ON A POINCARÉ LEMMA FOR SINGULAR FOLIATIONS AND GEOMETRIC QUANTIZATION

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Abstract. In this paper we prove a Poincaré lemma for forms tangent to a foliation with nondegenerate singularities given by an integrable system on a symplectic manifold. As a consequence, the Kostant complex in Geometric Quantization is a fine resolution of the sheaf of flat sections when the polarization is spanned by the Hamiltonian vector fields of the first integrals of this integrable system.

1. Introduction

In [17] Vu Ngoc and the first author of this paper proved a *singular Poincaré lemma* for the deformation complex of an integrable system with nondegenerate singularities. This complex is defined using a Chevalley-Eilenberg complex [4] associated to a representation by Hamiltonian vector fields of this integrable system on the set of functions (modulo basic functions). The initial motivation for [17] was to give a complete proof for a crucial lemma used in proving a deformation result for pairs of local integrable systems with compatible symplectic forms. This deformation proves a Moser path lemma which is a key point in establishing symplectic normal forms *à la Morse-Bott* for integrable systems with nondegenerate singularities ([5], [6], [13]). This normal form proof can be seen as a a *"infinitesimal stability theorem implies stability"* result in this context (see [14]). So the Poincaré

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lemma turns out to be an important ingredient in the study of the symplectic geometry of integrable systems with singularities.

In this paper, we prove a Poincaré lemma for foliated cohomology associated to the singular foliation defined by the Hamiltonian vector fields of an integrable system and we find applications to Geometric Quantization. The proof of the Poincaré lemma for foliated cohomology for 1-forms can be deduced from the singular Poincaré lemma of the deformation complex using a de Rham division lemma. One can implement these techniques to deduce the higher degree case from the degree 1-case.

Once this result is proved, we obtain as a corollary a singular Poincaré lemma in the context of Geometric Quantization. This Poincaré lemma turns out to be handy because it allows to compute a sheaf cohomology associated to Geometric Quantization. The foliation associated to an integrable system is a generically Lagrangian foliation, hence it makes sense to consider it as a real polarization with singularities. Polarizations are used in Geometric Quantization to make the necessary choices to define the representation space. As a first step to define a representation space a pair (L, ∇) of a complex line bundle over the manifold together with a compatible Hermitian connection is fixed: the curvature of this connection is $-i\omega$. A real polarization is just a Lagrangian foliation of the manifold and thus the connection is flat along the leaves of the polarization. This flatness condition allows to solve the equation $\nabla_X s = 0$ locally for sections of the line bundle and vector fields which are tangent to the polarization. This is the starting point for the definition of the representation space associated to this choice of polarization. As observed by Kostant, in the real polarization case, these sections are not globally defined along the leaves [1]. In this case it makes sense to define Geometric Quantization as the cohomology groups with coefficients in the sheaf of flat sections along the polarization. This is the point of view adopted in [9, 10, 11, 15]. The Poincaré lemma that we prove in this paper allows to compute this sheaf cohomology *à la de Rham* from a complex which is nothing but

[1] The leaves that admit global sections defined along them are called Bohr-Sommerfeld leaves.

the foliated cohomology complex twisted by the sheaf of sections of the bundle which are flat along this polarization.

Organization of this paper: In section 2 we describe the geometry of the singular foliations considered in this paper. We recall in section 3 the singular Poincaré lemma for a deformation complex contained in [17]. We revisit in section 4 the proof of Poincaré lemma using homotopy operators provided in [8], and we indicate how to apply these techniques to prove a Poincaré lemma for regular foliations. In section 5 we prove the main result in this paper: which is a Poincaré lemma for the complex of tangential forms to a singular foliation given by an integrable system with nondegenerate singularities. Finally, in Section 6, we give an application of this result to Geometric Quantization proving that the cohomology of line bundle valued polarized forms is a fine resolution of the sheaf of sections which are flat along the polarization, and therefore it can be used to compute Geometric Quantization with singularities via a de Rham approach.

2. Singular foliations given by nondegenerate integrable systems

An integrable system on a symplectic manifold (M^{2n}, ω) is given by a set of generically independent functions $F = (f_1, \dots, f_n)$ satisfying $\{f_i, f_j\} = 0, \forall i, j$. The mapping $F : M^{2n} \longrightarrow \mathbb{R}^n$ given by $F = (f_1, \dots, f_n)$ has been classically known as *moment map.*

The distribution generated by the Hamiltonian vector fields X_{f_i} is involutive because $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}}$. It spans an integrable distribution which has maximal rank at the points where the functions are functionally independent. In this case, the leaf of the foliation integrating the distribution is Lagrangian because the following equality holds $0 = \{f_1, f_j\} = \omega(X_{f_i}, X_{f_j})$, and the vector fields X_{f_i} are tangent to the fibers of $F = (f_1, \dots, f_n)$. At a singular point for F, the orbit of the foliation given by the Hamiltonian fields is isotropic.

There is a notion of nondegenerate singular points which was initially introduced by Eliasson ([5],[6]). We may consider different ranks for the singularity. To define the k-rank case we reduce to the 0-rank case considering a Marsden-Weinstein reduction associated to a natural

Hamiltonian $\mathbb{T}^k$-action ([29],[18]) given by the joint flow of the moment map F.

We denote by $(x_1, y_1, \dots, x_n, y_n)$ a set of coordinates centered at the origin of $\mathbb{R}^{2n}$ and by ω the Darboux symplectic form $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$ in this neighborhood.

In the rank zero case, since the functions f_i are in involution with respect to the Poisson bracket, the quadratic parts of the functions f_i commute, defining in this way an Abelian subalgebra of $Q(2n, \mathbb{R})$ (the set of quadratic forms on $2n$-variables). We say that these singularities are of nondegenerate type if this subalgebra is a Cartan subalgebra.

Cartan subalgebras of $Q(2n, \mathbb{R})$ were classified by Williamson in [28].

Theorem 2.1 (Williamson)**.** *For any Cartan subalgebra $\mathcal{C}$ of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, y_1, \dots, x_n, y_n)$ in $\mathbb{R}^{2n}$ and a basis $h_1, \dots, h_n$ of $\mathcal{C}$ such that each h_i is one of the following:*

(2.1)
$$\begin{array}{lcr}
h_i = x_i^2 + y_i^2 & \text{for } 1 \le i \le k_e \ , & \text{(elliptic)} \\
h_i = x_i y_i & \text{for } k_e + 1 \le i \le k_e + k_h \ , & \text{(hyperbolic)} \\
\begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1}, \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} & \begin{array}{c} \text{for } i = k_e + k_h + 2j - 1, \\ 1 \le j \le k_f \end{array} & \text{(focus-focus pair)}
\end{array}$$

Observe that the number of elliptic components k_e, hyperbolic components k_h and focus-focus components k_h is therefore an invariant of the algebra $\mathcal{C}$. The triple (k_e, k_h, k_f) is an invariant of the singularity and it is called the Williamson type of $\mathcal{C}$. We have that $n = k_e + k_h + 2k_f$. Let $h_1, \dots, h_n$ be a Williamson basis of this Cartan subalgebra. We denote by X_i the Hamiltonian vector field of h_i with respect to ω. Those vector fields are a basis of the corresponding Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$. We say that a vector field X_i is hyperbolic (resp. elliptic) if the corresponding function h_i is so. We say that a pair of vector fields X_i, X_{i+1} is a focus-focus pair if X_i and X_{i+1} are the Hamiltonian vector fields associated to functions h_i and h_{i+1} in a focus-focus pair.

In the local coordinates specified above, the vector fields X_i take the following form:

- X_i is an elliptic vector field,

$$X_i = 2\left(-y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}\right)\ ; \tag{2.2}$$

- X_i is a hyperbolic vector field,

$$X_i = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\ ; \tag{2.3}$$

- X_i, X_{i+1} is a focus-focus pair,

$$X_i = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}} + y_{i+1}\frac{\partial}{\partial y_{i+1}} \tag{2.4}$$

and

$$X_{i+1} = -x_i\frac{\partial}{\partial x_{i+1}} + y_{i+1}\frac{\partial}{\partial y_i} + x_{i+1}\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial y_{i+1}}\ . \tag{2.5}$$

Assume that $\mathcal{F}$ is a linear foliation on $\mathbb{R}^{2n}$ with a rank 0 singularity at the origin p. Assume that the Williamson type of the singularity is (k_e, k_h, k_f). The linear model for the foliation is then generated by the vector fields above, it turns out that these type of singularities are symplectically linearizable and we can read of the local symplectic geometry of the foliation from the algebraic data associated to the singularity (Williamson type).

This is the content of the following symplectic linearization result [5],[6] and [13],

Theorem 2.2. *Let ω be a symplectic form defined in a neighborhood U of the origin p for which $\mathcal{F}$ is generically Lagrangian, then there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i) = \omega$, with x_i, y_i local coordinates on $(\phi(U), p)$.*

Futhermore, if $\mathcal{F}'$ is a generically Lagrangian foliation and has $\mathcal{F}$ as a linear foliation model near a point, one can symplectic linearize $\mathcal{F}'$ (see [13]).

This is equivalent to Eliasson's theorem [5, 6] in the completely elliptic case.

There are normal forms for higher rank which have been obtained by the first author together with Nguyen Tien Zung [13, 18] also in the case of singular nondegenerate compact orbits. In the more general case, a collection of regular vector fields is attached to it.

3. A singular Poincaré lemma for a deformation complex

This section revisits the main results contained in [17].

Consider the family X_i of singular vector fields given by Williamson's theorem above which form a basis of a Cartan subalgebra of the Lie algebra $\mathfrak{sp}(2r, \mathbb{R})$ with $r \leq n$.

With all this notation at hand we can now state the main result of [17]

Theorem 3.1 (Miranda and Vu Ngoc)**.** *Let $g_1, \ldots g_r$, be a set of germs of smooth functions on $(\mathbb{R}^{2n}, 0)$ with $r \leq n$ fulfilling the following commutation relations*

$$X_i(g_j) = X_j(g_i), \quad \forall i, j \in \{1, \ldots, r\} \tag{3.1}$$

where the X_i's are the vector fields defined above. Then there exists a germ of smooth function G and r germs of smooth functions f_i such that,

$$X_j(f_i) = 0\ ,\ \forall\ i, j \in \{1, \ldots, r\} \quad \textit{and} \tag{3.2}$$

$$g_i = f_i + X_i(G)\ ,\ \forall\ i \in \{1, \ldots, r\}\ . \tag{3.3}$$

Vu Ngoc and the first author of this paper also included in [17] an interesting reinterpretation of this statement in terms of the deformation complex associated to an integrable system. We think that it is instructive to explain this succinctly here.

The deformation complex is defined in two steps: first we define a Chevalley-Eilenberg complex associated to a representation by Hamiltonian vector fields associated to the components of the moment map, and then we "quotient out" by the basic functions for the foliation. The cohomology groups associated to this complex are denoted by $H^k(\mathbf{h})$. We refer the reader to [26] and [17] for more details.

Using the same notation of the last section, let $\mathbf{h} = \langle h_1, \ldots, h_n \rangle_{\mathbb{R}}$ and $\mathcal{C}_{\mathbf{h}} = \{f \in C^\infty(\mathbb{R}^{2n})\ ;\ X_h(f) = 0,\ \forall\ h \in \mathbf{h}\}$. The set $\mathbf{h}$ is an Abelian Lie subalgebra of $(C^\infty(\mathbb{R}^{2n}), \{\cdot, \cdot\})$ and $\mathcal{C}_{\mathbf{h}}$ is its centralizer.

The components of the moment map induce a representation of the commutative Lie algebra $\mathbb{R}^n$ on $(C^\infty(\mathbb{R}^{2n}), \{\cdot,\cdot\})$,

$$\mathbb{R}^n \times C^\infty(\mathbb{R}^{2n}) \ni (v, f) \mapsto \{\mathbf{h}(v), f\} \in C^\infty(\mathbb{R}^{2n}) \ . \tag{3.4}$$

Where, denoting by $(e_1, \dots, e_n)$ a basis of $\mathbb{R}^n$, $v = v_1 e_1 + \cdots + v_n e_n$ and

$$\{\mathbf{h}(v), f\} = v_1 X_1(f) + \cdots + v_n X_n(f) \ . \tag{3.5}$$

We can consider two Chevalley-Eilenberg complexes with the above action in mind, and the deformation complex is built from them. The first is the Chevalley-Eilenberg complex of $\mathbb{R}^n$ with values in $C^\infty(\mathbb{R}^{2n})$. The second is the Chevalley-Eilenberg complex of $\mathbb{R}^n$ with values in $C^\infty(\mathbb{R}^{2n})/\mathcal{C}_{\mathbf{h}}$, with respect to this action, $\mathbb{R}^n$ acts trivially on $\mathcal{C}_{\mathbf{h}}$.

If α is a 1-cocycle, then for any smooth function g_i with $\alpha(e_i) = [g_i] \in C^\infty(\mathbb{R}^{2n})/\mathcal{C}_{\mathbf{h}}$ the commutation condition $X_i(g_j) = X_j(g_i)$ is fulfilled. Now Theorem 3.1 says that there exists a function G such that $g_i = f_i + X_i(G)$, so $[g_i] = [X_i(G)]$ and this is exactly the coboundary condition.

Theorem 3.1 combined with theorem 2.2 can be, then, reformulated as follows:

Theorem 3.2 (Miranda and Vu Ngoc)**.** *An integrable system with nondegenerate singularities is C^∞-infinitesimally stable at the singular point, that is,*

$$H^1(\mathbf{h}) = 0. \tag{3.6}$$

4. Homotopy operators and a regular Poincaré lemma for foliated cohomology

Let us recall the following construction due to Guillemin and Sternberg [8] which generalizes, in a way [2], the classical proof of Poincaré lemma.

Consider $Y \subset M$ an embedded submanifold and let ϕ_t be a smooth retraction from M to Y. Given any smooth k-form α, the following

[2]The proof contained in [27] makes a particular choice of retraction on star-shaped domains

formula holds,

$$\alpha - \phi_0^*(\alpha) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*(\alpha) = \int_0^1 \phi_t^*(\iota_{\xi_t}\mathrm{d}\alpha)\mathrm{d}t + \mathrm{d}\int_0^1 \phi_t^*(\iota_{\xi_t}\alpha)\mathrm{d}t \tag{4.1}$$

where ξ_t is the vector field associated to ϕ_t. Thus, defining $I(\alpha) = \int_0^1 \phi_t^*(\iota_{\xi_t}\alpha)\mathrm{d}t$, this gives the identification with the classical formula,

$$\alpha - \phi_0^*(\alpha) = I\mathrm{d}(\alpha) + \mathrm{d}(I(\alpha)) \ . \tag{4.2}$$

Now assume that α is a closed form, formula 4.2 yields $\alpha - \phi_0^*(\alpha) = \mathrm{d}(I(\alpha))$, and therefore $I(\alpha)$ is a primitive for the closed k-form $\alpha - \phi_0^*(\alpha)$.

This has been classically applied considering retractions to a point in contractible sets or to retractions to the base of a fiber bundle. In Geometry, this technique is extremely useful to produce homotopy of special closed forms which can be connected by a path, like closed two forms defining symplectic structures in a neighborhood of a distinguished submanifold. In the context of Symplectic and Contact Geometry, a refinement of this homotopy formula leads to the so-called Moser's path method [19]. As said before, fomula 4.2 does not, a priori, give a primitive for α but for the difference $\alpha - \phi_0^*(\alpha)$[3].

This approach using the general homotopy formula of Guillemin and Sternberg has the advantadge that some choices on the retraction can be done in such a way that the vector field ξ_t is tangent to special directions in M, thus, allowing an adaptation to the foliated cohomology case. In particular we can prove a Poincaré lemma for foliated cohomology of a regular foliation, since we can consider local coordinates in which the foliation is given by local equations $\mathrm{d}x_p = 0, \dots, \mathrm{d}x_n = 0$. And we can consider as homotopy $(x_1, \dots, tx_p, \dots, tx_n)$ and the vector field ξ_t is tangent to the relevant foliation. After applying this homotopy operators we need to take care of finding a primitive for $\phi_0^*(\alpha)$ which is constant along the foliation, but this can be done by a simple integration. A similar approach is considered in [21] to give vertical homotopy operators for basic cohomology of regular fibrations.

[3] When the retraction is $(tx_1, \dots, tx_n)$, the vector field ξ_t is the radial vector field and this formula gives a primitive for α and it coincides with the one of Warner [27].

Also observe that since we can add parameters in the formula 4.2, it gives a direct proof of the parametric Poincaré lemma.

One could try to mimic similar formulae to prove a singular Poincaré lemma for a foliation given by an integrable system with nondegenerate singularities. The main issue of adapting such a proof is the smoothness of the procedure.

5. A Poincaré lemma for singular foliations

The main objective of this section is to prove a Poincaré lemma for foliations.

5.1. **Foliated cohomology.** Let $(M,\mathcal{F})$ be a foliated manifold. The foliation can be thought as a integrable (in the Sussmann's sense [25]) distribution, i.e.: $\mathcal{F}=\langle X_1,\dots,X_m\rangle_{C^\infty(M)}$ and it is a Lie subalgebra of $(\Gamma(TM),[\cdot,\cdot])$. Only when the foliation is regular $\mathcal{F}$ defines a subbundle of TM, which is often denoted by $T\mathcal{F}$.

The foliated cohomology is the one associated to the cochain complex

(5.1)

$$0\longrightarrow C^\infty_{\mathcal{F}}(M)\hookrightarrow C^\infty(M)\xrightarrow{\mathrm{d}_\mathcal{F}}\Omega^1_{\mathcal{F}}(M)\xrightarrow{\mathrm{d}_\mathcal{F}}\cdots\xrightarrow{\mathrm{d}_\mathcal{F}}\Omega^m_{\mathcal{F}}(M)\xrightarrow{\mathrm{d}_\mathcal{F}}0\ ,$$

where $\Omega^k_{\mathcal{F}}(M)=\bigwedge^k\mathcal{F}^*$ and $\mathrm{d}_\mathcal{F}$ is the restriction of the exterior derivative, d, to the distribution directions.

Whilst the de Rham complex is a fine resolution of the constant sheaf $\mathbb{R}$ on M, when a Poincaré lemma exists, the foliated cohomology is a fine resolution of the sheaf of smooth functions which are constant along the leaves of the foliation.

5.2. **A singular Poincaré lemma for 1-forms in foliated cohomology.** In what follows when we say foliated cohomology, we mean foliated cohomology of the integrable system with nondegenerate singularities. We would also like to remark that we are using the statement 3.1 which was initially proved for nondegenerate singularities of rank 0 in [17], but this result admits an extension to singularities of higher rank (simply by applying the parametric trick explained in the proof of the regular Poincaré lemma). The statements are included here for rank 0 nondegenerate singularities, but they are valid for higher rank singularities.

The classical Poincaré lemma for de Rham 1-forms asserts that a closed 1-form on a smooth manifold is locally exact. In other words, given m-functions g_i on an m-dimensional manifold for which $\frac{\partial}{\partial x_i}(g_j) = \frac{\partial}{\partial x_j}(g_i)$ there exists a local smooth function G such that $g_i = \frac{\partial}{\partial x_i}(G)$.

Now assume that we have a set of r functions g_i and a set of r vector fields X_i of a Williamson type with a singularity at a point p and fulfilling a commutation relation of type $X_i(g_j) = X_j(g_i)$.

A priori, theorem 3.1 does not directly yield a Poincaré lemma for foliated 1-forms. We will try to apply this Poincaré lemma for deformation complex to obtain a Poincaré lemma for foliated cohomology.

First we need to understand how to express foliated forms as combination of distinguished singular one forms in order to see how both Poincaré lemmata for different complexes are related.

The following result from Moussu, which is a smooth version of de Rham's division lemma (see [20]), is a first step in this direction:

Theorem 5.1 (Moussu)**.** *Let η be a smooth 1-form on a neighborhood of the origin in $\mathbb{R}^n$ for which the origin is an algebraically isolated singularity. Then for any smooth p-form σ, $0 < p < n$, such that $\sigma \wedge \eta = 0$ we can factorize σ as $\sigma = \zeta \wedge \eta$ for a smooth $(p-1)$-form ζ.*

Define $\eta_i = \frac{1}{2}(x_i \mathrm{d}y_i - y_i \mathrm{d}x_i)$ for elliptic and hyperbolic components, and $\eta_i = \frac{1}{4}(x_i y_i + x_{i+1}y_{i+1})\mathrm{d}\left[\ln\left(\frac{y_i^2+y_{i+1}^2}{x_i^2+x_{i+1}^2}\right)\right]$ and $\eta_{i+1} = \frac{1}{2}(x_i y_{i+1} - x_{i+1}y_i)\mathrm{d}\left[\arctan\left(\frac{x_i y_i - x_{i+1}y_{i+1}}{x_i y_{i+1} + x_{i+1}y_i}\right)\right]$ for focus-focus pairs. Observe that if $\alpha \in \Omega^1_{\mathcal{F}}(\mathbb{R}^{2n})$ then $\alpha \wedge \eta_1 \cdots \wedge \eta_n = 0$. Outside the set $\cup\{x_i = 0, y_i = 0\}$, it is obvious that we can find smooth functions such that $\alpha = \sum_i A_i \eta_i$. The problem is to see that these smooth functions A_i extend to smooth funtions at $\{x_i = 0, y_i = 0\}$, for this we need to "divide out" by the singularities. This can be achieved thanks to an application of the division lemma above.

This is the content of the following lemma,

Lemma 5.1. *Given a local foliated* 1*-form, there exists a collection of local smooth functions A_i such that $\alpha = \sum_i A_i \eta_i$.*

Proof. Assume the dimension of the manifold is $2n$. Let us prove it by induction on n. If $n=1$, the origin $(0,0)$ is an algebraically isolated singularity and the result follows directly from theorem 5.1.

Now assume that the lemma holds for $n=k$ let us prove it for $k+1$. Let us first assume that f_{k+1} does not belong to a focus-focus pair. We cannot apply directly the division lemma because the singularity is not isolated but we can reduce the induction hypothesis by considering the following trick: take the difference $\beta=\alpha-\alpha_{|TS_{k+1}}$ where S_{k+1} is the submanifold given by equations $x_i=cte, y_i=cte.$ for $i=k+1$. We can apply the induction hypothesis to β since $\beta\wedge\eta_{k+1}=0$ by considering parametric version of the division lemma (with parameters $x_1,y_1,\dots,x_k,y_k$) to obtain $\beta=A_{k+1}\eta_{k+1}$(smoothness on (x_{k+1},y_{k+1}) is guaranteed by theorem 5.1 and smoothness on the other variables comes directly from parametric dependence). On the other hand since $\alpha_{|TS_{k+1}}\wedge\eta_1\wedge\cdots\wedge\eta_k=0$ and because the singularity is isolated (we play the parametric trick again) we can apply the induction hypothesis to $\alpha_{TS_{k+1}}$. This yields, $\alpha_{TS_{k+1}}=\sum_i^k A_i\eta_i$. Now adding this expression to β we obtain $\alpha=\sum_i^{k+1}A_i\eta_i$ for certain local smooth functions A_i.

In the case f_{k+1} belongs to a focus-focus pair, we can proceed in a similar way but applying the trick and then considering the restriction to a submanifold of type $x_{k+1}=0, y_{k+1}=0, x_{k+2}=0, y_{k+2}=0$.

This ends the proof of the lemma. ☐

We can apply this lemma to prove the following proposition,

Proposition 5.1. *With respect to a singular foliation given by a completely integrable system, any* $\alpha\in\Omega^1_{\mathcal{F}}(\mathbb{R}^{2n})$ *which is closed is indeed exact. That is to say, there exists a function* H *such that* $\mathrm{d}_{\mathcal{F}}(H)=\alpha$.

Proof. Because of lemma 5.1 we can write $\alpha=\sum_i A_i\eta_i$, and the condition of being closed implies that $X_i(h_jA_j)=X_j(h_iA_i)$, where $h_1,\dots,h_n$ is the basis for the Cartan subalgebras in Williamson's theorem 2.1. Thus, we apply theorem 3.1 to obtain $h_iA_i=f_i+X_i(G)$ with $X_j(f_i)=0$. Substituting this last expression in $\alpha=\sum_i A_i\eta_i$, one has $\alpha-\mathrm{d}_{\mathcal{F}}(G)=\sum_i\frac{f_i}{h_i}\eta_i$, and, on the other hand, lemma 5.1 gives $\alpha-\mathrm{d}_{\mathcal{F}}(G)=\sum_i B_i\eta_i$, which guarantees that the $\frac{f_i}{h_i}$ are smooth.

It is important to note that $\eta_i=h_i\mathrm{d}s_i$ with

- elliptic $s_i = \frac{1}{2}\mathrm{arctan}\left(\frac{y_i}{x_i}\right)$ for $i = k+1, ..., k_e$,
- hyperbolic $s_i = \frac{1}{2}\ln\left|\frac{y_i}{x_i}\right|$ for $i = k_e+1, ..., k_e+k_h$,
- focus-focus $s_i = \frac{1}{4}\ln\left(\frac{y_i^2+y_{i+1}^2}{x_i^2+x_{i+1}^2}\right)$ and $s_{i+1} = \frac{1}{2}\mathrm{arctan}\left(\frac{x_iy_i-x_{i+1}y_{i+1}}{x_iy_{i+1}+x_{i+1}y_i}\right)$ for $i = k_e+k_h+2m-1$, $m = 1, ..., k_f$.

Having reached this point, we can easily check that $\alpha = \mathrm{d}_{\mathcal{F}}(H)$ for

$$H = \sum_i f_i \cdot s_i + G \ , \tag{5.2}$$

where the functions $f_i \cdot s_i$ are smooth because $\frac{f_i}{h_i}$ are smooth functions.

□

5.3. **Higher degrees.** In this subsection we give a sketch of the proof of the Poincaré lemma for higher degrees. In [16] we will provide a detailed proof of this result, using a generalized division lemma for foliated forms and a generalization of decomposition results for functions with respect to a Williamson basis, as the ones contained in [17] and [13], to foliated forms.

Theorem 5.2. *A local closed foliated k-form α is exact when the singular, nondegenerate, foliation is given by a completely integrable system. That is to say, there exists a local $(k-1)$-form β such that $\mathrm{d}_{\mathcal{F}}\beta = \alpha$.*

Before proceeding with the proof, we claim here that we can easily prove a generalization of Lemma 5.1 to higher degrees (for details see [16]). The method of proof is again an induction from Moussu division's lemma 5.1 which works for k-forms. For the sake of simplicity , we just enclose the statement here:

Lemma 5.2. *Given a local foliated k-form α, there exists a collection of local smooth functions $A_{i_1,\ldots,i_k}$ such that*

$$\alpha = \sum_{i_1,\ldots,i_k} A_{i_1,\ldots,i_k}\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}. \tag{5.3}$$

The idea that we sketch here is to adapt similar techniques as the ones used for the standard Poincaré lemma to the singular case. There are two well-known proofs of the Poincaré lemma: one is based on explicit homotopy operators [8] and [27] and the other one uses an induction procedure [1][4].

As we explained in section 4,the homotopy formula 4.2 associates a $(k-1)$-form to a k-form.

In the singular case we could simply try to apply this formula considering as retraction $\phi_t = (h_1, \ldots, h_n, ts_1, \ldots, ts_n)$, where the s_i are the "singular coordinates" defined in the proof of Proposition 5.1. It is very important to point out that if we consider this retraction, the associated vector field ξ_t is tangent to the foliation.

Using lemma 5.2, any tangential form can be written as a smooth combination $\alpha = \sum_{i_1,\ldots,i_k} A_{i_1,\ldots,i_k} \eta_{i_1} \wedge \cdots \wedge \eta_{i_k}$. We can then check that this retraction, even if expressed in singular coordinates, when plugged in the integral formula gives a smooth $(k-1)$-form $\beta = I(\alpha)$.

Indeed, we could also argue here that we know that there is a smooth solution for 1-forms from the previous subsection and we know that two formal solutions just differ by a basic function. This would yield that formula works for 1-forms, an inductive proof would yield the rest.

In [16] we provide a complete proof which uses directly the inductive proof of Poincaré lemma(see for instance [1]) and develops the technical tools to obtain the necessary division lemmata.

6. An application to Geometric Quantization

6.1. **Prequantization.** This subsection deals with some concepts needed to define wave functions. The first attempt was to see them as sections of a complex line bundle over the symplectic manifold, the so-called prequantum line bundle. The other notion described here, the polarization, is a way to define a global distinction between momentum and position.

Definition 6.1. *A symplectic manifold (M, ω) such that $[\omega]$ is integral is called prequantizable. A prequantum line bundle of (M, ω) is a Hermitian line bundle over M with connection, compatible with the Hermitian structure, (L, ∇^ω) that satisfies curv$(\nabla^\omega) = -i\omega$.*

[4]We should say here that indeed this last one can also be deduced from [8].

Example 6.1. Any exact symplectic manifold satisfies $[\omega] = 0$; in particular cotangent bundles with the canonical symplectic structure. In that case the trivial line bundle is an example of a prequantum line bundle.

The following theorem[5] [12] provides a relation between the above definitions:

Theorem 6.1 (Kostant)**.** *A symplectic manifold (M, ω) admits a prequantum line bundle (L, ∇^{ω}) if and only if it is prequantizable.*

A real polarization $T\mathcal{F}$ is an integrable subbundle of TM whose leaves are Lagrangian submanifolds. But due to the examples above, another definition is considered.

Definition 6.2. *A real polarization $\mathcal{F}$ is an integrable (in the Sussmann's* [25] *sense) distribution of TM whose leaves are generically Lagrangian. The complexification of $\mathcal{F}$ is denoted by P and will be called polarization.*

From now on (L, ∇^{ω}) will be a prequantum line bundle and P the complexification of a real polarization of (M, ω).

6.2. **Geometric Quantization à la Kostant.** The original idea of Geometric Quantization is to associate a Hilbert space to a symplectic manifold via a prequantum line bundle and a polarization. Usually this is done using flat global sections of the line bundle. In case these global sections do not exist, one can define Geometric Quantization via higher cohomology groups by considering cohomology with coefficients in the sheaf of flat sections.

The existence of global flat sections is a nontrivial matter. Actually Rawnsley [22], and later Solha [24], showed that the existence of a S^1-action may be an obstruction for nonzero global flat sections.

In order to use flat sections as analogue for wave functions one is forced to work with delta functions with support over Bohr-Sommerfeld leaves, or deal with sheaves and higher order cohomology groups. Both

[5] This result is also attributed to André Weil, *Introduction à l'étude des variétés kählériennes* (1958).

approaches can be found in the literature[6], but here only the sheaf approach, as suggested by Kostant, is treated.

Definition 6.3. *Let $\mathcal{J}$ denotes the space of local sections s of a prequantum line bundle L such that $\nabla^{\omega}_X s = 0$ for all vector fields X of a polarization P. The space $\mathcal{J}$ has the structure of a sheaf and it is called the sheaf of flat sections.*

Considering the triplet: prequantizable symplectic manifold (M,ω), prequantum line bundle (L,∇^{ω}), and polarization P;

Definition 6.4. *The Quantization of $(M,\omega,L,\nabla^{\omega},P)$ is given by*

$$\mathcal{Q}(M) = \bigoplus_{k\geq 0} \check{H}^k(M;\mathcal{J}) \ , \tag{6.1}$$

where $\check{H}^k(M;\mathcal{J})$ are Čech cohomology groups with values in the sheaf $\mathcal{J}$.

Remark 6.1. Even though $\mathcal{Q}(M)$ is just a vector space and a priori has no Hilbert structure, it will be called Quantization. The true Quantization of the triplet $(M,\omega,L,\nabla^{\omega},P)$ shall be the completion of the vector space $\mathcal{Q}(M)$, after a Hilbert structure is given, together with a Lie algebra homomorphism (possibly defined over a smaller set) between the Poisson algebra of $C^{\infty}(M)$ and operators on the Hilbert space. In spite of the problems that may exist in order to define Geometric Quantization using $\mathcal{Q}(M)$, the first step is to compute this vector space.

6.3. **Line bundle valued polarized forms.** Following Rawnsley [22], given a prequantizable symplectic manifold with polarization, it is possible to construct a fine resolution for the sheaf of flat sections. Using the results presented in section 5 it is even possible to do it when the polarization has nondegenerate singularities, in the Morse-Bott sense.

The restriction of the connection ∇^{ω} to the polarization induces a linear operator

$$\nabla : \Gamma(L) \to P^* \otimes_{C^{\infty}(M;\mathbb{C})} \Gamma(L) \tag{6.2}$$

[6]Rawnsley cites works of Simms, Śniatycki and Keller in [22].

satisfying (by definition) the following property:

$$\nabla(fs) = \mathrm{d}_P f \otimes s + f\nabla s \ , \tag{6.3}$$

for $f \in C^\infty(M;\mathbb{C})$ and $s \in \Gamma(L)$, where d_P is the restriction of the exterior derivative to the distribution directions.

Definition 6.5. *The space of line bundle valued polarized forms is* $S_P{}^\bullet(L) = \bigoplus_{k\geq 0} S_P^k(L)$, *where* $S_P^k(L) = \bigwedge^k P^* \otimes_{C^\infty(M;\mathbb{C})} \Gamma(L)$.

So $\nabla : S_P^0(L) \to S_P^1(L)$ and $S_P{}^\bullet(L)$ has a module structure which enables an extension of ∇ to a derivation of degree $+1$, $\mathrm{d}^\nabla : S_P{}^\bullet(L) \to S_P{}^\bullet(L)$, as follows.

The space of polarized forms, $\Omega_P{}^\bullet(M) = \bigoplus_{k\geq 0} \bigwedge^k P^*$, is the complexification of $\Omega_{\mathcal{F}}{}^\bullet(M)$, and it acts on $S_P{}^\bullet(L)$ via wedge product: the space of line bundle valued polarized forms is a $\Omega_P{}^\bullet(M)$-module. Then we have for any $\alpha \in \Omega_P^k(M)$ and $\boldsymbol{\beta} = \beta \otimes s \in S_P^l(L)$,

$$\mathrm{d}^\nabla \boldsymbol{\beta} = \mathrm{d}^\nabla(\beta \otimes s) = \mathrm{d}_P\beta \otimes s + (-1)^l \beta \wedge \nabla s \ , \tag{6.4}$$

and

$$\mathrm{d}^\nabla \circ \mathrm{d}^\nabla \boldsymbol{\beta} = curv(\nabla^\omega)\big|_P \wedge \boldsymbol{\beta} \ . \tag{6.5}$$

Since $\omega = i \cdot curv(\nabla^\omega)$ vanishes along P, then $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla = 0$ and d^∇ is a coboundary operator.

6.4. **A Poincaré lemma for polarized forms.** Here there is a proof showing that the underlying complex given by d^∇ and the space of line bundle valued polarized forms, named the Kostant complex, is a fine resolution for the sheaf of flat sections when the polarization comes from a nondegenerate integrable system. This part of the paper was first announced in [23].

If $\mathcal{S}_P^k(L)$ denotes the associated sheaf of $S_P^k(L)$, one can extend d^∇ to a homomorphism of sheaves; $\mathrm{d}^\nabla : \mathcal{S}_P^k(L) \to \mathcal{S}_P^{k+1}(L)$. $\mathcal{S}_P^0(L) \cong \mathcal{S}$, the sheaf of sections of the line bundle L, and $\mathcal{J}$ is isomorphic to the kernel of $\mathrm{d}^\nabla : \mathcal{S} \to \mathcal{S}_P^1(L)$. Because $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla = 0$, one is able to build a sequence.

Definition 6.6. *The Kostant complex is*

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}^1_P(L) \xrightarrow{\mathrm{d}^\nabla} \cdots \xrightarrow{\mathrm{d}^\nabla} \mathcal{S}^n_P(L) \xrightarrow{\mathrm{d}^\nabla} 0 \ . \tag{6.6}$$

One proves that the above sequence of sheaves is exact using the following results:

Lemma 6.1. *For a polarization with nondegenerate singularities P there is always a local unitary flat section on each point of M.*

Proof. Let $W \subset M$ be a trivializing neighborhood of L with a unitary section $s : W \subset M \to L$. Since $\nabla s \in S^1_{P|_W}(L|_W)$ there is a $\alpha \in \Omega^1_{P|_W}(W)$ such that $\nabla s = \alpha \otimes s$. The condition $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla = 0$ implies $\mathrm{d}_P\alpha = 0$;

$$\begin{aligned} 0 = \mathrm{d}^\nabla(\nabla s) &= \mathrm{d}^\nabla(\alpha \otimes s) = \mathrm{d}_P\alpha \otimes s - \alpha \wedge \nabla s \\ &= \mathrm{d}_P\alpha \otimes s - (\alpha \wedge \alpha) \otimes s = \mathrm{d}_P\alpha \otimes s \ . \end{aligned} \tag{6.7}$$

By the Poincaré lemma for singular foliations (theorem 5.2) there exists a neighborhood $V \subset W$ and $f \in C^\infty(V;\mathbb{C})$ such that $\mathrm{d}_P f = \alpha|_V$. Setting $r = \mathrm{e}^{-f}s|_V$,

$$\nabla r = \mathrm{e}^{-f}\nabla s|_V + \mathrm{d}_P(\mathrm{e}^{-f}) \otimes s|_V = \mathrm{e}^{-f}(\alpha \otimes s)\big|_V - \mathrm{e}^{-f}\mathrm{d}_P f \otimes s|_V = 0 \ , \tag{6.8}$$

so r is a unitary flat section of $L|_V$. ☐

As a consequence of the existence of unitary flat sections, elements of $\mathcal{S}^k_P(L)$ which are closed can be interpreted as germs of closed polarized k-forms taking values on the sheaf $\mathcal{J}$.

Corollary 6.1. *Let $\boldsymbol{\Omega^k_P}$, $\mathcal{C}^\infty_{\mathbb{C}}$ and $\mathcal{C}^\infty_P$ be the sheaves associated to $\Omega^k_P(M)$, $C^\infty(M;\mathbb{C})$ and $C^\infty_P(M;\mathbb{C})$. Then $\mathcal{S}^k_P(L) \cong \boldsymbol{\Omega^k_P} \otimes_{\mathcal{C}^\infty_{\mathbb{C}}} \mathcal{J}$ and $\ker(\mathrm{d}^\nabla) \cong \ker(\mathrm{d}_P) \otimes_{\mathcal{C}^\infty_P} \mathcal{J}$.*

Proof. By lemma 6.1, for each point on M there exists a trivializing neighborhood $V \subset M$ of L with a unitary flat section $s : V \subset M \to L$. If $\boldsymbol{\alpha} \in S^k_P(L)$ it can be locally written as $\boldsymbol{\alpha}\big|_V = \alpha \otimes s$, where $\alpha \in \Omega^k_{P|_V}(V)$. If also $\mathrm{d}^\nabla\boldsymbol{\alpha} = 0$, then $\mathrm{d}_P\alpha = 0$, because $\mathrm{d}^\nabla(\alpha \otimes s) = \mathrm{d}_P\alpha \otimes s + (-1)^k\alpha \wedge \nabla s$, $s \neq 0$ and $\nabla s = 0$. ☐

Corollary 6.2. *The sheaves $\mathcal{S}^k_P(L)$ are fine and torsionless, and the sheaf $\mathcal{J}$ is torsionless.*

Proof. Lemma 6.1 implies that the stalks of $\mathcal{J}$ are free modules over the ring of smooth complex valued functions constant along P, so it is torsionless: $s = fr$ is flat if and only if $f \in C^\infty_P(V;\mathbb{C})$, supposing r unitary flat over V;

$$\nabla s = \nabla(fr) = \mathrm{d}_P f \otimes r + f \nabla r \ . \tag{6.9}$$

$\Gamma(L)$ and $\Omega^k_P(M)$ are free modules over the ring of smooth complex valued functions of M, and by that, it admits partition of unity. Thus the tensor product $\boldsymbol{\Omega^k_P} \otimes_{\mathcal{C}^\infty_{\mathbb{C}}} \mathcal{S} = \mathcal{S}^k_P(L)$ is fine and torsionless. $\square$

So, corollary 6.1 together with the Poincaré lemma for singular foliations (theorem 5.2) implies the exactness of (6.6).

Let us recall the abstract de Rham theorem [3].

Theorem 6.2 (**Abstract de Rham theorem**)**.** *Let M be a manifold (smooth and paracompact) and $\mathcal{J}$ a sheaf on it. For each fine (torsionless) resolution $\{\mathcal{S}^k_P(L), \mathrm{d}^\nabla\}$ of $\mathcal{J}$,*

$$\check{H}(M;\mathcal{J}) \cong \frac{\ker(\mathrm{d}^\nabla : S^k_P(L) \to S^{k+1}_P(L))}{\mathrm{im}(\mathrm{d}^\nabla : S^{k-1}_P(L) \to S^k_P(L))} \tag{6.10}$$

for all k.

Wherefore, applying corollary 6.2 with the abstract de Rham theorem, the following holds:

Theorem 6.3. *The Kostant complex is a fine torsionless resolution for $\mathcal{J}$. Therefore, each of its cohomology groups, $H^k({S_P}^\bullet(L))$, are isomorphic to $\check{H}^k(M;\mathcal{J})$.*

Remark 6.2. The only property of L that was used is the existence of flat connections along P. Any complex line bundle would do, not only a prequantum one, in particular the tensor product between a prequantum line bundle and a bundle of half forms normal to P: the results here work if metaplectic correction is included.

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