



## The group inverse of some circulant matrices

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**Abstract.** We present here necessary and sufficient conditions for the invertibility of some circulant matrices that depend on three parameters and moreover, we explicitly compute the inverse. Our study also encompasses a wide class of circulant symmetric matrices. The techniques we use are related with the solution of boundary value problems associated to second order linear difference equations. Consequently, we reduce the computational cost of the problem. In particular, we recover the inverses of some well known circulant matrices whose coefficients are arithmetic or geometric sequences, Horadam numbers among others. We also characterize when a general symmetric, circulant and tridiagonal matrix is invertible and in this case, we compute explicitly its inverse.

### 1. INTRODUCTION AND PRELIMINARIES

Many problems in applied mathematics and science lead to the solution of linear systems having circulant coefficients related to the periodicity of the problems, as the ones that appear when using the finite difference method to approximate elliptic equations with periodic boundary conditions, see [9]. Circulant matrices have a wide range of application in signal processing, image processing, digital image disposal, linear forecast, error correcting code theory, see [10, 20]. In the last years, there have been several papers on circulant matrices that attend to give an effective expression for the determinant, the eigenvalues and the inverse of the matrix, see for instance [14, 19, 21, 18].

Motivated by computing the Green function of some networks obtained by the addition of new vertices to a previously known one, see [7], in [6] we study some circulant matrices with few parameters, specifically circulant matrices of type  $\text{Circ}(a, b, c, \dots, c)$  and  $\text{Circ}(a, b, c, \dots, c, b)$ . We obtained necessary and sufficient condition for their invertibility and moreover we gave a closed formula for the inverse.

In this paper we study the circulant matrices of type  $\text{Circ}(a, b, c, \dots, c, b)$  in full generality and compute their group inverse. Therefore, the given results encompass those obtained in [6] when the corresponding matrix is non singular. Our methodology is the same than the one used in the above mentioned paper and it is related with linear difference equations, see also [17, 18], for a similar treatment in the case of circulant matrices with few parameters.

Throughout the paper  $\mathbb{N}$  denote the set of non-negative integers, whereas  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are the fields of rational, real and complex numbers, respectively. Given  $a \in \mathbb{C}$ , we define  $a^\#$  as

$$(1) \quad a^\# = \begin{cases} a^{-1}, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

Notice that  $(ab)^\# = a^\#b^\#$  for any  $a, b \in \mathbb{C}$ .

Given  $n \in \mathbb{N}$  we denote by  $R_n$  the multiplicative group of  $n$ -th roots of unity; that is, the set

$$(2) \quad R_n = \left\{ r \in \mathbb{C} : r^n = 1 \right\} = \left\{ e^{\frac{2\pi k}{n}i} : k = 0, \dots, n-1 \right\}.$$

Clearly  $\bar{r} = r^{-1} = r^{n-1}$  for any  $r \in R_n$  and  $-1 \in R_n$  iff  $n$  is even. We also consider the sets

$$(3) \quad \mathfrak{R}_n = \left\{ \Re(r) : r \in R_n \right\} = \left\{ \cos\left(\frac{2\pi k}{n}\right) : k = 0, \dots, \lceil \frac{n-1}{2} \rceil \right\}.$$

and

$$(4) \quad \mathfrak{R}_n^* = \mathfrak{R}_n \setminus \{\pm 1\} = \left\{ \cos\left(\frac{2\pi k}{n}\right) : k = 1, \dots, \lfloor \frac{n-1}{2} \rfloor \right\}.$$

Notice that  $0 \in \mathfrak{R}_n^*$  iff  $n \equiv 0 \pmod{4}$ .

For fixed  $n \in \mathbb{N}$ , we consider the vector space  $\mathbb{C}^n$ . We denote the components of the vector  $\mathbf{v} \in \mathbb{C}^n$  as  $v_j$ ,  $j = 1, \dots, n$ , *i.e.*,  $\mathbf{v} = (v_1, \dots, v_n)^\top$ . As usual,  $\mathbf{1}$  is the all ones vector and  $\mathbf{0}$  is the all zeroes vector in  $\mathbb{C}^n$ . Besides  $\mathbf{e}$  is the vector with a 1 in its first coordinate and 0's elsewhere. Besides,  $\hat{\mathbf{1}}$  is the vector whose components are  $(-1)^j$ ,  $j = 1, \dots, n$ .

Given  $\mathbf{v} \in \mathbb{C}^n$ ,  $\bar{\mathbf{v}}$  denotes the vector whose components are the conjugates of those of  $\mathbf{v}$ ; that is,  $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_n)^\top \in \mathbb{C}^n$ . Moreover, for any  $m \in \mathbb{N}$   $\mathbf{v}_m$  is the vector whose components are given by  $(v_m)_j = v_{(m-1+j) \pmod{n}}$ ,  $j = 1, \dots, n$ ; that is,

$$(5) \quad \mathbf{v}_m = (v_m, v_{m+1}, \dots, v_n, v_1, v_2, \dots, v_{m-1})^\top, \quad m = 1, \dots, n.$$

Notice that  $\mathbf{v}_1 = \mathbf{v}$  and that for any  $m \in \mathbb{N}$ ,  $\mathbf{e}_m \in \mathbb{C}^n$  is the vector with a 1 in the  $m$ -th coordinate and 0's elsewhere.

We consider  $\mathbb{C}^n$  endowed with the standard inner product  $\langle \cdot, \cdot \rangle$ ; that is,  $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^n a_j \bar{b}_j$ , whose associated norm is  $\|\cdot\|$ . Given  $V$  a subset of  $\mathbb{C}^n$ ,  $V^\perp$  denotes the subspace of  $\mathbb{C}^n$  orthogonal to  $V$ . In particular if  $\mathbf{v} \in \mathbb{C}^n$ ,  $\mathbf{v}^\perp$  is the subspace of  $\mathbb{C}^n$  orthogonal to  $\mathbf{v}$ .

Let  $\tau$  be the permutation of the set  $\{1, \dots, n\}$  defined as,

$$(6) \quad \tau(1) = 1, \quad \tau(j) = n+2-j, \quad j = 2, \dots, n.$$

Given  $\mathbf{a} \in \mathbb{C}^n$ , we define  $\mathbf{a}_\tau \in \mathbb{C}^n$  the vector whose components are  $(a_\tau)_j = a_{\tau(j)}$ ,  $j = 1, \dots, n$ . Thus,  $(\mathbf{e}_k)_\tau = \mathbf{e}_{\tau(k)}$ ,  $k = 1, \dots, n$ ,  $\mathbf{1}_\tau = \mathbf{1}$  whereas  $\hat{\mathbf{1}} = \hat{\mathbf{1}}_\tau$  iff  $n$  is even,  $(\mathbf{a}_\tau)_\tau = \mathbf{a}$  and  $\langle \mathbf{a}_\tau, \mathbf{b}_\tau \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$  for any  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ . In particular,  $\langle \mathbf{a}_\tau, \mathbf{1} \rangle = \langle \mathbf{a}, \mathbf{1} \rangle$ , for any  $\mathbf{a} \in \mathbb{C}^n$ .

If  $\mathbb{K}$  is any of the fields  $\mathbb{Q}$ ,  $\mathbb{R}$  or  $\mathbb{C}$ , the set of square matrices of order  $n$  with coefficients in  $\mathbb{K}$  is denoted  $\mathcal{M}_n(\mathbb{K})$ . If  $A \in \mathcal{M}_n(\mathbb{C})$ , its *transpose* and its *conjugate transpose* are denoted by  $A^\top$  and  $A^*$ , respectively. Moreover, we denote by  $\mathbf{O}, \mathbf{I}, \mathbf{J} \in \mathcal{M}_n(\mathbb{Q})$  the null matrix, the identity matrix and the all ones matrix, respectively.

Given  $A \in \mathcal{M}_n(\mathbb{K})$ , a matrix  $M \in \mathcal{M}_n(\mathbb{K})$  is called *generalized inverse* of  $A$  if it satisfies the identity

$$(7) \quad \mathbf{A}M\mathbf{A} = \mathbf{A}.$$

This kind of generalized inverses are also called *1-inverses* and also *system solver inverses*, since  $M$  satisfies (7) iff given  $\mathbf{b} \in \mathbb{K}^n$  in the range of  $A$ , then  $\mathbf{g} = M\mathbf{b}$  satisfies that  $A\mathbf{g} = \mathbf{b}$ . Clearly, if  $A$  is invertible, then any generalized inverse coincides with  $A^{-1}$ , the inverse of  $A$ , but when  $A$  is singular it has infinite generalized inverses. However, there exists a unique generalized inverse of  $A$ , known as *Moore-Penrose inverse* of  $A$  and denoted by  $A^\dagger$ , satisfying the so-called *Moore-Penrose equations*

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (A^\dagger A)^* = A^\dagger A, \quad (AA^\dagger)^* = AA^\dagger.$$

Moreover,  $(A^\dagger)^\dagger = A$ ,  $(aA)^\dagger = a^\# A^\dagger$ , for any  $a \in \mathbb{C}$ ,  $(A^\top)^\dagger = (A^\dagger)^\top$  and  $(A^*)^\dagger = (A^\dagger)^*$ .

When  $A$  has index 1, then there also exists a unique generalized inverse of  $A$  called the *group inverse* of  $A$  and denoted by  $A^\#$ , satisfying the identities

$$AA^\#A = A, \quad A^\#AA^\# = A^\#, \quad A^\#A = AA^\#.$$

When the group inverse of  $A$  exists, then  $(A^\#)^\# = A^\#$ ,  $A^\# \in \mathcal{M}_n(\mathbb{K})$ ,  $(aA)^\# = a^\#A^\#$ , for any  $a \in \mathbb{C}$ ,  $(A^\top)^\# = (A^\#)^\top$  and  $(A^*)^\# = (A^\#)^*$ . On the other hand, it coincides with the Moore-Penrose inverse of  $A$  iff  $A$  is *range-hermitian*; that is the nullity of  $A$  coincides with the nullity of  $A^*$ ; see [3, Chapter 4, Theorem 4]. Notice that  $O^\# = O$  and  $J^\# = \frac{1}{n^2}J$ , because  $J^2 = nJ$ . On the other hand, when  $n = 1$ , the identity (1) coincides with the group inverse of  $a \in \mathbb{C}$  understood as a matrix of order 1.

## 2. MATRICES $\text{Circ}(a, b, c, \dots, c, b)$

Our aim in this section is the computation of  $\text{Circ}(a, b, c, \dots, c, b)^\#$  where  $a, b, c \in \mathbb{C}$ , which requires that  $n \geq 3$ . Moreover, since for  $n = 3$  we have  $\text{Circ}(a, b, b)$  and for  $c = b$  we have  $\text{Circ}(a, b, \dots, b)$  and both matrices have been analyzed in the above section, in this one we always assume that  $n \geq 4$  and that  $c \neq b$ . Notice that  $\text{Circ}(a, b, c, \dots, c, b)$  is symmetric and hence its group inverse is also symmetric, which implies that  $\text{Circ}(a, b, c, \dots, c, b)^\# = \text{Circ}(\mathbf{g})$  where  $\mathbf{g} = \mathbf{g}_r$ . We remark that the case  $a, b \in \mathbb{R}$  and  $c = 0$  has been analyzed in [17] under the name of symmetric circulant tridiagonal matrix, but assuming the condition  $|a| > 2|b| > 0$ ; that is, that  $\text{Circ}(a, b, 0, \dots, 0, b)$  is a strictly diagonally dominant matrix.

We newly use the methodology of the above section, that corresponds to the adaptation of that in [6, Section 3] for complex parameters. It consists in analyzing the compatibility of the linear system  $\text{Circ}(a, b, c, \dots, c, b)\mathbf{h} = \mathbf{w}$  for a given  $\mathbf{w} \in \mathbb{C}^n$ . Notice that if  $\mathbf{h}$  satisfies the above identity, then

$$(8) \quad \langle \mathbf{w}, \mathbf{1} \rangle = \langle \mathbf{h}, \text{Circ}(\bar{a}, \bar{b}, \bar{c}, \dots, \bar{c}, \bar{b})\mathbf{1} \rangle = \langle \mathbf{h}, (\bar{a} + 2\bar{b} + (n-3)\bar{c})\mathbf{1} \rangle = (a + 2b + (n-3)c)\langle \mathbf{h}, \mathbf{1} \rangle,$$

From the identity  $\text{Circ}(a, b, c, \dots, c, b) = \text{Circ}(a-c, b-c, 0, \dots, 0, b-c) + cJ$ , we have that  $\mathbf{h} \in \mathbb{C}^n$  satisfies  $\text{Circ}(a, b, c, \dots, c, b)\mathbf{h} = \mathbf{w}$  iff  $\text{Circ}(a-c, b-c, 0, \dots, 0, b-c)\mathbf{h} = \mathbf{v} - c\langle \mathbf{h}, \mathbf{1} \rangle \mathbf{1}$  and hence

$$(9) \quad \text{Circ}(a, b, c, \dots, c, b)\mathbf{h} = \mathbf{w} \quad \text{iff} \quad \text{Circ}\left(\frac{a-c}{c-b}, -1, 0, \dots, 0, -1\right)\mathbf{h} = \frac{1}{c-b}(\mathbf{w} - c\langle \mathbf{h}, \mathbf{1} \rangle \mathbf{1})$$

The above identity justifies that we start our analysis with the case  $b = -1$  and  $c = 0$ . Specifically, for any  $r \in \mathbb{C}$ , we define the following vector in  $\mathbb{C}^n$ :

$$(10) \quad \mathbf{b}(r) = (2r, -1, 0, \dots, 0, -1)^\top.$$

Observe that

$$(11) \quad \text{Circ}(\mathbf{b}(1)) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

is the *Combinatorial Laplacian* of a  $n$ -cycle. Then, it is singular and moreover  $\text{Circ}(\mathbf{b}(1))^\# = (g_{ij})$  where

$$(12) \quad g_{ij} = \frac{1}{12n}(n^2 - 1 - 6|i-j|(n - |i-j|)), \quad i, j = 1, \dots, n,$$

is the Green function of the  $n$ -cycle, see [5, Corollary 6.1] and also [8, Section 3]. Therefore, for any  $r \in \mathbb{C}$  we have that

$$(13) \quad \text{Circ}(\mathbf{b}(r)) = \text{Circ}(\mathbf{b}(1)) + 2(r-1)\mathbf{l}$$

is the matrix associated with the Schrödinger operator on the  $n$ -cycle with constant complex potential  $q = 2(r-1)$ . This implies that  $\text{Circ}(\mathbf{b}(r))^\#$  is the Green's function of a  $n$ -cycle for the potential  $q = 2(r-1)$ ; or equivalently, it can be seen as the Green function associated with a  $n$ -path with periodic boundary conditions, see [4, Page 295]. The inverse of  $\text{Circ}(\mathbf{b}(r))$  when it is non singular has been obtained in [4, Proposition 3.12]. We summarize all these results below and moreover we also compute  $\text{Circ}(\mathbf{b}(r))^\#$  in the

singular case. Notice that  $\text{Circ}(\mathbf{b}(r))$  is also known as *periodic Jacobi matrix* and hence the computation of  $\text{Circ}(\mathbf{b}(r))^\#$  involves the solution of an specific kind of second order difference equations with constant coefficients. Before to start our analysis, let us define for any  $r \in \mathbb{C}$  the vector  $\mathbf{v}(r) \in \mathbb{C}^n$  whose components are given by

$$(14) \quad v_j(r) = U_{j-2}(r) + U_{n-j}(r), \quad j = 1, \dots, n$$

that together with  $\mathbf{t}(r), \mathbf{u}(r) \in \mathbb{C}^n$  defined in (??) will play a main role in our developments. Notice that  $\mathbf{v}(1) = n\mathbf{1}$  and that  $\mathbf{v}(-1) = n\hat{\mathbf{1}}$  when  $n$  is even.

**Lemma 2.1.** *Given  $n \in \mathbb{N}$  and  $r \in \mathbb{R}$ , then  $\mathbf{v}_\tau(r) = \mathbf{v}(r)$  and  $\mathbf{v}(r) = \mathbf{0}$  iff  $r \in \mathfrak{R}_n^*$ . Moreover,  $\langle \mathbf{v}(1), \mathbf{1} \rangle = n^2$  and  $\langle \mathbf{v}(r), \mathbf{1} \rangle = \frac{1}{r-1} [T_n(r) - 1]$  when  $r \neq 1$ . In particular,  $\langle \mathbf{v}(r), \mathbf{1} \rangle = 0$  when  $r \in \mathfrak{R}_n \setminus \{1\}$ .*

We define the vectors  $\mathbf{y}(\pm 1) = (y_1(\pm 1), \dots, y_n(\pm 1))^\top \in \mathbb{R}^n$ , where

$$(15) \quad y_j(\pm 1) = (\pm 1)^j (n^2 - 1 - 6(j-1)(n+1-j)), \quad j = 1, \dots, n,$$

whereas if  $r \in \mathfrak{R}_n^*$ , we define the vector  $\mathbf{y}(r) = (y_1(r), \dots, y_n(r))^\top \in \mathbb{R}^n$ , where

$$(16) \quad y_j(r) = \frac{3}{1-r^2} \left[ (n+3-2j)T_j(r) - (n+1-2j)T_{j-2}(r) \right], \quad j = 1, \dots, n,$$

Observe that  $\mathbf{y}(r) = \mathbf{y}_\tau(r)$  and that if  $r = \cos\left(\frac{2\pi k}{n}\right)$ ,  $k = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$  we have that

$$y_j(r) = \frac{3}{\sin^2\left(\frac{2\pi k}{n}\right)} \left[ (n+3-2j) \cos\left(\frac{2\pi k j}{n}\right) - (n+1-2j) \cos\left(\frac{2\pi k(j-2)}{n}\right) \right], \quad j = 1, \dots, n.$$

In addition, the identity (12) implies that  $\text{Circ}(\mathbf{b}(1))^\# = \frac{1}{12n} \text{Circ}(\mathbf{y}(1))$ .

**Corollary 2.2.** *For any  $r \in \mathbb{C}$ , the following properties hold:*

$$\text{Circ}(\mathbf{b}(r))^\# = \begin{cases} \frac{1}{2[T_n(r) - 1]} \text{Circ}(\mathbf{v}(r)), & \text{if } r \notin \mathfrak{R}_n, \\ \frac{1}{12n} \text{Circ}(\mathbf{y}(r)), & \text{if } r \in \mathfrak{R}_n. \end{cases}$$

Our last objective is to generalize the results to arbitrary  $a, b, c \in \mathbb{C}$ . We newly remark that the necessary and sufficient conditions for the invertibility of  $\text{Circ}(a, b, c, \dots, c, b)$  when  $a, b, c \in \mathbb{R}$ , and also the expression for its inverse were already obtained in [6, Theorem 3.5].

**Theorem 2.3.** *For  $a, b, c \in \mathbb{R}$ ,  $b \neq c$ , the circulant matrix  $\text{Circ}(a, b, c, \dots, c, b)$  is invertible iff*

$$(a + 2b + (n-3)c) \prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left[ a - c + 2(b-c) \cos\left(\frac{2\pi j}{n}\right) \right] \neq 0$$

and, in this case

$$\text{Circ}(a, b, c, \dots, c, b)^{-1} = \text{Circ}(\mathbf{g}(a, b, c)),$$

where if  $a \neq 3c - 2b$

$$g_j(a, b, c) = \frac{U_{j-2}(r) + U_{n-j}(r)}{2(c-b)[T_n(r) - 1]} + \frac{1}{n} \left[ \frac{1}{a + 2b + (n-3)c} - \frac{1}{a + 2b - 3c} \right], \quad j = 1, \dots, n,$$

with  $r = \frac{a-c}{2(c-b)}$ , whereas

$$g_j(3c - 2b, b, c) = \frac{1}{12n(c-b)} (n^2 - 1 - 6(j-1)(n+1-j)) + \frac{1}{n^2c}, \quad j = 1, \dots, n.$$

► If  $a \neq -2b + (3-n)c$  and  $r = \frac{a-c}{2(c-b)} = \cos\left(\frac{2\pi j}{n}\right)$  for some  $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$

$$g_j(a, b, c) = \frac{(n+1-2j)U_{j-2}(r)}{2n(b-c)} - \frac{2T_j(r)}{n(a+2b-3c)} + \frac{1}{n} \left[ \frac{1}{a+2b+(n-3)c} - \frac{1}{a+2b-3c} \right], \quad j = 1, \dots, n$$

► If  $n$  is even,  $a = 2b - c$  and  $b \neq \left(1 - \frac{n}{4}\right)c$ , then

$$g_j(2b-c, b, c) = \frac{(-1)^j}{12n(c-b)} \left[ n^2 - 1 - 6(j-1)(n+1-j) \right] + \frac{1}{n} \left[ \frac{1}{a+2b+(n-3)c} - \frac{1}{4(b-c)} \right]$$

► If  $a = -2b + (3-n)c$

◦ If  $c = 0$ , then  $g_j(-2b, b, 0) = -\frac{1}{12nb} \left[ n^2 - 1 - 6(j-1)(n+1-j) \right]$

◦ If  $c \neq 0$  and either  $bc^{-1} > 1 - \frac{n}{4}$  or  $\frac{b}{c} \leq 1 - \frac{n}{4}$  but  $n \arccos\left(\frac{a-c}{2(c-b)}\right) \neq 2\pi j$ ,  $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ ,

then  $g(a, b, c) = \frac{w(r)}{2(c-b)[T_n(r)-1]} - \frac{1}{n(a+2b-3c)} \mathbf{1}$ .

◦ If  $c \neq 0$ ,  $bc^{-1} < 1 - \frac{n}{4}$  and  $n \arccos\left(\frac{a-c}{2(c-b)}\right) \neq 2\pi j$  for some  $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ , then

$$g_j(a, b, c) = \frac{(n+1-2j)}{2n(b-c)} U_{j-2}(r) - \frac{2T_j(r)+1}{n(a+2b-3c)}$$

◦ If  $n$  even,  $c \neq 0$  and  $b = \left(1 - \frac{n}{4}\right)c$ , then

$$g_j(a, b, c) = \frac{(-1)^j}{12n(c-b)} \left[ n^2 - 1 - 6(j-1)(n+1-j) \right] + \frac{1}{4n(c-b)}$$

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