

Rainbow matchings in Dirac bipartite graphs

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Abstract

We show the existence of rainbow perfect matchings in μn -bounded edge colourings of Dirac bipartite graphs, for a sufficiently small $\mu > 0$. As an application of our results, we obtain several results on the existence of rainbow k -factors in Dirac graphs and rainbow spanning subgraphs of bounded maximum degree on graphs with large minimum degree.

1 Introduction

An $n \times n$ array of symbols where each symbol occurs precisely one in each row and column is called a *Latin square* of order n . A *partial transversal* of size k in a Latin square is a set of cells, including at most one from each row and each column that contains k distinct symbols. The question of finding the largest transversal in an arbitrary Latin square has attracted considerable attention. There exist Latin squares, such as the addition table of \mathbb{Z}_n for even n , whose largest transversal has size $n - 1$ [9, 27]. It has been conjectured that this is the worst case.

Conjecture 1 (Ryser, Brualdi, Stein [4, 23, 25]). *Every Latin square of order n contains a partial transversal of size at least $n - 1$.*

The best known lower bound is due to Hatami and Shor [11], who showed that every Latin square of order n has a partial transversal of size $n - O(\log^2(n))$.

There is a one-to-one correspondence between Latin squares $L = (L_{ij})_{i,j \in [n]}$ of order n and proper edge colourings of the complete bipartite graph $K_{n,n}$ on $2n$ vertices; simply, assign colour L_{ij} to the edge $a_i b_j$, where $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ are the stable sets of $K_{n,n}$. Given a graph G and an edge colouring $\chi : E(G) \rightarrow \mathbb{N}$ of G , the subgraph $H \subseteq G$ is *rainbow* if for every $c \in \mathbb{N}$, $|\chi^{-1}(c) \cap E(H)| \leq 1$. Under the above correspondence, a partial transversal of size k in a Latin square is equivalent to a partial rainbow matching of size k .

One can extend the problem to edge colourings of $K_{n,n}$ that satisfy a milder condition. An edge colouring is *k -bounded* if $|\chi^{-1}(c)| \leq k$ for every $c \in \mathbb{N}$. Stein [25] conjectured that Conjecture 1 still holds for n -bounded edge colourings. This was very recently disproved by Pokrovskiy and Sudakov [22]. However, positive results can be obtained if the size of each colour class is small enough.

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Theorem 2 (Erdős, Spencer [8]). *Let $K_{n,n}$ be the complete bipartite graph on $2n$ vertices, then any $(n - 1)/16$ -bounded edge colouring of $K_{n,n}$ contains a rainbow perfect matching.*

The goal of this paper is to obtain a sparse version of Theorem 2. A balanced bipartite graph G contains a perfect matching if and only if G satisfies Hall's condition. However, it is easy to see that Hall's condition is not a sufficient for the existence of a rainbow perfect matching if colour classes have linear size. For example, consider a graph consisting of a perfect matching which trivially satisfies Hall's Condition but has no rainbow perfect matching unless each colour class has size 1. Thus, we impose a stronger condition concerning the minimum degree of G . A *Dirac bipartite graph* on $2n$ vertices is a balanced bipartite graph with minimum degree at least $n/2$. The main result of this paper shows the existence of rainbow perfect matchings in Dirac bipartite graphs.

Theorem 3. *There exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and G is a Dirac bipartite graph on $2n$ vertices, then any μn -bounded edge colouring of G contains a rainbow perfect matching.*

The proof of Theorem 3 combines probabilistic and extremal ingredients. The main tool used to provide the existence of a rainbow matching is a lopsided version of the Lovász Local Lemma, which is standard in this setting. One of the novelties of our approach is the estimation of conditional probabilities in the uniform distribution on the set of perfect matchings of a Dirac bipartite graph, via a switching argument (see Section 2). However, this probability space often exhibits strong dependencies which limit the application of the local lemma.

In order to overcome this problem, in Section 3 we use a well-established dichotomy for Dirac bipartite graphs; either the graph has good expansion properties (*robust expander*) or the graph consists of two (possibly unbalanced) very dense bipartite graphs of order roughly n with few edges in-between (*extremal graph*). The notion of robust expanders was first introduced by Kühn et al. [18] in the context of Hamiltonian digraphs (see also [17]). A local lemma based argument provides the existence of a rainbow perfect matching in robust expanders (Section 4). However, this argument cannot be applied directly to extremal graphs. In Section 5, we construct a rainbow perfect matching by selecting a partial matching in-between the two dense bipartite graphs that balances the remainder, followed by extending it into a rainbow perfect matching using similar arguments to the ones displayed previously. In Section 6 we combine these two results, concluding that any Dirac bipartite graph with a μn -bounded edge colouring contains a rainbow perfect matching.

Our result can be extended to a more general setting by slightly strengthening the minimum degree condition. A *system of conflicts* for $E(G)$ is a set \mathcal{F} of unordered pairs of edges of G . If $\{e, f\} \in \mathcal{F}$ we say that e and f conflict and call $\{e, f\}$ a *conflict*. A system of conflicts \mathcal{F} for $E(G)$ is *k -bounded* if for each $e \in E(G)$, there are at most k conflicts that contain e .

Given a graph G and a system of conflicts \mathcal{F} for $E(G)$, the subgraph $H \subseteq G$ is *\mathcal{F} -conflict-free* if for each distinct $e, f \in E(H)$, we have $\{e, f\} \notin \mathcal{F}$.

Rainbow subgraphs correspond to conflict-free subgraphs of transitive systems of conflicts. Given an edge colouring χ of G , we define the system of conflicts \mathcal{F}_χ for $E(G)$ as follows

$$\mathcal{F}_\chi = \{\{e, f\} : e, f \in E(G) \text{ and } \chi(e) = \chi(f)\}$$

Note that χ is k -bounded if and only if \mathcal{F}_χ is $(k - 1)$ -bounded.

We obtain an asymptotic version of Theorem 3 for bounded systems of conflicts.

Theorem 4. *For all $\varepsilon > 0$ there exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and G is a balanced bipartite graph on $2n$ vertices with minimum degree $\delta(G) \geq (1/2 + \varepsilon)n$, then any μn -bounded system of conflicts \mathcal{F} for $E(G)$ contains a conflict-free perfect matching.*

Theorem 4 follows as a corollary of the proof of Theorem 3 for robust expanders (Section 6). Section 7 contains two applications of Theorem 3 and Theorem 4, providing the existence of rainbow Δ -factors in Dirac graphs and of rainbow spanning subgraphs with bounded maximum degree in graphs with large minimum degree. We conclude the paper in Section 8 with further remarks and open questions.

2 Switching over 6-cycles

Our main tool to find conflict-free matchings is a p -Lopsided form of the Lovász Local Lemma. In this section we introduce it and show how it will be applied.

Definition. Let $\mathcal{E} = \{E_1, E_2, \dots, E_q\}$ be a collection of events. A graph \mathcal{H} with vertex set $[q]$ is a p -dependency graph for \mathcal{E} if for every $i \in [q]$ and every set $S \subseteq [q] \setminus N_{\mathcal{H}}[i]$ such that $\mathbb{P}(\cap_{j \in S} E_j^c) > 0$ we have

$$\mathbb{P}(E_i | \cap_{j \in S} E_j^c) \leq p. \quad (1)$$

We will use the following version of the local lemma that uses p -dependency graphs.

Lemma 5 (p -Lopsided Lovász Local Lemma [8]). *Let \mathcal{E} be a collection of events and let \mathcal{H} be a p -dependency graph for \mathcal{E} . Let d be the maximum degree of \mathcal{H} . If $4pd \leq 1$, then*

$$\mathbb{P}(\cap_{E \in \mathcal{E}} E^c) \geq (1 - 2p)^{|\mathcal{E}|}.$$

While the statement in [8] does not specify the explicit lower bound, this is contained inside their proof.

The following notion will play a central role in showing the existence of conflict-free perfect matchings.

Definition. Let $G = (A \cup B, E)$ be a balanced bipartite graph on $2n$ vertices with at least one perfect matching. Suppose M is a perfect matching of G and let $x = a_1 b_1 \in M$. An edge $y = ab \in E(G)$ is (x, M) -switchable if $y \notin M$ and the 6-cycle $a_1 b_1 a_2 b a b_2$ is a subgraph of G , where $a_2 b, a b_2 \in M$.

The existence of many switchable edges in every perfect matching suffices to find a conflict-free perfect matching. In the following lemma and in many of the subsequent results of this paper we use hierarchies of the form $1/n \ll \alpha \ll \beta \ll 1$. We write $\alpha \ll \beta$ to mean that there exists an increasing function f such that our result holds whenever $\alpha \leq f(\beta)$. For simplicity we do not explicitly calculate such functions, however we could do so in principle should we so wish.

Lemma 6. *Let $n \in \mathbb{N}$ and suppose that $1/n \ll \mu \ll \gamma \leq 1$. Let $G = (A \cup B, E)$ be a balanced bipartite graph on $2n$ vertices with at least one perfect matching. Suppose that for every perfect matching M of G and for every $x = a_1 b_1 \in M$ there are at least γn^2 edges of G that are (x, M) -switchable. Given a μn -bounded system of conflicts for $E(G)$, the probability that a uniformly random perfect matching of G is conflict-free is at least $e^{-\mu^{1/2} n}$.*

Proof. Let $\Omega = \Omega(G)$ be the set of perfect matchings of G equipped with the uniform distribution. By assumption, note that $\Omega \neq \emptyset$. Let $M \in \Omega$ be a perfect matching chosen uniformly at random. Let \mathcal{F} be a μn -bounded system of conflicts for $E(G)$.

For each unordered pair of edges $x, y \in E(G)$ let $E(x, y) = \{x, y \in M\}$ be the event that both x and y are simultaneously in M . Define

$$Q = \{\{x, y\} \in \mathcal{F} : x, y \text{ non-incident}\} ,$$

and let $q = |Q|$. Consider the collection of events $\mathcal{E} = \{E(x, y) : \{x, y\} \in Q\}$.

Write $\mathcal{E} = \{E_i : i \in [q]\}$ and let \mathcal{H} be the graph with vertex set $[q]$ where $i, j \in [q]$ are adjacent if and only if the subgraph of G that contains the set of edges $\{x, y, w, z\}$ is not a matching, where $E_i = E(x, y)$ and $E_j = E(w, z)$.

Observe that given $\{x, y\} \in Q$, there are at most $4n$ ways to choose an edge $w \in E(G)$ that is incident either to x or to y , and at most μn ways to choose an edge $z \in E(G)$ with $\{w, z\} \in \mathcal{F}$. Hence, the maximum degree in \mathcal{H} is at most $d := 4\mu n^2$.

Our goal is to show that \mathcal{H} is a p -dependency graph for \mathcal{E} , for a suitably small $p > 0$. Given $i \in [q]$ and $S \subseteq [q] \setminus N_{\mathcal{H}}[i]$ with $\mathbb{P}(\cap_{j \in S} E_j^c) > 0$, it suffices to show that (1) holds.

Let $E_i = E(x, y)$. We say that a perfect matching is S -good if it belongs to $\cap_{j \in S} E_j^c$. Since $\mathbb{P}(\cap_{j \in S} E_j^c) > 0$, there is at least one S -good perfect matching. Let $\mathcal{M} = \mathcal{M}(S)$ be the set of S -good perfect matchings and let $\mathcal{M}_0 \subseteq \mathcal{M}$ the set of S -good perfect matchings that contain both x and y .

Construct an auxiliary bipartite graph $\mathcal{G} = (\mathcal{M}_0, \mathcal{M} \setminus \mathcal{M}_0, E(\mathcal{G}))$, where $M_0 \in \mathcal{M}_0$ and $M \in \mathcal{M}$ are adjacent (i.e. $M_0 M \in E(\mathcal{G})$) if there exist edges $x_1, x_2, y_1, y_2 \in M_0$ and $x_3, x_4, x_5, y_3, y_4, y_5 \in M$ such that $x, x_3, x_1, x_5, x_2, x_4$ and $y, y_3, y_1, y_5, y_2, y_4$ are vertex disjoint 6-cycles contained in G (see Figure 1).

By double-counting the edges of \mathcal{G} , we obtain

$$\delta(\mathcal{M}_0)|\mathcal{M}_0| \leq |E(\mathcal{G})| \leq \Delta(\mathcal{M} \setminus \mathcal{M}_0)|\mathcal{M} \setminus \mathcal{M}_0| ,$$

from which we deduce,

$$\mathbb{P}(E_i | \cap_{j \in S} E_j^c) = \frac{|\mathcal{M}_0|}{|\mathcal{M}|} \leq \frac{|\mathcal{M}_0|}{|\mathcal{M} \setminus \mathcal{M}_0|} \leq \frac{\Delta(\mathcal{M} \setminus \mathcal{M}_0)}{\delta(\mathcal{M}_0)} . \quad (2)$$

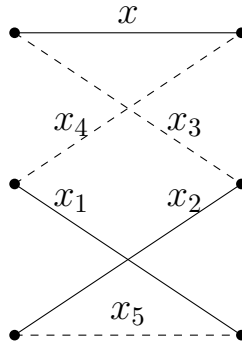


Figure 1: Switching for edge x .
 x_5 is (x, M) -switchable.

So, in order to prove (1) we need to bound $\Delta(\mathcal{M} \setminus \mathcal{M}_0)$ from above and $\delta(\mathcal{M}_0)$ from below.

We first bound $\Delta(\mathcal{M} \setminus \mathcal{M}_0)$ from above. Fix $M \in \mathcal{M} \setminus \mathcal{M}_0$ and let us count the number of 6-cycles of the form $xx_3x_1x_5x_2x_4$, with $x_3, x_4, x_5 \in M$. Since $x_5 \in M$ is not incident with x , once we have chosen x_5 the 6-cycle is completely determined, as the edges x_3 and x_4 are the ones in M that are incident to both endpoints of x . There are at most $|M| = n$ choices for x_5 , so there are at most n 6-cycles containing x . Similarly there are at most n 6-cycles containing y . It follows that $\Delta(\mathcal{M} \setminus \mathcal{M}_0) \leq n^2$.

In order to bound $\delta(\mathcal{M}_0)$ from below, fix $M_0 \in \mathcal{M}_0$. Note here that not all pairs of disjoint 6-cycles containing x and y , respectively, will generate an edge in \mathcal{G} as it may be that the perfect matching obtained by switching over the cycles is not S -good.

Define

$$F(z) = \{z' \in E(G) : z' \text{ is } (z, M_0)\text{-switchable and } \{w, z'\} \notin Q \text{ for all } w \in E(M_0)\} .$$

Let $F^*(x) \subseteq F(x)$ be the subset of edges that are not incident with x or y . By assumption, there are at least γn^2 edges that are (x, M_0) -switchable, from which at most μn^2 have conflicts with edges in M_0 and at most $2n$ are incident to y , implying $|F^*(x)| \geq \gamma n^2/2$. Each edge $x_5 \in F^*(x)$ uniquely determines a 6-cycle whose switching gives rise to a perfect matching. We claim that this matching is S -good. By adding x_3, x_4 and x_5 , we can only create conflicts which use one of these edges. By definition of S , if $j \in S$, then the two edges defining E_j are not incident with x . Thus, x_3 and x_4 cannot create any conflict. Moreover, by the properties of $F(x)$, x_5 does not share any colour with the edges in M_0 , so it cannot create any conflict. Given a choice of a 6-cycle of the form $xx_3x_1x_5x_2x_4$, let $F^{**}(y) \subseteq F(y)$ be the subset of edges that are not incident to the vertices of the fixed 6-cycle. Similarly as before, $|F^{**}(y)| \geq \gamma n^2/2$ and each edge $y_5 \in F^{**}(y)$ gives rise to a 6-cycle whose switching preserves the S -good condition. As for every choice of 6-cycle to switch out x and each choice of 6-cycle to switch out y we obtain an edge adjacent to M_0 , we conclude that $\delta(\mathcal{M}_0) \geq \gamma^2 n^4/4$.

Substituting into (2), we obtain the desired bound

$$\mathbb{P}(E_i | \cap_{j \in S} E_j^c) \leq \frac{4}{\gamma^2 n^2} =: p .$$

Note that $|\mathcal{E}| \leq \mu n^3$. Since, $4pd = \frac{64\mu}{\gamma^2} \leq 1$, by the p -Lopsided form of the Lovász Local Lemma (Lemma 5), the probability that a uniformly random perfect matching is conflict-free is

$$\mathbb{P}(\cap_{E \in \mathcal{E}} E^c) \geq \left(1 - \frac{8}{\gamma^2 n^2}\right)^{\mu n^3} \geq e^{-\mu^{1/2} n} ,$$

and the lemma follows. □

3 Dichotomy

In order to apply Lemma 6, we need to show that for every edge and every perfect matching containing it, there exist many switchable edges. However, this statement is not true for every Dirac bipartite graph. For instance, consider the graph where $n = 2m + 1$, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with $|A_1| = |B_2| = m + 1$, and where $G[A_1, B_1]$ and $G[A_2, B_2]$ induce complete bipartite graphs and $G[A_2, B_1]$ induces a perfect matching. Clearly, G is a Dirac bipartite graph.

However, for every edge in $x \in E(A_2, B_1)$, and independently of the choice of M containing x , there are at most n edges that are (x, M) -switchable as any such edge lies in $E(A_2, B_1)$.

Our proof proceeds by splitting the class of Dirac bipartite graphs into two cases: Robust Expanders, where we show the existence of many switchable edges, and Extremal Graphs, where we proceed carefully to handle the edges that produce a small number of switchings.

For $0 < \nu < 1$ and $X \subseteq V(G)$, the ν -robust neighbourhood of X in G is defined as

$$RN_\nu(X) := \{v \in V(G) : |N_G(v) \cap X| \geq \nu n\}.$$

Definition. Let $0 < \nu \leq \tau < 1$. A balanced bipartite graph $G = (A \cup B, E)$ on $2n$ vertices is a *bipartite robust (ν, τ) -expander* if for every set $X \subseteq V(G)$ with $\tau n \leq |X| \leq (1 - \tau)n$ and either $X \subseteq A$ or $X \subseteq B$, we have

$$|RN_\nu(X)| \geq |X| + \nu n.$$

For a bipartite graph $G = (A \cup B, E)$ if $X \subseteq A$ and $Y \subseteq B$, we let $E(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$ and $e(X, Y) = |E(X, Y)|$.

Definition. Let $0 < \varepsilon < 1$. A balanced bipartite graph $G = (A \cup B, E)$ on $2n$ vertices is an ε -*extremal graph* if there exist partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that the following properties are satisfied:

(P1) $||A_1| - |A_2|| \leq \varepsilon n;$

(P2) $||B_1| - |B_2|| \leq \varepsilon n;$

(P3) $e(A_1, B_2) \leq \varepsilon n^2.$

The following result establishes a dichotomy between these classes. Similar ideas have already appeared in previous work on Dirac graphs [7, 15, 16]. The proof follows the lines of the previous approaches and we include it here for the sake of completeness.

Theorem 7. *Let $n \in \mathbb{N}$ and suppose that $1/n \ll \nu \ll \varepsilon \ll \tau \ll 1$. Let $G = (A \cup B, E)$ be a Dirac bipartite graph on $2n$ vertices. Then one of the following holds:*

i) G is a bipartite robust (ν, τ) -expander;

ii) G is an ε -extremal graph.

Proof. Let $0 < \delta < 1$ be such that $\nu \ll \delta \ll \varepsilon$. Suppose that G is not a bipartite robust (ν, τ) -expander. Thus, we may assume without loss of generality that there exists a set $X \subseteq A$ with $\tau n \leq |X| \leq (1 - \tau)n$ and such that $|RN_\nu(X)| < |X| + \nu n$. We split the argument into three possible cases:

Case 1: $\tau n \leq |X| \leq \frac{n}{2} - \delta n$.

Since $e(X, N(X)) \geq |X|(n/2)$, we reach a contradiction:

$$e(X, N(X)) \leq |X||RN_\nu(X)| + \nu n^2 \leq |X| \left(\frac{n}{2} - (\delta - \nu)n \right) + \nu n^2 < |X| \frac{n}{2} \leq e(X, N(X)),$$

where the second last inequality holds because $\nu \ll \delta, \tau$.

Case 2: $\frac{n}{2} - \delta n \leq |X| \leq \frac{n}{2} + \delta n$.

Define $A_1 = X$, $A_2 = A \setminus X$, $B_1 = RN_\nu(X)$, $B_2 = B \setminus RN_\nu(X)$. Note that $e(A_1, B_2) \leq \nu n |B_2| \leq \varepsilon n^2$; thus, (P3) holds. Now, (P1) and (P2) follow immediately since the minimum degree of G is at least $n/2$. Hence G is an ε -extremal graph.

Case 3: $\frac{n}{2} + \delta n \leq |X| \leq (1 - \tau)n$.

Each vertex in B has at least δn neighbours in X . So $RN_\nu(X) = B$. Hence, $|RN_\nu(X)| = n \geq |X| + \nu n$, a contradiction with the choice of X . \square

4 Robust Expanders

As we will show below, the robust expansion property yields many (x, M) -switchable edges independently of the choice of x and M . Thus, Lemma 6 can be directly applied to obtain the existence of a conflict-free perfect matching in robust expanders.

Lemma 8. *Let $n \in \mathbb{N}$ and suppose $1/n \ll \gamma \ll \nu \ll \tau \ll 1$. Let $G = (A \cup B, E)$ be a bipartite robust (ν, τ) -expander on $2n$ vertices with minimum degree at least $n/2$. Let M be a perfect matching of G and let $x \in M$. Then, there are at least γn^2 edges of G that are (x, M) -switchable.*

Proof. Let $f_M : A \rightarrow B$ be a bijective map defined as $f(a) = b$ if and only if $ab \in M$. Given $x = a_1 b_1 \in M$ and an edge $y \notin M$ not incident to x , there is at most one 6-cycle in G that uses x , y and any two of the other edges in M . Also, note y is (x, M) -switchable if and only if there is such a 6-cycle. Therefore, to count the number of (x, M) -switchable edges y , we count the number of 6-cycles containing x and any two of the other edges in M .

Suppose that the 6-cycle is given by the sequence of vertices $a_1 b_1 a_2 b_3 a_3 b_2$, where $a_2 b_3, a_3 b_2 \in M$. To bound from below the number of ways to choose the 6-cycle, we compute a lower bound on the number of choices for a_3 and b_3 . Select,

$$a_3 \in RN_\nu(f_M(N(b_1))) \cap f_M^{-1}(N(a_1) \setminus \{b_1\}),$$

and let $b_2 = f_M(a_3)$. Given the choice of a_3 , select

$$b_3 \in f_M(N(b_1) \setminus \{a_1, a_3\}) \cap N(a_3),$$

and let $a_2 = f_M^{-1}(b_3)$. Recall that the minimum degree is at least $n/2$. As G is a bipartite robust (ν, τ) -expander, $|RN_\nu(f_M(N(b_1)))| \geq \frac{n}{2} + \nu n$, which implies that there are at least $\nu n - 1$ choices for a_3 . Again, by the expansion properties of G , a_3 has at least νn neighbours in $f_M(N(b_1))$, so there are at least $\nu n - 2$ choices for b_3 . In total, there are at least γn^2 choices of 6-cycles, $a_1 b_1 a_2 b_3 a_3 b_2$, or equivalently, γn^2 edges $y = a_3 b_3 \in E(G)$ that are (x, M) -switchable. \square

We can combine Lemma 6 and Lemma 8 together to conclude.

Corollary 9. *Let $n \in \mathbb{N}$ and suppose $1/n \ll \mu \ll \nu \ll \tau \ll 1$. Let $G = (A \cup B, E)$ be a bipartite robust (ν, τ) -expander on $2n$ vertices with minimum degree at least $n/2$. Then, any μn -bounded system of conflicts for $E(G)$ contains a conflict-free perfect matching.*

5 Extremal Graphs

In this section we study the existence of rainbow perfect matchings for extremal graphs.

The example displayed at the beginning of Section 3 suggests that extremal graphs have special edges that are difficult to switch; namely, the ones between A_1 and B_2 . Since the partitions $A = (A_1, A_2)$ and $B = (B_1, B_2)$ can be unbalanced, it may be unavoidable to select edges in $E(A_1, B_2)$ in a perfect matching of G . In fact, we may have to choose linearly many such edges.

A greedy approach for choosing the edges in $E(A_1, B_2)$ is likely to fail. By the properties of the edge colouring, the graph may contain vertices that only have a constant number of colours in the edges incident to them. If one selects a partial matching M^* in $E(A_1, B_2)$ and removes all the edges that have a colour in M^* , vertices that have few colours on their incident edges are likely to become isolated.

The way to handle this problem is given by Lemma 12, which shows that there is a way to select a rainbow partial matching M^* in $E(A_1, B_2)$ such that $|A_i \setminus V(M^*)| = |B_i \setminus V(M^*)|$, for $i \in \{1, 2\}$, and such that the degrees in the subgraph obtained after removing the colours in M^* are similar to the ones in the original graph.

5.1 A technical lemma

The core of the proof of Lemma 12 is a technical lemma that we present in this section.

We will be dealing both with multisets and with sets. For a multiset C and $k \in \mathbb{N}$, we denote by $m(k, C)$ the multiplicity of k in C . We define the operators \cap^+ and \setminus^+ both taking a multiset and a set and returning a multiset as follows: if A is a multiset and B is a set,

$$A \cap^+ B := \{\{x \in A : x \in B\}\} \quad A \setminus^+ B := A \setminus (A \cap^+ B)$$

where \setminus is the standard multiset difference.

Lemma 10. *Let $N \in \mathbb{N}$ and suppose that $1/N \ll \mu \ll \nu, 1/\alpha \ll \eta \leq 1$. Let C_1, \dots, C_N be multisets of \mathbb{N} such that:*

(B1) $\nu N \leq |C_i| \leq N$, for every $i \in [N]$;

(B2) $\sum_{i=1}^N m(k, C_i) \leq \mu N$, for every $i \in [N]$ and every $k \in \mathbb{N}$.

Let $\ell \in \mathbb{N}$ with $1 \leq \ell \ll \nu N$ and $\alpha \ell \in \mathbb{N}$. Let $U \subseteq \mathbb{N}$ be a set with $|U| = \alpha \ell$. Then, there exists $T \subseteq U$ such that:

(T1) $|T| \geq \ell$;

(T2) $|C_i \setminus^+ T| \geq (1 - \eta)|C_i|$, for every $i \in [N]$.

Proof. If $\ell \leq 2\alpha$, then let T be an arbitrary subset of U of size ℓ . Since for every $i \in [N]$ and as $\mu \ll \eta, \nu$, we have $\mu N \ell \leq 2\mu \alpha N \leq \nu \eta N \leq \eta |C_i|$, (T2) clearly holds. Throughout the proof we will assume that $\ell \geq 2\alpha$.

Let $0 < \varepsilon < 1$ such that $\mu \ll \varepsilon \ll \nu, 1/\alpha$ and let $m_* := \frac{\varepsilon}{10\alpha^2} \cdot \frac{N}{\log N}$ and let $s := \log(\mu N / m_*)$. For every $i \in [N]$ and every $j \in [s]$, define the (multi)sets

$$P_i^j = \{\{k \in C_i : 2^{-j} \mu N \leq m(k, C_i) \leq 2^{-(j-1)} \mu N\}\}$$

$$S_i^j = \{k \in P_i^j\}$$

Further, define

$$\begin{aligned} P_i &= \cup_{j \in [s]} P_i^j \\ S_i &= \cup_{j \in [s]} S_i^j \\ Q_i &= C_i \setminus P_i \end{aligned}$$

Note that for every $k \in Q_i$, $m(k, C_i) \leq m_*$. Let $p_i^j = |P_i^j|$, $s_i^j = |S_i^j|$, $p_i = |P_i|$, $s_i = |S_i|$, $q_i = |Q_i|$, $c_i = |C_i|$. Then, these parameters satisfy

$$2^{-j} \mu N s_i^j \leq p_i^j \leq 2^{-(j-1)} \mu N s_i^j \quad (3)$$

$$\sum_{j \in [s]} p_i^j = p_i \quad (4)$$

$$p_i + q_i = c_i$$

Let $T_0 \subseteq U$ be a random subset of U obtained by including each element of U independently at random with probability $\delta := 3\alpha^{-1}$. Note that $\mathbb{E}(|T_0|) = 3\ell$.

Claim. With probability $1 - o_N(1)$, for every $i \in [N]$, $|Q_i \setminus^+ T_0| \geq (1 - 4\alpha^{-1})q_i - \varepsilon N$.

Proof. Fix $i \in [N]$. If $q_i \leq \varepsilon N$ the statement is clearly true. So we may assume that $q_i \geq \varepsilon N$. For each $k \in Q_i$, define $m_k := m(k, C_i)$.

Then,

$$\sum_{k \in Q_i} m_k^2 \leq m_* \sum_{k \in Q_i} m_k = m_* q_i \leq \frac{\varepsilon N}{10\alpha^2 \log N} \cdot q_i. \quad (5)$$

Let $X_i = |Q_i \cap^+ T_0|$ and note that $\mathbb{E}(X_i) \leq \delta q_i$. By Azuma's Inequality (see e.g. [20]) with m_k satisfying (5) and the fact that $q_i \geq \varepsilon N$,

$$\mathbb{P}(X_i - \mathbb{E}(X_i) \geq \alpha^{-1} q_i) \leq 2 \exp\left(\frac{q_i^2}{2\alpha^2 \sum_{k \in Q_i} m_k^2}\right) \leq N^{-5}.$$

So, with probability $1 - o_N(1)$, for every $i \in [N]$, if $q_i \geq \varepsilon N$, then

$$|Q_i \setminus^+ T_0| \geq (1 - \alpha^{-1})q_i - \mathbb{E}(X_i) = (1 - 4\alpha^{-1})q_i \geq (1 - 4\alpha^{-1})q_i - \varepsilon N.$$

□

We now consider the sets P_i . For $\rho > 0$, a pair (i, j) is ρ -dense if $s_i^j \geq 2^{(j-1)/2} \rho$. Let R_i be the set of pairs (i, j) that are $\mu^{-1/2}$ -dense. The contribution of non-dense pairs is negligible; using (3), we have

$$\sum_{j \notin R_i} p_i^j \leq \mu N \sum_{j \notin R_i} 2^{-(j-1)} s_i^j \leq \mu^{1/2} N \sum_{j \notin R_i} 2^{-(j-1)/2} \leq \mu^{1/3} N. \quad (6)$$

We say that $i \in [N]$ is *susceptible* if $|C_i \cap^+ U| \geq \eta |C_i|$. Let $D = \{i \in [N] : i \text{ is susceptible}\}$. Note that (T2) is satisfied for every $i \notin D$, as we have

$$|C_i \setminus^+ T| \geq |C_i \setminus^+ U| = |C_i| - |C_i \cap^+ U| \geq (1 - \eta)|C_i|.$$

Since $|C_i| \geq \nu N$, we can bound the size of D as follows

$$|D| \leq \alpha \ell \cdot \mu N / (\eta \nu N) \leq \ell. \quad (7)$$

Finally, for every $S \subseteq \mathbb{N}$ and $j \in [s]$ we say that $i \in [N]$ is j -activated by S if $|S_i^j \cap S| \geq 2\delta s_i^j$.

Consider the set $T \subseteq T_0$ defined as follows: for each $i \in D$ and $j \in [s]$ remove S_i^j from T_0 if

- i) i is j -activated by T_0 , and
- ii) $j \in R_i$ (i.e., (i, j) is $\mu^{-1/2}$ -dense).

Observe that by removing elements from T_0 we only increase the size of $Q_i \setminus^+ T_0$. From the construction of T_0 and using (3) twice, it follows that for each $i \in D$, $j \in R_i$, we have

$$|P_i^j \cap^+ T| \leq \mu N 2^{-(j-1)} |S_i^j \cap T| \leq \mu N 2^{-(j-1)} \cdot 2\delta s_i^j \leq 4\delta p_i^j$$

By combining this with (6), we obtain

$$|P_i \cap^+ T| = \sum_{j \in [s]} |P_i^j \cap^+ T| = \sum_{j \in R_i} |P_i^j \cap^+ T| + \sum_{j \notin R_i} |P_i^j \cap^+ T| \leq 4\delta \sum_{j \in R_i} p_i^j + \sum_{j \notin R_i} p_i^j \leq 4\delta p_i + \mu^{1/3} N.$$

Therefore, with probability $1 - o_N(1)$, condition (T2) is satisfied; that is, for every $i \in [N]$,

$$\begin{aligned} |C_i \setminus^+ T| &= |P_i \setminus^+ T| + |Q_i \setminus^+ T| \\ &\geq |P_i \setminus^+ T| + |Q_i \setminus^+ T_0| \\ &\geq (1 - 4\delta)p_i - \mu^{1/3} N + (1 - 4\alpha^{-1})q_i - \varepsilon N \geq (1 - \eta)|C_i| \end{aligned}$$

In order to conclude the proof of the lemma, it suffices to show that condition (T1) holds with positive probability, from where we will deduce the existence of the desired set.

Claim. With probability at least $\frac{9}{10}$, we have $|T| \geq |T_0| - \ell$.

Proof. Since $|S_i^j \cap T_0|$ is stochastically dominated by a binomial random variable with parameters s_i^j and δ (there might be elements of S_i^j that are not in U), we can use Chernoff's inequality (see e.g. Corollary 2.3 in [12]) to show that

$$\mathbb{P}(i \text{ is } j\text{-activated}) \leq 2e^{-\frac{\delta s_i^j}{3}}.$$

If $j \in R_i$, then $s_i^j \geq 2^{(j-1)/2} \mu^{-1/2}$. Thus, $e^{-\frac{\delta s_i^j}{3}} \leq e^{-\frac{\delta 2^{(j-1)/2}}{3\mu^{1/2}}} \leq \mu^2 2^{-j}$. Hence, for $j \in R_i$

$$\mathbb{P}(i \text{ is } j\text{-activated}) \leq \mu^2 2^{-j}. \quad (8)$$

Recall the following inequality which follows from (3) and (4),

$$\sum_{j \in [s]} 2^{-j} s_i^j \leq \mu^{-1}. \quad (9)$$

Define the following random variable

$$Y := |T_0 \setminus T| \leq \sum_{i \in D} \sum_{j \in R_i} s_i^j \mathbb{1}(i \text{ is } j\text{-activated}).$$

Note that the sets D and R_i are fully determined by C_1, \dots, C_N . Then using (7), (8) and (9), it follows that

$$\mathbb{E}(Y) \leq \sum_{i \in D} \sum_{j \in R_i} s_i^j \mathbb{P}(i \text{ is } j\text{-activated}) \leq \mu^2 \sum_{i \in D} \sum_{j \in R_i} 2^{-j} s_i^j \leq \mu^2 \sum_{i \in D} \sum_{j \in [s]} 2^{-j} s_i^j \leq \mu |D| \leq \frac{\ell}{10}.$$

So, by Markov's inequality, $\mathbb{P}(Y \geq \ell) \leq 1/10$. \square

Recall that $\ell \geq 2\alpha$. Since $|T_0|$ is distributed as a binomial random variable with parameters $\alpha\ell$ and δ , Chernoff's inequality implies that $\mathbb{P}(|T_0| \leq 2\ell) \leq 2e^{-\frac{\ell^2}{2\alpha\ell}} = 2e^{-\frac{\ell}{2\alpha}} \leq \frac{2}{e}$. Thus, with positive probability, we have

$$|T| \geq 2\ell - \ell \geq \ell \tag{10}$$

We conclude that there exists $T \subseteq U$ satisfying (T1) and (T2), concluding the proof of the lemma. \square

5.2 Superextremal graphs

We will use Lemma 10 to control the effect of colour deletions in the degrees of G . If degrees do not shrink significantly, the graphs $G_i = G[A_i, B_i]$, $i \in \{1, 2\}$, will still be fairly dense, and by applying Lemma 6 we will get the existence of a rainbow perfect matching.

However, the ε -extremal condition does not ensure that the graphs G_i have large minimum degree; that is, G_i is not necessarily Dirac. In this section we refine the notion of extremality and we obtain a partition where the degrees of each vertex within its part is controlled. Eventually, this will allow us to count the number of switchable edges.

Definition. Let $0 < \nu_1 \leq \nu_2 < 1$. A balanced bipartite graph $G = (A \cup B, E)$ on $2n$ vertices is a (ν_1, ν_2) -superextremal graph if there exist partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that the following properties are satisfied for $i \in \{1, 2\}$:

(Q1) $e(v, B_i) \geq \frac{n}{2} - \nu_1 n$, for all but at most $\nu_1 n$ vertices $v \in A_i$;

(Q2) $e(v, B_i) \geq \nu_2 n$, for every $v \in A_i$;

(Q3) $e(v, A_i) \geq \frac{n}{2} - \nu_1 n$, for all but at most $\nu_1 n$ vertices $v \in B_i$;

(Q4) $e(v, A_i) \geq \nu_2 n$, for every $v \in B_i$;

(Q5) $\| |A_1| - |B_1| \|, \| |A_1| - |A_2| \| \leq \nu_1 n$;

(Q6) $e(v, B_2) \leq \nu_2 n$, for every $v \in A_1$, unless $|A_1| = |B_1|$;

(Q7) $e(v, A_1) \leq \nu_2 n$, for every $v \in B_2$, unless $|A_1| = |B_1|$;

(Q8) $|A_1| \geq |B_1|$;

(Q9) one of the following holds for $\ell := |A_1| - |B_1|$:

– $e(v, B_2) \geq \ell/2$, for every $v \in A_1$;

– $e(v, A_1) \geq \ell/2$, for every $v \in B_2$.

Lemma 11. *Let $n \in \mathbb{N}$ and suppose $1/n \ll \varepsilon \ll \nu_1 \ll \nu_2 \ll 1$. Let $G = (A \cup B, E)$ be an ε -extremal Dirac bipartite graph on $2n$ vertices. Then, G is a (ν_1, ν_2) -superextremal graph.*

Proof. Since G is an ε -extremal graph, there exist partitions $A = A_1^1 \cup A_2^1$ and $B = B_1^1 \cup B_2^1$ satisfying (P1), (P2) and (P3). Let $0 < \nu_3 \leq \nu_4 < 1$ such that $\varepsilon \ll \nu_3 \ll \nu_1 \ll \nu_4 \ll \nu_2$ and define

$$\begin{aligned} X_1^1 &= \left\{ v \in A_1^1 : e(v, B_1^1) \leq \frac{n}{2} - \nu_3 n \right\} & X_1^2 &= \left\{ v \in A_1^1 : e(v, B_1^1) \leq \frac{n}{4} \right\} \\ X_2^1 &= \left\{ v \in A_2^1 : e(v, B_2^1) \leq \frac{n}{2} - \nu_3 n \right\} & X_2^2 &= \left\{ v \in A_2^1 : e(v, B_2^1) \leq \frac{n}{4} \right\} \\ Y_1^1 &= \left\{ v \in B_1^1 : e(v, A_1^1) \leq \frac{n}{2} - \nu_3 n \right\} & Y_1^2 &= \left\{ v \in B_1^1 : e(v, A_1^1) \leq \frac{n}{4} \right\} \\ Y_2^1 &= \left\{ v \in B_2^1 : e(v, A_2^1) \leq \frac{n}{2} - \nu_3 n \right\} & Y_2^2 &= \left\{ v \in B_2^1 : e(v, A_2^1) \leq \frac{n}{4} \right\} \end{aligned}$$

We double count edges to bound the size of these sets. Note that $e(A_1^1, B_1^1) \geq \frac{n}{2}|A_1^1| - \varepsilon n^2$ by counting from A_1^1 . Alternately, we can also obtain that $e(A_1^1, B_1^1) \leq |X_1^1|(\frac{n}{2} - \nu_3 n) + (|A_1^1| - |X_1^1|)|B_1^1|$. Combining these two inequalities yields

$$|X_1^1| \left(|B_1^1| - \frac{n}{2} + \nu_3 n \right) \leq |A_1^1| \left(|B_1^1| - \frac{n}{2} \right) + \varepsilon n^2 \leq 2\varepsilon n^2.$$

Observe that $|B_1^1| \geq \frac{n}{2} - \varepsilon n$ and so $|B_1^1| - \frac{n}{2} + \nu_3 n \geq \frac{\nu_3 n}{2}$. Therefore $|X_1^1| \leq \nu_3 n$. Similarly, one can deduce that $|X_2^1| \leq 9\varepsilon n$. Analogous computations lead to $|X_2^1|, |Y_1^1|, |Y_2^1| \leq \nu_3 n$ and to $|X_2^2|, |Y_1^2|, |Y_2^2| \leq 9\varepsilon n$. Now, we define

$$\begin{aligned} A_1^2 &= (A_1^1 \setminus X_1^2) \cup X_2^2 & B_1^2 &= (B_1^1 \setminus Y_1^2) \cup Y_2^2 \\ A_2^2 &= (A_2^1 \setminus X_2^2) \cup X_1^2 & B_2^2 &= (B_2^1 \setminus Y_2^2) \cup Y_1^2 \end{aligned}$$

Without loss of generality, $|A_1^2| \geq |B_1^2|$; otherwise we swap the labels of A_1^2 and A_2^2 , and the labels of B_1^2 and B_2^2 . By swapping the labels we lose control of $e(A_1^2, B_2^2)$. However, at this point, this condition is no longer needed, as we have a bound on the size of the sets X_i^j and Y_i^j , for $i, j \in \{1, 2\}$.

Let

$$X_1^3 = \{v \in A_1^2 : e(v, B_2^2) \geq \nu_4 n\} \quad Y_2^3 = \{v \in B_2^2 : e(v, A_1^2) \geq \nu_4 n\}$$

If $|X_1^3| + |Y_2^3| \geq |A_1^2| - |B_1^2|$, choose $X_1^4 \subseteq X_1^3$ and $Y_2^4 \subseteq Y_2^3$ arbitrarily such that $|X_1^4| + |Y_2^4| = |A_1^2| - |B_1^2|$. Otherwise, let $X_1^4 = X_1^3$ and $Y_2^4 = Y_2^3$. Recall that, since G is ε -extremal, it satisfies $n/2 - \varepsilon n \leq |A_1^1|, |B_1^1| \leq n/2 + \varepsilon n$. Thus, we have

$$|X_1^4| \leq |A_1^2| - |B_1^2| \leq |A_1^1| - |B_1^1| + 18\varepsilon n \leq 20\varepsilon n.$$

and similarly for Y_2^4 .

We define

$$A_1 = A_1^2 \setminus X_1^4 \quad A_2 = A_2^2 \cup X_1^4 \quad B_1 = B_1^2 \cup Y_2^4 \quad B_2 = B_2^2 \setminus Y_2^4$$

We claim that the partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ satisfy properties (Q1)-(Q9), and so, G is a (ν_1, ν_2) -superextremal graph.

Let us first check that property (Q1) is satisfied. Observe that all the vertices in A_1^1 , excluding the ones in X_1^1 , have degree at least $n/2 - \nu_3 n$. Since $|X_1^2| \leq 9\varepsilon n$, all the vertices in A_1^2 have degree

at least $n/2 - \nu_3 n - 9\epsilon n$, excluding the ones in $X_1^1 \cup X_2^2$. Since $|X_1^4| \leq 20\epsilon n$, all the vertices in A_1 have degree at least $n/2 - \nu_3 n - 29\epsilon n \geq n/2 - \nu_1 n$, excluding the ones in $X_1^1 \cup X_2^2$. Moreover, $|X_1^1 \cup X_2^2| \leq \nu_3 n + 9\epsilon n \leq \nu_1 n$, so (Q1) follows. Similar arguments yield to properties (Q2)-(Q4) and (Q6)-(Q7).

Property (Q8) follows from the choice of X_1^4 and Y_2^4 , since $|A_1| = |A_1^2| - |X_1^4| \geq |B_1^2| + |Y_2^4| = |B_1|$. Property (Q5) follows since $|A_1| - |B_1| \leq |A_1^2| - |B_1^2| \leq 20\epsilon n \leq \nu_1 n$ (and since a similar computation bounds $||A_1| - |A_2||$).

Finally, Property (Q9) follows by noting that if $\ell = |A_1| - |B_1|$ (and $|A_2| = |B_2| - \ell$), then either $|B_1|$ or $|A_2|$ is at most $n/2 - \ell/2$ thus requiring minimum degree $\ell/2$ either from A_1 to B_2 or from B_2 to A_1 , respectively. □

5.3 Selecting a rainbow partial matching between parts

Given a superextremal graph G with partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, in this section we will show the existence of a rainbow partial matching M^* in $G[A_1, B_2]$ of size $\ell = |A_1| - |B_1|$ such that the graph H resulting from removing all edges incident to M^* and all edges with colours that appear in M^* , has similar degrees as the graph G .

Lemma 12. *Let $n, \ell \in \mathbb{N}$ and suppose $1/n \ll \mu \ll \nu_1 \ll \nu_2 \ll \nu_3 \ll \eta_1 \ll 1$. Let $G = (A \cup B, E)$ be a (ν_1, ν_3) -superextremal graph with partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then, any μn -bounded edge colouring χ of G admits a rainbow matching M^* of size $\ell = |A_1| - |B_1|$ such that the following holds. Let $H = (A^H \cup B^H, E^H)$ be the graph where $A^H = A \setminus V(M^*)$, $B^H = B \setminus V(M^*)$ and*

$$E^H = \{x = ab \in E(G) : a, b \notin V(M^*), \chi(x) \notin \chi(E(M^*))\}.$$

Let $n_H := n - \ell$. Then, there exist partitions $A^H = A_1^H \cup A_2^H$ and $B^H = B_1^H \cup B_2^H$ that satisfy the following properties for $i \in \{1, 2\}$:

$$(R1) \quad e_H(v, B_i^H) \geq (1 - \eta_1) \frac{n_H}{2}, \text{ for all but at most } \nu_1 n \text{ vertices } v \in A_i^H;$$

$$(R2) \quad e_H(v, B_i^H) \geq \nu_2 n_H, \text{ for every } v \in A_i^H;$$

$$(R3) \quad e_H(v, A_i^H) \geq (1 - \eta_1) \frac{n_H}{2}, \text{ for all but at most } \nu_1 n \text{ vertices } v \in B_i^H;$$

$$(R4) \quad e_H(v, A_i^H) \geq \nu_2 n_H, \text{ for every } v \in B_i^H;$$

$$(R5) \quad |A_1^H| = |B_1^H|, |A_2^H| = |B_2^H| \text{ and } |A_1^H| - |A_2^H| \leq \nu_1 n_H.$$

Proof. We first greedily select a large rainbow matching in $G[A_1, B_2]$. Let $E_0 = E(A_1, B_2)$ and $M_0 = \emptyset$. By (Q5) and (Q8), note that $|E_0| \geq \frac{\ell}{2} (\frac{n}{2} - \nu_1 n)$. For every $i \geq 1$ and while $E_{i-1} \neq \emptyset$, we arbitrarily choose $x_i = a_i b_i \in E_{i-1}$ and define the graph M_i with $V(M_i) = V(M_{i-1}) \cup \{a_i, b_i\}$ and $E(M_i) = E(M_{i-1}) \cup \{x_i\}$. We let

$$E_i = \{x = ab \in E_{i-1} : a, b \notin V(M_i), \chi(x) \notin \chi(E(M_i))\}.$$

Since χ is μn -bounded, $|A_1| - |B_1| = \ell \geq 1$ and using (Q6)-(Q7), we have $|E_i| \geq |E_{i-1}| - (2\nu_3 + \mu)n$. Let $i^* = \lfloor \ell / (10\nu_3) \rfloor$. It follows that $E_i \neq \emptyset$, for every $0 \leq i \leq i^*$.

We now apply Lemma 10 with parameters $N = 2n$, $\alpha = i^*/\ell$, $\nu = \nu_3/2$, $\eta = \eta_1/2$ and $U = \{\chi(x) : x \in M_{i^*}\}$. For every $v \in A \cup B$, we choose $C_v = \{\{\chi(x) : v \in x\}\}$ to be the multiset of colours

on edges incident with vertex v . By (Q2) and (Q4), we have $\nu N \leq |C_v| \leq N$, for each $v \in A \cup B$. As each edge has two endpoints and χ is μn -bounded, then $\sum_{v \in A \cup B} m(k, C_v) \leq 2\mu n = \mu N$. Hence, (B1) and (B2) hold.

Lemma 10 implies the existence of a set of colours $T \subseteq U$ of size ℓ satisfying (T1) and (T2). Let M^* be the subgraph of M_{i^*} induced by the colours in T . Then, M^* is a rainbow matching of size ℓ . It suffices to prove that H , as defined in the statement, satisfies (R1)-(R5).

For each $Z \in \{A, B\}$ and $i \in \{1, 2\}$, let $Z_i^H = Z_i \cap V(H)$. Property (R5) follows since $|B_1^H| = |B_1| = |A_1| - \ell = |A_1^H|$ and using (Q5). Then, for every $v \in Z_i^H$, we have

$$e_H(v, Z_i^H) \geq |C_v \setminus T| - \ell \geq (1 - \eta)|C_v| - \nu_1 n \geq \begin{cases} (1 - \eta)(\frac{n}{2} - \nu_1 n) - \nu_1 n \geq (1 - \eta_1)\frac{n_H}{2} & \text{if } v \text{ satisfies (Q1) or (Q3)} \\ (1 - \eta)\nu_3 n - \nu_1 n \geq \nu_2 n_H & \text{if } v \text{ satisfies (Q2) or (Q4)} \end{cases}$$

Thus H satisfies (R1)-(R4), completing the proof. \square

5.4 Completing the rainbow perfect matching

Consider the rainbow partial matching M^* and the graph H provided by Lemma 12. Note that H is vertex disjoint from M^* and has no edge with colour in $\chi(E(M^*))$. Thus, the union of any rainbow perfect matching of H and M^* will provide a rainbow perfect matching of G .

We will show that H satisfies the conditions of Lemma 6, to conclude the existence of a rainbow perfect matching there.

Of course, in order to have a rainbow perfect matching in H we need to ensure the existence of at least one perfect matching. We will use the Moon-Moser condition for the existence of Hamiltonian cycles in bipartite graphs to guarantee we can find a perfect matching.

Lemma 13. (Moon, Moser [21]) *Let $F = (R \cup S, E)$ be a balanced bipartite graph on $2m$ vertices with $R = \{r_1, \dots, r_m\}$ and $S = \{s_1, \dots, s_m\}$ that satisfies $d(r_1) \leq \dots \leq d(r_m)$ and $d(s_1) \leq \dots \leq d(s_m)$. Suppose that for every $1 \leq k \leq m/2$, we have $d(r_k) > k$ and $d(s_k) > k$. Then F has a Hamiltonian cycle.*

Lemma 14. *Let $n_H \in \mathbb{N}$ and suppose $1/n_H \ll \mu \ll \varepsilon \ll \nu_1 \ll \gamma \ll \nu_2 \ll \eta \ll 1$. Let $H = (A^H \cup B^H, E^H)$ be a bipartite graph with $|A^H| = |B^H| = n_H$ and satisfying properties (R1)-(R5). Consider the subgraph $H_* = (A^H \cup B^H, E_*^H)$ of H with $E_*^H = E_{*,1}^H \cup E_{*,2}^H$ and, for $i \in \{1, 2\}$,*

$$E_{*,i}^H = \{x = ab \in E(H) : a \in A_i, b \in B_i \text{ and } \max\{e_H(a, B_i), e_H(b, A_i)\} \geq (1 - \eta)n_H/2\} .$$

Then, H_ has at least one perfect matching.*

Moreover, if M_ is a perfect matching of H_* and $x = a_1 b_1 \in E(M_*)$, then there are at least γn_H^2 edges of H_* that are (x, M_*) -switchable.*

Proof. It is easy to check that H_* has only two connected components. We will show that it is true in $H_1 = H_*[A_1^H, B_1^H]$ and the same argument also applies to $H_2 = H_*[A_2^H, B_2^H]$. Note that H_1 is a balanced bipartite graph on $2m$ vertices for some $m \in (n_H(1/2 - \nu_1), n_H(1/2 + \nu_1))$.

We will use the Moon-Moser condition (Lemma 13) to show the existence of a Hamiltonian cycle in H_1 . Let $A_1^H = \{r_1, \dots, r_m\}$ and $B_1^H = \{s_1, \dots, s_m\}$ with $d(r_1) \leq \dots \leq d(r_m)$ and $d(s_1) \leq \dots \leq d(s_m)$.

If $1 \leq k \leq 2\nu_1 n_H < 5\nu_1 m - 1$, then, by (R2), $d(r_k) \geq \nu_2 n_H \geq 5\nu_1 m > k$, so there is nothing to prove. If $2\nu_1 n_H \leq k \leq m/2$, then, by (R1), $d(r_k) \geq (1 - \eta)m > k$. An identical argument works for s_k using (R3) and (R4). Thus we satisfy the Moon-Moser condition. So, H_1 has a Hamiltonian cycle, which implies the existence of a perfect matching.

Let M_* be a perfect matching of H_* . Consider the bijective map $f_{M_*} : A^H \rightarrow B^H$ defined as $f(a) = b$ if and only if $ab \in M_*$. Let $x = a_1 b_1 \in M_*$, and, without loss of generality, assume that $a_1 \in A_1^H$, so $b_1 \in B_1^H$. In order to prove the second part of the lemma, we need to show that there are many edges $y = ab$ that are (x, M_*) -switchable.

Let $0 < \delta < 1$ such that $\gamma \ll \delta \ll \nu_2$. Observe that the minimum degree in H_* is at least $(\nu_2 - \nu_1)n_H \geq \delta m$. By construction, there is no pair of vertices both of degree less than $(1 - \eta)m$ that are connected by an edge in H_* . Thus, without loss of generality, we may assume that $e_{H_*}(a_1, B_1^H) \geq \delta m$ and that $e_{H_*}(b_1, A_1^H) \geq (1 - \eta)m$.

Since $|f_{M_*}^{-1}(N_{H_*}(a_1))| \geq \delta m$ and since there are at most $2\nu_1 m$ vertices of degree less than $(1 - \eta)m$, there are at least $\delta m/2$ choices for $a \in f_{M_*}^{-1}(N_{H_*}(a_1) \setminus \{b_1\})$ that satisfies $e_{H_*}(a, B_1^H) \geq (1 - \eta)m$.

Fix such a vertex a and note that

$$e_{H_*}(a, B_1^H \setminus f_{M_*}(N_{H_*}(b_1) \setminus \{a_1, a\})) \leq |B_1^H| - (1 - \eta)m + 2 \leq \eta m + 2.$$

Therefore,

$$\begin{aligned} e_{H_*}(a, f_{M_*}(N_{H_*}(b_1) \setminus \{a_1, a\})) &= e_{H_*}(a, B_1^H) - e_{H_*}(a, B_1^H \setminus f_{M_*}(N_{H_*}(b_1) \setminus \{a_1, a\})) \\ &\geq (1 - \eta)m - (\eta m + 2) \\ &\geq (1 - 3\eta)m. \end{aligned}$$

Thus, there are at least $(1 - 3\eta)m$ choices for $b \in f_{M_*}(N_{H_*}(b_1) \setminus \{a_1, a\})$ with $ab \in E(H_*)$. It follows that there are at least $(\delta m/2)(1 - 3\eta)m \geq \gamma n_H^2$ choices of an edge $y = ab \in E(H_*)$ such that there exists a 6-cycle that contains x, y and two other edges of M_* . We conclude that there are at least γn_H^2 edges of H_* that are (x, M_*) -switchable. \square

The following corollary follows directly from the application of Lemma 6, Lemma 12 and Lemma 14.

Corollary 15. *Let $n \in \mathbb{N}$ and suppose $1/n \ll \mu \ll \varepsilon \ll 1$. Let $G = (A \cup B, E)$ be an ε -extremal Dirac bipartite graph on $2n$ vertices. Then, any μn -bounded edge colouring of G contains a rainbow perfect matching.*

6 Proofs of Theorem 3 and Theorem 4

We finally prove our main theorems.

Proof of Theorem 3. Let G be a Dirac bipartite graph on $2n$ vertices and suppose $1/n \ll \mu \ll \varepsilon \ll \nu \ll \tau \ll 1$. Consider a μn -bounded edge colouring χ of G . By Lemma 7, the graph G is either ε -extremal or a bipartite robust (ν, τ) -expander. If G is a bipartite robust (ν, τ) -expander, then G has a rainbow perfect matching by Corollary 9 with $\mathcal{F} = \mathcal{F}_\chi$. If G is an ε -extremal graph, then G has a rainbow perfect matching by Corollary 15. \square

Proof of Theorem 4. Let $G = (A \cup B, E)$ be a balanced bipartite graph on $2n$ vertices with minimum degree at least $(1/2 + \varepsilon)n$. Suppose that $1/n \ll \mu \ll \varepsilon \ll 1$. We will show that G is a bipartite robust $(\varepsilon/8, 1/4)$ -expander. Let X be a subset of either A or B with $n/4 \leq |X| \leq 3n/4$; without loss of generality, we may assume that $X \subseteq A$. By the minimum degree condition we have $e(X, B) \geq (1/2 + \varepsilon)n|X|$ and, by the definition of robust neighbourhood, we have $e(X, B) \leq |X||RN_{\varepsilon/8}(X)| + \varepsilon n(n - |RN_{\varepsilon/8}(X)|)/8$. Combining these inequalities yields $|X||RN_{\varepsilon/8}(X)| + \varepsilon n^2/8 \geq (1/2 + \varepsilon)n|X|$ and, as $|X| \geq n/4$, upon rearrangement, we have that $|RN_{\varepsilon/8}(X)| \geq (1/2 + \varepsilon/2)n$. If $n/4 \leq |X| \leq n/2$, then $|RN_{\varepsilon/8}(X)| \geq |X| + \varepsilon n/8$ and we are done. If $n/2 \leq |X| \leq 3n/4$, by the minimum degree condition, each $v \in B$ has at least εn neighbours in X . Thus $RN_{\varepsilon/8}(X) = B$ and $|RN_{\varepsilon/8}(X)| = n \geq |X| + \varepsilon n/8$. So G is a bipartite robust $(\varepsilon/8, 1/4)$ -expander. Corollary 9 completes the proof. \square

The following proposition shows that $\mu \leq 1/4$ (see Section 8 for a discussion).

Proposition 16. *For every $t \in \mathbb{N}$, there exists a Dirac bipartite graph G on $n = 4t(t + 1)$ vertices and a $(\frac{t+1}{4t}n)$ -bounded edge colouring of G such that G does not contain a rainbow perfect matching.*

Proof. Let $m = 2t$. Consider the bipartite graph $G = (A \cup B, E)$ constructed as follows. The vertex set is partitioned into $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, with

$$\begin{aligned} A_1 &= \{A_1^1, \dots, A_1^{m-1}\} \\ A_2 &= \{A_2^1, \dots, A_2^{m+1}\} \\ B_1 &= \{B_1^1, \dots, B_1^{m+1}\} \\ B_2 &= \{B_2^1, \dots, B_2^{m-1}\}, \end{aligned}$$

where $|A_k^i| = |B_k^i| = t + 1$. The edge set of G consists of two complete bipartite graphs induced by $G[A_1, B_1]$ and $G[A_2, B_2]$, and of $m + 1$ smaller complete bipartite graphs induced by $G[A_2^i, B_1^i]$, for $i \in [m + 1]$. Note that G is a Dirac bipartite graph.

Consider the edge colouring that assigns colour $c_{k,\ell}^{i,j}$ to the edges in $G[A_k^i, B_\ell^j]$. Since each set has size $t + 1$, the colouring is $(t + 1)^2 = (\frac{t+1}{4t}n)$ -bounded.

Suppose that G admits a rainbow perfect matching M . Note that M contains at most $m + 1$ edges in $G[A_2, B_1]$. Otherwise there exists $i \in [m + 1]$ such that M contains two edges in $E[A_2^i, B_1^i]$, contradicting the fact that it is rainbow, since both edges have colour $c_{2,1}^{i,i}$. Since all the edges incident to A_1 are also incident to B_1 , we must have $|A_1| \geq |B_1| - (m + 1)$. However

$$|A_1| = (m - 1)(t + 1) = (m + 1)(t + 1) - 2(t + 1) = |B_1| - (m + 2),$$

a contradiction. We conclude that, G has no rainbow perfect matching. \square

7 Applications

In the following section we provide some applications of our main theorems on the existence of rainbow spanning subgraphs of graphs with large minimum degree that are not necessarily bipartite.

We first discuss the existence of rainbow Δ -factors in Dirac graphs for a wide range of Δ . Recall that a *Dirac graph* on n vertices is a graph with minimum degree at least $n/2$. The existence of $(n/2)$ -factors in Dirac graphs was proved by Katerinis [13]. Our next result extends Theorem 3 to Δ -factors of Dirac graphs.

Theorem 17. *There exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and for every even $1 \leq \Delta \leq \mu n$ the following holds. Let G be a Dirac graph on n vertices, then any $(\mu n/\Delta)$ -bounded colouring of G contains a rainbow Δ -factor.*

Note that this theorem is tight in its dependence on n and Δ as a Δ factor contains $n\Delta/2$ edges.

Proof. We construct an auxiliary bipartite graph $Q = (V(Q), E(Q))$ as follows. The vertex set is $V(Q) = A \cup B$, where $A = \{u_{v,i} : v \in V(G), 1 \leq i \leq \Delta/2\}$ and $B = \{u_{v,i} : v \in V(G), \Delta/2 < i \leq \Delta\}$. The edge set is defined as

$$E(Q) = \{u_{v,i}u_{w,j} : u_{v,i} \in A, u_{w,j} \in B \text{ and } vw \in E(G)\}.$$

Note that Q is a bipartite Dirac graph on $2N = \Delta n$ vertices. Let $\chi : E(G) \rightarrow \mathbb{N}$ be a μn -bounded edge colouring of G . Construct the edge colouring $\chi_Q : E(Q) \rightarrow \mathbb{N}$ defined by $\chi_Q(u_{v,i}u_{w,j}) = \chi(vw)$, for every $u_{v,i}u_{w,j} \in E(Q)$. Since $2 \cdot (\Delta/2)^2 \cdot \mu n/\Delta = \mu N$, the colouring is μN -bounded. Thus, by Theorem 3, Q has a rainbow perfect matching M .

Consider the subgraph $H = (V(H), E(H))$ of G with $V(H) = V(G)$ and edge set

$$E(H) = \{vw \in E(G) : \text{there exist } 1 \leq i \leq \Delta/2 < j \leq \Delta \text{ such that } u_{v,i}u_{w,j} \in E(M)\}.$$

We claim that H is a rainbow Δ -factor of G . Since $u_{v,i} \in V(Q)$ for every $i \in [\Delta]$ and M is a perfect matching of Q , we have $d_H(v) = \Delta$. Since $u_{v,i}u_{v,j} \notin E(Q)$ for every $i, j \in [\Delta]$, H has no self loops. Finally, since M is a rainbow perfect matching of Q , and by definition of the colouring χ , H has no multiple edges and each colour in χ appears at most once in M . Thus, H is a simple rainbow Δ -regular spanning subgraph of G . \square

Our second corollary concerns bipartite subgraphs of graphs with large minimum degree. Consider two graphs G and H on n vertices with $\Delta(H) \leq \Delta$. The Bollobás-Eldridge-Catlin conjecture [2, 5], states that if $\delta(G) \geq (1 - 1/(\Delta + 1))n - 1/(\Delta + 1)$, then G contains a copy of H . Sauer and Spencer [24] showed that the conjecture holds if $\delta(G) \geq (1 - 1/2\Delta)n - 1$. This result has been improved for large values of Δ [14]. The existence of rainbow copies of H in k -bounded edge colourings of K_n was studied in [3], provided that $k = O(n/\Delta^2)$. In [26], it was observed that similar techniques allow to replace K_n by a graph G with $\delta(G) \geq (1 - c/\Delta)n$, for a sufficiently small constant $c > 0$.

Our last result partially extends the result in [3] at the Sauer-Spencer minimum degree threshold.

Theorem 18. *For every $\gamma > 0$ there exists $\mu > 0$ such that for every $\Delta \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for every even $n \geq n_0$ the following holds. Let G be a graph on n vertices with $\delta(G) \geq (1 - 1/2\Delta + \gamma)n$ and let H be a balanced bipartite graph on n vertices with $\Delta(H) \leq \Delta$, then any proper $(\mu n/\Delta^2)$ -bounded edge colouring of G contains a rainbow copy of H .*

Sudakov and Volec [26] showed that there exist a graph H with maximum degree at most Δ and a $3.9n/\Delta^2$ -bounded edge colouring of K_n which does not contain a rainbow copy of H . Therefore this theorem is also tight up to constant factors.

Proof. By Lemma 2.3 in [1] there is a balanced bipartite spanning subgraph $G' = (A \cup B, E)$ of G with minimum degree $\delta(G') \geq (1 - 1/2\Delta + \gamma/2)m$, where $2m = n$. By Theorem 3.5 in [10], the minimum degree condition ensures the existence of a subgraph J of G' that is isomorphic

to H . For each $a \in A$, let $N_a = \{b \in B : ab \in E(J)\}$ denote the neighbourhood of a in J . Construct an auxiliary bipartite graph $Q = (V(Q), E(Q))$. The vertex set is $V(Q) = A \cup \Gamma$, where $\Gamma = \{\{N_a : a \in A\}\}$ as a multiset. The edge set is defined as

$$E(Q) = \{a_1 N_{a_2} : N_{a_2} \subseteq N_G(a_1)\}.$$

Note that Q is a balanced bipartite graph on $2m$ vertices. We first show that $\delta(Q) \geq (1/2 + \varepsilon)m$, where $\varepsilon = \gamma/2$. For each $a \in A$, there at most $(1/2\Delta - \varepsilon)m$ vertices $b \in B$ such that $ab \notin E(G)$. Since $\Delta(J) \leq \Delta$, for each $b \in B$ there exist at most Δ vertices $a' \in A$ such that $b \in N_{a'}$. Thus, there are at most $(1/2 - \varepsilon\Delta)m$ vertices $a' \in A$ such that $N_{a'}$ is not included in $N_G(a)$. In particular, we have $d_Q(a) \geq (1/2 + \varepsilon)m$. For each $a \in A$, we have $N_Q(N_a) = \cap_{b \in N_a} N_G(b)$. So, by inclusion-exclusion,

$$d_Q(N_a) = |\cap_{b \in N_a} N_G(b)| \geq m - \sum_{b \in N_a} (m - |N_G(b)|) \geq (1/2 + \varepsilon)m.$$

Hence, $\delta(Q) \geq (1/2 + \varepsilon)n$.

Let χ be a proper μn -bounded edge colouring of G . Consider the following system of conflicts,

$$\mathcal{F}_Q = \{\{a_1 N_{a'_1}, a_2 N_{a'_2}\} : \exists x, y \in E(G) \text{ with } \chi(x) = \chi(y) \text{ and } \{x, y\} \subseteq E_G(a_1, N_{a'_1}) \cup E_G(a_2, N_{a'_2})\}.$$

Fix an edge $a_1 N_{a'_1} \in E(Q)$. For each $b_1 \in N_{a'_1}$, there are at most $\mu n / \Delta^2$ edges $a_2 b_2$ with $\chi(a_2 b_2) = \chi(a_1 b_1)$. Again, since $\Delta(J) \leq J$, b_2 is in at most Δ neighbourhoods $\Gamma_{a'_2}$. So, for each $b \in N_{a'_1}$, there are most $\mu n / \Delta$ edges $a_2 N_{a'_2}$ conflicting with $a_1 N_{a'_1}$. Since $|N_{a'_1}| \leq \Delta$, the total number of conflicts involving edge $a_1 N_{a'_1}$ is at most $\mu n = 2\mu m$. So \mathcal{F} is $2\mu m$ -bounded.

We can apply Theorem 4 to the balanced bipartite graph Q and the system of conflicts \mathcal{F}_Q to deduce the existence of a \mathcal{F}_Q -conflict-free perfect matching M in Q . Define the subgraph $R = (V(R), E(R))$ of G as follows. The vertex set is $V(R) = V(G)$ and edge set is

$$E(R) = \{ab \in E(G) : \text{there exist } a' \in A \text{ such that } a N_{a'} \in E(M) \text{ and } b \in N_{a'}\}.$$

We claim that R is a rainbow subgraph of G isomorphic to H . Consider a bijective map $f : V(G) \rightarrow V(G)$, such that $f(u) = v$ if and only if $u N_v \in M$ for $u \in A$ and f is the identity map on B . We claim that f is an isomorphism from R to J . To see this, first observe that f is an automorphism of $V(G)$. Now, consider an edge $ab \in E(R)$ and note that $f(a)f(b) = f(a)b$ where $f(a)$ is such that $a N_{f(a)} \in M$. As M is a matching, there is only one choice $N_{a'}$ such that $a N_{a'} \in E(M)$, implying that $a' = f(a)$. By definition of $E(R)$, we have that $b \in N_{f(a)}$, so $f(a)b = f(a)f(b) \in E(J)$. Similarly, one can show that for all edges $ab \in E(J)$, $f^{-1}(a)f^{-1}(b) = f^{-1}(a)b \in E(R)$. Thus f is an isomorphism between R and J , and since J is isomorphic to H , so is R .

Finally, suppose for contradiction that there exist $x, y \in E(R)$ with $\chi(x) = \chi(y)$. If $x = a_1 b_1$ and $y = a_2 b_2$, let $a'_1, a'_2 \in A$ be such that $a_1 N_{a'_1}, a_2 N_{a'_2} \in E(M)$. Then, as $x, y \in E(R)$, we have $b_1 \in N_{a'_1}$ and $b_2 \in N_{a'_2}$, implying that $a_1 N_{a'_1}$ and $a_2 N_{a'_2}$ conflict under \mathcal{F}_Q . This is a contradiction as M is a \mathcal{F}_Q -conflict-free perfect matching. So R is rainbow. \square

8 Further remarks

We conclude the paper with a number of remarks and open questions.

- 1) The condition on the minimum degree in Theorem 3 is best possible. However, the value of μ that follows from our proof is far from being optimal. In Section 6, we showed that the statement is not true if $\mu > 1/4$. Obtaining the best possible value for μ is a difficult problem, since it would imply a minimum degree version of the Ryser-Brualdi-Stein conjecture, which is wide open.
- 2) We believe that the statement of Theorem 3 should also hold for system of conflicts. The only obstacle in our proof is Lemma 10, which, in its current form, cannot be adapted to deal with conflicts instead of colours.
- 3) As shown in Section 7, the methods presented in this paper are of potential interest to embed other conflict-free spanning structures in graphs with large minimum degree, beyond perfect matchings. A natural candidate is to embed Hamiltonian cycles in Dirac graphs. Krivelevich et al.t [15] proved the existence of \mathcal{F} -conflict-free Hamiltonian cycles in Dirac graphs, provided that the conflicts in \mathcal{F} are local. Their proof is substantially different from ours and relies on Pósa rotations.
- 4) Lu and Székely generalised the idea of system of conflicts to include, not only unordered pairs of edges, but sets of edges of any size [19]. Under some sparsity conditions on the set of conflicts, they proved the existence of conflict-free perfect matchings in $K_{n,n}$. Our results can be seen as a first step towards extending the Lu-Székely framework to Dirac graphs.
- 5) Csaba [6] proved the Bollobás-Eldridge-Catlin conjecture for embedding bipartite graphs of maximum degree Δ into any graph G of minimum degree at least $(1 - \beta)(1 - 1/(\Delta + 1))n$ for some $\beta > 0$. It would be of interest to determine whether a form of Theorem 18 holds in this setting, since it does not follow as a direct consequence of Theorem 3.

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