Bazarra, N. [et al.]. Numerical analysis of a dual-phase-lag model involving two temperatures. "Mathematical methods in the applied sciences", 27 Desembre 2019. doi: 10.1002/mma.6082

Received 26 July 2018; | Revised ; | Accepted

DOI: xxx/xxxx

ARTICLE TYPE

# Numerical analysis of a dual-phase-lag model involving two temperatures

Noelia Bazarra[1] | José R. Fernández[1] | Antonio Magaña[2] | Ramón Quintanilla[2]

[1]Departamento de Matemática Aplicada I, Universidade de Vigo, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain
[2]Departamento de Matemáticas, E.S.E.I.A.A.T., Universitat Politécnica de Catalunya, Colom 11, 08222 Terrassa, Barcelona, Spain

**Correspondence**
*J.R. Fernández, Departamento de Matemática Aplicada I, Universidade de Vigo, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain. Email: jose.fernandez@uvigo.es

**Summary**

In this paper we numerically analyse a phase-lag model with two temperatures which arises in the heat conduction theory. The model is written as a linear partial differential equation of third order in time. The variational formulation, written in terms of the thermal acceleration, leads to a linear variational equation, for which we recall an existence and uniqueness result and an energy decay property. Then, using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives, fully discrete approximations are introduced. A discrete stability property is proved, and a priori error estimates are obtained, from which the linear convergence of the approximation is derived. Finally, some one-dimensional numerical simulations are described to demonstrate the accuracy of the approximation and the behaviour of the solution.

**KEYWORDS:**
Heat conduction, phase-lag with two temperatures, finite elements, a priori estimates, numerical simulations

## 1 | INTRODUCTION

Problems involving thermal effects are usually analyzed using the Fourier heat conduction theory. However, when we adjoin this relation with the usual energy equation

$$c\,\dot{\theta} + \operatorname{div}\mathbf{q} = 0, \quad c > 0, \tag{1}$$

we obtain that the heat waves are propagated instantaneously. This is a drawback of the model because the infinite speed of propagation is not compatible with basic axioms of physics. In the previous equation, $\boldsymbol{q} = (q_i)$ is the heat flux vector, $\theta$ is the temperature, $c$ is the thermal capacity and a dot over the function denotes its time derivative. To overcome this inconvenience, several alternative proposals have been considered for the propagation of heat. The most known brings to the hyperbolic damped equation proposed by Cattaneo and Maxwell[2], which eliminates this drawback. Green and Nagdhi[9,10,11] also proposed three thermoelastic theories, where the heat conduction is described in alternative forms by means of the constitutive variables.

In 1995, Tzou[15,16] suggested another theory where the heat flux and the gradient of the temperature have a delay in the constitutive equations, which are given by

$$q_i(\boldsymbol{x}, t+\tau_1) = -k_2\theta_{,i}(\boldsymbol{x}, t+\tau_2), \quad k_2 > 0, \tag{2}$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are the delay parameters. As usual, the notation $\theta_{,i}$ means the derivative of $\theta$ with respect to the variable $x_i$, and repeated subscripts mean summation. This equation suggests that the temperature gradient established across a material volume at position $\boldsymbol{x}$ and time $t+\tau_2$ results in a heat flux to flow at a different time $t+\tau_1$. These delays can be understood in terms of the microstructure of the material. This theory has several derivations when the heat flux and the gradients of the temperature and the thermal displacement are replaced by Taylor approximations.

Unfortunately, the proposal of Tzou leads to *ill-posed* problems in the sense of Hadamard. It can be shown that combining equation (2) with the energy equation (1) implies the existence of a sequence of elements in the point spectrum with real parts tending to infinity [7]. At the same time, Tzou's theory is not compatible with the basic axioms of thermomechanics [8]. This suggests that this theory cannot be accepted nor from the mathematical point of view neither from the thermomechanical point of view.

To obtain a heat conduction theory with delays but without such an explosive behaviour, Quintanilla [13] combined the delay parameters of Tzou with the two-temperatures theory proposed by Chen and Gurtin [3,4,5,17]. The basic constitutive equation reads

$$q_i(\boldsymbol{x}, t + \tau_1) = -k_2 T_{,i}(\boldsymbol{x}, t + \tau_2),$$

where $\theta = T - a\Delta T$, being $T$ and $a$ the inductive temperature and a positive constant, respectively. This heat conduction theory was studied by Quintanilla and Jordan [14]. However, it seems very complicated to study it directly and it is usual to consider several approximations using Taylor developments. In this paper, we consider the following approximations:

$$\begin{aligned} \boldsymbol{q}(\boldsymbol{x}, t + \tau_1) &\approx \boldsymbol{q}(\boldsymbol{x}) + \tau_1 \dot{\boldsymbol{q}}(\boldsymbol{x}) + \frac{\tau_1^2}{2}\ddot{\boldsymbol{q}}(\boldsymbol{x}), \\ T(\boldsymbol{x}, t + \tau_2) &\approx T(\boldsymbol{x}) + \tau_2 \dot{T}(\boldsymbol{x}). \end{aligned} \tag{3}$$

It is worth recalling that the field equation obtained with these approximations has been studied recently in [12]. In particular, the exponential stability of solutions was proved when $2\tau_2 > \tau_1$. The numerical study corresponding to this problem will be the aim of this paper, providing a fully discrete algorithm for the approximation of the variational formulation of the problem, proving a discrete stability property and an error estimates result, and performing some numerical simulations.

The paper is outlined as follows. The mathematical model is described in Section 2 following Magaña et al. [12], deriving its variational formulation and recalling an existence and uniqueness result and an energy decay property proved by the same authors. Then, in Section 3 a numerical scheme is introduced, based on the finite element method to approximate the spatial domain and the backward Euler scheme to discretize the time derivatives. A discrete stability property is proved, a discrete version of the energy decay property is shown, a priori error estimates are deduced for the approximative solutions and, under suitable regularity assumptions, the linear convergence of the algorithm is obtained. Finally, some one-dimensional numerical simulations are presented in Section 4.

## 2 | THE MATHEMATICAL MODEL AND ITS VARIATIONAL FORMULATION: EXISTENCE AND UNIQUENESS

In this section, we describe the model and the conditions on the given data, we provide its variational formulation and we recall an existence and uniqueness result and an energy decay property (see [12] for details).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the thermal domain, assumed to be bounded, and denote by $[0, T_f]$, $T_f > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$. For the sake of simplicity, we assume that the temperature is zero on the whole boundary $\Gamma$ (i.e. null Dirichlet boundary conditions are used). Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T_f]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate, in general, the dependence of the functions on $\boldsymbol{x}$ and $t$.

Let us denote by $T(\boldsymbol{x}, t)$ the inductive temperature (from now on, simply "the temperature") of the body at point $\boldsymbol{x} \in \overline{\Omega}$ and time $t \in [0, T_f]$.

Following [12] and assuming that the body is isotropic and homogeneous, we set the following thermal problem.

**Problem P**. *Find the temperature* $T : \overline{\Omega} \times [0, T_f] \to \mathbb{R}$ *such that*

$$c(\dot{T} - a\Delta\dot{T}) + c\tau_1(\ddot{T} - a\Delta\ddot{T}) + c\frac{\tau_1^2}{2}(\dddot{T} - a\Delta\dddot{T}) = k_2(\Delta T + \tau_2 \Delta\dot{T}) \quad in \quad \Omega \times (0, T_f), \tag{4}$$

$$T(\boldsymbol{x}, t) = 0 \quad for\ a.e. \quad \boldsymbol{x} \in \Gamma,\ t \in [0, T_f], \tag{5}$$

$$T(\boldsymbol{x}, 0) = T_0(\boldsymbol{x}), \quad \dot{T}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}), \quad \ddot{T}(\boldsymbol{x}, 0) = \xi_0(\boldsymbol{x}) \quad for\ a.e. \quad \boldsymbol{x} \in \Omega. \tag{6}$$

*Remark 1.* We note that, proceeding as in the next section, we could also analyse the second model proposed in [12], replacing partial differential equation (4) by the following

$$c(\ddot{T} - a\Delta\ddot{T}) + c\tau_1(\dddot{T} - a\Delta\dddot{T}) = k_1\Delta T + \tau_4\Delta\dot{T} + k_2\tau_2\Delta\ddot{T} \quad \text{in} \quad \Omega\times(0,T_f).$$

However, we skip the details for the sake of simplicity.

In order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and denote by $(\cdot,\cdot)_Y$ and $(\cdot,\cdot)_H$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$ and $\|\cdot\|_H$. Moreover, let us define the variational space $E$ as follows,

$$E = \{z \in H^1(\Omega)\,;\; z = 0 \quad \text{on} \quad \Gamma\},$$

with respective scalar product $(\cdot,\cdot)_E$ and norm $\|\cdot\|_E$.

By using Green's formula and defining the thermal velocity $v = \dot{T}$ and the thermal acceleration $\xi = \ddot{T}$, we obtain the variational formulation of Problem P.

**Problem VP**. *Find the thermal acceleration* $\xi : [0,T_f] \to E$ *such that* $\xi(0) = \xi_0$*, and, for all* $\eta \in E$ *and for a.e.* $t \in (0,T_f)$*,*

$$c(v(t),\eta)_Y + ca(\nabla v(t),\nabla\eta)_H + c\tau_1(\xi(t),\eta)_Y + c\tau_1\, a(\nabla\xi(t),\nabla\eta)_H + c\frac{\tau_1^2}{2}(\dot{\xi}(t),\eta)_Y$$
$$+c\frac{\tau_1^2}{2}a(\nabla\dot{\xi}(t),\nabla\eta)_H + k_2(\nabla T(t),\nabla\eta)_H + k_2\tau_2(\nabla v(t),\nabla\eta)_H = 0, \tag{7}$$

*where the temperature* $T$ *and the thermal velocity* $v$ *are obtained from the respective equations*

$$v(t) = \int_0^t \xi(s)\,ds + v_0, \quad T(t) = \int_0^t v(s)\,ds + T_0. \tag{8}$$

Problem VP was studied and the following result, which states the existence of a unique solution and an energy decay property, was proved in [12].

**Theorem 1.** Let the assumptions

$$c > 0, \quad a > 0, \quad \tau_1 > 0, \quad \tau_2 > 0, \quad k_2 > 0, \quad T_0,\, v_0,\, \xi_0 \in H^2(\Omega) \tag{9}$$

hold. Therefore, Problem VP has a unique solution $\xi$ with the following regularity

$$\xi \in C^1([0,T_f]; H^2(\Omega)).$$

Moreover, if

$$2\tau_2 - \tau_1 > 0 \tag{10}$$

then the energy given by

$$E(t) = \frac{1}{2}\Big\{ \|T(t) + \tau_1 v(t) + \frac{\tau_1^2}{2}\xi(t)\|_Y^2 + k_2(\tau_1+\tau_2)\left(\|\nabla T(t)\|_H^2 + a\|\Delta T(t)\|_Y^2\right) + k_2\tau_1^2(\nabla T(t),\nabla v(t))_H$$
$$+\frac{k_2\tau_1^2\tau_2}{2}\left(\|\nabla v(t)\|_H^2 + a\|\Delta v(t)\|_Y^2\right) + ak_2\tau_1^2(\Delta T(t),\Delta v(t))_Y \Big\}$$

decays exponentially; i.e. there exist $\omega > 0$ and $M > 0$ such that

$$E(t) \le M E(0)e^{-\omega t}, \quad t \ge 0.$$

## 3 | FULLY DISCRETE APPROXIMATIONS: AN A PRIORI ERROR ANALYSIS

In this section, we introduce a finite element algorithm for approximating solutions to variational problem VP. This is done in two steps. First, we consider the finite element space $E^h \subset E$ given by

$$E^h = \{\eta^h \in C(\overline{\Omega})\,;\; \eta^h_{|_{Tr}} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad \eta^h = 0 \text{ on } \Gamma\}, \tag{11}$$

where $\Omega$ is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to 1 in $Tr$. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $T_0^h$, $v_0^h$ and $\xi_0^h$, are given by

$$T_0^h = \mathcal{P}^h T_0, \quad v_0^h = \mathcal{P}^h v_0, \quad \xi_0^h = \mathcal{P}^h \xi_0, \tag{12}$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over $E^h$ (see, e.g.,[6]).

Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0, T_f]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T_f$, and let $k$ be the time step size, $k = T_f/N$. Moreover, for a continuous function $f(t)$ let $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the classical backward Euler scheme, the fully discrete approximation of Problem VP is the following.

**Problem VP$^{hk}$.** *Find the discrete thermal acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ such that $\xi_0^{hk} = \xi_0^h$ and, for all $\eta^h \in E^h$ and $n = 1, \ldots, N$,*

$$\begin{aligned} c(v_n^{hk}, \eta^h)_Y + ca(\nabla v_n^{hk}, \nabla \eta^h)_H + c\tau_1(\xi_n^{hk}, \eta^h)_Y + c\tau_1 a(\nabla \xi_n^{hk}, \nabla \eta^h)_H + c\frac{\tau_1^2}{2}(\delta \xi_n^{hk}, \eta^h)_Y \\ + ca\frac{\tau_1^2}{2}(\nabla \delta \xi_n^{hk}, \nabla \eta^h)_H + k_2(\nabla T_n^{hk}, \nabla \eta^h)_H + k_2\tau_2(\nabla v_n^{hk}, \nabla \eta^h)_H = 0, \end{aligned} \tag{13}$$

*where the discrete temperature and the discrete thermal velocity are then recovered from the relations*

$$v_n^{hk} = k\sum_{j=1}^n \xi_j^{hk} + v_0^h, \quad T_n^{hk} = k\sum_{j=1}^n v_j^{hk} + T_0^h. \tag{14}$$

The classical Lax-Milgram lemma allows to prove that discrete problem VP$^{hk}$ admits a unique discrete solution. Therefore, the aim of the section is to provide its numerical analysis.

We have the following discrete stability result.

**Lemma 1.** Under the assumptions of Theorem 1, it follows that the sequences $\{T^{hk}, v^{hk}, \xi^{hk}\}$ generated by Problem VP$^{hk}$ satisfy the stability estimate:

$$\|\xi_n\|_Y^2 + \|\nabla \xi_n\|_H^2 + \|v_n\|_Y^2 + \|\nabla v_n\|_H^2 + \|T_n\|_Y^2 + \|\nabla T_n\|_H^2 \le C,$$

where $C$ is a positive constant which is independent of the discretization parameters $h$ and $k$.

*Proof.* For the sake of clarity in the writing of this proof, we remove the superscripts $h$ and $k$ in all the variables.

Taking $\eta^h = \xi_n$ as a test function in discrete variational equation (13) we have

$$\begin{aligned} c(v_n, \xi_n)_Y + ca(\nabla v_n, \nabla \xi_n)_H + c\tau_1(\xi_n, \xi_n)_Y + c\tau_1\, a(\nabla \xi_n, \nabla \xi_n)_H + c\frac{\tau_1^2}{2}(\delta \xi_n, \xi_n)_Y \\ + c\frac{\tau_1^2}{2} a(\nabla \delta \xi_n, \nabla \xi_n)_H + k_2(\nabla T_n, \nabla \xi_n)_H + k_2\tau_2(\nabla v_n, \nabla \xi_n)_H = 0. \end{aligned}$$

Thus, keeping in mind that

$$\begin{aligned} (\delta \xi_n, \xi_n)_Y &\ge \frac{1}{2k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\}, \\ (\nabla \delta \xi_n, \nabla \xi_n)_H &\ge \frac{1}{2k}\left\{\|\nabla \xi_n\|_H^2 - \|\nabla \xi_{n-1}\|_H^2\right\}, \\ (v_n, \xi_n)_Y &\ge \frac{1}{2k}\left\{\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2\right\}, \\ (\nabla v_n, \nabla \xi_n)_H &\ge \frac{1}{2k}\left\{\|\nabla v_n\|_H^2 - \|\nabla v_{n-1}\|_H^2\right\}, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{2k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\nabla \xi_n\|_H^2 - \|\nabla \xi_{n-1}\|_H^2\right\} + \frac{1}{2k}\left\{\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2\right\} \\ + \frac{1}{2k}\left\{\|\nabla v_n\|_H^2 - \|\nabla v_{n-1}\|_H^2\right\} \le C\left(\|\nabla v_n\|_H^2 + \|\nabla T_n\|_H^2\right). \end{aligned}$$

By induction we find that

$$\begin{aligned} \|\xi_n\|_Y^2 + \|\nabla \xi_n\|_H^2 + \|v_n\|_Y^2 + \|\nabla v_n\|_H^2 \le Ck\sum_{j=1}^n \left(\|\nabla v_j\|_H^2 + \|\nabla T_j\|_H^2\right) \\ + C\left(\|\xi_0\|_Y^2 + \|\nabla \xi_0\|_H^2 + \|v_0\|_Y^2 + \|\nabla v_0\|_H^2\right), \end{aligned}$$

and so, taking into account that

$$\|T_n\|_Y^2 \leq Ck\sum_{j=1}^{n}\|v_j\|_Y^2 + C\|T_0\|_Y^2,$$
$$\|\nabla T_n\|_H^2 \leq Ck\sum_{j=1}^{n}\|\nabla v_j\|_H^2 + C\|\nabla T_0\|_H^2,$$

we have

$$\begin{aligned}&\|\xi_n\|_Y^2 + \|\nabla\xi_n\|_H^2 + \|v_n\|_Y^2 + \|\nabla v_n\|_H^2 + \|T_n\|_Y^2 + \|\nabla T_n\|_H^2\\ &\quad\leq Ck\sum_{j=1}^{n}\left(\|\nabla v_j\|_H^2 + \|v_j\|_Y^2\right) + C\Big(\|\xi_0\|_Y^2 + \|\nabla\xi_0\|_H^2 + \|v_0\|_Y^2 + \|\nabla v_0\|_H^2 + \|T_0\|_Y^2 + \|\nabla T_0\|_H^2\Big).\end{aligned}$$

Finally, the desired stability estimates are a straightforward consequence of the application of a discrete version of Gronwall's inequality (see, e.g.,[1]). □

Now, we have a discrete version of the energy decay property.

**Lemma 2.** Under the assumptions of Theorem 1, if we define the discrete energy $E_n^{hk}$ in the following form:

$$\begin{aligned}E_n^{hk} = \frac{1}{2}\Big\{&c\|T_n^{hk} + \tau_1 v_n^{hk} + \frac{\tau_1^2}{2}\xi_n^{hk}\|_Y^2 + ca\|\nabla T_n^{hk} + \tau_1\nabla v_n^{hk} + \frac{\tau_1^2}{2}\nabla\xi_n^{hk}\|_H^2 + k_2(\tau_1+\tau_2)\|\nabla T_n^{hk}\|_H^2\\ &+k_2\tau_2\frac{\tau_1^2}{2}\|\nabla v_n^{hk}\|_H^2 + k_2\frac{\tau_1^2}{2}(\nabla T_n^{hk},\nabla\xi_n^{hk})_H\Big\},\end{aligned} \tag{15}$$

then it decays, i.e. $\dfrac{E_n^{hk}-E_{n-1}^{hk}}{k} \leq 0$.

*Proof.* In order to simplify the writing of the calculations, we define the discrete function

$$\tilde{T}_n^{hk} = T_n^{hk} + \tau_1 v_n^{hk} + \frac{\tau_1^2}{2}\xi_n^{hk}.$$

Taking $w^h = \tilde{T}_n^{hk}$ as a test function in discrete variational equation (13), it follows that

$$\frac{c}{2k}\Big(\|\tilde{T}_n^{hk}\|_Y^2 - \|\tilde{T}_{n-1}^{hk}\|_Y^2\Big) + \frac{ca}{2k}\Big(\|\nabla\tilde{T}_n^{hk}\|_H^2 - \|\nabla\tilde{T}_{n-1}^{hk}\|_H^2\Big) + k_2(\nabla T_n^{hk},\nabla\tilde{T}_n^{hk})_H + k_2\tau_2(\nabla v_n^{hk},\nabla\tilde{T}_n^{hk})_H \leq 0,$$

where we use $v_n^{hk} + \tau_1\xi_n^{hk} + \frac{\tau_1^2}{2}\delta\xi_n^{hk} = \delta\tilde{T}_n^{hk}$.

Taking into account that

$$(\nabla T_n^{hk},\nabla\tilde{T}_n^{hk})_H \geq \frac{\tau_1}{2k}\left\{\|\nabla T_n^{hk}\|_H^2 - \|\nabla T_{n-1}^{hk}\|_H^2\right\} + \frac{\tau_1^2}{2}(\nabla T_n^{hk},\nabla\xi_n^{hk})_H,$$
$$(\nabla v_n^{hk},\nabla\tilde{T}_n^{hk})_H \geq \frac{1}{2k}\left\{\|\nabla T_n^{hk}\|_H^2 - \|\nabla T_{n-1}^{hk}\|_H^2\right\} + \frac{\tau_1^2}{4k}\left\{\|\nabla v_n^{hk}\|_H^2 - \|\nabla v_{n-1}^{hk}\|_H^2\right\},$$

after easy algebraic manipulations we obtain the decay of the discrete energy. □

Now, we obtain the following a priori error estimates result.

**Theorem 2.** Under the assumptions of Theorem 1, if we denote by $(T,v,\xi)$ and $(T^{hk},v^{hk},\xi^{hk})$ the respective solutions to problems VP and VP$^{hk}$, then we have the following a priori error estimates, for all $\eta^h = \{\eta_j^h\}_{j=0}^N \subset E^h$,

$$\begin{aligned}&\max_{0\leq n\leq N}\left\{\|\xi_n - \xi_n^{hk}\|_Y^2 + \|\nabla(\xi_n-\xi_n^{hk})\|_H^2 + \|\nabla(v_n - v_n^{hk})\|_H^2 + \|v_n - v_n^{hk}\|_Y^2 + \|T_n - T_n^{hk}\|_Y^2 + \|\nabla(T_n - T_n^{hk})\|_H^2\right\}\\ &\leq Ck\sum_{j=1}^{N}\Big(\|\xi_j-\eta_j^h\|_Y^2 + \|\nabla(\xi_j-\eta_j^h)\|_H^2 + \|\dot{\xi}_j - \delta\xi_j\|_Y^2 + \|\nabla(\dot{\xi}_j-\delta\xi_j)\|_H^2 + \|\dot{v}_j - \delta v_j\|_Y^2 + \|\nabla(\dot{v}_j - \delta v_j)\|_H^2 + I_j^2\Big)\\ &\quad +C\max_{0\leq n\leq N}\|\xi_n - \eta_n^h\|_Y^2 + \frac{C}{k}\sum_{j=1}^{N-1}\|\xi_j - \eta_j^h - (\xi_{j+1}-\eta_{j+1}^h)\|_Y^2 + C\max_{0\leq n\leq N}\|\nabla(\xi_n-\eta_n^h)\|_H^2\end{aligned}$$

$$+\frac{C}{k}\sum_{j=1}^{N-1}\|\nabla(\xi_j-\eta_j^h-(\xi_{j+1}-\eta_{j+1}^h))\|_H^2+C\Big(\|\xi_0-\xi_0^h\|_Y^2+\|\nabla(\xi_0-\xi_0^h)\|_H^2+\|v_0-v_0^h\|_Y^2+\|\nabla(v_0-v_0^h)\|_H^2$$
$$+\|\nabla(T_0-T_0^h)\|_H^2+\|T_0-T_0^h\|_Y^2\Big), \qquad (16)$$

where $I_j$ denotes the integration error defined by

$$I_j=\left\|\int_0^{t_j}\nabla v(s)\,ds-k\sum_{l=1}^{j}\nabla v_l\right\|_H,$$

and $C$ is a positive constant which is independent of the discretization parameters $h$ and $k$.

*Proof.* First, we subtract variational equation (7) at time $t=t_n$ for a test function $\eta=\eta^h\in E^h\subset E$ and discrete variational equation (13) to obtain, for all $\eta^h\in E^h$,

$$\begin{aligned}&c(v_n-v_n^{hk},\eta^h)_Y+ca(\nabla(v_n-v_n^{hk}),\nabla\eta^h)_H+c\tau_1(\xi_n-\xi_n^{hk},\eta^h)_Y\\&\quad+c\tau_1\,a(\nabla(\xi_n-\xi_n^{hk}),\nabla\eta^h)_H+c\frac{\tau_1^2}{2}(\dot\xi_n-\delta\xi_n^{hk},\eta^h)_Y+k_2(\nabla(T_n-T_n^{hk}),\nabla\eta^h)_H\\&\quad+c\frac{\tau_1^2}{2}a(\nabla(\dot\xi_n-\delta\xi_n^{hk}),\nabla\eta^h)_H+k_2\tau_2(\nabla(v_n-v_n^{hk}),\nabla\eta^h)_H=0,\end{aligned}$$

and therefore, we find that

$$\begin{aligned}&c(v_n-v_n^{hk},\xi_n-\xi_n^{hk})_Y+ca(\nabla(v_n-v_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H+c\tau_1(\xi_n-\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\\&\quad+c\tau_1\,a(\nabla(\xi_n-\xi_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H+c\frac{\tau_1^2}{2}(\dot\xi_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\\&\quad+k_2(\nabla(T_n-T_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H+c\frac{\tau_1^2}{2}a(\nabla(\dot\xi_n-\delta\xi_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\&\quad+k_2\tau_2(\nabla(v_n-v_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\&=c(v_n-v_n^{hk},\xi_n-\eta^h)_Y+ca(\nabla(v_n-v_n^{hk}),\nabla(\xi_n-\eta^h))_H+c\tau_1(\xi_n-\xi_n^{hk},\xi_n-\eta^h)_Y\\&\quad+c\tau_1\,a(\nabla(\xi_n-\xi_n^{hk}),\nabla(\xi_n-\eta^h))_H+c\frac{\tau_1^2}{2}(\dot\xi_n-\delta\xi_n^{hk},\xi_n-\eta^h)_Y\\&\quad+k_2(\nabla(T_n-T_n^{hk}),\nabla(\xi_n-\eta^h))_H+c\frac{\tau_1^2}{2}a(\nabla(\dot\xi_n-\delta\xi_n^{hk}),\nabla(\xi_n-\eta^h))_H\\&\quad+k_2\tau_2(\nabla(v_n-v_n^{hk}),\nabla(\xi_n-\eta^h))_H.\end{aligned}$$

Keeping in mind that

$$\begin{aligned}&(\dot\xi_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\ge(\dot\xi_n-\delta\xi_n,\xi_n-\xi_n^{hk})_Y+\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\},\\&(\nabla(\dot\xi_n-\delta\xi_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\ge(\nabla(\dot\xi_n-\delta\xi_n),\nabla(\xi_n-\xi_n^{hk}))_H+\frac{1}{2k}\left\{\|\nabla(\xi_n-\xi_n^{hk})\|_H^2-\|\nabla(\xi_{n-1}-\xi_{n-1}^{hk})\|_H^2\right\},\\&(v_n-v_n^{hk},\xi_n-\xi_n^{hk})_Y\ge(v_n-v_n^{hk},\dot v_n-\delta v_n)_Y+\frac{1}{2k}\left\{\|v_n-v_n^{hk}\|_Y^2-\|v_{n-1}-v_{n-1}^{hk}\|_Y^2\right\},\\&(\nabla(v_n-v_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\ge(\nabla(v_n-v_n^{hk}),\nabla(\dot v_n-\delta v_n))_H+\frac{1}{2k}\left\{\|\nabla(v_n-v_n^{hk})\|_H^2-\|\nabla(v_{n-1}-v_{n-1}^{hk})\|_H^2\right\},\end{aligned}$$

it follows that, for all $\eta^h\in E^h$,

$$\begin{aligned}&\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\nabla(\xi_n-\xi_n^{hk})\|_H^2-\|\nabla(\xi_{n-1}-\xi_{n-1}^{hk})\|_H^2\right\}\\&\quad+\frac{1}{2k}\left\{\|v_n-v_n^{hk}\|_Y^2-\|v_{n-1}-v_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\nabla(v_n-v_n^{hk})\|_H^2-\|\nabla(v_{n-1}-v_{n-1}^{hk})\|_H^2\right\}\\&\le C\Big(\|v_n-v_n^{hk}\|_Y^2+\|\nabla(v_n-v_n^{hk})\|_H^2+\|\xi_n-\xi_n^{hk}\|_Y^2+\|\nabla(\xi_n-\xi_n^{hk})\|_H^2\\&\quad+\|\xi_n-\eta^h\|_Y^2+\|\nabla(\xi_n-\eta^h)\|_H^2+\|\dot\xi_n-\delta\xi_n\|_Y^2+\|\nabla(\dot\xi_n-\delta\xi_n)\|_H^2\\&\quad+\|\dot v_n-\delta v_n\|_Y^2+\|\nabla(\dot v_n-\delta v_n)\|_H^2+\|\nabla(T_n-T_n^{hk})\|_H^2\\&\quad+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-\eta^h)_Y+(\nabla(\delta\xi_n-\delta\xi_n^{hk}),\nabla(\xi_n-\eta^h))_H\Big),\end{aligned}$$

where we used several times Cauchy's inequality

$$ab\le\epsilon a^2+\frac{1}{4\epsilon}b^2,\ \text{for all} a,b,\epsilon\in\mathbb{R},\ \text{with} \epsilon>0,$$

and the notations $\delta v_n = (v_n - v_{n-1})/k$ and $\delta\xi_n = (\xi_n - \xi_{n-1})/k$, and we recall that $\xi_n^{hk} = \delta v_n^{hk}$ and $v_n^{hk} = \delta T_n^{hk}$.

Therefore, by induction it leads

$$
\begin{aligned}
&\|\xi_n - \xi_n^{hk}\|_Y^2 + \|\nabla(\xi_n - \xi_n^{hk})\|_H^2 + \|v_n - v_n^{hk}\|_Y^2 + \|\nabla(v_n - v_n^{hk})\|_H^2 \\
&\quad \leq Ck\sum_{j=1}^{n}\Big(\|v_j - v_j^{hk}\|_Y^2 + \|\nabla(v_j - v_j^{hk})\|_H^2 + \|\xi_j - \xi_j^{hk}\|_Y^2 + \|\nabla(\xi_j - \xi_j^{hk})\|_H^2 \\
&\qquad + \|\xi_j - \eta_j^h\|_Y^2 + \|\nabla(\xi_j - \eta_j^h)\|_H^2 + \|\dot{\xi}_j - \delta\xi_j\|_Y^2 + \|\nabla(\dot{\xi}_j - \delta\xi_j)\|_H^2 \\
&\qquad + \|\dot{v}_j - \delta v_j\|_Y^2 + \|\nabla(\dot{v}_j - \delta v_j)\|_H^2 + \|\nabla(T_j - T_j^{hk})\|_H^2 \\
&\qquad + (\delta\xi_j - \delta\xi_j^{hk}, \xi_j - \eta_j^h)_Y + (\nabla(\delta\xi_j - \delta\xi_j^{hk}), \nabla(\xi_j - \eta_j^h))_H\Big) \\
&\qquad + C\Big(\|\xi_0 - \xi_0^h\|_Y^2 + \|\nabla(\xi_0 - \xi_0^h)\|_H^2 + \|v_0 - v_0^h\|_Y^2 + \|\nabla(v_0 - v_0^h)\|_H^2\Big).
\end{aligned}
$$

Finally, taking into account that

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\xi_j - \delta\xi_j^{hk}, \xi_j - \eta_j^h)_Y &= \sum_{j=1}^{n}(\xi_j - \xi_j^{hk} - (\xi_{j-1} - \xi_{j-1}^{hk}), \xi_j - \eta_j^h)_Y \\
&= (\xi_n - \xi_n^{hk}, \xi_n - \eta_n^h)_Y + (\xi_0^h - \xi_0, \xi_1 - \eta_1^h)_Y \\
&\quad + \sum_{j=1}^{n-1}(\xi_j - \xi_j^{hk}, \xi_j - \eta_j^h - (\xi_{j+1} - \eta_{j+1}^h))_Y,
\end{aligned}
$$

$$
\begin{aligned}
&k\sum_{j=1}^{n}(\nabla(\delta\xi_j - \delta\xi_j^{hk}), \nabla(\xi_j - \eta_j^h))_H \\
&\quad = (\nabla(\xi_n - \xi_n^{hk}), \nabla(\xi_n - \eta_n^h))_H + (\nabla(\xi_0^h - \xi_0), \nabla(\xi_1 - \eta_1^h))_H \\
&\qquad + \sum_{j=1}^{n-1}(\nabla(\xi_j - \xi_j^{hk}), \nabla(\xi_j - \eta_j^h - (\xi_{j+1} - \eta_{j+1}^h)))_H,
\end{aligned}
$$

$$
\|\nabla(T_n - T_n^{hk})\|_H^2 \leq C\Big(I_n^2 + \sum_{j=1}^{n} k\|\nabla(v_j - v_j^{hk})\|_H^2 + \|\nabla(T_0 - T_0^h)\|_H^2\Big),
$$

where we recall that $I_n$ is the integration error defined previously, using the above estimates and a discrete version of Gronwall's inequality (see, again, [1]) we conclude the proof. □

We note that estimates (16) are the basis to obtain the convergence order of the approximations given by Problem VP$^{hk}$. Therefore, as an example, if we assume the following additional regularity

$$
T \in C^2([0, T_f]; H^2(\Omega)) \cap H^3(0, T_f; E), \tag{17}
$$

using the classical results on the approximation by finite elements (see, e.g., [6]) we have the following result.

**Corollary 1.** Let the assumptions of Theorem 2 hold. Under the additional regularity (17) it follows that the approximations obtained by Problem VP$^{hk}$ are linearly convergent; that is, there exists a positive constant $C$, independent of the discretization parameters $h$ and $k$, such that

$$
\max_{0\leq n\leq N}\Big\{\|\xi_n - \xi_n^{hk}\|_E + \|v_n - v_n^{hk}\|_E + \|T_n - T_n^{hk}\|_E\Big\} \leq C(h + k).
$$

## 4 | NUMERICAL SIMULATIONS

In this final section, we describe the numerical scheme implemented in MATLAB for solving discrete problem VP$^{hk}$, and we show some numerical examples to demonstrate the accuracy of the approximations and the behaviour of the solutions.

Let $E^h$ be the finite element space defined in (11) and $n = 1, 2, \dots, N$. Given $\theta^{hk}_{n-1}$, $e^{hk}_{n-1}$, $\xi^{hk}_{n-1} \in E^h$, the discrete thermal acceleration $\xi^{hk}_n$ for Problem VP$^{hk}$ is obtained from equation (13), that is, we solve the following linear problem, for all $\eta^h \in E^h$,

$$
\begin{aligned}
&ck^2(\xi^{hk}_n, \eta^h)_Y + cak^2(\nabla\xi^{hk}_n, \nabla\eta^h)_H + c\tau_1 k(\xi^{hk}_n, \eta^h)_Y + c\tau_1 ak(\nabla\xi^{hk}_n, \nabla\eta^h)_H + c\frac{\tau_1^2}{2}(\xi^{hk}_n, \eta^h)_Y \\
&\quad + ca\frac{\tau_1^2}{2}(\nabla\xi^{hk}_n, \nabla\eta^h)_H + k_2 k^3(\nabla\xi^{hk}_n, \nabla\eta^h)_H + k_2\tau_2 k^2(\nabla\xi^{hk}_n, \nabla\eta^h)_H \\
&= c\frac{\tau_1^2}{2}(\xi^{hk}_{n-1}, \eta^h)_Y + ca\frac{\tau_1^2}{2}(\nabla\xi^{hk}_{n-1}, \nabla\eta^h)_H - cak(\nabla v^{hk}_{n-1}, \nabla\eta^h)_H - ck(v^{hk}_{n-1}, \eta^h)_Y - k_2 k(\nabla T^{hk}_{n-1}, \nabla\eta^h)_H \\
&\quad - k_2 k^2(\nabla v^{hk}_{n-1}, \nabla\eta^h)_H - k_2\tau_2 k(\nabla v^{hk}_{n-1}, \nabla\eta^h)_H.
\end{aligned}
$$

The discrete temperature and the discrete thermal velocity are then recovered from the relations:

$$v^{hk}_n = k\xi^{hk}_n + v^{hk}_{n-1}, \quad T^{hk}_n = k^2\xi^{hk}_n + kv^{hk}_{n-1} + T^{hk}_{n-1}.$$

The above numerical scheme was implemented using MATLAB on a Intel Core $i7 - 3337U$ @ 2.20GHz and a typical run (1000 step times and 1000 nodes) took about 0.979 seconds of CPU time.

## 4.1 | Numerical convergence for the approximation of Problem P

In order to show the numerical convergence, we will consider the following academic problem.

**Problem $P^{ex}$**. *Find the temperature $T : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$
\begin{aligned}
&\dot T - \dot T_{xx} + 5(\ddot T - \ddot T_{xx}) + \frac{25}{2}(\dddot T - \dddot T_{xx}) = 2(\Delta T + 3\dot T_{xx}) + F \quad in \quad (0,1)\times(0,1), \\
&T(0,t) = T(1,t) = 0 \quad for\ a.e.\ t \in (0,1), \\
&T(x,0) = \dot T(x,0) = \ddot T(x,0) = x(x-1) \quad for\ a.e.\ x \in (0,1),
\end{aligned}
$$

where the artificial volume force $F$ is given by

$$F(x,t) = e^t\frac{5}{2}(x^2 - x - 2).$$

We note that Problem P$^{ex}$ corresponds to Problem $P$ with the following data:

$$\Omega = (0,1), \quad T_f = 1, \quad k_2 = 2, \quad \tau_1 = 5, \quad \tau_2 = 3, \quad a = 1, \quad c = 1,$$

and the initial conditions:

$$T_0(x) = v_0(x) = \xi_0(x) = x(x-1) \quad \text{for} \quad x \in (0,1).$$

The exact solution to Problem P$^{ex}$ is the following:

$$T(x,t) = e^t x(x-1) \quad \text{for all} \quad (x,t) \in (0,1)\times(0,1).$$

The numerical errors given by

$$\max_{0\le n\le N}\left\{ \|\xi_n - \xi^{hk}_n\|_E + \|v_n - v^{hk}_n\|_E + \|T_n - T^{hk}_n\|_E \right\},$$

and obtained with different discretization parameters $h$ and $k$, are depicted in TABLE 1 . The convergence of the algorithm is clearly found. Moreover, the evolution of the error depending on $h + k$ is plotted in FIGURE 1 . We observe that the linear convergence stated in Corollary 1 is achieved.

If we assume now that there are not volume forces, and we use the final time $T_f = 5$, the following data

$$\Omega = (0,1), \quad k_2 = 2, \quad \tau_1 = 0.3, \quad \tau_2 = 0.2, \quad a = 1, \quad c = 1$$

and the initial conditions

$$T_0(x) = x(x-1) \quad \text{for} \quad x \in (0,1), \quad v_0 = \xi_0 = 0,$$

taking the discretization parameters $h = k = 10^{-3}$, the evolution in time of the discrete energy $E^{hk}_n$, defined in (15), is plotted in FIGURE 2 in both natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

| $h \downarrow k \rightarrow$ | 0.01 | 0.005 | 0.002 | 0.001 | 0.0005 | 0.0002 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $1/2^3$ | 0.5129493 | 0.5131588 | 0.5133687 | 0.5134525 | 0.5134970 | 0.5135245 | 0.5135338 |
| $1/2^4$ | 0.2546921 | 0.2545361 | 0.2546299 | 0.2546910 | 0.2547270 | 0.2547504 | 0.2547584 |
| $1/2^5$ | 0.1281855 | 0.1269754 | 0.1268024 | 0.1268294 | 0.1268558 | 0.1268756 | 0.1268828 |
| $1/2^6$ | 0.0665330 | 0.0641768 | 0.0633975 | 0.0632941 | 0.0632958 | 0.0633096 | 0.0633159 |
| $1/2^7$ | 0.0373524 | 0.0332942 | 0.0319545 | 0.0317141 | 0.0316372 | 0.0316212 | 0.0316248 |
| $1/2^8$ | 0.0250363 | 0.0186773 | 0.0163810 | 0.0159838 | 0.0158627 | 0.0158158 | 0.0158059 |
| $1/2^9$ | 0.0207857 | 0.0125074 | 0.0088742 | 0.0081922 | 0.0079935 | 0.0079238 | 0.0079090 |
| $1/2^{10}$ | 0.0195690 | 0.0103769 | 0.0055511 | 0.0044371 | 0.0040963 | 0.0039832 | 0.0039619 |
| $1/2^{11}$ | 0.0192518 | 0.0097669 | 0.0043203 | 0.0027748 | 0.0022181 | 0.0020249 | 0.0019911 |
| $1/2^{12}$ | 0.0191716 | 0.0096078 | 0.0039493 | 0.0021589 | 0.0013866 | 0.0010685 | 0.0010109 |
| $1/2^{13}$ | 0.0191515 | 0.0095678 | 0.0038504 | 0.0019729 | 0.0010777 | 0.0006260 | 0.0005337 |

**TABLE 1** Example 1: Numerical errors for some $h$ and $k$.

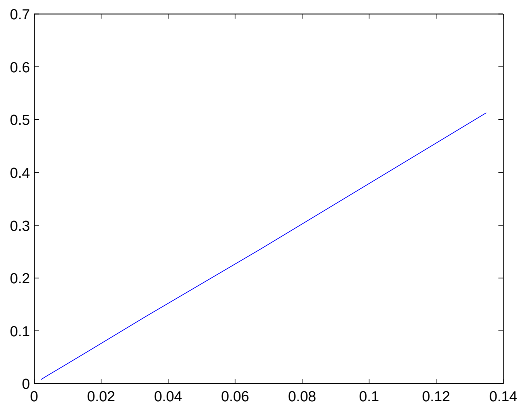


**FIGURE 1** Example : Asymptotic constant error.

## 4.2 | Example 2: dependence on the diffusion parameter $k_2$

As a second example, we will analyse the dependence on the diffusion parameter $k_2$. Thus, we use the following data:

$$\Omega = (0,1), \quad T_f = 1, \quad a = 1, \quad \tau_1 = 1, \quad \tau_2 = 4, \quad c = 1,$$

and the initial conditions:

$$T_0(x) = x(x-1) \quad \text{for} \quad x \in (0,1), \qquad v_0 = \xi_0 = 0.$$

Using the discretization parameters $h = k = 0.001$, in FIGURE 3 we plot the thermal velocity (left) and the thermal acceleration (right) at final time for some values of parameter $k_2$. As can be seen, the solutions increase when the parameter decreases and they have a quadratic behaviour as expected. Moreover, the solution tends to zero when parameter $k_2$ becomes large.

Finally, in FIGURE 4 the temperature is shown at final time for those values of parameter $k_2$. Now, the solutions are closer for the different values and they have a similar shape.

## 4.3 | Example 3: comparison with the dual-phase-lag model

As a final example, we will compare the solution with that obtained for the usual dual-phase-lag model (that is, when we assume $a = 0$). Thus, we use the following data:

$$\Omega = (0,1), \quad T_f = 1, \quad k_2 = 3, \quad \tau_1 = 1, \quad \tau_2 = 4, \quad c = 1,$$

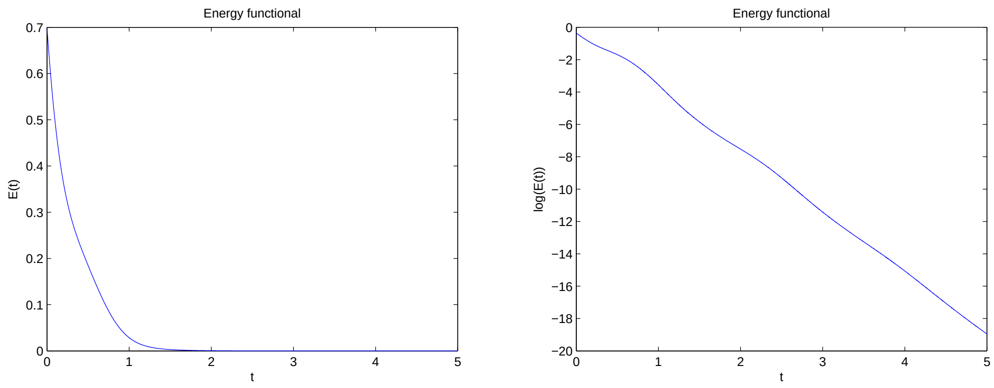


**FIGURE 2** Example 1: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales.

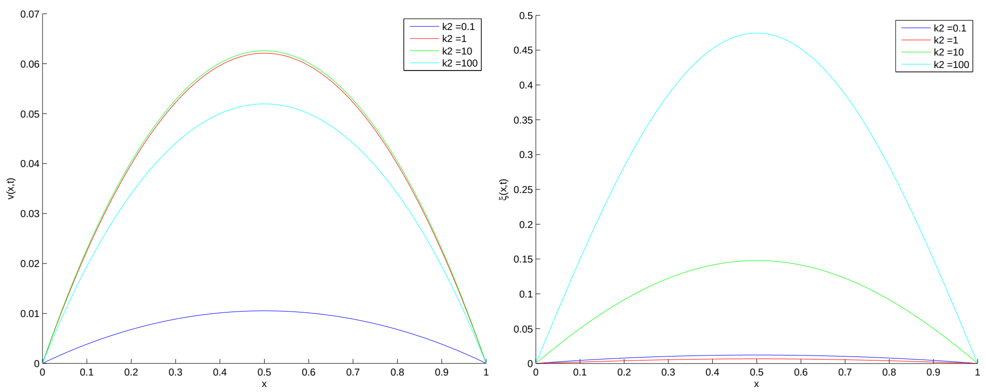


**FIGURE 3** Example 2: Thermal velocity and thermal acceleration at final time for different diffusion parameters.

and the initial conditions:

$$T_0(x) = x(x-1) \quad \text{for} \quad x \in (0,1), \qquad v_0 = \xi_0 = 0.$$

Using the discretization parameters $h = k = 0.001$, we plot the thermal velocity (left) and the thermal acceleration (right) at final time in FIGURE 5 for two values of parameter $a$. As can be seen, the thermal velocity increases when the diffusion parameter $a$ is not neglected although both velocities are positive and they have a quadratic behaviour. Instead, the thermal accelerations are really different, being quadratic and negative the solution corresponding to the dual-phase-lag case, and positive that one corresponding to the case with viscous diffusion.

Finally, in FIGURE 6 the temperature is shown at final time for these values of parameter $a$. We can see that both solutions almost coincide.

## ACKNOWLEDGEMENTS

The work of N. Bazarra and J.R. Fernández has been supported by the Ministerio de Economía y Competitividad under the research project MTM2015-66640-P (with the participation of FEDER) and the Ministerio de Ciencia, Innovación y Universidades under the research project PGC2018-096696-B-I00 (with the participation of FEDER).

The work of A. Magaña and R. Quintanilla has been supported by the Ministerio de Economía y Competitividad under the research project “Análisis Matemático de Problemas de la Termomecánica” (MTM2016-74934-P), (AEI/FEDER, UE).

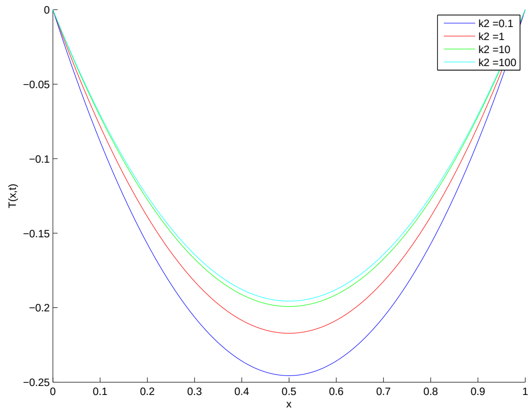


**FIGURE 4** Example 2: Temperature at final time for different diffusion parameters.

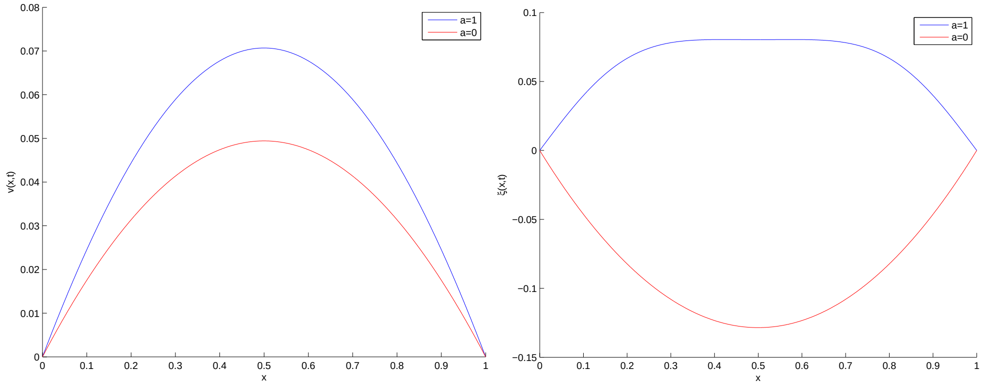


**FIGURE 5** Example 3: Thermal velocity and thermal acceleration at final time for two values of viscous diffusion parameter *a*.

## Financial disclosure

None reported.

## Conflict of interest

The authors declare no potential conflict of interests.

## References

1. Campo M, Fernández JR, Kuttler KL, Shillor M, Viaño JM. Numerical analysis and simulations of a dynamic frictionless contact problem with damage. *Comput Methods Appl Mech Engrg*. 2006;196(1-3):476–488.

2. Cattaneo C. On a form of heat equation which eliminates the paradox of instantaneous propagation. *C R Acad Sci Paris*. 1958;247:431–433.

3. Chen PJ, Gurtin ME. On a theory of heat involving two temperatures. *J Appl Math Phys.* 1968;19:614-627.

4. Chen PJ, Gurtin ME, Williams WO. A note on non-simple heat conduction. *J Appl Math Phys.* 1968;19:969-970.

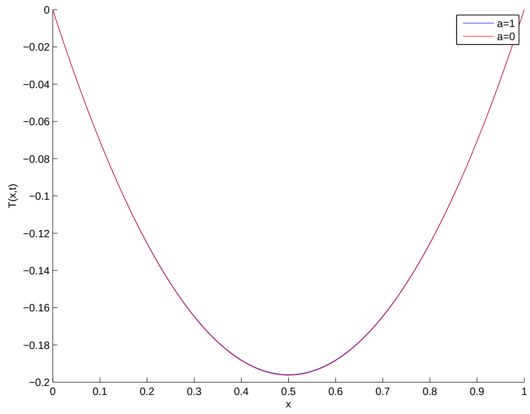


**FIGURE 6** Example 3: Temperature at final time for two values of viscous diffusion parameter *a*.


5. Chen PJ, Gurtin ME, Williams WO. On the thermodynamics of non-simple materials with two temperatures. *J Appl Math Phys.* 1969;20: 107-112.

6. Ciarlet PG. *Basic error estimates for elliptic problems*. In: Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds. 1993;vol II:17-351.

7. Dreher M, Quintanilla R, Racke R. Ill-posed problems in thermomechanics. *Appl Math Letters* 2009;22:1374–1379.

8. Fabrizio M, Franchi F. Delayed thermal models: stability and thermodynamics. *J Thermal Stresses*. 2014;37:160–173.

9. Green AE, Naghdi PM. On undamped heat waves in an elastic solid. *J Thermal Stresses*. 1992;15:253–264.

10. Green AE, Naghdi PM. Thermoelasticity without energy dissipation. *J Elasticity*. 1993;31:189–208.

11. Green AE, Naghdi PM. A unified procedure for contruction of theories of deformable media, I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua. *Proc Royal Society London A*. 1995;448:335–356, 357–377, 379–388.

12. Magaña A, Miranville A, Quintanilla R. On the stability in phase-lag heat conduction with two temperatures. *J Evol Equ.* https://doi.org/10.1007/s00028-018-0457-z

13. Quintanilla R. A well-posed problem for the dual-phase-lag heat conduction. *J Thermal Stresses*. 2008;31:260-269.

14. Quintanilla R, Jordan PM. A note on the two temperature theory with dual-phase-lag delay: some exact solutions. *Mech Res Comm*. 2009;36:796–803.

15. Tzou DY. A unified approach for heat conduction from macro to micro-scales. *ASME J Heat Transfer*. 1995;117:8–16.

16. Tzou DY. The generalized lagging response in small-scale and high-rate heating. *Int J Heat Mass Transfer*. 1995;38:3231–3240.

17. Warren WE, Chen PJ. Wave propagation in two temperatures theory of thermoelaticity. *Acta Mechanica*. 1973;16:83–117.