computation of the regularized local periods $\mathcal{I}^\sharp_p(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})$ at primes dividing M_g and of the archimedean period $\mathcal{I}^\sharp_\infty(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})$.

6.1. **The regularized local period at primes $p \mid M_g$.** Let p be a prime dividing $M_g = N_f/N_g$, as in Section 5.3. In this case, the three vectors $\breve{\mathbf{h}}_p$, $\breve{\mathbf{g}}_p$ and $\breve{\boldsymbol{\phi}}_p$ are fixed under the action of $\Gamma = \Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$. Therefore, the matrix coefficients involved in the computation of the local integral (8) will be Γ-biinvariant. In particular, one can compute $\alpha_p^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})$ as a sum

$$\alpha_p^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \sum_{r \in \mathcal{R}} \overline{\Phi_{\breve{\mathbf{h}}_p}(r)} \Phi_{\breve{\mathbf{g}}_p}(r) \Phi_{\breve{\boldsymbol{\phi}}_p}(r) \mathrm{vol}(\Gamma r \Gamma),$$

where $\mathcal{R}$ is a set of representatives for a decomposition of $\mathrm{SL}_2(\mathbb{Q}_p)$ into double cosets for Γ, and we abbreviate

$$\Phi_{\breve{\mathbf{h}}_p}(r) = \frac{\langle \tilde{\pi}_p(r)\breve{\mathbf{h}}_p, \breve{\mathbf{h}}_p \rangle}{||\breve{\mathbf{h}}_p||^2} \quad \text{for } r \in \mathrm{SL}_2(\mathbb{Q}_p),$$

and similarly for $\breve{\mathbf{g}}_p$ and $\breve{\boldsymbol{\phi}}_p$. A set of representatives $\mathcal{R}$ as required above is furnished by the elements

$$\alpha_n = \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix}, \quad \beta_m = s\alpha_m = \begin{pmatrix} 0 & p^{-m} \\ -p^m & 0 \end{pmatrix},$$

with n and m varying over all the integers, and where $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Indeed, by combining the Cartan decomposition for $\mathrm{SL}_2(\mathbb{Q}_p)$ relative to the maximal compact open subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$ with the so-called Bruhat decomposition for SL_2 over $\mathbb{F}_p$ yields a double coset decomposition

$$\mathrm{SL}_2(\mathbb{Q}_p) = \bigsqcup_{n \in \mathbb{Z}} \Gamma\alpha_n\Gamma \sqcup \bigsqcup_{m \in \mathbb{Z}} \Gamma\beta_m\Gamma.$$

Hence,

$$\alpha_p^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \sum_{n \in \mathbb{Z}} \overline{\Phi_{\breve{\mathbf{h}}_p}(\alpha_n)} \Phi_{\breve{\mathbf{g}}_p}(\alpha_n) \Phi_{\breve{\boldsymbol{\phi}}_p}(\alpha_n) \mathrm{vol}(\Gamma\alpha_n\Gamma) + \sum_{m \in \mathbb{Z}} \overline{\Phi_{\breve{\mathbf{h}}_p}(\beta_m)} \Phi_{\breve{\mathbf{g}}_p}(\beta_m) \Phi_{\breve{\boldsymbol{\phi}}_p}(\beta_m) \mathrm{vol}(\Gamma\beta_m\Gamma). \tag{13}$$

For later reference, let us also add that the volumes of these double cosets are given by the following formulae:

$$\mathrm{vol}(\Gamma\alpha_n\Gamma) = \begin{cases} p^{2n-2}(p-1) & \text{if } n > 0, \\ p^{-2n-2}(p-1) & \text{if } n \le 0, \end{cases} \quad \mathrm{vol}(\Gamma\beta_m\Gamma) = \begin{cases} p^{2m-3}(p-1) & \text{if } m > 0, \\ p^{-2m-1}(p-1) & \text{if } m \le 0. \end{cases} \tag{14}$$

Since p divides M_g, note that $\breve{\mathbf{h}}_p = \mathbf{h}_p \in \tilde{\pi}_p^\Gamma$ is a newvector in the one-dimensional subspace $\tilde{\pi}_p^\Gamma \subset \tilde{\pi}_p$ of Γ-invariant vectors, and so the same computation as in [PdVP19, Propositions 7.9, 7.12] applies for $\Phi_{\breve{\mathbf{h}}_p}(\alpha_n)$ and $\Phi_{\breve{\mathbf{h}}_p}(\beta_m)$. Similarly, the values $\Phi_{\breve{\boldsymbol{\phi}}_p}(\alpha_n)$ and $\Phi_{\breve{\boldsymbol{\phi}}_p}(\beta_m)$ were computed in [PdVP19, Proposition 7.13]. We collect these computation for later reference:

Proposition 6.2. *If p divides M_g and $n, m \in \mathbb{Z}$, then*

$$\Phi_{\breve{\boldsymbol{\phi}}_p}(\alpha_n) = \chi_{\overline{\psi}_p}(p^n)p^{-|n|/2}, \quad \Phi_{\breve{\boldsymbol{\phi}}_p}(\beta_m) = \chi_{\overline{\psi}_p}(p^m)p^{-|m|/2},$$

and

$$\Phi_{\breve{\mathbf{h}}_p}(\alpha_n) = (-1)^n \chi_{\overline{\psi}_p^D}(p^n)p^{-3|n|/2}, \quad \Phi_{\breve{\boldsymbol{\phi}}_p}(\beta_m) = (-1)^{m+1}\chi_{\overline{\psi}_p^D}(p^m)p^{-|3m/2-1|}.$$

It thus remains to compute the normalized matrix coefficients $\Phi_{\breve{\mathbf{g}}_p}(\alpha_n)$ ($n \in \mathbb{Z}$) and $\Phi_{\breve{\mathbf{g}}_p}(\beta_m)$ ($m \in \mathbb{Z}$). Notice that, since $||\breve{\mathbf{g}}_p||^2 = 1$ by Corollary 5.3, we have $\Phi_{\breve{\mathbf{g}}_p}(g) = \langle \tau_p(g)\breve{\mathbf{g}}_p, \breve{\mathbf{g}}_p \rangle$. Recall that $\tau_p = \pi(\chi, \chi^{-1})$ is the (unramified) induced representation associated with an unramified character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, and that $\breve{\mathbf{g}}_p \in \tau_p^{K_0}$ is fixed by $K_0 = K_0(p) \supseteq \Gamma_0(p) = \Gamma$. To simplify the notation, in what follows we set $\xi := \chi(p)\overline{\chi}(p)^{-1} = \chi(p)^2$.

In the computations below, we will need to use certain subsets of K_0 and its complement

$$\mathrm{GL}_2(\mathbb{Z}_p) \setminus K_0 = K_0 w K_0, \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On the one hand, for each integer $j \ge 1$ we put

$$\mathcal{C}_j := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 : \mathrm{val}_p(c) = j \right\},$$

and notice that K_0 is the (disjoint) union of all the sets $\mathcal{C}_j$, $j \ge 1$. As a piece of notation, we will use $\mathcal{C}_{\ge j}$, $\mathcal{C}_{>j}$, $\mathcal{C}_{\le j}$, $\mathcal{C}_{<j}$ with the obvious meaning. Recalling that we normalize the Haar measure on $\mathrm{GL}_2(\mathbb{Q}_p)$ so that $\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z}_p)) = 1$, and hence $\mathrm{vol}(K_0) = (p+1)^{-1}$, an easy recursive argument shows that $\mathrm{vol}(\mathcal{C}_j) = p^{-j}(p-1)(p+1)^{-1}$. One also gets

$$\mathrm{vol}(\mathcal{C}_{\ge j}) = p^{1-j}(p+1)^{-1}, \qquad \mathrm{vol}(\mathcal{C}_{\le j}) = p^{-j}(p+1)^{-1}(p^j - 1).$$

On the other hand, for $i \ge 0$ we also define

$$\mathcal{D}_i := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 w K_0 : \mathrm{val}_p(d) = i \right\},$$

thus K_0wK_0 is the (disjoint) union of all the sets $\mathcal{D}_i$, $i \geq 0$. We will also use the notation $\mathcal{D}_{\geq i}$, $\mathcal{D}_{>i}$, $\mathcal{D}_{\leq i}$, $\mathcal{D}_{<i}$. If $i \geq 0$, then one can see easily that $\mathrm{vol}(\mathcal{D}_i) = p^{-i}(p-1)(p+1)^{-1}$. Similarly as above, the following formulas will be also useful:

$$\mathrm{vol}(\mathcal{D}_{\geq i}) = p^{1-i}(p+1)^{-1}, \qquad \mathrm{vol}(\mathcal{D}_{\leq i}) = p^{-i}(p+1)^{-1}(p^{i+1}-1).$$

We start by considering the case of the elements α_n. Observe that for an arbitrary element

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$$

we have

$$x\alpha_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix} = \begin{pmatrix} ap^n & bp^{-n} \\ cp^n & dp^{-n} \end{pmatrix}.$$

In order to evaluate $\breve{\mathbf{g}}_p$ at elements of the form $x\alpha_n$ we will need an Iwasawa decomopsition of such elements. And such a decomposition depends on how the p-adic valuations of the bottom row entries c and d compare one to each other. To be precise, if $\mathrm{val}_p(c) + 2n > \mathrm{val}_p(d)$, then we may use the Iwasawa decomposition

$$x\alpha_n = \begin{pmatrix} p^n u & * \\ 0 & p^{-n}d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cp^{2n}d^{-1} & 1 \end{pmatrix}, \tag{IW1-α_n}$$

where $u = d^{-1}\det(x) \in d^{-1}\mathbb{Z}_p^\times$ and the rightmost element belongs to K_0. In contrast, when $\mathrm{val}_p(c) + 2n \leq \mathrm{val}_p(d)$, we can instead use the Iwasawa decomposition

$$x\alpha_n = \begin{pmatrix} p^{-n} v & * \\ 0 & p^{n}c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & dp^{-2n}c^{-1} \end{pmatrix}, \tag{IW2-α_n}$$

where now $v = -c^{-1}\det(x) \in c^{-1}\mathbb{Z}_p^\times$ and the rightmost element belongs to K_0wK_0. When deciding whether we use (IW1-α_n) or (IW2-α_n), observe that since $x \in \mathrm{GL}_2(\mathbb{Z}_p)$ at least one of the non-negative integers $\mathrm{val}_p(c)$, $\mathrm{val}_p(d)$ must be zero.

Proposition 6.3. *With the above notation, for all integers n it holds*

$$\Phi_{\breve{\mathbf{g}}_p}(\alpha_n) = \frac{p^{-|n|}}{p+1}\left(\frac{\xi^{|n|}(p\xi-1)+\xi^{-|n|}(\xi-p)}{\xi-1}\right).$$

Proof. As pointed out above, we just need to compute

$$\langle \tau_p(\alpha_n)\breve{\mathbf{g}}_p, \breve{\mathbf{g}}_p\rangle = \int_{\mathrm{GL}_2(\mathbb{Z}_p)} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = \int_{K_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx + \int_{K_0wK_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx. \tag{15}$$

Suppose first that $n \geq 0$, and let us compute the first integral on the right hand side of (15). Let $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0$ be an arbitrary element, thus $c \in p\mathbb{Z}_p$ and $d \in \mathbb{Z}_p^\times$. Since $n \geq 0$, in this case we may use (IW1-α_n) together with Lemma 5.2 to deduce that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\chi(p)^n\overline{\chi}(p)^{-n}|p^{2n}|^{1/2} = p^{1/2-n}\xi^n\chi(p)^{-1}.$$

Besides, being $x \in K_0$ we have $\overline{\breve{\mathbf{g}}_p(x)} = p^{1/2}\chi(p)\overline{\mathbf{g}_p(x)} = p^{1/2}\chi(p)$, hence we deduce that

$$\int_{K_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{1-n}\xi^n\mathrm{vol}(K_0) = \frac{p^{-n}}{p+1}\cdot p\xi^n.$$

To compute the second integral on the right hand side of (15), we decompose K_0wK_0 as the disjoint union of $\mathcal{D}_{\geq 2n}$ and $\mathcal{D}_{<2n}$, so that

$$\int_{K_0wK_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = \int_{\mathcal{D}_{\geq 2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx + \int_{\mathcal{D}_{<2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx.$$

In both integrals, by Lemma 5.2 we have $\overline{\breve{\mathbf{g}}_p(x)} = p^{-1/2}\overline{\chi}(p)\overline{\mathbf{g}_p(x)} = p^{-1/2}\overline{\chi}(p)$. If $x \in \mathcal{D}_j$ with $0 \leq j < 2n$, we can still use (IW1-α_n) to obtain that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\chi(p^nd^{-1})\overline{\chi}(p^{-n}d)|p^{2n}d^{-2}|^{1/2} = p^{1/2+j-n}\xi^{n-j}\chi(p)^{-1},$$

hence

$$\int_{\mathcal{D}_{<2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{-n}\xi^{n-1}\sum_{j=0}^{2n-1} p^j\xi^{-j}\mathrm{vol}(\mathcal{D}_j) = \frac{p^{-n}(p-1)}{p+1}\xi^{n-1}\sum_{j=0}^{2n-1}\xi^{-j} = \frac{p^{-n}}{p+1}\cdot\frac{(p-1)(\xi^n-\xi^{-n})}{\xi-1}.$$

In contrast, when $x \in \mathcal{D}_{\geq 2n}$ we may invoke (IW2-α_n) to deduce that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\chi(p)^{-n}\overline{\chi}(p)^n|p^{-2n}|^{1/2} = p^{-1/2+n}\xi^{-n}\overline{\chi}(p)^{-1},$$

and therefore

$$\int_{\mathcal{D}_{\geq 2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{n-1}\xi^{-n}\mathrm{vol}(\mathcal{D}_{\geq 2n}) = \frac{p^{-n}}{p+1}\cdot\xi^{-n}.$$

Summing up the contributions obtained by integrating over K_0, $\mathcal{D}_{\geq 2n}$, and $\mathcal{D}_{<2n}$, we get

$$\langle \tau_p(\alpha_n)\breve{\mathbf{g}}_p, \breve{\mathbf{g}}_p\rangle = \frac{p^{-n}}{p+1}\left(p\xi^n + \xi^{-n} + \frac{(p-1)(\xi^n - \xi^{-n})}{\xi - 1}\right) = \frac{p^{-n}}{p+1}\left(\frac{\xi^n(p\xi-1) + \xi^{-n}(\xi - p)}{\xi - 1}\right).$$

We now proceed similarly for $n < 0$, and consider first the integral over K_0wK_0 on the right hand side of (15). When $x \in K_0wK_0$, we may use (IW2-α_n) to deduce that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\chi(p)^{-n}\overline{\chi}(p)^n|p^{-2n}|^{1/2} = p^{-1/2+n}\xi^{-n}\overline{\chi}(p)^{-1}.$$

Besides, for $x \in K_0wK_0$ we have $\overline{\breve{\mathbf{g}}_p(x)} = p^{-1/2}\overline{\chi}(p)\overline{\mathbf{g}_p(x)} = p^{-1/2}\overline{\chi}(p)$, and therefore

$$\int_{K_0wK_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{n-1}\xi^{-n}\mathrm{vol}(K_0wK_0) = \frac{p^n}{p+1}\cdot \xi^{-n}.$$

Let us now work out the integral over K_0. To do so, we use that $K_0 = \mathcal{C}_{>-2n} \sqcup \mathcal{C}_{\leq -2n}$, hence

$$\int_{K_0} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx = \int_{\mathcal{C}_{>-2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx + \int_{\mathcal{C}_{\leq -2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\breve{\mathbf{g}}_p(x)}dx.$$

We will compute separately each of the two integrals on the right hand side. In both cases, we have $\overline{\breve{\mathbf{g}}_p(x)} = p^{1/2}\chi(p)\overline{\mathbf{g}_p(x)} = p^{1/2}\chi(p)$ because $x \in K_0$, thus it remains to study the value of $\breve{\mathbf{g}}_p(x\alpha_n)$ in each case. For $x \in \mathcal{C}_{>-2n}$, using (IW1-α_n) we obtain that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{1/2}\chi(p)^{-1}\chi(p)^n\overline{\chi}(p)^{-n}|p^{2n}|^{1/2} = p^{1/2-n}\xi^n\chi(p)^{-1},$$

and therefore

$$\int_{\mathcal{C}_{>-2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\mathbf{g}_p(x)}dx = p^{1-n}\xi^n\mathrm{vol}(\mathcal{C}_{>-2n}) = \frac{p^n}{p+1}\cdot p\xi^n.$$

At last, we consider the integral over $\mathcal{C}_{\leq -2n}$. We may write $\mathcal{C}_{\leq -2n}$ as the disjoint union of the subsets $\mathcal{C}_j$ for $j = 1, \ldots, -2n$. For $x \in \mathcal{C}_j$, we may use (IW2-α_n) to deduce that

$$\breve{\mathbf{g}}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\mathbf{g}_p(x\alpha_n) = p^{-1/2}\overline{\chi}(p)^{-1}\chi(p)^{-n-j}\overline{\chi}(p)^{n+j}|p^{-2(n+j)}|^{1/2} = p^{n+j-1/2}\xi^{-n-j}\overline{\chi}(p)^{-1},$$

and therefore

$$\int_{\mathcal{C}_j} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\mathbf{g}_p(x)}dx = p^{n+j}\chi(p)^{1-n-j}\overline{\chi}(p)^{n+j-1}\mathrm{vol}(\mathcal{C}_j) = \frac{p^n}{p+1}\cdot (p-1)\xi^{1-n-j}.$$

It follows that

$$\int_{\mathcal{C}_{\leq -2n}} \breve{\mathbf{g}}_p(x\alpha_n)\overline{\mathbf{g}_p(x)}dx = \frac{p^n}{p+1}(p-1)\xi^{1-n}\sum_{j=1}^{-2n}\xi^{-j} = \frac{p^n}{p+1}(p-1)\xi^{1-n}\cdot\frac{1-\xi^{2n}}{\xi-1} = \frac{p^n}{p+1}\cdot\frac{(p-1)(\xi^{1-n} - \xi^{n+1})}{\xi - 1}.$$

Finally, summing up the different contributions we conclude that for $n < 0$ one has

$$\langle \tau_p(\alpha_n)\breve{\mathbf{g}}_p, \breve{\mathbf{g}}_p\rangle = \frac{p^n}{p+1}\left(\xi^{-n} + p\xi^n + \frac{(p-1)(\xi^{1-n} - \xi^{n+1})}{\xi - 1}\right) = \frac{p^n}{p+1}\left(\frac{\xi^{-n}(p\xi-1) + \xi^n(\xi - p)}{\xi - 1}\right).$$

□

Next we proceed analogously for the computation of the matrix coefficients for the elements of the form β_m, $m \in \mathbb{Z}$. We continue to use the notation $\xi = \chi(p)^2$. As we did in the previous case, observe now that if $x = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is an arbitrary element in $\mathrm{GL}_2(\mathbb{Z}_p)$, then

$$x\beta_m = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & p^{-m} \\ -p^m & 0\end{pmatrix} = \begin{pmatrix} -bp^m & ap^{-m} \\ -dp^m & cp^{-m}\end{pmatrix}.$$

In order to evaluate $\breve{\mathbf{g}}_p$ at elements of the form $x\beta_m$, it will be again useful to have an explicit Iwasawa decomposition of the above matrix. We will have to distinguish two cases. If $\mathrm{val}_p(d) + 2m > \mathrm{val}_p(c)$, then we will use that

$$\text{(IW1-}\beta_m\text{)} \qquad x\beta_m = \begin{pmatrix} p^m u & * \\ 0 & p^{-m}c\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -dp^{2m}c^{-1} & 1\end{pmatrix},$$

where $u = c^{-1}\det(x) \in c^{-1}\mathbb{Z}_p^\times$, and the rightmost element belongs to K_0. In contrast, when $\mathrm{val}_p(d) + 2m \leq \mathrm{val}_p(c)$, we will instead use that

$$\text{(IW2-}\beta_m\text{)} \qquad x\beta_m = \begin{pmatrix} p^{-m} v & * \\ 0 & -p^m d\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & cp^{-2m}d^{-1}\end{pmatrix},$$

where now $v = d^{-1}\det(x) \in d^{-1}\mathbb{Z}_p^\times$, and the rightmost element lies now in K_0wK_0. As in the previous case, observe that when deciding whether to use (IW1-β_m) or (IW2-β_m) at least one of the non-negative integers $\mathrm{val}_p(c)$, $\mathrm{val}_p(d)$ is necessarily zero.

Proposition 6.4. *With notation as above, for all integers m it holds*

$$\Phi_{\breve{\mathbf{g}}_p}(\beta_m) = \frac{p^{-|m-1|}}{p+1}\left(\frac{\xi^{|m-1|}(p\xi-1)+\xi^{-|m-1|}(\xi-p)}{\xi-1}\right).$$

Proof. We are going to compute

$$\langle\tau_p(\beta_m)\breve{\mathbf{g}}_p,\breve{\mathbf{g}}_p\rangle = \int_{\mathrm{GL}_2(\mathbb{Z}_p)}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx = \int_{K_0}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx + \int_{K_0wK_0}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx. \quad (16)$$

We consider first the case $m>0$. On K_0wK_0, the lower-left entry of x is a unit, and (IW1-β_m) implies that $\breve{\mathbf{g}}_p(x\beta_m) = p^{1/2-m}\xi^m\chi(p)^{-1}$. Besides, we have $\overline{\mathbf{g}_p(x)} = p^{-1/2}\overline{\chi}(p)$, and therefore

$$\int_{K_0wK_0}\breve{\mathbf{g}}_p(x\beta_m)\overline{\mathbf{g}_p(x)}dx = p^{-m}\xi^{m-1}\mathrm{vol}(K_0wK_0) = \frac{p^{1-m}}{p+1}\cdot\xi^{m-1}.$$

To compute the integral over K_0, we use that $K_0 = \mathcal{C}_{\geq 2m}\sqcup\mathcal{C}_{<2m}$. When $x\in\mathcal{C}_{\geq 2m}$, we can use (IW2-$\beta_m$) to obtain that $\breve{\mathbf{g}}_p(x\beta_m) = p^{m-1/2}\xi^{-m}\overline{\chi}(p)^{-1}$. Since $\overline{\breve{\mathbf{g}}_p(x)} = p^{1/2}\chi(p)$ for $x\in K_0$, we find

$$\int_{\mathcal{C}_{\geq 2m}}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx = p^m\xi^{1-m}\mathrm{vol}(\mathcal{C}_{\geq 2m}) = p^m\xi^{1-m}\frac{1}{p^{2m-1}(p+1)} = \frac{p^{1-m}}{p+1}\cdot\xi^{1-m}.$$

It thus remains to consider the integral over the region $\mathcal{C}_{<2m}$, which can be written as the disjoint union of the sets $\mathcal{C}_j$ for $j=1,\dots,2m-1$. For $x\in\mathcal{C}_j$, we use now (IW1-β_m) to compute

$$\breve{\mathbf{g}}_p(x\beta_m) = p^{1/2}\chi(p)^{-1}\mathbf{g}_p(x\beta_m) = p^{1/2}\chi(p)^{-1}\chi(p)^{m-j}\overline{\chi}(p)^{j-m}|p^{2m-2j}|^{1/2} = p^{1/2+j-m}\xi^{m-j}\chi(p)^{-1}.$$

Besides, we still have $\overline{\breve{\mathbf{g}}_p(x)} = p^{1/2}\chi(p)\overline{\mathbf{g}_p(x)} = p^{1/2}\chi(p)$, and therefore

$$\int_{\mathcal{C}_j}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}}_p(x)dx = p^{1+j-m}\xi^{m-j}\mathrm{vol}(\mathcal{C}_j) = p^{1+j-m}\xi^{m-j}\frac{(p-1)}{p^j(p+1)} = \frac{p^{1-m}}{p+1}\cdot(p-1)\xi^{m-j},$$

and altogether

$$\int_{\mathcal{C}_{<2m}}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}}_p(x)dx = \frac{p^{1-m}}{p+1}(p-1)\xi^m\sum_{j=1}^{2m-1}\xi^{-j} = \frac{p^{1-m}}{p+1}(p-1)\xi^m\cdot\frac{1-\xi^{1-2m}}{\xi-1} = \frac{p^{1-m}}{p+1}\cdot\frac{(p-1)(\xi^m-\xi^{1-m})}{\xi-1}.$$

Finally, summing up the three contributions obtained, we conclude that for $m>0$ one has

$$\langle\tau_p(\beta_m)\breve{\mathbf{g}}_p,\breve{\mathbf{g}}_p\rangle = \frac{p^{1-m}}{p+1}\left(\xi^{m-1}+\xi^{1-m}+\frac{(p-1)(\xi^m-\xi^{1-m})}{\xi-1}\right) = \frac{p^{1-m}}{p+1}\left(\frac{\xi^{m-1}(p\xi-1)+\xi^{1-m}(\xi-p)}{\xi-1}\right).$$

Suppose now that $m\leq 0$. For $x\in K_0$, (IW2-β_m) implies that $\breve{\mathbf{g}}_p(x\beta_m) = p^{m-1/2}\xi^{-m}\overline{\chi}(p)^{-1}$. Besides, $\overline{\breve{\mathbf{g}}_p(x)} = p^{1/2}\chi(p)$, and hence

$$\int_{K_0}\breve{\mathbf{g}}_p(x\beta_m)\overline{\mathbf{g}}_p(x)dx = p^m\xi^{1-m}\mathrm{vol}(K_0) = \frac{p^m}{p+1}\cdot\xi^{1-m}.$$

To compute the integral over K_0wK_0, we use now that $K_0wK_0 = \mathcal{D}_{>-2m}\sqcup\mathcal{D}_{\leq -2m}$. For elements $x\in\mathcal{D}_{>-2m}$, we can use the explicit Iwasawa decomposition in (IW1-β_m) to find that $\breve{\mathbf{g}}_p(x\beta_m) = p^{1/2-m}\xi^m\chi(p)^{-1}$. Since $\overline{\breve{\mathbf{g}}_p(x)} = p^{-1/2}\overline{\chi}(p)$ in this case, we obtain that

$$\int_{\mathcal{D}_{>-2m}}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{-m}\xi^{m-1}\mathrm{vol}(\mathcal{D}_{>-2m}) = p^{-m}\xi^{m-1}\frac{p^{2m}}{p+1} = \frac{p^m}{p+1}\cdot\xi^{m-1}.$$

In contrast, if $x\in\mathcal{D}_j$, with $0\leq j\leq -2m$, we may use (IW2-β_m) to get $\breve{\mathbf{g}}_p(x\beta_m) = p^{m+j-1/2}\xi^{-m-j}\overline{\chi}(p)^{-1}$. Using again that $\overline{\breve{\mathbf{g}}_p(x)} = p^{-1/2}\overline{\chi}(p)$, one deduces that

$$\int_{\mathcal{D}_j}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx = p^{m+j-1}\xi^{-m-j}\mathrm{vol}(\mathcal{D}_j) = p^{m+j-1}\xi^{-m-j}\frac{1}{p^j(p+1)} = \frac{p^{m-1}}{p+1}\cdot(p-1)\xi^{-m-j},$$

and altogether

$$\int_{\mathcal{D}_{\leq -2m}}\breve{\mathbf{g}}_p(x\beta_m)\overline{\breve{\mathbf{g}}_p(x)}dx = \frac{p^{m-1}(p-1)\xi^{-m}}{p+1}\sum_{j=0}^{-2m}\xi^{-j} = \frac{p^{m-1}(p-1)\xi^{-m}}{p+1}\cdot\frac{\xi-\xi^{2m}}{\xi-1} = \frac{p^{m-1}}{p+1}\cdot\frac{(p-1)(\xi^{1-m}-\xi^m)}{\xi-1}.$$

Summing up all the contributions, we find that for $m\leq 0$ it holds

$$\langle\tau_p(\beta_m)\breve{\mathbf{g}}_p,\breve{\mathbf{g}}_p\rangle = \frac{p^{m-1}}{p+1}\left(p\xi^{1-m}+p\xi^{m-1}+\frac{(p-1)(\xi^{1-m}-\xi^m)}{\xi-1}\right) = \frac{p^{m-1}}{p+1}\left(\frac{\xi^{1-m}(p\xi-1)+\xi^{m-1}(\xi-p)}{\xi-1}\right).$$

□

Remark 6.5. *Observe from the previous propositions that for every integer n one has $\Phi_{\breve{\mathbf{g}}_p}(\beta_n) = \Phi_{\breve{\mathbf{g}}_p}(\alpha_{n-1})$.*

Having computed the matrix coefficients that were missing for this case, we can finally tackle the computation of the regularized local period. First, we compute the local integral (cf. (13))

$$\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \sum_{n\in\mathbb{Z}} \Omega_p(\alpha_n)\mathrm{vol}(\Gamma\alpha_n\Gamma) + \sum_{m\in\mathbb{Z}} \Omega_p(\beta_m)\mathrm{vol}(\Gamma\beta_m\Gamma)$$

where we abbreviate $\Omega_p(g) := \overline{\Phi_{\breve{\mathbf{h}}_p}(g)}\Phi_{\breve{\mathbf{g}}_p}(g)\Phi_{\breve{\boldsymbol{\phi}}_p}(g)$ for $g\in\mathrm{SL}_2(\mathbb{Q}_p)$.

Proposition 6.6. *Let p be a prime dividing M_g. Then the local integral $\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ vanishes if $w_p=-1$. And if $w_p=1$, then one has*

$$\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \frac{2(p-1)^2(p-\xi)(p\xi-1)}{p^2(p+1)(p+\xi)(p\xi+1)}.$$

Proof. We focus first on the computation of $\Omega_p(\alpha_n)\mathrm{vol}(\Gamma\alpha_n\Gamma)$. From Proposition 6.2, using that $\chi_{\overline{\psi}_p^D} = \chi_{\overline{\psi}_p}\cdot\chi_D$ and $(D,p)_p = w_p$, we have

$$\overline{\Phi_{\breve{\mathbf{h}}_p}(\alpha_n)}\Phi_{\breve{\boldsymbol{\phi}}_p}(\alpha_n) = p^{-2|n|}(-1)^n w_p^n.$$

Since $\mathrm{vol}(\Gamma\alpha_n\Gamma) = p^{|2n|-2}(p-1)$ (cf. (14)), we get

$$\Omega_p(\alpha_n)\mathrm{vol}(\Gamma\alpha_n\Gamma) = p^{-2}(p-1)(-1)^n w_p^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n).$$

Now we now look at $\Omega_p(\beta_m)\mathrm{vol}(\Gamma\beta_m\Gamma)$. Similarly as before, from Proposition 6.2 one has $\overline{\Phi_{\breve{\mathbf{h}}_p}(\beta_m)}\Phi_{\breve{\boldsymbol{\phi}}_p}(\beta_m) = p^{-|2m-1|}(-1)^{m+1}w_p^m$, and using that $\mathrm{vol}(\Gamma\beta_m\Gamma) = p^{|2m-1|}p^{-2}(p-1)$ we find

$$\Omega_p(\beta_m)\mathrm{vol}(\Gamma_0\beta_m\Gamma_0) = p^{-2}(p-1)(-1)^{m+1}w_p^m\Phi_{\breve{\mathbf{g}}_p}(\beta_m).$$

Altogether, the above yields (using Remark 6.5)

$$\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \frac{p-1}{p^2}\sum_{n\in\mathbb{Z}}((-1)^n w_p^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n) + (-1)^{n-1}w_p^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_{n-1})) = (1+w_p)\frac{p-1}{p^2}\sum_{n\in\mathbb{Z}}(-1)^n w_p^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n).$$

Here we see that the desired local integral *vanishes if* $w_p=-1$. Assume in the following that $w_p=1$. By using that $\Phi_{\breve{\mathbf{g}}_p}(\alpha_{-n}) = \Phi_{\breve{\mathbf{g}}_p}(\alpha_n)$ (cf. Proposition 6.3), we have

$$\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \frac{2(p-1)}{p^2}\sum_{n\in\mathbb{Z}}(-1)^n w_p^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n) = \frac{2(p-1)}{p^2}\left(\Phi_{\breve{\mathbf{g}}_p}(1) + 2\sum_{n>0}(-1)^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n)\right).$$

Now, $\Phi_{\breve{\mathbf{g}}_p}(1)=1$, and by Proposition 6.3

$$\sum_{n>0}(-1)^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n) = \frac{p\xi-1}{(p+1)(\xi-1)}\sum_{n>0}(-p^{-1}\xi)^n + \frac{\xi-p}{(p+1)(\xi-1)}\sum_{n>0}(-p^{-1}\xi^{-1})^n.$$

These geometric series are computed easily, and one eventually finds

$$\sum_{n>0}(-p^{-1}\xi)^n = \frac{-1}{(p+1)}\cdot\frac{(\xi+p^2\xi^2+p^2\xi+p^2)}{(p+\xi)(p\xi+1)}.$$

Back to the computation of $\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$, the above yields

$$\Phi_{\breve{\mathbf{g}}_p}(1) + 2\sum_{n>0}(-1)^n\Phi_{\breve{\mathbf{g}}_p}(\alpha_n) = 1 - \frac{2(\xi+p^2\xi^2+p^2\xi+p^2)}{(p+1)(p+\xi)(p\xi+1)} = \frac{(p-1)(p-\xi)(p\xi-1)}{(p+1)(p+\xi)(p\xi+1)}.$$

This gives the claimed formula. □

Finally, in the next proposition we bring the local L-values into the picture to conclude the computation of the regularized local period $\mathcal{I}_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$:

Proposition 6.7. *Let p be a prime dividing N_f but not N_g. Then the regularized local period $\mathcal{I}_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ vanishes if $w_p=-1$. And if $w_p=1$, one has*

$$\mathcal{I}_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \frac{2}{p+1}.$$

Proof. The vanishing statement in the case $w_p=-1$ follows from the previous proposition, so we may assume that $w_p=1$. Let us look at the local L-values involved in the definition of $\mathcal{I}_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$. On the one hand, using that π_p is a special representation and τ_p is an unramified principal series representation, we have (see [Hid86, Section 10] or [GJ78])

$$L(1,\pi_p,\mathrm{ad}) = \frac{p^2}{p^2-1} = \frac{p^2}{(p+1)(p-1)},\quad L(1,\tau_p,\mathrm{ad}) = \frac{p^3}{(p-1)(p-\xi)(p-\xi^{-1})}.$$

Besides, it is well-known that $L(\pi_p,1/2)=\frac{p}{p+w_p}=\frac{p}{p+1}$, whereas for the triple product L-function we have (see [Kud94, Section 3], for example)

$$L(\pi_p\otimes\tau_p\otimes\tau_p,1/2)=\frac{p^4}{(p+w_p)^2(p+w_p\xi)(p+w_p\xi^{-1})}=\frac{p^4}{(p+1)^2(p+\xi)(p+\xi^{-1})}.$$

Therefore, we have

$$L(\pi_p\otimes\mathrm{ad}(\tau_p),1/2)=\frac{L(\pi_p\otimes\tau_p\otimes\tau_p,1/2)}{L(\pi_p,1/2)}=\frac{p^3}{(p+1)(p+\xi)(p+\xi^{-1})},$$

and as a consequence

$$\frac{L(1,\pi_p,\mathrm{ad})L(1,\tau_p,\mathrm{ad})}{L(\pi_p\otimes\mathrm{ad}(\tau_p),1/2)}=\frac{p^5(p+1)(p+\xi)(p+\xi^{-1})}{p^3(p+1)(p-1)^2(p-\xi)(p-\xi^{-1})}=\frac{p^2(p+\xi)(p\xi+1)}{(p-1)^2(p-\xi)(p\xi-1)}.$$

Multiplying with the value of $\alpha_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ we get as claimed

$$\mathcal{I}_p^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})=\frac{2(p-1)^2(p-\xi)(p\xi-1)}{p^2(p+1)(p+\xi)(p\xi+1)}\cdot\frac{p^2(p+\xi)(p\xi+1)}{(p-1)^2(p-\xi)(p\xi-1)}=\frac{2}{p+1}.$$

□

6.2. **The regularized local period at the archimedean place.** To address the computation of the regularized period $\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$, we follow the approach of Xue [Xue19]. We fulfill some details missing in loc. cit. in order to provide an explicit formula.

In order to lighten the notation, let us write in this paragraph $\psi=\overline{\psi_\infty}$, so that $\psi(x)=e^{-2\pi\sqrt{-1}x}$ and $\omega_\infty=\omega_\psi$. By Iwasawa decomposition, recall that every element $g\in\mathrm{SL}_2(\mathbb{R})$ can be written as

$$g=\begin{pmatrix} y & 0\\ 0 & y^{-1}\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}k$$

for some $y\in\mathbb{R}_{>0}$, $x\in\mathbb{R}$ and $k\in\mathrm{SO}(2)$. We consider the Haar measure $dg=y^{-2}dxdydk$, where dx and dy are the usual Lebesgue measure on $\mathbb{R}$, and dk is the Haar measure on $\mathrm{SO}(2)$ for which the volume of $\mathrm{SO}(2)$ is π.

Recall from Section 5.4 that τ_∞ is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of weight $\ell+1$, and that $\breve{\mathbf{g}}_\infty\in\tau_\infty$ is a lowest weight vector. Similarly, recall that $\tilde\pi_\infty$ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest $\widetilde{\mathrm{SO}}(2)$-type $k+1/2$, and $\breve{\mathbf{h}}_\infty=\tilde V_+^m\mathbf{h}_\infty$ with $\mathbf{h}_\infty$ a lowest weight vector in $\tilde\pi_\infty$.

Let J be the Jacobi group, which arises as the semidirect product of SL_2 with the so-called Heisenberg group H, and it can be realized as a subgroup of Sp_2 (see [BS98, Section 1.1]). In explicit terms, elements in J can be written as products

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(\lambda,\mu,\xi)=\begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & \mu\\ \lambda & 1 & \mu & \xi\\ & & 1 & -\lambda\\ & & & 1\end{pmatrix},\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2,\ (\lambda,\mu,\xi)\in H.$$

By virtue of [BS98, Theorem 7.3.3], $\tilde\pi_\infty\otimes\omega_\infty$ is isomorphic to a discrete series representation ρ_∞ of $J(\mathbb{R})$ of lowest K-type $k+1$. In particular, the vector $\mathbf{h}_\infty\otimes\boldsymbol{\phi}_\infty\in\tilde\pi_\infty\otimes\omega_\infty$ is then identified under the previous isomorphism with a lowest weight vector in ρ_∞, which we shall call $\mathbf{J}_\infty\in\rho_\infty$. By an abuse of notation, we will simply write $\mathbf{J}_\infty=\mathbf{h}_\infty\otimes\boldsymbol{\phi}_\infty$, keeping in mind that this equality is through the isomorphism between $\tilde\pi_\infty\otimes\omega_\infty$ and ρ_∞.

Before entering in the computation of the archimedean period $\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$, it will be useful to fix once and for all an explicit model $D(k+1,N_f)$ of the discrete series representation ρ_∞, which can be found in [BS98, Chapter 3], and to describe its main features. As vector spaces, one has

$$D(k+1,N_f)=\bigoplus_{r,s\geq 0,s\text{ even}}\mathbb{C}\cdot v_{r,s},$$

and $\mathrm{SO}_2(\mathbb{R})$ acts on $v_{r,s}$ through the character $u\mapsto u^{k+1+r+s}$. The element $v_{0,0}$ is a lowest weight vector, and $\mathrm{SO}_2(\mathbb{R})$ acts on the line spanned by $v_{0,0}$ through the character $u\mapsto u^{k+1}$. Let $\mathfrak{r}$ be the Lie algebra of $J(\mathbb{R})$, and denote by $\mathfrak{r}_\mathbb{C}$ its complexification. There are certain operators X_+, X_-, Y_+, Y_- acting on $\mathfrak{r}_\mathbb{C}$ (see loc. cit. for the precise definition) satisfying $\mathrm{d}\rho_\infty X_-\mathbf{J}_\infty=\mathrm{d}\rho_\infty Y_-\mathbf{J}_\infty=0$. The action of these operators on the above model is given by the following recipe:

$$\mathrm{d}\rho_\infty Y_+v_{r,s}=v_{r+1,s},\quad \mathrm{d}\rho_\infty X_+v_{r,s}=-\frac{1}{2\pi N_f}v_{r+2,s},$$
$$\mathrm{d}\rho_\infty Y_-v_{r,s}=-2\pi N_f r v_{r-1,s},\quad \mathrm{d}\rho_\infty X_-v_{r,s}=\pi N_f r(r-1)v_{r-2,s}-\frac{s}{4}(2k+s-1)v_{r,s-2}.$$

The space $D(k+1,N_f)$ is further endowed with an inner product $\langle\ ,\ \rangle$, and the vectors $v_{r,s}$ form an orthogonal basis with respect to this inner product. Setting $||v||^2 = \langle v, v\rangle$, from [BS98, pages 46, 47] we know that

$$||v_{r,s+2}||^2 = \frac{(s+2)(2k+s+1)}{4}||v_{r,s}||^2, \quad ||v_{r+1,s}||^2 = 2\pi N_f(r+1)||v_{r,s}||^2. \tag{17}$$

From now on, we normalize the inner product by requiring that $||v_{2m,0}||^2 = \langle v_{2m,0}, v_{2m,0}\rangle = 1$.

Lemma 6.8. *With the above notation, if s is an even integer with $2 \le s \le 2m$, then*

$$||v_{2m-s,s}||^2 = (4\pi N_f)^{-s} \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} \frac{(j+2)(2k+j+1)}{(2m-j-1)(2m-j)}.$$

Proof. The claimed identity follows by applying recursively the relations in (17). Indeed, using the first of them one easily gets

$$||v_{2m-s,s}||^2 = 4^{-s/2}||v_{2m-s,0}||^2 \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} (s-j)(2k+s-j-1) = 2^{-s}||v_{2m-s,0}||^2 \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} (j+2)(2k+j+1). \tag{18}$$

In a similar manner, we can now use recursively the second identity in (17) to deduce that

$$||v_{2m-s,0}||^2 = (2\pi N_f)^{-s} \prod_{0\le i\le s-1} \frac{1}{2m-s+i+1}||v_{2m,0}||^2 = (2\pi N_f)^{-s} \prod_{0\le i\le s-1} \frac{1}{2m-s+i+1} = \tag{19}$$

$$= \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} \frac{1}{(2m-(s-j-2)-1)(2m-(s-j-2))} = \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} \frac{1}{(2m-j-1)(2m-j)},$$

using that $||v_{2m,0}||^2 = 1$ according to our normalization. The statement follows by combining (18) and (19). □

We shall now focus our attention on the space

$$D(k+1,N_f;2m) = \bigoplus_{\substack{r+s=2m,\\ s \text{ even}}} \mathbb{C}\cdot v_{r,s},$$

which is the largest subspace of $D(k+1,N_f)$ on which $SO_2(\mathbb{R})$ acts through the character $u \mapsto u^{\ell+1}$.

Proposition 6.9. *Up to a scalar, there is a unique non-zero vector $v_{2m}^{\mathrm{hol}} \in D(k+1,N_f;2m)$ such that $\mathrm{d}\rho_\infty X_- v_{2m}^{\mathrm{hol}} = 0$. Such a vector is given, up to scalar, by*

$$v_{2m}^{\mathrm{hol}} = \sum_{\substack{0\le s\le 2m,\\ s \text{ even}}} c_s v_{2m-s,s}, \quad c_0 = 1, \quad c_s = (4\pi N_f)^{s/2} \prod_{\substack{0\le j\le s-2,\\ j \text{ even}}} \frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)} \quad (s\ge 2).$$

Proof. Let $v_{2m}^{\mathrm{hol}} \in D(k+1,N_f;2m)$ be a putative solution of $\mathrm{d}\rho_\infty X_- v_{2m}^{\mathrm{hol}} = 0$, and let

$$v_{2m}^{\mathrm{hol}} = \sum_{\substack{0\le s\le 2m,\\ s \text{ even}}} c_s v_{2m-s,s}$$

be its representation in terms of the basis $\{v_{2m-s,s} : 0 \le s \le 2m \text{ even}\}$. From the description of X_-,

$$d\rho_\infty X_- v_{2m-s,s} = \pi N_f(2m-s)(2m-s-1)v_{2m-s-2,s} - \frac{s}{4}(2k+s-1)v_{2m-s,s-2}.$$

By linearity, we can then write down explicitly $d\rho_\infty X_- v_{2m}^{\mathrm{hol}}$ in the form

$$d\rho_\infty X_- v_{2m}^{\mathrm{hol}} = \sum_{\substack{0\le s\le 2m,\\ s \text{ even}}} d_s v_{2m-(s+2),s},$$

where we understand that $v_{-2,2m} = 0$. Imposing that $d\rho_\infty X_- v_{2m}^{\mathrm{hol}} = 0$ then means that d_s must be zero for all s. From the description of $d\rho_\infty X_-$, one easily checks that

$$d_s = \pi N_f(2m-s)(2m-s-1)c_s - \frac{s+2}{4}(2k+s+1)c_{s+2},$$

hence $d_s = 0$ if and only if the recursive formula

$$c_{s+2} = 4\pi N_f \frac{(2m-s)(2m-s-1)}{(s+2)(2k+s+1)} c_s$$

holds. In particular, for each non-zero c_0 one can solve recursively all the c_s for $2 \le s \le 2m$ even. Setting $c_0 = 1$, this yields the expression in the statement. □

Corollary 6.10. *For the vector v_{2m}^{hol} in the previous proposition, one has*

$$||v_{2m}^{\mathrm{hol}}||^2 = \sum_{\substack{0\leq s\leq 2m,\\ s\ even}} \prod_{\substack{0\leq j\leq s-2,\\ j\ even}} \frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)}.$$

Proof. With notation as in Lemma 6.8 and Proposition 6.9, observe that if s is an even integer with $0\leq s\leq 2m$, then $c_2 = ||v_{2m-s,s}||^{-2}(4\pi N_f)^{-s/2}$. Using that the basis $v_{r,s}$ is orthogonal with respect to the inner product, we find out that

$$||v_{2m}^{\mathrm{hol}}||^2 = \sum_{\substack{0\leq s\leq 2m,\\ s\text{ even}}} c_s^2||v_{2m-s,s}||^2 = \sum_{\substack{0\leq s\leq 2m,\\ s\text{ even}}} (4\pi N_f)^{-s}||v_{2m-s,s}||^{-2} = \sum_{\substack{0\leq s\leq 2m,\\ s\text{ even}}} \prod_{\substack{0\leq j\leq s-2,\\ j\text{ even}}} \frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)}.$$

□

Now we come back to the isomorphism $\tilde{\pi}_\infty\otimes\omega_\infty\simeq\rho_\infty$, under which we identify $\mathbf{J}_\infty = \mathbf{h}_\infty\otimes\boldsymbol{\phi}_\infty$. One can check further that $\breve{\mathbf{h}}_\infty\otimes\breve{\boldsymbol{\phi}}_\infty = \tilde{V}_+^m\mathbf{h}_\infty\otimes\breve{\boldsymbol{\phi}}_\infty$ is identified with a multiple of $\breve{\mathbf{J}}_\infty := Y_+^{2m}\mathbf{J}_\infty$. Since the local regularized period we want to compute does not depend on replacing $\breve{\mathbf{h}}_\infty\otimes\breve{\boldsymbol{\phi}}_\infty$ by a multiple, and $\tilde{\pi}_\infty\otimes\omega_\infty\simeq\rho_\infty$ is an isometry, we may assume that $\breve{\mathbf{h}}_\infty\otimes\breve{\boldsymbol{\phi}}_\infty$ is identified exactly with $\breve{\mathbf{J}}_\infty$, so that we will simply write $\breve{\mathbf{J}}_\infty = \breve{\mathbf{h}}_\infty\otimes\breve{\boldsymbol{\phi}}_\infty$. Therefore,

$$\alpha_\infty^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \int_{\mathrm{SL}_2(\mathbb{R})}\frac{\langle\tau(g)\breve{\mathbf{g}}_\infty,\breve{\mathbf{g}}_\infty\rangle}{||\breve{\mathbf{g}}_\infty||^2}\frac{\overline{\langle\tilde{\pi}(g)\breve{\mathbf{h}}_\infty,\breve{\mathbf{h}}_\infty\rangle}}{||\breve{\mathbf{h}}_\infty||^2}\frac{\langle\omega_\infty(g)\breve{\boldsymbol{\phi}}_\infty,\breve{\boldsymbol{\phi}}_\infty\rangle}{||\breve{\boldsymbol{\phi}}_\infty||^2}dg = \int_{\mathrm{SL}_2(\mathbb{R})}\frac{\langle\tau(g)\breve{\mathbf{g}}_\infty,\breve{\mathbf{g}}_\infty\rangle}{||\breve{\mathbf{g}}_\infty||^2}\frac{\overline{\langle\rho(g)\breve{\mathbf{J}}_\infty,\breve{\mathbf{J}}_\infty\rangle}}{||\breve{\mathbf{J}}_\infty||^2}dg.$$

In view of this, we will compute the local integral $\alpha_\infty^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$ by actually computing the integral on the right hand side, which we will denote by $\alpha_\infty^\sharp(\breve{\mathbf{g}}_\infty,\breve{\mathbf{J}}_\infty)$.

Proposition 6.11. *With the above notation, we have*

$$\alpha_\infty^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = \frac{2\pi^2}{\ell||v_{2m}^{hol}||^2}.$$

Proof. With respect to the above model, τ_∞ might be realized as a subrepresentation of $\rho_\infty|_{\mathrm{SL}_2(\mathbb{R})}$, spanned by v_{2m}^{hol}, and hence we can assume the inner product for τ_∞ to be given by the restriction of the inner product for ρ_∞. Besides, $\alpha_\infty^\sharp(\breve{\mathbf{g}}_\infty,\breve{\mathbf{J}}_\infty)$ is invariant when replacing $\breve{\mathbf{g}}_\infty$ and $\breve{\mathbf{J}}_\infty$ by multiples of them, so that we may choose $\breve{\mathbf{g}}_\infty = v_{2m}^{\mathrm{hol}}$ and $\breve{\mathbf{J}}_\infty = v_{2m,0}$. Therefore,

$$\alpha_\infty^\sharp(\breve{\mathbf{g}}_\infty,\breve{\mathbf{J}}_\infty) = \int_{\mathrm{SL}_2(\mathbb{R})}\frac{\langle\tau(g)v_{2m}^{\mathrm{hol}},v_{2m}^{\mathrm{hol}}\rangle}{||v_{2m}^{\mathrm{hol}}||^2}\frac{\overline{\langle\rho(g)v_{2m,0},v_{2m,0}\rangle}}{||v_{2m,0}||^2}dg = \frac{1}{||v_{2m}^{\mathrm{hol}}||^2}\int_{\mathrm{SL}_2(\mathbb{R})}\langle\tau(g)v_{2m}^{\mathrm{hol}},v_{2m}^{\mathrm{hol}}\rangle\overline{\langle\rho(g)v_{2m,0},v_{2m,0}\rangle}dg,$$

where we have used that $||v_{2m,0}||^2=1$ according to our normalization of the inner product. Now, the orthogonal projection of $v_{2m,0}$ to the line generated by v_{2m}^{hol} is $\mathrm{pr}_{2m}^{\mathrm{hol}}(v_{2m,0}) = ||v_{2m}^{\mathrm{hol}}||^{-2}v_{2m}^{\mathrm{hol}}$. Therefore,

$$\langle\rho(g)v_{2m,0},v_{2m,0}\rangle = \langle\tau(g)\mathrm{pr}_{2m}^{\mathrm{hol}}(v_{2m,0}),\mathrm{pr}_{2m}^{\mathrm{hol}}(v_{2m,0})\rangle = \frac{1}{||v_{2m}^{\mathrm{hol}}||^4}\langle\tau(g)v_{2m}^{\mathrm{hol}},v_{2m}^{\mathrm{hol}}\rangle,$$

and we deduce that

$$\alpha_\infty^\sharp(\breve{\mathbf{g}}_\infty,\breve{\mathbf{J}}_\infty) = \frac{1}{||v_{2m}^{\mathrm{hol}}||^6}\int_{\mathrm{SL}_2(\mathbb{R})}|\langle\tau(g)v_{2m}^{\mathrm{hol}},v_{2m}^{\mathrm{hol}}\rangle|^2dg.$$

At this point, recall that using Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$, which tells us that any element $g\in\mathrm{SL}_2(\mathbb{R})$ is written in the form

$$g = \begin{pmatrix} y & 0\\ 0 & y^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k$$

for some $y\in\mathbb{R}_{>0}$, $x\in\mathbb{R}$ and $k\in\mathrm{SO}_2(\mathbb{R})$, we have chosen dg to be the Haar measure $y^{-2}dxdydk$, where dx and dy are the usual Lebesgue measure on $\mathbb{R}$ and dk is the Haar measure on $\mathrm{SO}_2(\mathbb{R})$ for which the total volume is π. Define now

$$A^+ := \left\{\begin{pmatrix} e^t & \\ & e^{-t}\end{pmatrix} : t\geq 0\right\},$$

and consider the map

$$(\mathrm{SO}_2(\mathbb{R})\times A^+\times\mathrm{SO}_2(\mathbb{R}))/\{\pm1\} \longrightarrow \mathrm{SL}_2(\mathbb{R}),$$
$$\left(k,\begin{pmatrix} e^t & \\ & e^{-t}\end{pmatrix},k'\right)\longmapsto k\begin{pmatrix} e^t & \\ & e^{-t}\end{pmatrix}k',$$

where on the left hand side $-1=(-1,1,-1)$. This map is a bijection outside the boundary of A^+, by virtue of Cartan decomposition. The product measure $dkdtdk'$ on $\mathrm{SO}_2(\mathbb{R})\times A^+\times\mathrm{SO}_2(\mathbb{R})$, where dt is the Lebesgue measure on $\mathbb{R}$ and both dk and dk' are the Haar measure on $\mathrm{SO}_2(\mathbb{R})$ for which the total volume is π, induces a measure on the quotient $(\mathrm{SO}_2(\mathbb{R})\times A^+\times\mathrm{SO}_2(\mathbb{R}))/\{\pm1\}$. Under the above bijection, one deduces (by a similar argument as the one in [II10, Section 12]) that $dg = 2 \cdot \sinh(2t)dkdtdk'$. On the other hand, it is well-known (cf. [Kna]) that

$$\left\langle \tau_\infty\left(\begin{pmatrix} e^t & \\ & e^{-t}\end{pmatrix}\right) v_{2m}^{\mathrm{hol}}, v_{2m}^{\mathrm{hol}}\right\rangle = ||v_{2m}^{\mathrm{hol}}||^2 \cosh(t)^{-(\ell+1)},$$

and therefore

$$\alpha_\infty^\sharp(\breve{\mathbf{g}}_\infty, \breve{\mathbf{J}}_\infty) = \frac{1}{||v_{2m}^{\mathrm{hol}}||^6}\int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau(g)v_{2m}^{\mathrm{hol}}, v_{2m}^{\mathrm{hol}}\rangle|^2 dg = \frac{2\pi^2}{||v_{2m}^{\mathrm{hol}}||^2}\int_0^\infty \cosh(t)^{-2(\ell+1)}\sinh(2t)dt = \frac{2\pi^2}{\ell ||v_{2m}^{\mathrm{hol}}||^2}.$$

□

Proposition 6.12. *We have $\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \pi^{2m}2^{2m}C_\infty(k,\ell)^{-1}$, where the constant $C_\infty(k,\ell) \in \mathbb{Q}^\times$ is given (setting $\ell - k = 2m$) by*

$$C_\infty(k,\ell) := (2m)!\frac{(\ell+k-1)!(k-1)!}{(2k-1)!(\ell-1)!}\sum_{\substack{0\leq s\leq 2m,\\ s \text{ even}}}\prod_{\substack{0\leq j\leq s-2,\\ j \text{ even}}}\frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)}.$$

Proof. From the previous proposition, we know that $\alpha_\infty^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = 2\pi^2\ell^{-1}||v_{2m}^{\mathrm{hol}}||^{-2}$. Besides, we have

$$\begin{aligned}\frac{L(1,\pi_\infty,\mathrm{ad})L(1,\tau_\infty,\mathrm{ad})}{L(\pi_\infty\otimes \mathrm{ad}(\tau_\infty),1/2)} &= \frac{2(2\pi)^{-\ell-1}\Gamma(\ell+1)\pi^{-1}\Gamma(1)2(2\pi)^{-2k}\Gamma(2k)\pi^{-1}\Gamma(1)}{2^2(2\pi)^{-2\ell-1}\Gamma(\ell+k)\Gamma(\ell-k+1)2(2\pi)^{-k}\Gamma(k)} = \\ &= \frac{2^{1-\ell-2k}\pi^{-\ell-2k-3}\ell!(2k-1)!}{2^{2-2\ell-k}\pi^{-2\ell-k-1}(\ell+k-1)!(\ell-k)!(k-1)!} = \frac{\pi^{\ell-k-2}2^{\ell-k-1}\ell!(2k-1)!}{(\ell+k-1)!(\ell-k)!(k-1)!}.\end{aligned}$$

Recalling that $2m = \ell - k$, it follows from the definition of $\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}})$ that

$$\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = \pi^{2m}2^{2m}\frac{1}{||v_{2m}^{\mathrm{hol}}||^2\cdot(2m)!}\frac{(2k-1)!(\ell-1)!}{(\ell+k-1)!(k-1)!},$$

and the claimed expression results from replacing $||v_{2m}^{\mathrm{hol}}||^2$ by its expression computed in Corollary 6.10. □

Remark 6.13. *Observe that when $\ell = k$, i.e. $m = 0$, the above expression reduces to $\mathcal{I}_\infty^\sharp(\breve{\mathbf{h}}, \breve{\mathbf{g}}, \breve{\boldsymbol{\phi}}) = 1$ (when $m = 0$, one has $v_{2m}^{\mathrm{hol}} = v_{0,0}$, and the inner product is normalized in this case so that $||v_{0,0}||^2 = 1$), which is coherent with* [PdVP19, Proposition 9.4].

7. Global computations and proof of Theorem 4.1

After the computation of regularized local SL$_2$-periods, this section is devoted to prove the explicit (global) theta identities that will allow us to conclude the proof of Theorem 4.1 following the strategy explained in Section 4.

7.1. **An explicit theta identity for the pair** $(\mathrm{GL}_2, \mathrm{GO}_{2,2})$**.** Let τ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated with g. We can regard $\tau \boxtimes \tau$ as a representation of $\mathrm{GSO}_{2,2}(\mathbb{A})$, and it extends to a unique automorphic representation Υ of $\mathrm{GO}_{2,2}(\mathbb{A})$ having a non-zero $\mathrm{O}(V_4')(\mathbb{A})$-invariant distribution, where $V_4' = \{x \in V_4 : \mathrm{tr}(x) = 0\}$. Then, the representations τ and Υ are in theta correspondence for the pair $(\mathrm{GL}_2, \mathrm{GO}_{2,2})$:

$$\Theta(\tau) = \Upsilon, \quad \Theta(\Upsilon) = \tau.$$

As in previous sections, write $\mathbf{g} \in \tau$ for the adelization of the newform g. The cusp form $\mathbf{g}\otimes\mathbf{g} \in \tau\boxtimes\tau$ extends to a cusp form $\mathbf{G} \in \Upsilon$ on $\mathrm{GO}_{2,2}(\mathbb{A})$ satisfying $\mathbf{G}(hh') = \mathbf{G}(h)$ for all $h \in \mathrm{GO}_{2,2}(\mathbb{A})$ and $h' \in \mu_2(\mathbb{A})$, where μ_2 is the subgroup of $\mathrm{O}_{2,2}$ generated by the involution $*$ on V_4. Observe also that, by construction,

$$\mathbf{G}_{|\mathrm{GL}_2\times\mathrm{GL}_2} = \mathbf{g}\otimes\mathbf{g} \in \tau\boxtimes\tau.$$

Associated with $\mathbf{g}$, we define a Bruhat–Schwartz function $\phi_{\mathbf{g}} = \otimes_v\phi_{\mathbf{g},v} \in \mathcal{S}(V_4(\mathbb{A}))$ by describing its local components as follows:

i) $\phi_{\mathbf{g},q} = \mathbf{1}_{\mathrm{M}_2(\mathbb{Z}_q)}$ at all primes $q \nmid N_g$;
ii) at primes $p \mid N_g$,

$$\phi_{\mathbf{g},p}\left(\left(\begin{smallmatrix} x_1 & x_2\\ x_3 & x_4\end{smallmatrix}\right)\right) = \mathbf{1}_{\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p\mathbb{Z}_p}(x_3)\left(\mathbf{1}_{\mathbb{Z}_p}(x_2) - p^{-1}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_2)\right);$$

iii) at the archimedean place,

$$\phi_{\mathbf{g},\infty}\left(\left(\begin{smallmatrix} x_1 & x_2\\ x_3 & x_4\end{smallmatrix}\right)\right) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)^{\ell+1}\exp(-\pi\mathrm{tr}(x^tx)).$$

By using the rules of the Weil representation of $\widetilde{\mathrm{SL}}_2\times\mathrm{GO}_{2,2}(\mathbb{A})$, one can easily check the following:

Lemma 7.1. *Let p be a finite prime. Then the following properties hold.*

- *If $p \nmid N_g$, then $\phi_{\mathbf{g},p}$ is fixed by $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$ and by $\mathrm{GL}_2(\mathbb{Z}_p)\times\mathrm{GL}_2(\mathbb{Z}_p) \subseteq \mathrm{GO}_{2,2}(\mathbb{Q}_p)$.*
- *If $p \mid N_g$, then $\phi_{\mathbf{g},p}$ is fixed by $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ and by $K_0(p)\times K_0(p) \subseteq \mathrm{GL}_2(\mathbb{Z}_p)\times\mathrm{GL}_2(\mathbb{Z}_p)$.*

With this, [PdVP19, Corollary 5.4] shows that

(20) $$\theta(\mathbf{G},\phi_{\mathbf{g}}) = 2^{\ell+1}\zeta_{\mathbb{Q}}(2)^{-2}\mu_{N_g}^{-1}\langle g,g\rangle\mathbf{g},$$

where $\mu_{N_g} = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_g)]$. We want to derive an explicit theta identity analogous to (20) involving the old form $\breve{\mathbf{g}}$ instead of $\mathbf{g}$. To begin with, define $\mathbf{g}^\sharp = \tau(t(2^{-2})_2)\mathbf{g} \in \tau$, where the element $t(2^{-1})_2$ is concentrated at the place 2. In parallel, we also define a Bruhat–Schwartz function by $\phi_{\mathbf{g}^\sharp} = 2^{-2}\omega(t(2^{-1})_2,1)\phi_{\mathbf{g}}$. That is, if $\phi_{\mathbf{g}^\sharp} = \otimes_v\phi_{\mathbf{g}^\sharp,v}$, then we keep $\phi_{\mathbf{g}^\sharp,v} = \phi_{\mathbf{g},v}$ for all $v \neq 2$ and set $\phi_{\mathbf{g}^\sharp,2} = 2^{-2}\omega_2(t(2^{-1})_2,1)\phi_{\mathbf{g},2}$. One can easily check that $\phi_{\mathbf{g}^\sharp,2} = \mathbf{1}_{V_4(2\mathbb{Z}_2)}$. With this slight modification at the prime 2, [PdVP19, Corollary 5.5] shows that

(21) $$\theta(\mathbf{G},\phi_{\mathbf{g}^\sharp}) = 2^{\ell-1}\zeta_{\mathbb{Q}}(2)^{-2}\mu_{N_g}^{-1}\langle g,g\rangle\mathbf{g}^\sharp.$$

Next, with this modification at $p=2$ observe from Section 5 that $\breve{\mathbf{g}}$ is obtained from $\mathbf{g}^\sharp$ by applying the level raising operator on τ defined by

$$\mathbf{V}_{M_g} : \varphi \mapsto \tau(\varpi_{M_g})\varphi,$$

where $M_g = N_f/N_g$ and $\varpi_{M_g} \in \mathrm{GL}_2(\mathbb{A})$ is 1 away from M_g, and equals $\varpi_p = \left(\begin{smallmatrix} p^{-1} & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ at primes $p \mid M_g$, hence $\breve{\mathbf{g}} = \mathbf{V}_{M_g}\mathbf{g}^\sharp$. Besides, consider the element $h_p = (1,\varpi_p) \in \mathrm{GL}_2(\mathbb{Q}_p)\times\mathrm{GL}_2(\mathbb{Q}_p)$ for each prime $p \mid M_g$, and identify it with its image $\rho(h_p) \in \mathrm{GSO}_{2,2}(\mathbb{Q}_p) \subseteq \mathrm{GO}_{2,2}(\mathbb{Q}_p)$. Let $\mathbf{Y}_p$ be the operator acting on Υ by $\Upsilon(h_p)$, and $\mathbf{Y}_{M_g}$ be defined as the product $\prod_{p\mid M_g}\mathbf{Y}_p$ (each acting on the corresponding component). Equivalently, we may write $h_{M_g} \in \mathrm{GL}_2(\mathbb{A})\times\mathrm{GL}_2(\mathbb{A})$ for the element which is trivial at all places $v \nmid M_g$ and which equals h_p at each prime $p \mid M_g$, and identify it with its image $\rho(h_{M_g}) \in \mathrm{GSO}_{2,2}(\mathbb{A})$. Then $\mathbf{Y}_{M_g}$ is the operator acting on Υ by $\Upsilon(h_{M_g})$. With this, consider the automorphic form

$$\breve{\mathbf{G}} := \mathbf{Y}_{M_g}\mathbf{G} \in \Upsilon,$$

and observe that

$$\breve{\mathbf{G}}_{|\mathrm{GL}_2\times\mathrm{GL}_2} = \mathbf{g}\otimes\mathbf{V}_{M_g}\mathbf{g} \in \tau\boxtimes\tau.$$

For each prime $p \mid M_g$ the automorphic form $\breve{\mathbf{G}}$ is fixed by the action of $\mathrm{GL}_2(\mathbb{Z}_p)\times K_0(p) \subseteq \mathrm{GO}_{2,2}(\mathbb{Q}_p)$.

Along similar lines, define a Bruhat–Schwartz function $\phi_{\breve{\mathbf{g}}} = \otimes_v\phi_{\breve{\mathbf{g}},v} \in \mathcal{S}(V_4(\mathbb{A}))$ by keeping $\phi_{\breve{\mathbf{g}},v} = \phi_{\mathbf{g}^\sharp,v}$ at places $v \nmid M_g$, and setting

$$\phi_{\breve{\mathbf{g}},p} := p^{-1}\omega_p(\varpi_p,h_p)\phi_{\mathbf{g}^\sharp,p}$$

at each prime $p \mid M_g$. Note that in this definition we are using the extended Weil representation. Again, a routinary check easily shows the following:

Lemma 7.2. *Let p be a prime dividing M_g. Then $\phi_{\breve{\mathbf{g}},p}$ is fixed by the action of $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$, and by the action of $\mathrm{GL}_2(\mathbb{Z}_p)\times K_0(p) \subseteq \mathrm{GO}_{2,2}(\mathbb{Q}_p)$.*

Proof. The proof just uses the invariance properties of $\phi_{\mathbf{g}^\sharp,p} = \phi_{\mathbf{g},p}$, the definition of $\phi_{\breve{\mathbf{g}},p}$, and the fact that if $\gamma \in K_0(p) \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ (resp. $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$), then $\gamma\varpi_p \in \varpi_p\mathrm{GL}_2(\mathbb{Z}_p)$ (resp. $\varpi_p\mathrm{SL}_2(\mathbb{Z}_p)$). □

With these definitions, the above explicit theta identities recalled from [PdVP19] can be adapted easily to identities relating $\breve{\mathbf{g}}$ and $\breve{\mathbf{G}}$ through the theta correspondence with respect to $\phi_{\breve{\mathbf{g}}}$. Indeed, most importantly for our purposes we have the following:

Proposition 7.3. *With the above notation,*

(22) $$\theta(\breve{\mathbf{G}},\phi_{\breve{\mathbf{g}}}) = 2^{\ell-1}M_g^{-1}\mu_{N_g}^{-1}\zeta_{\mathbb{Q}}(2)^{-2}\langle g,g\rangle\breve{\mathbf{g}}.$$

Proof. If $x \in \mathrm{GL}_2(\mathbb{A})$ and $y' \in \mathrm{GO}_{2,2}(\mathbb{A})$ is such that $\det(x) = \nu(y')$, then notice that $\det(x\varpi_{M_g}) = \nu(y'h_{M_g})$, hence we can write by applying the definitions

$$\theta(\breve{\mathbf{G}},\phi_{\breve{\mathbf{g}}})(x) = M_g^{-1}\int_{[\mathrm{O}_{2,2}]}\left(\sum_{v\in V_4(\mathbb{Q})}\omega(x\varpi_{M_g},y'yh_{M_g})\phi_{\mathbf{g}^\sharp}(v)\right)\mathbf{G}(y'yh_{M_g})dy =$$

$$= M_g^{-1}\int_{[\mathrm{O}_{2,2}]}\left(\sum_{v\in V_4(\mathbb{Q})}\omega(x\varpi_{M_g},y'h_{M_g}y)\phi_{\mathbf{g}^\sharp}(v)\right)\mathbf{G}(y'h_{M_g}y)dy,$$

and from this we deduce that $\theta(\breve{\mathbf{G}},\phi_{\breve{\mathbf{g}}}) = M_g^{-1}\tau(\varpi_{M_g})\theta(\mathbf{G},\phi_{\mathbf{g}^\sharp})$. The statement follows directly from (21). □

For later use, we compute in the following lemma the precise description of $\phi_{\breve{\mathbf{g}}}$ at primes p dividing N_f. Recall that $N_f = N_gM_g$, and $\gcd(N_g,M_g) = 1$.

Lemma 7.4. *With the above notation, if p is a prime dividing $N_f = N_gM_g$ we have*

$$\phi_{\breve{\mathbf{g}},p}\left(\left(\begin{smallmatrix} x_1 & x_2\\ x_3 & x_4\end{smallmatrix}\right)\right) = \begin{cases} \mathbf{1}_{\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p\mathbb{Z}_p}(x_3)\left(\mathbf{1}_{\mathbb{Z}_p}(x_2) - p^{-1}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_2)\right) & \text{if } p \mid N_g,\\ \mathbf{1}_{p\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{p\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4) & \text{if } p \mid M_g.\end{cases}$$

Proof. The case $p \mid N_g$ was already recalled above. When $p \mid M_g$, one just has to compute

$$\phi_{\breve{\mathbf{g}},p} = p^{-1}\omega_p(\varpi_p, h_p)\phi_{\mathbf{g}^\sharp,p} = p^{-1}\omega_p(\varpi_p, h_p)\phi_{\mathbf{g},p} = p^{-1}\omega(\varpi_p, h_p)\mathbf{1}_{\mathrm{M}_2(\mathbb{Z}_p)}$$

using the rules of the (extended) Weil representation. Recall that if $g \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and $h \in \mathrm{GO}_{2,2}(\mathbb{Q}_p)$, then

$$\omega_p(g,h)\phi = \omega\left(g\left(\begin{smallmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{smallmatrix}\right), 1\right) L(h)\phi,$$

where $L(h)\phi(x) = |\nu(h)|_p^{-1}\phi(h^{-1}\cdot x)$. Applying this for $(g,h) = (\varpi_p, h_p)$, where $h_p = (1, \varpi_p) \in \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)$ and $\nu(h_p) = p^{-1}$, one obtains the expression in the statement. □

7.2. **An explicit theta identity for the pair** $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}_2)$**.** We now focus on an explicit theta identity for the pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}_2)$. In [PdVP19, Proposition 5.10] we proved an explicit theta identity relating the adelization $\mathbf{h}$ of the newform

$$h = \sum_{n\geq 1} c(n)q^n \in S^+_{k+1/2}(N_f)$$

with the adelization $\mathbf{F} \in \Pi$ of its Saito–Kurokawa lift $F \in S_{k+1}(\Gamma_0^{(2)}(N_f))$. We will now proceed along the same lines to prove an analogous identity between $\breve{\mathbf{h}}$ and the adelization of $\Delta^m_{k+1}F$. Before doing so, let us first recall some properties concerning the classical forms h and F, as well as of their adelizations.

Let $\xi \in \mathbb{Q}_{>0}$, and write $\xi = \mathfrak{d}_\xi\mathfrak{f}_\xi^2$, where $\mathfrak{d}_\xi \in \mathbb{N}$ is such that $-\mathfrak{d}_\xi$ is the discriminant of $\mathbb{Q}(\sqrt{-\xi})/\mathbb{Q}$ and $\mathfrak{f}_\xi > 0$. If ξ is an integer, then it is well-known that

$$c(\xi) = c(\mathfrak{d}_\xi)\sum_{\substack{0<d\mid\mathfrak{f}_\xi,\\ (d,N_f)=1}} \mu(d)\chi_{-\xi}(d)d^{k-1}a_f(\mathfrak{f}_\xi/d),$$

where we write $a_f(n)$ for the Fourier coefficients of f. If p is a prime not dividing N_f, we let $\{\alpha_p, \alpha_p^{-1}\}$ be the Satake parameter of f at p. If p is a prime dividing N_f, we instead define $\alpha_p := p^{1/2-k}a_f(p) = -p^{-1/2}w_p$.

Besides, writing $\xi = \mathfrak{d}_\xi\mathfrak{f}_\xi^2 \in \mathbb{Q}_{>0}$ as before, let $e_p := \mathrm{val}_p(\xi)$ and define $\Psi_p(\xi; X) \in \mathbb{C}[X, X^{-1}]$ by

$$\Psi_p(\xi;X) = \begin{cases} \frac{X^{e_p+1}-X^{-e_p-1}}{X-X^{-1}} - p^{-1/2}\chi_{-\xi}(p)\frac{X^{e_p}-X^{-e_p}}{X-X^{-1}} & \text{if } p\nmid N_f, e_p\geq 0,\\ \chi_{-\xi}(p)(\chi_{-\xi}(p)+w_p)X^{e_p} & \text{if } p\mid N_f, e_p \geq 0,\\ 0 & \text{if } e_p<0.\end{cases} \tag{23}$$

As explained in [PdVP19, Lemma 3.1], one has the identity

$$c(\xi) = 2^{-\nu(N)}c(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{k-1/2}\prod_p \Psi_p(\xi;\alpha_p), \tag{24}$$

where one reads $c(\xi) = 0$ if ξ is not an integer. On the other hand, one also has $c(\xi) = e^{2\pi\xi}W_{\mathbf{h},\xi}(1)$, where

$$W_{\mathbf{h},\xi}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \mathbf{h}(u(x)g)\overline{\psi(\xi x)}dx$$

is the ξ-th Fourier coefficient of $\mathbf{h}$ with respect to the standard additive character ψ of $\mathbb{A}$.

As for the Saito–Kurokawa lift $F = \mathrm{SK}(h) \in S_{k+1}(\Gamma_0^{(2)}(N_f))$ of h, its Fourier expansion

$$F(Z) = \sum_B A_F(B)e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}, \quad Z = X+\sqrt{-1}Y \in \mathcal{H}_2,$$

can be explicitly given in terms of the coefficients $c(n)$. Indeed, for each symmetric, half-integral two-by-two matrix $B = \left(\begin{smallmatrix} b_1 & b_2/2\\ b_2/2 & b_3\end{smallmatrix}\right)$ one has

$$A_F(B) = \sum_{\substack{0<d\mid\gcd(b_1,b_2,b_3),\\ (d,N_f)=1}} d^k c(4\xi/d^2), \tag{25}$$

where $\xi = \det(B)$. The adelization of F is the automorphic form $\mathbf{F} : \mathrm{GSp}_2(\mathbb{A}) \to \mathbb{C}$ determined by

$$\mathbf{F}(\gamma g_\infty k) = \det(g_\infty)^{(k+1)/2}\det(C\sqrt{-1}+D)^{-k-1}F(g_\infty\sqrt{-1}),$$

whenever $\gamma \in \mathrm{GSp}_2(\mathbb{Q})$, $k \in K_0^{(2)}(N_f)$, and $g_\infty = \left(\begin{smallmatrix} * & *\\ C & D\end{smallmatrix}\right) \in \mathrm{GSp}_2^+(\mathbb{R})$. Here, $K_0^{(2)}(N_f) = \prod_p K_0^{(2)}(N_f;\mathbb{Z}_p)$ with

$$K_0^{(2)}(N_f;\mathbb{Z}_p) = \left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathrm{GSp}_2(\mathbb{Z}_p) : C \equiv 0 \pmod{N_f}\right\}$$

If $B \in \mathrm{Sym}_2(\mathbb{Q})$ is a two-by-two symmetric matrix, then the B-th Fourier coefficient of $\mathbf{F}$ is defined as the function

$$\mathcal{W}_{\mathbf{F},B}(h) = \int_{\mathrm{Sym}_2(\mathbb{Q})\backslash\mathrm{Sym}_2(\mathbb{A})} \mathbf{F}(n(X)h)\overline{\psi(\mathrm{Tr}(BX))}dX, \quad h \in \mathrm{GSp}_2(\mathbb{A}).$$

This Fourier coefficient is determined by its values at elements

$$h_\infty = n(X)m(A,1) = \begin{pmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{R}), \tag{26}$$

with $X \in \mathrm{Sym}_2(\mathbb{R})$ and $A \in \mathrm{GL}_2^+(\mathbb{R})$, and one has

$$\mathcal{W}_{\mathbf{F},B}(h_\infty) = A_F(B)\det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)},$$

where $Y = A^tA$ and $Z = X + \sqrt{-1}Y \in \mathcal{H}_2$.

Finally, let $\Delta_{k+1} : S^{nh}_{k+1}(\Gamma_0^{(2)}(N_f)) \to S^{nh}_{k+3}(\Gamma_0^{(2)}(N_f))$ be the Maass differential operator sending nearly holomorphic Siegel forms of weight $k+1$ (and level $\Gamma_0^{(2)}(N_f)$) to nearly holomorphic Siegel forms of weight $k+3$ (and level $\Gamma_0^{(2)}(N_f)$). Writing

$$Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix}, \quad \tau_i = x_i + \sqrt{-1}y_i,\ z = u + \sqrt{-1}v, \quad Z = X + \sqrt{-1}Y,$$

the Maass differential operator Δ_{k+1} is defined as

$$\Delta_{k+1} = \frac{1}{32\pi^2}\left[\frac{(k+1)(2k+1)}{\det(Y)} - 8\frac{\partial^2}{\partial\tau_1\partial\tau_2} + 2\frac{\partial^2}{\partial^2 z} + \frac{(2k+2)\sqrt{-1}}{\det(Y)}\left(y_1\frac{\partial}{\partial\tau_1} + y_2\frac{\partial}{\partial\tau_2} + v\frac{\partial}{\partial z}\right)\right], \tag{27}$$

and $\Delta^m_{k+1}F \in S^{nh}_{\ell+1}(\Gamma_0^{(2)}(N_f))$ has Fourier expansion

$$\Delta^m_{k+1}F(Z) = \sum_B A_F(B)C(B,Y)e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)},$$

where for each B one has

$$\begin{aligned} C(B,Y) = &\sum_{j=0}^{m}(-4\pi)^{j-m}\frac{\Gamma(\ell-m+\frac12)}{\Gamma(\ell-2m+j+\frac12)}\binom{m}{j}\det(B)^j\det(Y)^{j-m}\times \\ &\sum_{i=0}^{m-j}\frac{(2m-2j-i)!}{i!(m-j-i)!}(4\pi)^{i+j-m}\times\sum_{n=0}^{i}\frac{(\ell+1)!(-4\pi)^{-n}}{(\ell+1-n)!}\binom{i}{n}\mathrm{Tr}(BY)^{i-n}. \end{aligned} \tag{28}$$

The adelization of $\Delta^m_{k+1}F$ is the automorphic form $\tilde{D}^m_+\mathbf{F}$, where $\tilde{D}_+ = -\frac{1}{64\pi^2}D_+$ for a certain standard weight raising element $D_+ \in \mathcal{U}(\mathfrak{sp}(2,\mathbb{R})_{\mathbb{C}})$ (see [PSS]). One defines analogously the B-th Fourier coefficients of $\tilde{D}^m_+\mathbf{F}$, which are again determined by their values at elements h_∞ as before, and one has

$$\mathcal{W}_{\tilde{D}^m_+\mathbf{F},B}(h_\infty) = A_F(B)C(B,Y)\det(Y)^{(\ell+1)/2}e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}. \tag{29}$$

Having collected these facts, we now proceed with our main goal of this paragraph. We need to define a Bruhat–Schwartz function $\phi_{\breve{\mathbf{h}}} \in \mathcal{S}(V_5(\mathbb{A}))$ with respect to which we will compute the theta lift of $\breve{\mathbf{h}}$. To do so, we use the same model for V_5 as explained above, together with the embedding $V_4 \subset V_5$ obtained by identifying the former with the four-dimensional subspace $\langle v_3\rangle^\perp$ of V_5. With respect to this embedding, we define the Bruhat–Schwartz function $\phi_{\breve{\mathbf{h}}}$ as a product of two Bruhat–Schwartz functions, namely

$$\phi_{\breve{\mathbf{h}}} = \phi^{(1)}_{\breve{\mathbf{h}}}\phi^{(4)}_{\breve{\mathbf{h}}},$$

where $\phi^{(1)}_{\breve{\mathbf{h}}} = \otimes_v\phi^{(1)}_{\breve{\mathbf{h}},v} \in \mathcal{S}(\langle v_3\rangle) \simeq \mathcal{S}(\mathbb{A})$ is given by

$$\phi^{(1)}_{\breve{\mathbf{h}},v}(x) = \begin{cases} \mathbf{1}_{\mathbb{Z}_q}(x) & \text{if } v = q, \\ e^{-2\pi x^2} & \text{if } v = \infty, \end{cases}$$

and $\phi^{(4)}_{\breve{\mathbf{h}}} = \phi_{\breve{\mathbf{g}}} \in \mathcal{S}(V_4)$. In precise terms, with respect to the basis $v_1,\dots,v_5$ of V_5, for an arbitrary element $z = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5$, we have

$$\phi_{\breve{\mathbf{h}}}(z) := \phi^{(1)}_{\breve{\mathbf{h}}}(x_3)\phi^{(4)}_{\breve{\mathbf{h}}}\left(\left(\begin{smallmatrix} x_2 & x_1 \\ x_5 & x_4 \end{smallmatrix}\right)\right) = \phi^{(1)}_{\breve{\mathbf{h}}}(x_3)\phi_{\breve{\mathbf{g}}}\left(\left(\begin{smallmatrix} x_2 & x_1 \\ x_5 & x_4 \end{smallmatrix}\right)\right).$$

Recalling the description of the Bruhat–Schwartz function $\phi_{\breve{\mathbf{g}}}$ (see Section 7.1, especially Lemma 7.4), the function $\phi_{\breve{\mathbf{h}}} = \otimes_v\phi_{\breve{\mathbf{h}},v}$ is described locally at each place as follows.

i) At $v = 2$,

$$\phi_{\breve{\mathbf{h}},2}(z) = \mathbf{1}_{\mathbb{Z}_2}(x_3)\phi_{\breve{\mathbf{g}},2}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right) = \mathbf{1}_{2\mathbb{Z}_2}(x_1)\mathbf{1}_{2\mathbb{Z}_2}(x_2)\mathbf{1}_{\mathbb{Z}_2}(x_3)\mathbf{1}_{2\mathbb{Z}_2}(x_4)\mathbf{1}_{2\mathbb{Z}_2}(x_5).$$

ii) If $v = p$ is a prime not dividing $2N_f$, then

$$\phi_{\breve{\mathbf{h}},p}(z) = \mathbf{1}_{\mathbb{Z}_p}(x_3)\phi_{\breve{\mathbf{g}},p}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right) = \mathbf{1}_{\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{\mathbb{Z}_p}(x_5).$$

iii) If $v = p$ is a prime dividing N_g, then

$$\phi_{\breve{\mathbf{h}},p}(z) = \mathbf{1}_{\mathbb{Z}_p}(x_3)\phi_{\breve{\mathbf{g}},p}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right) = \left(\mathbf{1}_{\mathbb{Z}_p}(x_1) - p^{-1}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_1)\right)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p\mathbb{Z}_p}(x_5).$$

iv) If $v = p$ is a prime dividing $M_g = N_f/N_g$, then

$$\phi_{\breve{\mathbf{h}},p}(z) = \mathbf{1}_{\mathbb{Z}_p}(x_3)\phi_{\breve{\mathbf{g}},p}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right) = \mathbf{1}_{\mathbb{Z}_p}(x_1)\mathbf{1}_{p\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p\mathbb{Z}_p}(x_5).$$

v) If $v = \infty$, then

$$\phi_{\breve{\mathbf{h}},\infty}(z) = e^{-2\pi x_3^2}\phi_{\breve{\mathbf{g}},\infty}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{\ell+1}\exp(-\pi(x_1^2 + x_2^2 + 2x_3^2 + x_4^2 + x_5^2)).$$

At each finite prime p, the invariance properties of the Bruhat–Schwartz function $\phi_{\breve{\mathbf{h}},p}$ with respect to the actions of $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{GSp}_2(\mathbb{Q}_p)$ are collected in the following lemma.

Lemma 7.5. *Let p be an odd finite prime. Then the following assertions hold.*

- *If $p \nmid N_f$, then $\phi_{\breve{\mathbf{h}},p}$ is fixed by $\mathrm{SL}_2(\mathbb{Z}_p) \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$ and by $\mathrm{Sp}_2(\mathbb{Z}_p)$.*
- *If $p \mid N_f$, then $\phi_{\breve{\mathbf{h}},p}$ is fixed by $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ and by $\Gamma_0^{(2)}(p) \subseteq \mathrm{Sp}_2(\mathbb{Z}_p)$.*

By construction, it follows that the theta lift $\theta(\breve{\mathbf{h}}, \phi_{\breve{\mathbf{h}}})$ belongs to the space of $K_0^{(2)}(N_f)$-fixed vectors in Π, and hence it is the adelization of a classical (nearly holomorphic) Siegel modular form in $S_{\ell+1}^{nh}(\Gamma_0^{(2)}(N_f))$. In order to prove a relation between $\theta(\breve{\mathbf{h}}, \phi_{\breve{\mathbf{h}}})$ and the adelization $\mathbf{F}$ of the Saito–Kurokawa lift F, we will compute the B-th Fourier coefficients

$$h \mapsto \mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h) = \int_{\mathrm{Sym}_2(\mathbb{Q})\backslash \mathrm{Sym}_2(\mathbb{A})} \theta(\breve{\mathbf{h}}, \phi_{\breve{\mathbf{h}}})(n(X)h)\overline{\psi(\mathrm{tr}(BX))}dX, \quad h \in \mathrm{GSp}_2(\mathbb{A}),$$

of the theta lift $\theta(\breve{\mathbf{h}}, \phi_{\breve{\mathbf{h}}})$, for each positive definite rational symmetric two-by-two matrix

$$B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q}).$$

The Fourier coefficients $\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}$ are completely determined by their value at elements $h_\infty \in \mathrm{GSp}_2(\mathbb{R})$ as in (26). Setting $\xi = \det(B)$ and $\beta = (b_3, b_2/2, -b_1)$, it follows from [Ich05, Lemma 4.2] that

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h) = \int_{U(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \hat{\omega}(g,h)\hat{\phi}_{\breve{\mathbf{h}}}(\beta; 0, 1)W_{\breve{\mathbf{h}},\xi}(g)dg,$$

where

$$g \mapsto W_{\breve{\mathbf{h}},\xi}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \breve{\mathbf{h}}(u(x)g)\overline{\psi(\xi x)}dx$$

is the ξ-th Fourier coefficient of $\breve{\mathbf{h}}$, $\hat{\phi}_{\breve{\mathbf{h}}} = \otimes_v \hat{\phi}_{\breve{\mathbf{h}},v} \in \mathcal{S}(V_3(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}^2)$ is the Bruhat–Schwartz function obtained from $\phi_{\breve{\mathbf{h}}}$ by applying a change of polarization, and $\hat{\omega}$ denotes the Weil representation acting on $\mathcal{S}(V_3(\mathbb{A}))\otimes\mathcal{S}(\mathbb{A}^2)$ (by the rule $\hat{\omega}(g,h)\hat{\phi}(x) = (\omega(g,h)\phi)\hat{}\,$).

If $\xi = \det(B) > 0$, we write $\xi = \mathfrak{d}_\xi \mathfrak{f}_\xi^2$ with $\mathfrak{f}_\xi \in \mathbb{Q}_{>0}$ and $\mathfrak{d}_\xi \in \mathbb{N}$ such that $-\mathfrak{d}_\xi$ is the discriminant of the quadratic field $\mathbb{Q}(\sqrt{-\xi})$. Then we have (compare with [PdVP19, Lemma 5.14])

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B} = \begin{cases} 2^{-\nu(N_f)}c(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{k-1/2}\zeta_{\mathbb{Q}}(2)^{-1}\prod_v \mathcal{W}_{B,v} & \text{if } \xi > 0, \\ 0 & \text{if } \xi \leq 0, \end{cases} \tag{30}$$

where the local functions $\mathcal{W}_{B,v}$ are defined as the integrals

$$\mathcal{W}_{B,v}(h) = \int_{U(\mathbb{Q}_v)\backslash \mathrm{SL}_2(\mathbb{Q}_v)} \hat{\omega}_v(g,h)\hat{\phi}_{\breve{\mathbf{h}},v}(\beta; 0, 1)W_{v,\xi}(g)dg \times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SO}_2)^{-1} & \text{if } v = \infty. \end{cases} \tag{31}$$

Here, for each place v the function $W_{v,\xi}$ is a suitably normalized local Whittaker function associated with $\breve{\mathbf{h}}$. Namely, at the archimedean place $v = \infty$ we consider $W_{\infty,\xi} = \tilde{V}_+^m W_{\mathbf{h}_\infty,\xi}$, where $W_{\mathbf{h}_\infty,\xi}$ is the Whittaker function of $\widetilde{\mathrm{SO}}(2)$-type $k + 1/2$ defined by

$$W_{\mathbf{h}_\infty,\xi}(u(x)t(a)\tilde{k}_\theta) = e^{2\pi\sqrt{-1}\xi x}a^{k+1/2}e^{-2\pi\xi a^2}e^{\sqrt{-1}(k+1/2)\theta}, \qquad x \in \mathbb{R}, a \in \mathbb{R}_{>0}^\times, \theta \in \mathbb{R}/4\pi\mathbb{Z},$$

where for $\theta \in \mathbb{R}/4\pi\mathbb{Z}$ the elements $k_\theta \in \mathrm{SO}(2)$, $\tilde{k}_\theta \in \widetilde{\mathrm{SO}}(2)$ are defined by

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \tilde{k}_\theta = \begin{cases} [k_\theta, 1] & \text{if } -\pi < \theta \leq \pi, \\ [k_\theta, -1] & \text{if } \pi < \theta \leq 3\pi. \end{cases}$$

Observe that $W_{\mathbf{h}_\infty,\xi}(1) = e^{-2\pi\xi}$. And if $v = p$ is a finite prime, then $W_{p,\xi}$ is the non-zero multiple of the local Whittaker function $W_{\breve{\mathbf{h}}_p,\xi}$ determined by requiring that $W_{p,\xi}(1) = \Psi_p(\xi;\alpha_p)$. That is to say, $W_{p,\xi} := W_{\breve{\mathbf{h}}_p,\xi}(1)^{-1}\Psi_p(\xi;\alpha_p)\cdot W_{\breve{\mathbf{h}}_p,\xi}$. In Appendix A below we recall the definition of the local Whittaker functions $W_{\breve{\mathbf{h}}_p,\xi}$ and collect some special values of them that will be used in this section.

We will determine $\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}$ by computing via (31) the local values $\mathcal{W}_{B,p}(1)$ at all finite places, and the values $\mathcal{W}_{B,\infty}(h_\infty)$ at special elements $h_\infty \in \mathrm{GSp}_2(\mathbb{R})$ as in (26). We start dealing with the case of finite places. At rational primes $p \nmid M_g$, the computation of $\mathcal{W}_{B,p}(1)$ was already carried out in [PdVP19, Section 5]. With the same notation as before, if $\xi \neq 0$ and $\mu_p = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]$, then (check Equations (35), (36), and (37) in loc. cit.):

$$\mathcal{W}_{B,p}(1) = \begin{cases} \mathbf{1}_{\mathbb{Z}_p}(b_1,b_2,b_3)\sum_{n=0}^{\min(\mathrm{val}_p(b_i))} p^{\frac{n}{2}}\Psi_p(p^{-2n}\xi;\alpha_p) & \text{if } p\nmid 2N_f,\\ \mathbf{1}_{\mathbb{Z}_2}(b_1,b_2,b_3)2^{\frac{-7}{2}}\sum_{n=0}^{\min(\mathrm{val}_p(b_i))} 2^{\frac{n}{2}}\Psi_2(2^{-2n+2}\xi;\alpha_2) & \text{if } p=2,\\ \mathbf{1}_{\mathbb{Z}_p}(b_1,b_2,b_3)\mu_p^{-1}\Psi_p(\xi;\alpha_p) & \text{if } p\mid N_g.\end{cases} \tag{32}$$

Thus we assume from now on that p is a prime dividing $M_g = N_f/N_g$, and first study the change of polarization.

Lemma 7.6. *Let p be a prime dividing M_g, and let $x = (x_1,x_2,x_3) \in V_3(\mathbb{Q}_p)$, and $y = (y_1,y_2) \in \mathbb{Q}_p^2$. Then*

$$\hat{\phi}_{\breve{\mathbf{h}},p}(x;y) = \phi_{p,1}(x)\cdot\phi_{p,2}(y)$$

where the functions $\phi_{p,1} \in \mathcal{S}(V_3(\mathbb{Q}_p))$ and $\phi_{p,2} \in \mathcal{S}(\mathbb{Q}_p^2)$ are given by

$$\phi_{p,1}(x) = \mathbf{1}_{p\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3), \qquad \phi_{p,2}(y) = \mathbf{1}_{p\mathbb{Z}_p}(y_1)\mathbf{1}_{\mathbb{Z}_p}(y_2).$$

Proof. This follows straightforward from the definition of the partial Fourier transform,

$$\hat{\phi}_{\breve{\mathbf{h}},p}(x;y) = \int_{\mathbb{Q}_p}\phi_{\breve{\mathbf{h}},p}(z;x;y_1)\psi_p(-y_2z)dz.$$

□

With this change of polarization, one has

$$\hat{\omega}_p(g,h)\hat{\phi}_{\breve{\mathbf{h}},p}(\beta;0,1) = \omega_p(g,h)\phi_{p,1}(\beta)\cdot\phi_{p,2}((0,1)g),$$

and using equation (31) we can now compute $\mathcal{W}_{B,p}(1)$. A first key observation is to use the $\Gamma_0(p)$-invariance of $\breve{\mathbf{h}}_p$ and of $\hat{\phi}_{\breve{\mathbf{h}},p}$, together with the fact that $\mathrm{SL}_2(\mathbb{Q}_p) = U(\mathbb{Q}_p)T(\mathbb{Q}_p)\mathrm{SL}_2(\mathbb{Z}_p)$, to rewrite $\mathcal{W}_{B,p}(1)$ as

$$\mathcal{W}_{B,p}(1) = \mu_p^{-1}\sum_{r\in R_p}\int_{\mathbb{Q}_p^\times}\hat{\omega}_p(t(a)r,1)\hat{\phi}_{\breve{\mathbf{h}},p}(\beta;0,1)W_{p,\xi}(t(a)r)|a|_p^{-2}d^\times a = \mu_p^{-1}\sum_{r\in R_p} I(B,r), \tag{33}$$

where R_p is a set of representatives for $\mathrm{SL}_2(\mathbb{Z}_p)/\Gamma_0(p)$. We can choose the set R_p to consist of the elements

$$r_b = \begin{pmatrix}1 & 0\\ b & 1\end{pmatrix} \text{ with } b\in\{0,1,\dots,p-1\}, \text{ and } s = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

and so we are reduced to compute the values $I(B,1)$, $I(B,s)$, and $I(B,r_b)$ for $b = 1,\dots,p-1$. To compute these values, first we point out that the local Whittaker functions $W_{p,\xi}$ satisfy

$$W_{p,\xi}(t(a)g) = \chi_\psi(a^{-1})\chi_\delta(a^{-1})|a|_p^{1/2}W_{p,a^2\xi}(g), \qquad a\in\mathbb{Q}_p^\times,\ g\in\mathrm{SL}_2(\mathbb{Q}_p),$$

and recall also that $\chi_\psi(x) = (-1,x)_p\gamma(x,\psi)$. In particular, using this one easily finds that

$$I(B,r) = \sum_{n\in\mathbb{Z}} p^{n/2}\chi_\delta(p^n)\chi_\psi(p^n)\int_{\mathbb{Z}_p^\times}(a,p^n)_p\omega_p(t(p^na)r,1)\phi_{p,1}(\beta)\phi_{p,2}((0,1)t(p^na)r)W_{p,p^{2n}a^2\xi}(r)d^\times a.$$

We fix B and split the discussion according to whether $r = 1$, $r = s$, or $r = r_b$ for some $b\in\mathbb{Z}_p^\times$. To compute the terms $\omega_p(t(p^na)r,1)\phi_{p,1}(\beta)$, we will need to apply the rules for the Weil representation, relative to V_3 and $\psi = \psi_p^{-D}$. It is easy to check that $\gamma(\psi,V_3) = 1$ and $\chi_{\psi,V_3}(x) = (-2,x)_p\chi_\psi(x)^3$ for $x\in\mathbb{Q}_p^\times$ (cf. Section 3.3). In the discussion below, we put $\nu_i = \mathrm{val}_p(b_i)$.

Case $r = 1$**.** Start observing that for $n\in\mathbb{Z}$ and $a\in\mathbb{Z}_p^\times$ we have $(0,1)t(p^na) = (0,p^{-n}a^{-1})$, and hence $\phi_{p,2}((0,1)t(p^na)) = 1$ if $n\leq 0$, and vanishes otherwise. Therefore, we obtain

$$I(B,1) = \sum_{n\leq 0} p^{n/2}\chi_\delta(p^n)\chi_\psi(p^n)\int_{\mathbb{Z}_p^\times}(a,p^n)_p\omega_p(t(p^na),1)\phi_{p,1}(\beta)W_{p,p^{2n}a^2\xi}(1)d^\times a.$$

By applying the rules of the Weil representation, one finds

$$\omega_p(t(p^na),1)\phi_{p,1}(\beta) = (-2,p^n)_p(a,p^n)_p\chi_\psi(p^n)^3p^{-3n/2}\mathbf{1}_{\mathbb{Z}_p}(p^nb_1)\mathbf{1}_{\mathbb{Z}_p}(p^nb_2)\mathbf{1}_{p\mathbb{Z}_p}(p^nb_3).$$

Therefore, using that $W_{p,p^{2n}a^2\xi}(1) = W_{p,p^{2n}\xi}(1)$ for all $a \in \mathbb{Z}_p^\times$ and that $\mathrm{vol}(\mathbb{Z}_p^\times, d^\times a) = 1$ (and also that $\chi_\psi(p^n)^4 = 1$),

$$I(B,1) = \sum_{n\le 0} p^{-n}\chi_\delta(p^n)(-2,p^n)_p \mathbf{1}_{\mathbb{Z}_p}(p^n b_1)\mathbf{1}_{\mathbb{Z}_p}(p^n b_2)\mathbf{1}_{p\mathbb{Z}_p}(p^n b_3)W_{p,p^{2n}\xi}(1) =$$

$$= \sum_{n=0}^{\min(\nu_1,\nu_2,\nu_3-1)} p^n\chi_\delta(p^n)(-2,p^n)_p W_{p,p^{-2n}\xi}(1).$$

Case $r = s$. Similarly as before, we now have $(0,1)t(p^n a)s = (-p^{-n}a^{-1}, 0)$, thus $\phi_{p,2}((0,1)t(p^n a)s) = 1$ if $n \le -1$, and vanishes otherwise. Therefore,

$$I(B,s) = \sum_{n\le -1} p^{n/2}\chi_\delta(p^n)\chi_\psi(p^n)\int_{\mathbb{Z}_p^\times}(a,p^n)_p\omega_p(t(p^n a)s,1)\phi_{p,1}(\beta)W_{p,p^{2n}a^2\xi}(s)d^\times a.$$

Now applying the rules of the Weil representation yields

$$\omega_p(t(p^n a)s,1)\phi_{p,1}(\beta) = (-2,p^n)_p(a,p^n)_p\chi_\psi(p^n)^3 p^{-3n/2-1}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(p^n b_1)\mathbf{1}_{\mathbb{Z}_p}(p^n b_2)\mathbf{1}_{\mathbb{Z}_p}(p^n b_3).$$

From this we conclude, using again that $W_{p,p^{2n}a^2\xi}(s) = W_{p,p^{2n}\xi}(s)$ for all $a \in \mathbb{Z}_p^\times$, that

$$I(B,s) = \sum_{n=1}^{\min(\nu_1+1,\nu_2,\nu_3)} p^{n-1}\chi_\delta(p^n)(-2,p^n)_p W_{p,p^{-2n}\xi}(s).$$

Case $r = r_b$, $b \in \mathbb{Z}_p^\times$. We have now $(0,1)t(p^n a)r_b = (p^{-n}a^{-1}b, p^{-n}a^{-1})$, and hence $\phi_{p,2}((0,1)t(p^n a)r_b) = 1$ if $n \le -1$, and vanishes otherwise. Thus again we can rewrite

$$I(B,r_b) = \sum_{n\le -1} p^{n/2}\chi_\delta(p^n)\chi_\psi(p^n)\int_{\mathbb{Z}_p^\times}(a,p^n)_p\omega_p(t(p^n a)r_b,1)\phi_{p,1}(\beta)W_{p,p^{2n}a^2\xi}(r_b)d^\times a.$$

We can now use that $W_{p,p^{2n}a^2\xi}(r_b) = \psi_p(b^{-1}p^{2n}a^2\xi)W_{p,p^{2n}a^2\xi}(s)$ to rewrite this as

$$I(B,r_b) = \sum_{n\le -1} p^{n/2}\chi_\delta(p^n)\chi_\psi(p^n)\int_{\mathbb{Z}_p^\times}(a,p^n)_p\psi_p(b^{-1}p^{2n}a^2\xi)\omega_p(t(p^n a)r_b,1)\phi_{p,1}(\beta)W_{p,p^{2n}a^2\xi}(s)d^\times a.$$

To deal with the term $\omega_p(t(p^n a)r_b,1)\phi_{p,1}(\beta)$, we first notice that

$$r_b = u(b^{-1})s\begin{pmatrix} -b & -1 \\ 0 & -b^{-1}\end{pmatrix}.$$

The rightmost element belongs to $\Gamma_0(p)$, and hence leaves invariant the function $\phi_{p,1}$. We must therefore compute $\omega_p(t(p^n a)u(b^{-1})s)\phi_{p,1}(\beta)$. By applying the rules of the Weil representation, we have

$$\omega_p(t(p^n a)u(b^{-1})s)\phi_{p,1}(\beta) = (-2,p^n)_p(a,p^n)_p\chi_\psi(p^n)^3 p^{-3n/2-1}\overline{\psi_p(b^{-1}a^2p^{2n}\xi)}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(p^n b_1)\mathbf{1}_{\mathbb{Z}_p}(p^n b_2)\mathbf{1}_{\mathbb{Z}_p}(p^n b_3).$$

From this, it follows that

$$I(B,r_b) = \sum_{n=1}^{\min(\nu_1+1,\nu_2,\nu_3)} p^{n-1}\chi_\delta(p^n)(-2,p^n)_p W_{p,p^{-2n}\xi}(s),$$

and hence $I(B,r_b) = I(B,s)$, independently on b.

Putting all the above discussion together, defining $m(B) := \min(\nu_1+1,\nu_2,\nu_3)$ and $n(B) := \min(\nu_1,\nu_2,\nu_3-1)$ we can rewrite (33) as

$$\mathcal{W}_{B,p}(1) = \mu_p^{-1}\left(\sum_{n=0}^{n(B)} p^n\chi_\delta(p^n)(-2,p^n)_p W_{p,p^{-2n}\xi}(1) + \sum_{n=1}^{m(B)} p^n\chi_\delta(p^n)(-2,p^n)_p W_{p,p^{-2n}\xi}(s)\right). \tag{34}$$

Proposition 7.7. *Let $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3\end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q})$ be a two-by-two symmetric matrix. If $b_1 \notin \mathbb{Z}_p$, or $b_2 \notin \mathbb{Z}_p$, or $b_3 \notin p\mathbb{Z}_p$, then $\mathcal{W}_{B,p}(1) = 0$. Otherwise, let $\xi = \det(B)$ and define integers*

$$m(B) := \min(\nu_1+1,\nu_2,\nu_3), \quad n(B) := \min(\nu_1,\nu_2,\nu_3-1),$$

where $\nu_i = \mathrm{val}_p(b_i)$. Then $m(B) \ge n(B) \ge 0$, and

$$\mathcal{W}_{B,p}(1) = \mathcal{E}_p(B)\mu_p^{-1}\Psi_p(\xi;\alpha_p),$$

where $\mu_p = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]$ and

$$\mathcal{E}_p(B) = 1 + (1-p^{-1})\sum_{n=1}^{n(B)} p^{2n}\chi_{-2\delta}(p)^n + (n(B)-m(B))p^{2n(B)+1}\chi_{-2\delta}(p)^{n(B)+1}$$

Proof. It is clear from (34) that $\mathcal{W}_{B,p}(1)=0$ if either $b_1\notin\mathbb{Z}_p$, $b_2\notin\mathbb{Z}_p$, or $b_3\notin p\mathbb{Z}_p$. Let us thus assume that $b_1\in\mathbb{Z}_p$, $b_2\in\mathbb{Z}_p$, and $b_3\in p\mathbb{Z}_p$, and notice that then we have $m(B),n(B)\geq 0$. It is also clear from their definition that $m(B)\geq n(B)$, and one can easily check that $2n(B)\leq \mathrm{val}_p(\xi)$. In addition, when $m(B)>n(B)$ one necessarily has $m(B)=n(B)+1$. In particular, observe that the factor $n(B)-m(B)$ in the definition of $\mathcal{E}_p(B)$ is either 0 or -1, according to whether $m(B)=n(B)$ or $m(B)>n(B)$, respectively.

From our computations in Appendix A, for $n=1,\dots,m(B)$ we have $W_{p,p^{-2n}\xi}(s)=pW_{p,p^{-2(n-1)}\xi}(s)$. Using this and reindexing the second sum in (34), we get

$$\mathcal{W}_{B,p}(1)=\left(\sum_{n=0}^{n(B)}p^n\chi_{-2\delta}(p)^nW_{p,p^{-2n}\xi}(1)+p^2\chi_{-2\delta}(p)\sum_{n=0}^{m(B)-1}p^n\chi_{-2\delta}(p)^nW_{p,p^{-2n}\xi}(s)\right)\mu_p^{-1}.$$

Now, for $n=0,\dots,m(B)-1$, we also have $W_{p,p^{-2n}\xi}(s)=-p^{-1}W_{p,p^{-2n}\xi}(1)$, and hence we deduce

$$\mathcal{W}_{B,p}(1)=\left(\sum_{n=0}^{n(B)}p^n\chi_{-2\delta}(p)^nW_{p,p^{-2n}\xi}(1)-p\chi_{-2\delta}(p)\sum_{n=0}^{m(B)-1}p^n\chi_{-2\delta}(p)^nW_{p,p^{-2n}\xi}(1)\right)\mu_p^{-1}=$$
$$=\left(\sum_{n=0}^{n(B)}p^{2n}\chi_{-2\delta}(p)^n-p\chi_{-2\delta}(p)\sum_{n=0}^{m(B)-1}p^{2n}\chi_{-2\delta}(p)^n\right)\mu_p^{-1}\Psi_p(\xi;\alpha),$$

where we have used that $W_{p,p^{-2n}\xi}(1)=\Psi_p(p^{-2n}\xi;\alpha_p)=p^n\Psi_p(\xi;\alpha_p)$. With this, suppose first that $m(B)=n(B)$. In this case, we get

$$\mathcal{W}_{B,p}(1)=\left((1-p\chi_{-2\delta}(p))\sum_{n=0}^{m(B)-1}p^{2n}\chi_{-2\delta}(p)^n+p^{2m(B)}\chi_{-2\delta}(p)^{m(B)}\right)\mu_p^{-1}\Psi_p(\xi;\alpha_p)=$$
$$=\left(1+(1-p^{-1})\sum_{n=1}^{n(B)}p^{2n}\chi_{-2\delta}(p)^n\right)\mu_p^{-1}\Psi_p(\xi;\alpha_p).$$

If $m(B)>n(B)$ instead, then $m(B)-1=n(B)$ and the above yields

$$\mathcal{W}_{B,p}(1)=\left((1-p\chi_{-2\delta}(p))\sum_{n=0}^{n(B)}p^{2n}\chi_{-2\delta}(p)^n\right)\mu_p^{-1}\Psi_p(\xi;\alpha_p)=$$
$$=\left(1+(1-p^{-1})\sum_{n=1}^{n(B)}p^{2n}\chi_{-2\delta}(p)^n-p^{2n(B)+1}\chi_{-2\delta}(p)^{n(B)+1}\right)\mu_p^{-1}\Psi_p(\xi;\alpha_p).$$

□

We must emphasize that the quantities $\mathcal{E}_p(B)$ in the proposition are non-zero rational numbers, and that they depend only on p and B, as the notation suggests.

Now we deal with the computation of $\mathcal{W}_{B,\infty}(h_\infty)$, for an arbitrary $h_\infty=n(X)m(A,1)$ as in (26), with $X\in\mathrm{Sym}_2(\mathbb{R})$ and $A\in\mathrm{GL}_2^+(\mathbb{R})$. We recall that by definition

$$\mathcal{W}_{B,\infty}(h_\infty)=\mathrm{vol}(\mathrm{SO}_2(\mathbb{R}))^{-1}\int_{U(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})}\hat{\omega}(g,h)\hat{\phi}_{\check{\mathbf{h}},\infty}(\beta;0,1)W_{\infty,\xi}(g)dg,$$

where $W_{\infty,\xi}$ is a Whittaker function of $\widetilde{\mathrm{SO}}(2)$-type $\ell+1/2$, and satisfies

$$W_{\infty,\xi}(t(a))=a^{k+1/2}e^{-2\pi\xi a^2}\sum_{j=0}^{m}(-4\pi)^{j-m}a^{2j}\frac{\Gamma(k+1/2+m)}{\Gamma(k+1/2+j)}\binom{m}{j}\qquad\text{for } a\in\mathbb{R}_{>0}.$$

Proposition 7.8. *With the above notation, one has*

$$\mathcal{W}_{B,\infty}(n(X)m(A,1))=\begin{cases}2^{\ell+1}\det(Y)^{(\ell+1)/2}C(B,Y)e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)} & \text{if } B>0,\\ 0 & \text{otherwise,}\end{cases}$$

where $Y=A^tA^{-1}$, $Z=X+\sqrt{-1}Y$, *and* $C(B,Y)$ *is defined as in* (28).

Proof. This is Lemma 5.6 in [Che19]. □

We can now finish the computation of $\mathcal{W}_{\theta(\check{\mathbf{h}},\phi_{\check{\mathbf{h}}}),B}(h_\infty)$, where $h_\infty=n(X)m(A,1)\in\mathrm{GSp}_2(\mathbb{R})$ is as in (26). So fix $B\in\mathrm{Sym}_2(\mathbb{Q})$ be as usual, with entries given by b_1, $b_2/2$ and b_3, and set $\xi=\det(B)$. If $\xi\leq 0$, the above proposition implies that $\mathcal{W}_{\theta(\check{\mathbf{h}},\phi_{\check{\mathbf{h}}}),B}(h_\infty)=0$, so let us assume that $\xi>0$ and write $\xi=\mathfrak{d}_\xi\mathfrak{f}_\xi^2$ with conventions as above. To simplify the notation in the computation, for each prime p dividing $M_g = N_f/N_g$ we let $\mathcal{E}_p(B)$ be as in Proposition 7.7, and define

$$\mathcal{E}(B) := \prod_{p|M_g} \mathcal{E}_p(B).$$

With this, it follows from (30), (32) and Proposition 7.7 that $\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h_\infty) = 0$ if either $b_1 \notin \mathbb{Z}$, $b_2 \notin \mathbb{Z}$, or $b_3 \notin M_g\mathbb{Z}$. And assuming that $b_1, b_2 \in \mathbb{Z}$ and $b_3 \in M_g\mathbb{Z}$, combining (30), (32), Proposition 7.7, and Proposition 7.8 we find

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h_\infty) = 2^{-\nu(N_f)}\zeta_{\mathbb{Q}}(2)^{-1}c(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{k-1/2}\mathcal{W}_{B,\infty}(h_\infty)\prod_p \mathcal{W}_{B,p}(1) =$$

$$= 2^{-\nu(N_f)-7/2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}c(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{k-1/2}\mathcal{E}(B)\mathcal{W}_{B,\infty}(h_\infty)\prod_{p\nmid N_f}\sum_{n=0}^{\min(\nu_i)} p^{\frac{n}{2}}\Psi_p\left(\frac{4\xi}{p^{2n}};\alpha_p\right)\prod_{p|N_f}\Psi_p(\xi;\alpha_p),$$

where we have used that for an odd prime $p \nmid N_f$ one has $\Psi_p(p^{-2n}\xi;\alpha_p) = \Psi_p(4p^{-2n}\xi;\alpha_p)$. We also have $\mathfrak{d}_{4\xi} = \mathfrak{d}_\xi$ and $\mathfrak{f}_\xi^{k-1/2} = 2^{-k+1/2}\mathfrak{f}_{4\xi}^{k-1/2}$, hence we can rewrite the above expression as

$$2^{-k-\nu(N_f)-3}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}c(\mathfrak{d}_{4\xi})\mathfrak{f}_{4\xi}^{k-1/2}\mathcal{E}(B)\mathcal{W}_{B,\infty}(h_\infty)\sum_{\substack{d|(b_1,b_2,b_3),\\(d,N_f)=1}} d^{1/2}\prod_{p\nmid N_f}\Psi_p\left(\frac{4\xi}{d^2};\alpha_p\right)\prod_{p|N_f}\Psi_p(\xi;\alpha_p).$$

Now, for each integer d with $(d, N_f) = 1$ and each $p \mid N_f$ we have $\Psi_p(\xi;\alpha_p) = \Psi_p(4d^{-2}\xi;\alpha_p)$. We also have $\mathfrak{d}_{4\xi} = \mathfrak{d}_{4\xi/d^2}$ and $\mathfrak{f}_{4\xi}^{k-1/2} = d^{k-1/2}\mathfrak{f}_{4\xi/d^2}$, hence

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h_\infty) = 2^{-k-3}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\mathcal{E}(B)\mathcal{W}_{B,\infty}(h_\infty)\sum_{\substack{d|(b_1,b_2,b_3),\\(d,N_f)=1}} d^k 2^{-\nu(N_f)}c(\mathfrak{d}_{4\xi/d^2})\mathfrak{f}_{4\xi/d^2}^{k-1/2}\prod_p \Psi_p\left(\frac{4\xi}{d^2};\alpha_p\right) =$$

$$= 2^{\ell-k-2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\mathcal{E}(B)\det(Y)^{(\ell+1)/2}C(B,Y)e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}\sum_{\substack{d|(b_1,b_2,b_3),\\(d,N_f)=1}} d^k c\left(\frac{4\xi}{d^2}\right),$$

where in the second equality we have plugged in the value of $\mathcal{W}_{B,\infty}(h_\infty)$ from Proposition 7.8 and we have used (24). But now the last sum is exactly the B-th Fourier coefficient of $F = \mathrm{SK}(h)$, hence

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h_\infty) = \begin{cases} 2^{\ell-k-2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\mathcal{E}(B)A_F(B)C(B,Y)\det(Y)^{(\ell+1)/2}e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)} & \text{if } b_3 \in M_g\mathbb{Z}, \\ 0 & \text{otherwise.}\end{cases}$$

Comparing with (29), we see that

$$\mathcal{W}_{\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),B}(h_\infty) = \begin{cases} 2^{\ell-k-2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\mathcal{E}(B)\mathcal{W}_{\tilde{D}_+^m\mathbf{F},B}(h_\infty) & \text{if } b_3 \in M_g\mathbb{Z}, \\ 0 & \text{otherwise,}\end{cases}$$

and hence we have proved the following statement:

Proposition 7.9. *With the above notation,*

$$\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}) = 2^{2m-2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\breve{\mathbf{F}},$$

where $\breve{\mathbf{F}}$ *is the adelization of the nearly holomorphic Siegel form* $\breve{F} \in S_{\ell+1}^{nh}(\Gamma_0^{(2)}(N_f))$ *whose Fourier coefficients are given by*

$$A_{\breve{F}}(B) = \begin{cases} \mathcal{E}(B)A_{\Delta_{k+1}^m F}(B) & \textit{if } b_3 \in M_g\mathbb{Z}, \\ 0 & \textit{otherwise.}\end{cases}$$

By recalling that $A_{\Delta_{k+1}^m F}(B) = C(B,Y)A_F(B)$, with $C(B,Y)$ as in (28), we see that

$$A_{\breve{F}}(B) = \begin{cases} \mathcal{E}(B)C(B,Y)A_F(B) & \text{if } b_3 \in M_g\mathbb{Z}, \\ 0 & \text{otherwise.}\end{cases}$$

7.3. **Conclusion of the proof of Theorem 4.1.** We can now finish the proof of Theorem 4.1. Suppose first that $\Lambda(f \otimes \mathrm{Ad}(g), k) \neq 0$, so that $\mathcal{Q}$ is non-vanishing by Corollary 4.3. Then we know from (9) that

$$\Lambda(f\otimes\mathrm{Ad}(g),k) = \frac{4\Lambda(1,\pi,\mathrm{ad})\Lambda(1,\tau,\mathrm{ad})}{\langle\breve{\mathbf{h}},\breve{\mathbf{h}}\rangle\langle\breve{\mathbf{g}},\breve{\mathbf{g}}\rangle\langle\breve{\boldsymbol{\phi}},\breve{\boldsymbol{\phi}}\rangle}\left(\prod_v \mathcal{I}_v^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})^{-1}\right)\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}), \tag{35}$$

where $\breve{\mathbf{h}}\otimes\breve{\mathbf{g}}\otimes\breve{\boldsymbol{\phi}}\in\tilde{\pi}\otimes\tau\otimes\omega$ is our test-vector as chosen in Section 5. Now we can compute all the terms on the right hand side. First of all, by Propositions 6.1, 6.7, 6.12, we have

$$\prod_v \mathcal{I}_v^\sharp(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})^{-1} = \pi^{-2m}2^{-2m}C_\infty(k,\ell)2^{-\nu(M_g)}N_g\prod_{p\mid M_g}(p+1) = \pi^{-2m}2^{-2m-\nu(M_g)}C_\infty(k,\ell)N_g\mu_{M_g}.$$

Secondly, we have $\langle\breve{\boldsymbol{\phi}},\breve{\boldsymbol{\phi}}\rangle = 2^{-1}$, $\langle\breve{\mathbf{g}},\breve{\mathbf{g}}\rangle=\zeta_{\mathbb{Q}}(2)^{-1}\langle g,g\rangle$ and (cf. [Wal80, page 22])

$$\begin{aligned}\langle\breve{\mathbf{h}},\breve{\mathbf{h}}\rangle &= (4\pi)^{-2m}\frac{\Gamma(k+m+1/2)\Gamma(m+1)}{\Gamma(k+1/2)}\langle\mathbf{h},\mathbf{h}\rangle = 2^{-1}\zeta_{\mathbb{Q}}(2)^{-1}(4\pi)^{-2m}\frac{\Gamma(k+m+1/2)\Gamma(m+1)}{\Gamma(k+1/2)}\langle h,h\rangle\\ &= 2^{-1-2m}\zeta_{\mathbb{Q}}(2)^{-1}(4\pi)^{-2m}\frac{(\ell+k-1)!(k-1)!m!}{(k+m-1)!(2k-1)!}\langle h,h\rangle.\end{aligned}$$

Besides, by applying [Hid00, Theorem 5.15], [Wat02, §3.2.1], we find

$$\Lambda(1,\pi,\mathrm{ad}) = 2^{2k}N_f^{-1}\mu_{N_f}\langle f,f\rangle,\quad \Lambda(1,\tau,\mathrm{ad}) = 2^{\ell+1}N_g^{-1}\mu_{N_g}\langle g,g\rangle,$$

and altogether the first term on the right hand side of (35) reads

$$\frac{4\Lambda(1,\pi,\mathrm{ad})\Lambda(1,\tau,\mathrm{ad})}{\langle\breve{\mathbf{h}},\breve{\mathbf{h}}\rangle\langle\breve{\mathbf{g}},\breve{\mathbf{g}}\rangle\langle\breve{\boldsymbol{\phi}},\breve{\boldsymbol{\phi}}\rangle} = \pi^{2m}\frac{2^{8m+3k+5}\zeta_{\mathbb{Q}}(2)^2\mu_{N_f}\mu_{N_g}}{N_fN_g}\frac{(k+m-1)!(2k-1)!}{(\ell+k-1)!(k-1)!m!}\frac{\langle f,f\rangle}{\langle h,h\rangle}.$$

Finally, it remains to compute the value of the global SL_2-period evaluated on our test vector, $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})$. By Proposition 7.3 we have

$$\theta(\breve{\mathbf{G}},\phi_{\breve{\mathbf{g}}}) = C_1^{-1}\breve{\mathbf{g}},\quad C_1 = 2^{1-\ell}M_g\mu_{N_g}\zeta_{\mathbb{Q}}(2)^2\langle g,g\rangle^{-1},$$

and hence [Qiu14, Theorem 5.3] implies that $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = C_1^2\mathcal{P}(\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),\breve{\mathbf{G}})$, where

$$\mathcal{P}:\Pi\otimes\Pi\otimes\Upsilon\otimes\Upsilon\longrightarrow\mathbb{C}$$

is the $\mathrm{SO}(V_4)$-period defined by associating any choice of decomposable vectors $\mathbf{F}_1,\mathbf{F}_2\in\Pi$, $\mathbf{G}_1,\mathbf{G}_2\in\Upsilon$ the product of integrals

$$\mathcal{P}(\mathbf{F}_1,\mathbf{F}_2,\mathbf{G}_1,\mathbf{G}_2) := \left(\int_{[\mathrm{SO}(V_4)]}\mathbf{F}_1(h)\overline{\mathbf{G}_1(h)}dh\right)\left(\int_{[\mathrm{SO}(V_4)]}\overline{\mathbf{F}_2(h)}\mathbf{G}_2(h)dh\right),$$

and we abbreviate $\mathcal{P}(\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),\breve{\mathbf{G}}) = \mathcal{P}(\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}),\breve{\mathbf{G}},\breve{\mathbf{G}})$. But from Proposition 7.9 we know that $\theta(\breve{\mathbf{h}},\phi_{\breve{\mathbf{h}}}) = C_2\breve{\mathbf{F}}$ with $C_2 = 2^{2m-2}\mu_{N_f}^{-1}\zeta_{\mathbb{Q}}(2)^{-1}$, hence $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = C_1^2C_2^2\mathcal{P}(\breve{\mathbf{F}},\breve{\mathbf{G}})$. Now, $\breve{\mathbf{F}}$ is the adelization of $\breve{F}$ and $\breve{\mathbf{G}}_{|\mathrm{GL}_2\times\mathrm{GL}_2} = \mathbf{g}\otimes\mathbf{V}_{M_g}\mathbf{g}$, which is the adelization of $g\times V_{M_g}g$, hence $\mathcal{P}(\breve{\mathbf{F}},\breve{\mathbf{G}}) = C_3^2|\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}},g\times V_{M_g}g\rangle|^2$ with $C_3 = 2^{-1}\zeta_{\mathbb{Q}}(2)^{-2}$ (cf. [II10, Section 9]). Altogether,

$$\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}}) = (C_1C_2C_3)^2|\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}},g\times V_Mg\rangle|^2 = 2^{4m-2\ell-4}\zeta_{\mathbb{Q}}(2)^{-2}M_g^2\mu_{N_f}^{-2}\mu_{N_g}^2\frac{|\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}},g\times V_{M_g}g\rangle|^2}{\langle g,g\rangle^2}.$$

Combining all the terms, we obtain that

$$\Lambda(f\otimes\mathrm{Ad}(g),k) = 2^{6m+k+1-\nu(M_g)}\frac{M_g^2\mu_{N_g}^3\mu_{M_g}}{N_f\mu_{N_f}}\frac{(k+m-1)!(2k-1)!}{(\ell+k-1)!(k-1)!m!}C_\infty(k,\ell)\cdot\frac{\langle f,f\rangle}{\langle h,h\rangle}\frac{|\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}},g\times V_{M_g}g\rangle|^2}{\langle g,g\rangle^2}.$$

By using the definition of $C_\infty(k,\ell)$, we see

$$\frac{(k+m-1)!(2k-1)!}{(\ell+k-1)!(k-1)!m!}C_\infty(k,\ell) = \frac{(2m)!}{m!}\frac{(k+m-1)!}{(\ell-1)!}\sum_{\substack{0\le s\le 2m,\\ s\text{ even}}}\prod_{\substack{0\le j\le s-2,\\ j\text{ even}}}\frac{(2m-j)(2m-j-1)}{(j+2)(2k+j+1)} = C_\infty(f,g),$$

and so the claimed formula in Theorem 4.1 follows by noticing that $\mu_{N_f} = \mu_{N_g}\mu_{M_g}$.

If $\Lambda(f\otimes\mathrm{Ad}(g),k)=0$, then Corollary 4.3 tells us that the functional $\mathcal{Q}$ is identically zero, and hence the central formula stated in Theorem 4.1 holds trivially because all the local periods continue to be non-zero whereas $\mathcal{Q}(\breve{\mathbf{h}},\breve{\mathbf{g}},\breve{\boldsymbol{\phi}})=0$, and hence $\langle\breve{F}_{|\mathcal{H}\times\mathcal{H}},g\times V_{M_g}g\rangle = 0$. □

One can use the explicit formula in Theorem 1.2 to prove Deligne's algebraicity conjecture for $\Lambda(f\otimes\mathrm{Ad}(g),k)$:

Corollary 7.10. *Let $f\in S_{2k}(N_f)$ and $g\in S_{\ell+1}(N_g)$ be as in Theorem 4.1. If $\sigma\in\mathrm{Aut}(\mathbb{C})$, then*

$$\left(\frac{\Lambda(f\otimes\mathrm{Ad}(g),k)}{\langle g,g\rangle^2c^+(f)}\right)^\sigma = \frac{\Lambda(f^\sigma\otimes\mathrm{Ad}(g^\sigma),k)}{\langle g^\sigma,g^\sigma\rangle^2c^+(f^\sigma)},$$

where $c^+(f)$ is the 'plus' period associated with f as in [Shi77]. In particular, if $\mathbb{Q}(f,g)$ denotes the number field generated by the Fourier coefficients of f and g, then

$$\Lambda(f\otimes\mathrm{Ad}(g),k)^{\mathrm{alg}} := \frac{\Lambda(f\otimes\mathrm{Ad}(g),k)}{\langle g,g\rangle^2c^+(f)}\in\mathbb{Q}(f,g).$$

The corollary can be proved along the same lines as [PdVP19, Corollary 6.5] or [Che19, Corollary 8.2], using Kohnen's formula [Koh85] relating Fourier coefficients of h and central values of twisted L-series for f (see also [CC19, Theorem A] for a different approach). We leave the details for the reader.

8. Application to subconvexity

This section is devoted to derive a partial result towards the subconvexity problem stated in (1) in the Introduction, as a direct consequence of the computation of local SL_2-periods in Section 6 (some of those computations already carried out in [PdVP19]).

As a piece of motivation, let us recall that the subconvexity problems for the families of automorphic L-functions

$$L(\pi\otimes\tau, s), \qquad \pi \text{ on } GL_2 \text{ fixed}, \ \tau \text{ on } GL_2 \text{ varying}, \tag{36}$$

$$L(\pi\otimes \mathrm{ad}(\tau), s), \qquad \pi \text{ on } PGL_2 \text{ fixed}, \ \tau \text{ on } GL_2 \text{ varying}, \tag{37}$$

are closely related to fundamental arithmetic equidistribution questions. Indeed, the subconvexity problem in (36) is related for instance to the distribution of integral points on spheres, representations of integers by ternary quadratic forms, Heegner points and closed geodesics on modular surfaces, etc. (see, e.g., [IS00], [MV06], [Mic07]). Besides, the subconvexity problem in (37) has applications towards the limiting mass distribution of automorphic forms, also referred to as the 'arithmetic quantum unique ergodicity' (see [Sar95], [IS00], [HS10], [NPS14], [Sar11]). This motivated and pushed the efforts to tackle these problems by many authors. The major achievement of these efforts was the solution for the subconvexity problem in (36) given by Michel–Venkatesh in [MV10], relying crucially on Michel's observation that (36) can be reduced to the subconvexity problem for $L(\pi\otimes\chi, s)$ with π on GL_2 fixed and χ on GL_1 varying (cf. [Mic04]).

Concerning the subconvexity problem in (37), much less progress has been done until very recently (except for the case where π is dihedral, see [Sar01]). We focus our attention here in the work of Nelson [Nel19], who reduces the subconvexity problem in (37), under important local assumptions, to the subconvexity problem for

$$L(\tau\otimes\tau^\vee\otimes\chi, s), \qquad \chi \text{ on } GL_1 \text{ fixed}, \ \tau \text{ on } GL_2 \text{ varying}. \tag{38}$$

Thanks to the factorization

$$L(\tau\otimes\tau^\vee\otimes\chi, s) = L(\chi, s)L(\mathrm{ad}(\tau)\otimes\chi, s),$$

one can further reduce the subconvexity problem in (38) to that for

$$L(\mathrm{ad}(\tau)\otimes\chi, s) \qquad \chi \text{ on } GL_1 \text{ fixed}, \ \tau \text{ on } GL_2 \text{ varying}.$$

Recent work of Munshi [Mun] on this latter problem, which can be seen as a specialization of (37) upon restricting π to an Eisenstein series, motivates Hypothesis (H) below in Nelson's study of (37).

As commented in the Introduction, our contribution to the subconvexity problem in (37) relies on providing the bounds for local SL_2-periods required in the main result of Nelson [Nel19]. Let us fix an odd integer $\ell \geq 1$, and let q traverse an infinite sequence $\mathfrak{Q}$ of (odd) primes. For each prime $q\in\mathfrak{Q}$, choose an automorphic representation τ of $GL_2(\mathbb{A})$ such that

$$\text{the local component } \tau_q \text{ is a twist of the special representation,}$$

and let $\mathcal{G}$ be the infinite family of all such representations τ, when varying q in $\mathfrak{Q}$. We may also refer to elements in $\mathcal{G}$ as pairs (q,τ), in order to keep track of the distinguished prime q of each automorphic representation τ in the family.

We consider the following hypothesis on the family $\mathcal{G}$, namely the existence of a subconvex bound for $L(\tau\otimes\tau^\vee\otimes\chi, 1/2)$ with polynomial dependence upon the character χ:

Hypothesis (H): there exist an absolute constant $\delta_0 = \delta_0(\mathcal{G}) > 0$ such that for all $\tau\in\mathcal{G}$ and all unitary characters χ of $\mathbb{A}^\times/\mathbb{Q}^\times$ one has

$$L(\tau\otimes\tau^\vee\otimes\chi, 1/2) \ll C(\tau\otimes\tau^\vee\otimes\chi)^{1/4-\delta_0}C(\chi)^{O(1)}.$$

Here, we use the usual 'big O' and Vinogradov notation, so that the above hypothesis is equivalent to the existence of absolute constants $c_0, A_0 \geq 0$ and $\delta_0 > 0$ (depending only on the family $\mathcal{G}$) such that

$$|L(\tau\otimes\tau^\vee\otimes\chi, 1/2)| \leq c_0 C(\tau\otimes\tau^\vee\otimes\chi)^{1/4-\delta_0}C(\chi)^{A_0}$$

for all $\tau\in\mathcal{G}$ and all unitary characters χ of $\mathbb{A}^\times/\mathbb{Q}^\times$.

Theorem 8.1. *Fix an odd integer $\ell\geq 1$. With the above notation, suppose that every $\tau\in\mathcal{G}$ is the automorphic representation associated with some newform $g\in S_{\ell+1}(N_g)$ of odd squarefree level N_g (and trivial nebentypus), and that $\mathcal{G}$ satisfies Hypothesis (H). Then, there exists an absolute constant $\delta=\delta(\mathcal{G})$ with the following property: if $\pi=\pi(f)$ is an automorphic representation of $PGL_2(\mathbb{A})$ associated with a newform $f\in S_{2k}(N_f)$ of weight $2k$, with $1\leq k\leq\ell$ odd, and odd squarefree level, then*

$$L(\pi\otimes\mathrm{ad}(\tau), 1/2) \ll C(\pi\otimes\mathrm{ad}(\tau))^{1/4-\delta}P^{O(1)},$$

where $P = C(\pi) \cdot \prod_{p \neq q} C(\tau_p)$.

Proof. The theorem follows by checking that the hypotheses of [Nel19, Theorem 2] hold. Indeed, it is enough to check that for every $\tau = \tau(g) \in \mathcal{G}$ and every $\pi = \pi(f)$ as in the statement, and every rational prime p, either π_p is unramified or the conclusion of the conjecture in [Nel19, §2.15.1] is satisfied. If p does not divide N_f, then π_p is unramified and there is nothing to say. If $p \mid N_f$, then we must prove that (with notations as in Section 6) there are unit vectors $\varphi_1 \in \tilde{\pi}_p$, $\varphi_2 \in \tau$, $\varphi_3 \in \omega_p$ such that

$$\alpha_p^\sharp(\varphi_1, \varphi_2, \varphi_3)(C(\pi_p)C(\tau_p))^{O(1)} \gg 1 \quad \text{and} \quad \mathcal{S}(\varphi_i) \ll (C(\pi_p)C(\tau_p))^{O(1)} \; (i = 1, 2, 3), \tag{39}$$

where $\alpha_p^\sharp(\varphi_1, \varphi_2, \varphi_3)$ is the regularized local integral as defined in Section 4, $C(\pi_p)$ and $C(\tau_p)$ denote the analytic conductor of π_p and τ_p, respectively, and $\mathcal{S}(\varphi_i)$ are the Sobolev norms of φ_i (on the corresponding representation, in each case), as defined[2] in [MV10, Section 2] (see also [Nel, Sections 4.6, 5.3]). We let φ_1, φ_2, φ_3 be the p-th components $\breve{\mathbf{h}}_p$, $\breve{\mathbf{g}}_p$, $\breve{\boldsymbol{\phi}}_p$ of our test vector chosen in Section 5, normalizing them so that each of the φ_i has norm 1. Since each of the φ_i is fixed by $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$, it is well–known that the Sobolev norm of φ_i satisfies $\mathcal{S}(\varphi_i) = ||\varphi_i||p^{O(1)} = p^{O(1)}$. Having this into account, we divide the discussion in two cases.

a) If $p \mid N_g$, then $C(\pi_p) = C(\tau_p) = p$. Besides, from [PdVP19, Proposition 7.14] we have
$$\alpha_p^\sharp(\varphi_1, \varphi_2, \varphi_3) = \frac{p - w_p}{p + w_p}\zeta_p(2)^{-1} = \frac{(p - w_p)^2}{p^2},$$
and therefore we clearly see that both conditions in (39) hold.

b) If $p \nmid N_g$, we have $C(\pi_p) = p$ and $C(\tau_p) = 1$. In this case, we may invoke instead Proposition 6.6, which tells us that
$$\alpha_p^\sharp(\varphi_1, \varphi_2, \varphi_3) = \frac{2(p-1)^2(p-\xi)(p\xi - 1)}{p^2(p+1)(p+\xi)(p\xi+1)},$$
where $\xi = \chi(p)^2$ with $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ the unramified character such that $\tau_p = \pi(\chi, \chi^{-1})$. Again, it follows that both conditions in (39) are satisfied.

□

Some final comments are in order:

i) For a family $\mathcal{G}$ as in the Introduction, we immediately see that $P = C(\pi)$, and hence Theorem 1.2 is just a particular instance of the above statement.

ii) If the quantity P in the statement satisfies $\log P \gg \log q$, then the conclusion is worse than the convex bound. So the theorem becomes interesting only under the assumption that
$$C(\pi) \cdot \prod_{p \neq q} C(\tau_p) = q^{o(1)},$$
where $o(1)$ is a quantity tending to 0 as q tends to ∞. One may hence assume this, which implies in particular that π_q is unramified, and that τ is 'essentially unramified away from q'.

iii) As hinted in the introduction of [Nel19], modulo the Hypothesis (H) the above theorem should lead to strong quantitative forms of the arithmetic quantum unique ergodicity conjecture in the prime level aspect.

Appendix A. Computation of local Whittaker functions at special elements

We collect here some special values of Whittaker functions attached to the local components of $\breve{\mathbf{h}}$, needed in Section 7.2. If p is a prime, and $\xi \in \mathbb{Q}$, we define
$$W_{\breve{\mathbf{h}}_p,\xi}(g) = \int_{\mathbb{Q}_p} \breve{\mathbf{h}}_p(s^{-1}u(x)g)\overline{\psi_p(\xi x)}dx = \int_{\mathbb{Q}_p} \breve{\mathbf{h}}_p(s^{-1}u(x)g)\psi_p(-\xi x)dx, \quad g \in \mathrm{SL}_2(\mathbb{Q}_p),$$
where $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and ψ_p denotes the standard additive character of $\mathbb{Q}_p$. Assume that p divides N_f. By using the definition of $\breve{\mathbf{h}}_p \in \tilde{\pi}_p$ given in Section 5, together with the transformation property spelled out in (10), one can prove the following statements. The details are left to the reader.

Proposition A.1. *With the above notation,*
$$W_{\breve{\mathbf{h}}_p,\xi}(1) = \begin{cases} p^{-r}(1 + (-p\xi, p)_p) & \text{if } \mathrm{val}_p(\xi) = 2r, r \geq 0, \\ p^{-r-1}(p+1) & \text{if } \mathrm{val}_p(\xi) = 2r+1, r \geq 0, \\ 0 & \text{if } \mathrm{val}_p(\xi) < 0. \end{cases}$$

[2] One actually defines a family of Sobolev norms $\mathcal{S}_d$, for each integer d, and then $\mathcal{S}$ denotes a Sobolev norm of the form $\mathcal{S}_d$ for some fixed large enough d (the "implied index").

Proposition A.2. *With the above notation,*

$$W_{\breve{\mathbf{h}}_p,\xi}(s) = \begin{cases} -p^{-r-1}(1+(-p\xi,p)_p) & \text{if } \mathrm{val}_p(\xi) = 2r, r \geq 0, \\ -p^{-r-2}(p+1) & \text{if } \mathrm{val}_p(\xi) = 2r+1, r \geq -1, \\ 0 & \text{if } \mathrm{val}_p(\xi) < -1. \end{cases}$$

Proposition A.3. *With the above notation, if $b \in \mathbb{Z}_p^\times$ and $r_b = \left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$, then*

$$W_{\breve{\mathbf{h}}_p,\xi}(r_b) = \begin{cases} -\psi_p(b^{-1}\xi)p^{-r-1}(1+(-p\xi,p)_p) & \text{if } \mathrm{val}_p(\xi) = 2r, r \geq 0, \\ -\psi_p(b^{-1}\xi)p^{-r-2}(p+1) & \text{if } \mathrm{val}_p(\xi) = 2r+1, r \geq -1, \\ 0 & \text{if } \mathrm{val}_p(\xi) < -1. \end{cases}$$

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