Characterizing identifying codes from the spectrum of a graph or digraph

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Extended Abstract

1 Introduction

We consider simple digraphs (or directed graphs) without loops or multiple edges.

Given a vertex subset $U \subset V$, let $N^-[U] = \bigcup_{u\in U} N^-[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subset V$ is a $(1, \leq \ell)$-*identifying code* in D when, for all distinct subsets $X, Y \subset V$, with $1 \leq |X|, |Y| \leq \ell$, the corresponding closed in-neighborhoods within the set C are different, that is

$$N^-[X] \cap C \neq N^-[Y] \cap C. \tag{1}$$

A $(1, \leq 1)$-identifying code is referred to as an *identifying code.*

Laihonen [6] proved the following result for graphs.

Theorem 1 *[6] Let $k \geq 2$ be an integer.*

1. *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*
2. *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

Besides, Laihonen and Ranto [7] showed that if G is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then $\ell \leq \delta$, where δ is the minimum degree of G.

Regarding digraphs, the authors proved in [3] that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if its girth is at least 5. They also characterized the 2-in-regular digraphs having a $(1, \leq 2)$-identifying code or a $(1, \leq 3)$-identifying code. Moreover, they gave some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 2$ to admit a $(1, \leq \delta^-)$-identifying code. As a corollary of this result, they proved that a graph of minimum degree $\delta \geq 2$ and girth at least 7 admits a $(1, \leq \delta)$-identifying code.

Recall that if D admits a $(1, \leq \ell)$-identifying code, then it admits a $(1, \leq \ell')$-identifying code for any $\ell' < \ell$.

A digraph with adjacency matrix $A = (a_{uv})$ has eigenvalue λ and eigenvector $\boldsymbol{x} = (x_u)$ if and only if

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \quad \Leftrightarrow \quad \sum_{v\in V} a_{uv}x_v = \sum_{v\in N^+(u)} x_v = \lambda x_u \qquad \text{for all } u \in V. \tag{2}$$

This last equation leads to the charge displacement interpretation; for more information about it, see Fiol and Mitjana [4] . Moreover, the spectral radius of A is the largest among the absolute values of its eigenvalues.

Recall also that a *transitive tournament* TT_3 if formed by vertices u, v, and w, and arcs (u, v), (u, w), and (v, w). Besides, we called *bipartite tournament* $BT_{2,2}$ to the digraph formed by vertices u, v, w and x, and arcs (u, w), (u, x), (v, w) and (v, x). See both digraphs in Figure 1 (b) and Figure 1 (c), respectively.

Our first lemma is the only non-spectral result of this paper.

Lemma 2 *Let D be a d-in-regular digraph on n vertices, without any of the subdigraphs of Figure 1. If D admits a $(1, \leq \ell)$-identifying code, then $\ell \in \{d, d+1\}$.*

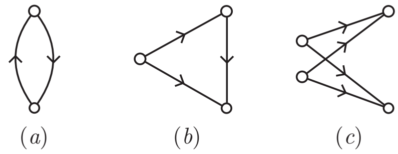

Figure 1: The subdigraphs forbidden by Lemma 2: (a) The digon, (b) TT_3, and (c) $BT_{2,2}$.

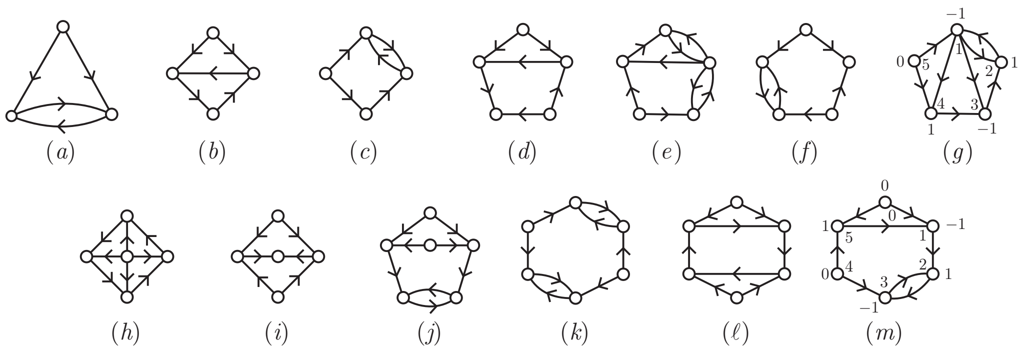

Figure 2: The forbidden subdigraphs characterizing a 2-in-regular digraph admitting a $(1, \leq 1)$-identifying code (only (a)), or a $(1, \leq 2)$-identifying code (all of them). The subdigraph (g) and (m) have their vertices numbered in the interior, and the entries of the eigenvector corresponding to eigenvalue -1 in the exterior.

2 Main results

We begin with a result that gives a sufficient (spectral) condition for a digraph to admit a $(1, \leq 1)$-identifying code.

Lemma 3 *Let D be a digraph with adjacency matrix A and with a set of eigenvalues denoted by* $\mathrm{ev}(A)$*. If $-1 \notin \mathrm{ev}(A)$, then D admits a $(1, \leq 1)$-identifying code.*

Observe that the converse is not true since, if $-1 \in ev(A)$, this does not imply that some of its corresponding eigenvectors are of the form $\boldsymbol{e_i} - \boldsymbol{e_j}$. For example, the digraph in Figure 2 (j) has -1 as an eigenvalue, but it does admit a $(1, \leq 1)$-identifying code.

In [3] the authors gave the following theorem, which is a combinatorial characterization of a 2-in-regular digraph admitting a $(1, \leq 1)$-, $(1, \leq 2)$-, or $(1, \leq 3)$-identifying code.

Theorem 4 ([3]) *Let D be a 2-in-regular digraph. Then,*

(i) D admits a $(1, \leq 1)$-identifying code if and only if it does not contain any subdigraph isomorphic to Figure 2 (a).

(ii) D admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to one of the digraphs of Figure 2.

(iii) D admits a $(1, \leq 3)$-identifying code if and only if it is oriented, TT_3-free and does not contain any subdigraph isomorphic to one of the digraphs of Figure 3.

Next, we present an algebraic-combinatorial sufficient condition for a 2-in-regular digraph to admit a $(1, \leq 2)$- or $(1, \leq 3)$-identifying code, but first we need the following lemma.

Lemma 5 *Let D' be a digraph with maximum in-degree Δ^- having an eigenvalue λ with eigenvector $\boldsymbol{x}' = (x'_u)$, such that $x'_v = 0$ for any vertex $v \in V(D')$ with $d^-(v) < \Delta^-$. Then, any Δ^--in-regular digraph D containing D' as a subdigraph has also the eigenvalue λ.*

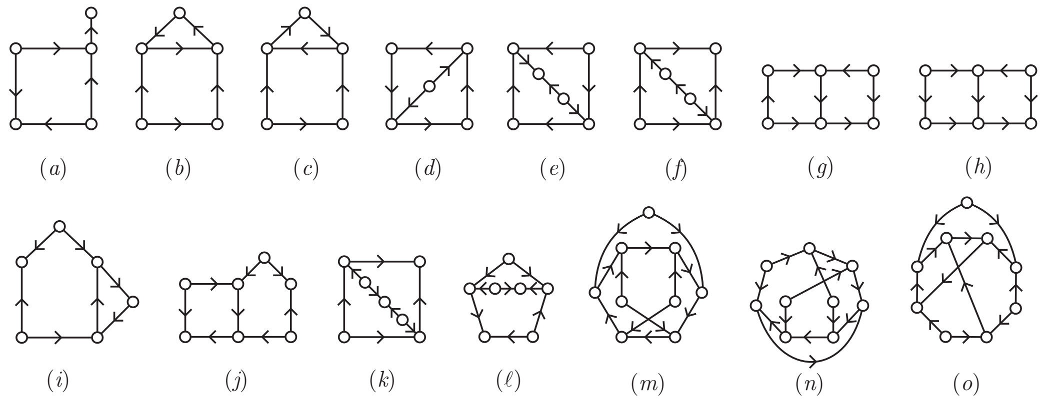

Figure 3: The forbidden subdigraphs characterizing a TT_3-free, 2-in-regular and oriented digraph admitting a $(1, \leq 3)$-identifying code.

Now we give an algebraic-combinatorial sufficient condition for a 2-in-regular digraph to admit a $(1, \leq 2)$-, or $(1, \leq 3)$-identifying code. By appealing to the eigenvalues of the digraphs we can reduce the number of forbidden subdigraphs considered in Theorem 4.

Theorem 6 *Let D be a 2-in-regular digraph with adjacency matrix A.*

(i) If $-1 \notin \mathrm{ev}(A)$ and D does not contain any subdigraph isomorphic to (b), (c), (d), (f) and (i) of Figure 2, then D admits a $(1, \leq 2)$-identifying code.

(ii) If $-1, 0 \notin \mathrm{ev}(A)$ and D does not contain any subdigraph isomorphic to (b)-(ℓ) of Figure 3, then D admits a $(1, \leq 3)$-identifying code.

We provide some necessary notation introduced by Powers [8]. Let $\boldsymbol{x} = (x_i)$ be an eigenvector associated with an eigenvalue λ different from the spectral radius, and let $\boldsymbol{z}$ be an eigenvector associated with the spectral radius. We denote by $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{N}(\boldsymbol{x})$ the set of its positive and negative entries, respectively. That is, $\mathcal{P}(\boldsymbol{x}) = \{i : x_i > 0\}$ and $\mathcal{N}(\boldsymbol{x}) = \{i : x_i < 0\}$. Since $\boldsymbol{x}$ and $\boldsymbol{z}$ are orthogonal, $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{N}(\boldsymbol{x})$ are nonempty, because all the entries of $\boldsymbol{z}$ are positive.

Let us show the meaning of the sign of a real eigenvalue on the sets of in-neighborhoods of vertices.

Proposition 7 *Let $D = (V, E)$ be a digraph with adjacency matrix A having some real eigenvalue, say $\lambda \in \mathrm{ev}(A)$, different from the spectral radius. Let $\boldsymbol{x} = (x_u)_{u \in V}$ be an eigenvector of A associated with λ such that $X = \mathcal{P}(\boldsymbol{x})$ and $Y = \mathcal{N}(\boldsymbol{x})$. Then, depending on the sign of λ, the following holds:*

(a) If $\lambda < 0$, then $X \cup N^-(X) = Y \cup N^-(Y) \quad (\Leftrightarrow N^-[X] = N^-[Y])$.

(b) If $\lambda > 0$, then $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) If $\lambda = 0$, then $N^-(X) = N^-(Y)$.

The same result holds for graphs by changing $N^-(X)$ and $N^-(Y)$ by $N(X)$ and $N(Y)$, respectively. Moreover, a similar result concerning out-neighborhoods (instead of in-neighborhoods) can be obtained by applying Proposition 7 to the converse digraph of D or, equivalently, considering the left (instead of right) eigenvectors of D. The next result gives an upper bound for ℓ in a digraph D having a $(1, \leq \ell)$-identifying code.

Corollary 8 *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. Let A be its adjacency matrix having at least one negative eigenvalue $-\lambda$ (with $\lambda > 0$) with $\boldsymbol{x} = (x_1, \ldots, x_n)$ any associated eigenvector. Then $\ell < \min_{\boldsymbol{x}} \max\{|\mathcal{P}(\boldsymbol{x})|, |\mathcal{N}(\boldsymbol{x})|\}$.*

Example 9 *Consider the digraph of Figure 2 (m). Its spectrum is* $\{0^4, 1^1, -1^1\}$*. An eigenvector corresponding to the eigenvalue* -1 *is* $(0, -1, 1, -1, 0, 1)$*. The positions of the positive entries of this eigenvector give us vertex subset* $X = \{2, 5\}$*, and the positions of the negatives entries give* $Y = \{1, 3\}$*. We can check that* $N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5\}$*. Then, this digraph does not admit a* $(1, \leq 2)$*-identifying code.*

To give a general result not depending on some specific eigenvector, but only on the multiplicity of the corresponding eigenvalue, we give the following lemma. More precisely, the next result shows that, given a real eigenvalue with geometric multiplicity m, some of its eigenvectors can be chosen with at least $m - 1$ zero entries.

Theorem 10 *Let* D *be a digraph on* n *vertices with adjacency matrix* A*, and let* λ *be a real eigenvalue of* A *with geometric multiplicity* m*. For any given index set* $I \subset \{1, 2, \ldots, n\}$ *with* $|I| = m - 1$*, there exists an eigenvector* $\boldsymbol{x}$ *with eigenvalue* λ *and entries* $x_i = 0$ *for every* $i \in I$*.*

Observe that Corollary 8 can also be applied to graphs, which always have real eigenvalues.

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