CONFINED ROTATING CONVECTION IN WATER-ETHANOL MIXTURES

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1 Introduction

Flows driven by temperature gradients in the presence of a gravity field have interest in geophysical flows and in many engineering applications (heat transfer, crystal growth, storage tanks, etc). Special attention deserves the case of binary fluids. Thermal convection in a binary fluid layer heated from below is a system that exhibits a great variety of pattern-forming phenomena when driven away from equilibrium. In binary mixtures, the concentration flux depends both on concentration and temperature gradients (Soret effect, quantified by the Soret coefficient). When rotation is introduced in the system, the competition between rotation and thermal buoyancy gives rise to a rich dynamics of great importance to atmospheric and oceanic circulations. In rotating pure fluid enclosed flows, the onset of convection takes place at temperature differences much smaller than the expected onset for an infinitely extended system. This is due to the presence of wallinduced instabilities, resulting in wall modes. The convective (bulk modes) instabilities happen as the temperature difference is increased.

While there is a lot of previous work on rotating pure fluid Rayleigh-Bénard convection [1] and on non-rotating convection in binary fluid layers heated from below, rotating convection with a binary mixture is an almost unexplored subject. The present project is directly related to the work of Net, Mercader et al.[2], where the onset of convection in a rotating vertical cylinder for a 3He-4He mixture was analysed. The eigenfunctions in that case are rigidly precessing spirals which can take the form of spatially extended body modes which fill the container (typically precessing in the retrograde direction), or of wall modes confined to its boundary (prograde modes). However, the nonlinear problem was not considered. The work of Alonso, Mercader and Batiste [3] included both a linear stability analysis and a nonlinear time-evolution simulation of the patterns arising after the subcritical oscillatory instability in a vertical cylinder filled with a water-ethanol mixture, but rotation was not taken into account. Finally, among the works in rotating pure fluid convection relevant to the present project, Marques et al.[4] studied the interplay between the three forces of the problem, Coriolis, gravitational and centrifugal buoyancy, by taking into account the (usually neglected) dependence of density with temperature in the centrifugal term. When the centrifugal buoyancy is not neglected the trivial basic state (conductive solution, velocity corresponds to solid body rotation) is replaced by a large-scale circulation, which has not been considered in this work because its contribution is minimal on the rotation ranks we are making the study.

So basically, on this final-grade work we are going to study the behaviour of a water-ethanol mixture confined in a rotating vertical cylinder, where a temperature gradient is applied in the vertical direction. In order to do this we have opted to use numerical simulation, more specifically we have used a code developed by the Fluid Dynamics Group of the UPC, based on a pseudo-spectral
method (collocation). We focused on the stability linear analysis and did not study the non-linear time evolution. Much of the work consisted specifically in understanding the modelling of the problem and understanding how the code works. Then there is a part where the results are presented and extensively discussed. In particular, we studied the linear stability of the water-ethanol mixture in two different aspect ratios (ratio between cylinder radius and its height): A small one, $\Gamma = 1$, in which walls make an important influence producing wall modes dominate for small rotations, and a bigger one, $\Gamma = 2.76$, where walls make a lower influence and a few body modes dominate before rotation is large enough to produce a wall mode. In addition we made a characterisation of the dominating modes by plotting temperature and concentration eigenfunctions from several different rotation rates where each mode dominates. With this characterisation, we are able to visualise the shape of the dominating mode of the perturbation that will be arising at the onset of convection in a real experiment with the same characteristics.
2 Fundamentals

2.1 Binary mixtures

A binary mixture consists of two different chemical mixtures, as could be water and ethanol or 3He-4He. The main characteristic of binary mixtures which makes its study interesting, is the Soret effect, also known as thermophoresis or thermal migration. It consists on a concentration flux produced by a temperature gradient, which has been broadly studied and can be quantified by the Soret coefficient ($S_T$). This phenomenon is observed in mixtures of mobile particles where the different particle types exhibit different responses to the force of a temperature gradient. Thermodiffusion is considered "positive" when the denser component moves from a hot to cold region and "negative" when the reverse is true.

As previously mentioned, the interplay of Rayleigh-Bénard convection with the Soret effect (with a negative coefficient) in binary fluids results in totally different patterns of instability than those seen in pure fluid cases.

The density of the binary fluid will be considered to depend linearly on both concentration and temperature, in the following form:

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(c - c_0))$$  \hspace{1cm} (1)

$$\alpha = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial T}$$ \hspace{1cm} (2)

$$\beta = \frac{1}{\rho_0} \frac{\partial \rho}{\partial c}$$ \hspace{1cm} (3)

where $c$ refers to the mass fraction of the denser component, $\rho_0$ is the reference density in the temperature and concentration conditions $T_0$ and $c_0$. The $\alpha$ and $\beta$ coefficients refer to the thermal and solutal expansion respectively.

2.2 Development of equations

We consider a cylinder radius $R$ and height $H$, and we define the aspect ratio of the cylinder as their ratio, $\Gamma = R/H$. We also consider gravity defined as $\vec{g} = -g\hat{k}$. The temperature boundary conditions defined as $T_{\text{top-plate}} = T_0 - \frac{\Delta T}{2}$ and $T_{\text{bottom-plate}} = T_0 + \frac{\Delta T}{2}$ where $\Delta T$ is the temperature difference between the plates and $T_0$ is the temperature at the medium height of the cylinder of the conductive profile. We also name the rotation speed of the cylinder as $\Omega = \Omega \hat{k}$, considered positive when rotating anticlockwise direction seen from above. Last we will define the velocity field in cylindrical coordinates: $\vec{v} = v_r \hat{r} + v_\theta \hat{\theta} + v_z \hat{k} = (u, v, w)$. 


2.2.1 Heat and concentration fluxes

We start from heat flux and concentration flux equations extracted from De Groot & Mazur\cite{5}.

\[
\vec{J}_q = -\lambda \nabla T - \rho \frac{\partial \mu}{\partial c} T D'' \nabla c
\] (4)

\[
\vec{J}_c = -\rho \left[ c(1-c)D' \nabla T + D \nabla C \right] = -\rho D \left[ c(1-c)S_T \nabla T + \nabla C \right]
\] (5)

where \( T \) is the temperature, \( c \) the solute mass fraction, \( \rho \) is the density, \( \mu \) is the chemical potential of the solute, \( \lambda \) is the thermal conductivity, \( D \) is the mass diffusion coefficient, \( D' \) the thermal diffusion coefficient, \( S_T \) is the Soret coefficient that we define as \( S_T = \frac{D'}{D} \) (which can be calculated in a laboratory by various methods \cite{6}\cite{7}) and \( D'' \) is the Dufour coefficient.

The mass diffusion coefficient (which is the one appearing in the classical Fick law) represents the case with which a particular solute moves within the solvent. On the other hand, the thermal diffusion coefficient describes how the solute behaviour depends on the temperature gradient. So we have a positive \( D' \) when the solute moves from a 'hot' region to a 'cold' one and negative when it happens inversely. The Dufour coefficient shows the ease with which there is a heat flow due to a concentration gradient.
It is clearly observable that in the flow equations seen, apart from Fick’s law and the typical Fourier law, cross-effects are also considered as Thermophoresis (Soret effect) and Dufour effect. In our study case Dufour effect will be neglected since in comparison to the other terms this one is much smaller when referring to liquids.

Inserting the definitions of the fluxes (4) and (5) in the phenomenological equations, which are the conservation of mass (continuity equation) and the conservation of energy, we arrive to these two equations which define the thermal and concentration temporal evolution respectively. Furthermore, we considered $D' = D''$ according to Onsager Reciprocal relations as done in Esteban’s Mell’s dissertation [8].

\[
\frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T = \kappa \nabla^2 T \tag{6}
\]

\[
\frac{\partial c}{\partial t} + (\vec{v} \cdot \nabla) c = D(c_0(1 - c_0)S_T \nabla^2 T + \nabla^2 c) \tag{7}
\]

where $\kappa = \lambda / \rho c_p$ is the thermal diffusivity and $c_p$ is the heat capacity measured at constant pressure.

From this point we will introduce new variables to simplify the development of the equations by using variables which have in consideration the problem geometry. These new variables define the temperature and the concentration as the summation of the basic field plus a perturbation which we will call $T_D'$ and $\Sigma'$ respectively.

\[
T = T_0 + \Delta T(\frac{1}{2} - \frac{z_D}{H}) + T_D' \\
c = c_0 + c_0(1 - c_0)S_T \Delta T(\frac{z_D}{H} - \frac{1}{2}) + \Sigma'
\] (8)

By introducing these new variables (8) in the temporal evolution equations of temperature (6) and concentration (7), we obtain the equations:

\[
\frac{\partial T_D'}{\partial t_D} - v_{zD} \Delta T \frac{H}{H} + (\vec{v}_D \cdot \nabla_D) T_D' = \kappa \nabla^2_D T_D' \tag{9}
\]

\[
\frac{\partial \Sigma'}{\partial t_D} + v_{zD} \frac{c_0(1 - c_0)S_T \Delta T}{H} + (\vec{v}_D \cdot \nabla_D) \Sigma' = D[c_0(1 - c_0)S_T \nabla^2_D T_D' + \nabla^2_D \Sigma'] \tag{10}
\]

where we have added the subscript D to emphasise that these values are dimensional. In the case of the concentration, we have defined it as the mass fraction of the denser component of the fluid, hence is dimensionless.
2.2.2 Momentum equation

We start from the Navier-Stokes equations for incompressible fluid taking into account the Boussinesq approximation (very small density variations while the other properties of the fluid remain constant). In addition, we also consider that we are in the cylinder system of reference, then appearing the terms of the apparent forces, such as the Coriolis force and the Centrifugal force. We are interested in working on this system of reference because it will facilitate the simulation. For example when defining boundary conditions or when neglecting the centrifugal term, provoking the basic solution of the problem to be $\vec{v} = 0$ greatly simplifying the whole problem. The Navier-Stokes equation in Boussinesq aproximation and in the rotating frame is given by:

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} + 2 \rho \Omega \times \vec{k} + \rho \Omega^2 \vec{r} \tag{11}$$

where $\mu$ is the viscosity of a fluid which measures its resistance to deformation at a given rate.

Applying the Boussinesq approximation implies treating all fluid properties as constant, except for the density. This density will be treated as we defined in (1), where the density was split in three terms: the leading order term with only a factor $\rho_0$ and the terms depending on $T$ and $C$. Hence, we introduce equation (1) into equation (11). If the leading-term order term is not a gradient the other two terms may be neglected, if it is a gradient, it can be absorbed into the pressure gradient and the other two terms have to be maintained. In our case we have introduced two terms into the gradient which are: the gravity term and the centrifugal force term. Moreover we also apply the change of variables (8), already used previously. We get:

$$\left( \frac{\partial \vec{v}_D}{\partial t_D} + (\vec{v}_D \cdot \nabla)\vec{v}_D \right) = -\nabla_D p'_D + \nu \nabla^2_D \vec{v}_D + [\alpha (\Delta T \left( \frac{1}{2} - \frac{z_D}{H} \right) + T'_D)] g \hat{k}
- [\beta (c_0 (1 - c_0) S_T \Delta T \left( \frac{z_D}{H} - \frac{1}{2} \right) + \Sigma' \right] g \hat{k} + 2 \Omega_D \vec{v}_D \times \hat{k}
+ \left[ (\alpha (\Delta T \left( \frac{1}{2} - \frac{z_D}{H} \right) + T'_D)
+ (\beta (c_0 (1 - c_0) S_T \Delta T \left( \frac{z_D}{H} - \frac{1}{2} \right) + \Sigma' \right] \Omega^2_D r_D \hat{r} \tag{12}$$

where $p'_D$ is the new pressure term which includes the gravitational and centrifugal contributions (which have been included into the gradient) apart from the original pressure term. The $\nu$ refers to the kinematic viscosity defined as the ratio of the viscosity $\mu$ to the density of the fluid $\rho$. Note that we have, again, introduced sub-index D to emphasise the dimensional character of the variables.
In order to simplify this equation we apply another change of variables to it.

\[
T_D = \Delta T \left( \frac{1}{2} - \frac{z_D}{H} \right) + T'_D \\
C_D = c_0(1 - c_0)S_T \Delta T \left( \frac{z_D}{H} - \frac{1}{2} \right) + \Sigma'
\]  

(13)

Finally we obtain the dimensional equation in which we have applied the Boussinesq approximation in those new variables that simplify the equations.

\[
\frac{\partial \tilde{v}_D}{\partial t_D} + (\tilde{v}_D \cdot \nabla_D)\tilde{v}_D = -\frac{\nabla_D p'_D}{\rho_0} + \nu \nabla^2_D \tilde{v}_D + (\alpha T_D - \beta C_D)g \hat{k} + 2\Omega_D \tilde{v}_D \times \hat{k} - (\alpha T_D - \beta C_D)\Omega^2_D r_D \hat{r}
\]

(14)

2.2.3 Adimensionalisation

Once we have obtained the equations that define the temporal evolution of the temperature field (9), concentration field (10) and velocity field (14) in the form we wanted, it is time to adimensionalize them.

To develop those equations without dimensions we define new variables in non-dimensional units. We use \( L \) (cylinder height) as length scale and its associated thermal diffusion time \( \tau_T = \frac{L^2}{\kappa} \) as time scale. We use the temperature differences between the two plates of the cylinder \( \Delta T \) for the temperature, and for the concentration a re-escalation will be done using \( \Delta c = -c_0(c_0 - 1)S_T \Delta T \) (difference of concentration in the conductive state due to the Soret effect). Here there are some of the transformations used:

\[
\begin{align*}
  r &= \frac{r_D}{L} \\
  \bar{v} &= \frac{\tilde{v}_D}{\kappa/L} \\
  t &= \frac{t_D}{L^2/\kappa} \\
  T &= \frac{T_D}{\Delta T} \\
  C &= \frac{C_D}{-c_0(1 - c_0)S_T \Delta T} \\
  \nabla &= L \nabla_D
\end{align*}
\]
We also define two new dimensionless variables \( \Theta = \frac{T_D}{\Delta T} \) and \( \Sigma = -\frac{\Sigma' c_0}{c_0(1 - c_0)S_T \Delta T} \). Adimensionalising equations (8) and knowing \( z_D = Lz \) we obtain these expressions for non dimensional T and C:

\[
T = \frac{T_D}{\Delta T} = \left( \frac{1}{2} - \frac{zL}{H} \right) + \Theta
\]
\[
C = \frac{C_D}{c_0(1 - c_0)S_T \Delta T} = \left( \frac{1}{2} - \frac{zL}{H} \right) + \Sigma
\]

Substituting all the dimensional variables for the expressions as function of the dimensionless ones on equations (9), (10) and (14) and simplifying them we arrive at the non dimensional time-evolution equations for our problem:

\[
\frac{\partial \Theta}{\partial t} + \vec{v} \cdot \nabla \Theta = \frac{L}{H} v_z + \nabla^2 \Theta
\]  
(16)

\[
\frac{\partial \Sigma}{\partial t} + \vec{v} \cdot \nabla \Sigma = \frac{L}{H} v_z + L e \left[ -\nabla^2 \Theta + \nabla^2 \Sigma \right]
\]  
(17)

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \pi + \sigma \nabla^2 \vec{v}
\]
\[
+ Ra e \left[ \left( \frac{1}{2} - \frac{zL}{H} \right)(1 + S) + \Theta + S \Sigma \right] \hat{k}
\]
\[
+ 2 \Omega_t \vec{v} \times \hat{k}
\]
\[
- Ra e \Omega_t^2 Ce \left[ \left( \frac{1}{2} - \frac{zL}{H} \right)(1 + S) + \Theta + S \Sigma \right] \hat{r}
\]  
(18)

where we have defined defined the following parameters:

- Separation ratio: \( S = \frac{\beta}{\alpha} c_0(1 - c_0)S_T \)
- Prandtl number: \( \sigma = \frac{\nu}{\kappa} \)
- Rayleigh: \( Ra = \frac{\alpha \Delta T g L^3}{\kappa \nu} \)
- Lewis number: \( Le = \frac{D}{\kappa} \)
- Adimensionalized rotation: \( \Omega_t = \Omega D \frac{L^2}{\kappa} \)
- Centrifugal number: \( Ce = \frac{\kappa^2}{g L^3} \)
- Pressure term: \( \pi = \frac{\nu' L^2}{\rho_0 \kappa} \)

Other parameters could have been chosen leading to the same results. One of the main reasons we chose to use \( Ce \) and \( \Omega_t \) instead of Coriolis (= \( \frac{\Omega L^2}{\nu} \))
and Froude (= $\frac{\Omega^2 R}{g}$) used at [4], is that we are not able to change Coriolis and Froude independly, since they both change when we vary $\Omega$.

Eventually, we do a final change of variables to simplify a bit more our equations. This change of variables is mainly applied to simplify the contour conditions. One of our contour condition is that we have impermeable walls, disabling the concentration flux through the contours. So starting off this assumption, using (5) and adimensionalising using $\Theta$ and $\Sigma$:

$$
\vec{J}_c \cdot \hat{n} = 0 \Rightarrow c_0(1 - c_0)S_T \partial_n T'_D + \partial_n \Sigma' = 0 \Rightarrow \partial_n (\Theta - \Sigma) = \partial_n \eta = 0 \quad (19)
$$

where we defined the variable change we were looking for $\eta = \Sigma - \Theta$. Finally, after applying this and doing some manipulation, we obtain the definitive non dimensional time-evolution equations for our problem, which we will try to solve numerically in the rest of this work. Equation (21) results from subtracting equation (16) to (17), while (22) results from applying the change of variable to (18).

$$
\begin{align*}
\frac{\partial \Theta}{\partial t} + \vec{v} \nabla \Theta &= \frac{L}{H} v_z + \nabla^2 \Theta \\
\frac{\partial \eta}{\partial t} + \vec{v} \cdot \nabla \eta &= Le \nabla^2 \eta - \nabla^2 \Theta
\end{align*}
$$

$$
\begin{align*}
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} &= - \nabla \pi + \sigma \nabla^2 \vec{v} + Ra \sigma [(\frac{1}{2} - \frac{z L}{H})(1 + S) + \Theta (S + 1) + S \eta] \hat{k} \\
&\quad + 2\Omega_t \vec{v} \times \hat{k} \\
&\quad - Ra \sigma \Omega^2 Ce [(\frac{1}{2} - \frac{z L}{H})(1 + S) + \Theta (S + 1) + S \eta] r \hat{r}
\end{align*}
$$

### 2.2.4 Boundary conditions

In this section we will talk about the boundary conditions we applied to our problem. We had to set different conditions for the velocity, the temperature and the concentration.

- We applied the most common boundary for velocity, the no-slip boundary, which states that the fluid will have zero velocity relative to the boundary, in our case (cylindrical system of reference) $\vec{v} = 0$ at the walls.

- The appropriate boundary conditions for the concentration are impervious walls. Otherwise, the gradient of concentration between the walls due to ordinary diffusion (Fick’s Law) would mask the effect of thermal diffusion. So we applied no-flux condition and we deduced its mathematical expression at (19): $\partial_n \eta = 0$.
• For temperature we will use two different boundary conditions. We will use a fixed temperature, perfectly conducting boundary at the top and bottom plates, and perfectly insulating boundary on the lateral wall.

Putting all together we have the following boundary condition for the top and bottom plates (23) and the boundary conditions on the lateral wall (24):

\[
\vec{v} = \Theta = \partial_z \eta = 0 \quad \text{on} \quad z = 0, 1
\]
\[
\vec{v} = \partial_r \Theta = \partial_z \eta = 0 \quad \text{on} \quad r = \Gamma
\]

2.2.5 Conductive state (Ce=0)

As we commented in the introduction, for the ranks of parameters we are using on our work we are able to neglect the centrifugal buoyancy force. In Section 4.1 there are some calculations which prove that for our ranks of parameters Ce is very small.

What we are doing is to neglect the term in equation (22) containing Ce, which represents the centrifugal force, provided that the product \( \Omega^2 Ce \) is \( \ll 1 \). An important consequence of doing this is that in this case the simplified equations admit as a solution the conductive state \( \Theta, \eta, v_r, v_\theta, v_z = 0 \). Notice that in equation (22) the term \(-Ra\sigma \Omega^2 Ce[(\frac{1}{2} - \frac{zL}{H})(1 + S)]r \hat{r}\) cannot be added into the gradient pressure term while \(-Ra\sigma[(\frac{1}{2} - \frac{zL}{H})(1 + S)]k \) term can (we did not write it inside the pressure term so that it is easy to visualise the differences in the contributions of each of the terms). The temperature and concentration fields for the basic conductive state are:

\[
T_c = T_0 + \Delta T \left( \frac{1}{2} - \frac{zD}{H} \right)
\]
\[
C_c = C_0 + (-c_0(1 - c_0)S_T \Delta T) \left( \frac{1}{2} - \frac{zD}{H} \right)
\]

Notice that if \( S_T < 0 \) the conductive state has higher temperature and higher concentration on the bottom plate \( (z = 0) \), which will give rise a competition between a destabilizing thermal effect and a stabilizing concentration effect.
3 Simulation methods

In this section we will discuss the simulation methods used, the reasons of using them instead of other options and, in a detailed way, how they work. We decided to use a code based on spectral methods formulated on primitive variables[9] developed by Fluid Dynamics group (UPC). This code is specifically created to solve the incompressible Navier-Stokes equations in cylinder type boundings. Basically the code is based on a projection method formulated in primitive variables to maintain the incompressibility constraint, with a second-order semi-implicit scheme for the time integration, and a pseudospectral approximation for the space variables. We chose this method because it presents some important advantages in comparison with other methods:

- Simplicity of implementation
- Better resolution and accuracy using spectral methods than using finite difference/volume methods.
- Absence of clustering on the origin, letting us to do large-scale simulations without restrictions.

Nevertheless, this doesn’t mean that this is the only and the best method we can use. There are other methods that use totally different points of view. For instance, apart from Primitive Variables methods, there are the streamfunction-vorticity formulations (in 2D) and the velocity decomposition into toroidal and poloidal potentials (in 3D) that use scalar potentials rather than the velocity field itself. This may have some advantages such as having the divergence-free condition satisfied by construction or the pressure not appearing on the equations. However, this brings to a high order PDE with coupled boundary conditions, more difficult to solve.

3.1 The temporal discretization and the projection scheme

The code uses a second order stiffy-stable scheme for the time discretization. So writing the equations we obtained plus divergence free condition, and using the scheme proposed we have:

\[
\nabla \cdot \vec{v}^{n+1} = 0
\]

\[
\frac{3\vec{v}^{n+1} - 4\vec{v}^n + \vec{v}^{n-1}}{2\Delta t} = -2NL(\vec{v}^n) + NL(\vec{v}^{n-1}) - \nabla \pi^{n+1} + \sigma \nabla^2 \vec{v}^{n+1} + Ra\sigma \Pi^{n+1}\hat{k} + \vec{F}_1^{n+1} + 2\vec{F}_2^n - \vec{F}_2^{n-1}
\]
\[
\frac{3\Theta^{n+1} - 4\Theta^n + \Theta^{n-1}}{2\Delta t} = -2NL(\vec{v}^n, \Theta^n) + NL(\vec{v}^{n-1}, \Theta^{n-1}) \\
+ \frac{L}{H}(2v_z^n - v_z^{n-1}) + \nabla^2 \Theta^{n+1}
\]
(29)

\[
\frac{3\eta^{n+1} - 4\eta^n + \eta^{n-1}}{2\Delta t} = -2NL(\vec{v}^n, \eta^n) + NL(\vec{v}^{n-1}, \eta^{n-1}) \\
+ Le \nabla^2 \eta^{n+1} - \nabla^2 \Theta^{n+1}
\]
(30)

where NL refers to the non-linear terms of the equations, \( \Pi \) is an abbreviation for \( \Pi = \left( \frac{1}{2} - \frac{1}{2H} \right)(1 + S) + \Theta(S + 1) + S\eta \). We also have split \( \vec{F} \) into two parts: \( \vec{F}_1 = -Ra\sigma \Omega^2 Ce\Pi \hat{k} \) which is independent of the velocity field and treated implicitly, while \( \vec{F}_2 = 2\Omega_t \vec{v} \times \hat{k} \) depends on the velocity and is treated explicitly.

From this and following certain steps we will be able to find all the variables at \( n+1 \) temporal step. Basically, the projection scheme consists of a predictor for the pressure derived from the Navier-Stokes equations with the Neumann boundary condition. Using this pressure, a predictor created from the momentum equation will be used to find an intermediate velocity field. Finally, a projection step with an explicit evaluation of the final divergence-free velocity field. While following this projection scheme we will need to solve sequences of two-dimensional Helmholtz and Poisson equations, which will be faced by a diagonalization technique.

Now we will try to explain with the maximum possible detail how the projection steps used in this work is applied:

1. \( \Theta^{n+1} \) is obtained from the following Helmholtz equation derived from (25).

\[
(\nabla^2 - \frac{3}{2\Delta t})\Theta^{n+1} = 2NL(\vec{v}^n, \Theta^n) - NL(\vec{v}^{n-1}, \Theta^{n-1}) \\
- \frac{L}{H}(2v_z^n - v_z^{n-1}) - \frac{4\Theta^n - \Theta^{n-1}}{2\Delta t}
\]
(31)

2. Using the previous result and (26), we search for \( \eta^{n+1} \).

\[
(Le\nabla^2 - \frac{3}{2\Delta t})\eta^{n+1} = 2NL(\vec{v}^n, \eta^n) - NL(\vec{v}^{n-1}, \eta^{n-1}) \\
- \frac{4\eta^n - \eta^{n-1}}{2\Delta t} + \nabla^2 \Theta^{n+1}
\]
(32)

3. Now a preliminary pressure field(\( \bar{\pi} \)) is obtained from (24).

\[
\nabla^2 \bar{\pi}^{n+1} = \nabla \cdot \left[ -2NL(\vec{v}^n) + NL(\vec{v}^{n-1}) + Ra\sigma \Pi^{n+1} \hat{k} + \vec{F}_1^{n+1} + 2\vec{F}_2^n - \vec{F}_2^{n-1} \right]
\]
(33)
This equation results from applying a divergence operator at each side of (24). A consistent boundary condition is required to solve this elliptic equation (29), because no natural condition exists. A straight consistent Neumann condition can be used by looking for the normal component of the pressure term at the walls. Applying this on (24) and decomposing the Laplacian operator into an irrotational part approximated by an implicit scheme (which cancels) and a solenoidal part treated explicitly, you obtain the necessary boundary condition.

4. Then a predictor of the velocity field \( \vec{v}^* = (u^*, v^*, w^*) \) is calculated from de Navier-Stokes equation by using the preliminary pressure field \( \bar{\pi}_{n+1} \). 

\[
\left( \sigma \nabla^2 - \frac{3}{2\Delta t} \right) \vec{v}^* = \nabla \pi_{n+1} + 2NL(\vec{v}^*) - NL(\vec{v}_{n-1}) - Ra \sigma \Pi_{n+1} \hat{k} - \vec{F}_n^{+1} - 2\vec{F}_2^{n-1} + \frac{4\vec{v}^n - \vec{v}^{n-1}}{2\Delta t} 
\]

(34)

Since the velocity components \((u, v)\) are coupled in the linear viscous term (see laplacian in cylindrical coordinates) it is convenient to introduce a new set of complex variables. This new variables \((u^*_+, u^*_-, w)\) give rise to decoupled equations allowing us to solve (30) easily.

\[
u^*_+ = u^* + iv^* \\
u^*_- = u^* - iv^*
\]

5. It should be noted that this predicted velocity does not meet the divergence free condition. To finish we do the correction step, in which we subtract the Navier-Stokes equation with \( \vec{v}_{n+1} \) and \( \pi_{n+1} \) to the same equation but with the preliminary pressure field \( \bar{\pi} \) and the prediction of the velocity field \( \vec{v}^* \), obtaining:

\[
\frac{3(\vec{v}^{n+1} - \vec{v}^*)}{2\Delta t} = -\nabla(\pi_{n+1} - \bar{\pi}) \implies \vec{v}^{n+1} - \vec{v}^* = -\nabla \phi 
\]

(35)

where \( \phi = 2\Delta t(\pi_{n+1} - \bar{\pi})/3 \). Now we apply a divergence operator at both sides of (31) and considering \( \nabla \cdot \vec{v}^{n+1} = 0 \) we arrive to a system formed by a Poisson equation for variable \( \phi \), with Neumann boundary conditions \( \frac{\partial \phi}{\partial n} = 0 \).

\[
\nabla^2 \phi = \nabla \cdot \vec{v}^*
\]

(36)

Once we solve this and the value of \( \phi \) is known we can finally obtain the corrected pressure and velocity fields.

\[
\pi_{n+1} = \bar{\pi} + \frac{3\phi}{2\Delta t} \\
\vec{v}^{n+1} = \vec{v}^* - \nabla \phi 
\]

(37)

(38)
3.2 Spatial discretization

The components of the velocity field and scalar fields $T$ and $\eta$ are functions of the cylindrical coordinates $(r, \theta, z)$. We will expand the azimuthal dependence using a Fourier expansion while for the $r$ and $z$ we will use a Chebyshev collocation method.

We will make two different types of Fourier expansions depending on the components we are expanding so that only purely real or imaginary parts of the unknowns appear in the equations. This can be seen for example when solving the divergence free equations for the Fourier mode $k$.

- For the components $u$, $w$, $\Theta$ and $\eta$ we will use the next expansion type (we will use $u$ for this example):
  \[
  u(r, \theta, z) = \sum_{k=-n_\theta/2}^{n_\theta/2-1} F_k(r, z)e^{ik\theta}
  \]  
  where the complex functions $F_k(r, z)$ satisfy:
  \[
  F_0(r, z) = f_0(r, z) \\
  F_{-n_\theta/2}(r, z) = f_{n_\theta-1} \\
  F_k(r, z) = f_{2k-1}(r, z) + if_{2k}(r, z) \quad \text{for} \quad k = 1 : n_\theta/2 - 1 \\
  F_{-k}(r, z) = \bar{F}_k(r, z) \quad \text{for} \quad k = 1 : n_\theta/2 - 1
  \]

  It’s easy to see that using this premises we obtain $n_\theta$ independent real-valued functions $f_l$ so that we don’t need to compute $n_\theta$ complex numbers. We just need to compute the real-valued functions $f_l$ and construct our coefficients as seen at the two previous final equations.

- For the azimuthal component $v$ we will use another type of expansion:
  \[
  v(r, \theta, z) = i \sum_{k=-n_\theta/2}^{n_\theta/2-1} F_k(r, z)e^{ik\theta}
  \]  
  where the complex functions $F_k(r, z)$ satisfy:
  \[
  F_0(r, z) = if_0(r, z) \\
  F_{-n_\theta/2}(r, z) = if_{n_\theta-1} \\
  F_k(r, z) = f_{2k-1}(r, z) + if_{2k}(r, z) \quad \text{for} \quad k = 1 : n_\theta/2 - 1 \\
  F_{-k}(r, z) = -\bar{F}_k(r, z) \quad \text{for} \quad k = 1 : n_\theta/2 - 1
  \]

  And we will be able to construct our coefficients analogously, as we did for the other case.
On the other hand, we will treat the radial and height dependence using the Chebyshev collocation method. The principle of these methods is that the solution is represented by a finite Chebyshev series with unknown coefficients; this expression is substituted into the differential equation and the coefficients are determined so that the differential equation is satisfied at certain points within the range under consideration. We chose an appropriated discretization thought so that the appropriated parity of the Fourier coefficients is forced and clustering at the origin avoided (besides no point coincides with \( r=0 \)).

So we define them the following way:

\[
\begin{align*}
  r_j &= \cos \left( \frac{\pi j}{2n_r + 1} \right) \quad \text{with} \quad j = 0 : n_r \\
  z_j &= \cos \left( \frac{\pi j}{n_z} \right) \quad \text{with} \quad j = 0 : n_z
\end{align*}
\]

Where \( n_r \) and \( n_z \) are the \( n-1 \) collocation points we choose. Notice that \( r_j \in [1, 0) \) and \( z_j \in [1, -1] \).

Figure 2: Grid generated with \( n_r = 20 \) and \( n_z = 20 \). Notice that there isn’t any point at \( r=0 \), and how there are more density of points on the vicinity of the walls avoiding the clustering at the problematic origin.

Chebyshev quadrature is a Gaussian quadrature which is a form of numerical integration that involves the approximation of the integral by a weighted sum at specific points. In particular this quadrature uses the Chebyshev polynomials.
So the approximation of a 2D function using this method is of the form:

\[
f_{NM}(r,z,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} f_{nm}(t) T_n(r) T_m(z) \quad (41)
\]

where \( t \) makes reference to time, \( T_n \) and \( T_m \) are the \( n \)-order Chebyshev polynomials and \( f_{nm}(t) \), which can be expressed as a matrix, are the coefficients.

So basically we will need to solve the Helmholtz and Poisson equations (as we saw on the previous section). And more specifically, we will need to solve their real and imaginary parts of its every Fourier mode \( k \). Each one of this partial differential equations will be solved by using the Chebyshev collocation. This method consists in imposing that the differential equations are satisfied identically in a sufficient number of collocation points inside the domain. Hence the unknowns of our problem will be the values of the variable \( f_{NM}(r,z,t) \) inside the collocations points. It is useful to treat this kind of problems using matrices and there exist methods to express the differential operators depending on the collocation points known as Chebyshev derivative matrices, extensively explained in the Appendix of Oriol Batiste’s PhD dissertation [10].

It will be useful to notice that \((r, \theta + \pi, z)\) is the exact same point than \((-r, \theta, z)\). So any scalar function, like the temperature, will satisfy \( F(r, \theta + \pi, z) = F(-r, \theta, z) \). For the vector fields, like the velocity one, we will need to consider that radial and azimuthal basis are reversed: \( \hat{r}(r, \theta + \pi, z) = -\hat{r}(-r, \theta, z) \) and \( \hat{\theta}(r, \theta + \pi, z) = -\hat{\theta}(-r, \theta, z) \). Knowing this, we can expand the previous equalities using the Fourier expansion previously explained. Working a little bit the expressions the following parities for the complex Fourier Coefficients are obtained:

- \( u \) and \( v \) coefficients have to fulfill:

\[
F_k(-r, z) = (-1)^{k+1} F_k(r, z) \quad (42)
\]

- \( \Theta, \eta \) and \( w \) coefficients have to fulfill:

\[
F_k(-r, z) = (-1)^k F_k(r, z) \quad (43)
\]

Taking advantage of these parities the radial derivatives that in the usual way would use a \((2n_r + 2) \times (2n_r + 2)\) Chebyshev differentiation matrix can be split into two smaller different Chebyshev differentiation matrices of dimensions \((n_r + 1) \times (n_r + 1)\): one for odd parity functions (even Fourier coefficients of \( u \) and \( v \) and odd Fourier coefficients of \( \Theta, \eta \) and \( w \)), and another for functions of even parity (odd Fourier coefficients of \( u \) and \( v \) and even Fourier coefficients of \( \Theta, \eta \) and \( w \)).
3.3 Linear Stability Analysis

In this section we will explain how the Linear Stability Analysis code works and its theoretical background. The linear stability of the steady states has been studied computing the leading eigenvalues of the Jacobian by using Arnoldi’s method, in the implementation found in the ARPACK routines. Code developed by Fluid Dynamics Group (UPC) taking advantage of the ARPACK subroutines which are designed to solve large scale eigenvalue problems. As you will see later, in this case we don’t have an explicit Jacobian so we needed to use a clever method. This, uses the linearized time evolution code to find its action, making us capable to solve it by Arnoldi’s Method.

3.3.1 First order semi-implicit scheme

Before entering into the complexities of the problem, we will present a well-known scheme which will be used afterwards, the first order semi-implicit scheme. We have:

\[
\partial_t X = L(X) + N(X) \tag{44}
\]

This assumes that the time evolution of the field \( X \) can be split in a linear operator \( L(X) \) and a quadratic nonlinear operator \( N(X) \). Now we apply the simple Euler’s algorithm treating the linear part of the equation implicitly and the non-linear part explicitly\[8\].

\[
\frac{X(t + \Delta t) - X(t)}{\Delta t} = L(X(t + \Delta t)) + N(X(t))
\]

\[
= (I - \Delta tL)^{-1}(L(X(t)) + N(X(t))) \tag{45}
\]

where \( I \) is the identity and the way to obtain the last equality consists of adding and subtracting the term \( L(X(t)) \) followed by an easy manipulation.

3.3.2 Linear Stability Analysis

From here we will focus on our problem at hand. We decompose our solution into a known basic state plus a small perturbation:

\[
X = X_B + X^*(\vec{r}, t) \tag{46}
\]

So we apply to our problem \( (\partial_t X = L(X) + N(X)) \) the following transformations:

- \( L X_B + N(X_B, X_B) = 0 \) cause it is a stationary solution.
Knowing that our basic solution does not depend on time:

\[
\frac{\partial X}{\partial t} = \frac{\partial (X_B + X^*)}{\partial t} = \frac{\partial X^*}{\partial t}
\]  

(47)

- \(N(X)\) being quadratic lets us write:

\[
N(X) = N(X_B, X_B) + N(X^*, X^*) + N(X^*, X_B) + N(X_B, X^*)
\]  

(48)

Applying a linearization about \(X_B\) we obtain:

\[
N(X) \simeq N(X_B, X_B) + N(X^*, X_B) + N(X_B, X^*)
\]

\[
= N(X_B, X_B) + N_{X_B}(X^*)
\]

(49)

were we defined \(N_{X_B}(X^*) = N(X_B, X^*) + N(X^*, X_B)\). The term \(N(X^*, X^*)\) can be neglected because of small perturbation considered \((X^* \ll 1)\).

Having into account these considerations we can write the following form of the evolution equation:

\[
\frac{\partial X^*}{\partial t} = (L + N_{X_B})X^*
\]

(50)

We can decompose our perturbation solution \((X^*(\vec{r}, t))\) into two functions, separating variables; having a part depending on \(t\) and an other part depending on \(\vec{r}\) like: \(X^*(\vec{r}, t) = \tilde{X}(\vec{r})e^{\lambda t}\). And applying this decomposition into equation (50) making the temporal derivative of a product and simplifying we arrive to the typical form of a standard eigenvalue problem:

\[
\lambda \tilde{X}(\vec{r}) = (L + N_{X_B})\tilde{X}(\vec{r}) = J_{X_B}\tilde{X}(\vec{r})
\]

(51)

where \(\lambda\) is eigenvalue, \(\tilde{X}(\vec{r})\) is its corresponding eigenvector and \((L + N_{X_B})\) is the Jacobian.

If we knew the form of the Jacobian matrix we would be able to find each eigenvalue and eigenvector by looking for Jacobian’s spectrum. And we would denote if we had an stable solution (the real part of all eigenvalues are negative) or an unstable one (there is at least one eigenvalue with a positive real part). But as we have already state, we do not know the Jacobian explicitly, so what do we do next? We take advantage of another iterative method developed called Arnoldi iteration. Arnoldi finds an approximation to the leading eigenvalues and its eigenvectors by constructing an orthonormal basis of the Krylov subspace, which makes it particularly useful when dealing with large sparse matrices. This method will not be explained in this work, for further information check Esteban Meca’s Thesis [8]. Last but not least, the use of Arnoldi’s method avoids the use of matrices, only requesting the action of the matrix on a given vector. We
don’t explicitly know the matrix $J_{X_B}$ but we will be able to solve the problem by using a mathematical trick, finding the action of a function containing this Jacobian, as you can see below.

From (50) and taking advantage of the development made in the First Order semi-implicit scheme section, more precisely in (45), we can write:

$$\frac{X^{*(n+1)} - X^{*(n)}}{\Delta t} = (I - \Delta tL)^{-1}(L + N_{X_B})X^{*(n)}$$

(52)

where $\Delta t$ is the time step corresponding to the linearized problem.

Knowing that $(L + N_{X_B})$ is lineal is clear that the time-depending part of $X^*(r,t)$ is cancelled at each side of the equality. Obtaining:

$$\frac{\tilde{X}^{(n+1)} - \tilde{X}^{(n)}}{\Delta t} = (I - \Delta tL)^{-1}(L + N_{X_B})\tilde{X}^{(n)}$$

(53)

Isolating and supposing $\Delta t << 1$:

$$\tilde{X}^{(n+1)} = \tilde{X}^{(n)} + \Delta t(L + N_{X_B})\tilde{X}^{(n)} = \tilde{X}^{(n)} + \Delta tJ\tilde{X}^{(n)} \simeq e^{\Delta tJ}\tilde{X}^{(n)}$$

(54)

where the Jacobian ($J = L + N_{X_B}$) and where we applied the Taylor approximation ($e^x \simeq (1 + x)$ where $x \ll 1$). Finally, we can assert that if we use a sufficiently small time step for integrating our linearized problem is equivalent to the action of $e^{\Delta tJ}$:

$$\tilde{X}^{(n+1)} = e^{\Delta tJ}\tilde{X}^{(n)}$$

(55)

So we did not found the action of the Jacobian, but we found the action of an exponential function with the Jacobian as a variable. And lucky us, we know this action! We can calculate it by applying our linearized temporal evolution code. Which is the same explained in this work but using linearized equations and using a first order scheme instead of a second order stiffy-stable scheme for time discretizion. Therefore, we are able to find the spectrum of $e^{\Delta tJ}$ thanks to Arnoldi’s iteration method. More specifically, we have used an ARPACK subroutines package. Once the eigenvalue problem is solved we obtain the leading eigenvalues and its corresponding eigenvectors. Notice that this eigenvalues may be complex implying its eigenvector to be complex too (due to $e^{\Delta tJ} \in \mathbb{R}$).

The eigenvalues that are computed correspond to the action of the exponential of the Jacobian, but we are interested in the ones corresponding to the Jacobian of our problem $(J = L + N_{X_B})$. Consequently we apply the following calculations, easy to deduct, to obtain the desired eigenvalues. Be aware that
the eigenvectors of the exponential are exactly the same as the ones in the Jacobian. If \( \lambda_i = a + bi \) is eigenvalue of \( e^{\Delta t J} \) and \( \lambda_i = c + di \) is eigenvalue of \( J \), then:

\[
\begin{align*}
  c &= \frac{\ln(a^2 + b^2)}{2\Delta t} \\
  d &= \frac{1}{\Delta t} \arctan\left(\frac{b}{a}\right)
\end{align*}
\]

Note that we have five evolving fields in our problem (three components of velocity, \( \Theta \) and \( \eta \) and all this procedures have to be performed for each of them at each iteration. So in reality:

\[
X^* = \begin{bmatrix}
  v^*_r(\vec{r}, t) \\
  v^*_\theta(\vec{r}, t) \\
  v^*_z(\vec{r}, t) \\
  \Theta^*(\vec{r}, t) \\
  \eta^*(\vec{r}, t)
\end{bmatrix}
\]

(57)

### 3.4 Prograde and retrograde identification

In this section we will try to explain the method used to determine the direction of rotation of the perturbation appearing on the onset of the instability. It is a very powerful tool because it lets us know what will be its rotation direction without the need of doing a time evolution simulation.

Before explaining the theoretic fundamentals that lies behind this method we should talk a little of how our code returns the results. It doesn’t returns a complex eigenfunction (\( Z \)) for each complex eigenvalue (\( \lambda \)) as we could expect. It returns real data with which we construct the solution:

\[
\begin{align*}
  \lambda &= \lambda_R \pm iw \\
  Z &= Z_R + iZ_I
\end{align*}
\]

(58)

The ARPACK subroutines return \( Z_R \) and \( Z_I \) so that each conjugated eigenvalue has associated the real or the imaginary part of the total eigenvector, depending on the sign of the eigenvalue imaginary part (we assumed \( w > 0 \)). Just as can be seen below:

\[
\begin{align*}
  \lambda_R + iw &\implies Z_R \\
  \lambda_R - iw &\implies Z_I
\end{align*}
\]
Once we know this we are ready to construct the real eigenvector. Since it consists of a physics magnitude, it must be a real value. So we can construct the real solution by summing the conjugated pair eigenvalues multiplied by its eigenvector:

\[ X = (Z_R + iZ_I)e^{(\lambda_R + iw)t} + (Z_R - iZ_I)e^{(\lambda_R - iw)t} \]  

(60)

As we are on the onset of the instability (on the bifurcation), we have \( \lambda_R = 0 \), so:

\[ X = (Z_R + iZ_I)e^{iwt} + (Z_R - iZ_I)e^{-iwt} \]

\[ = 2Z_R \cos(wt) - 2Z_I \sin(wt) \]  

(61)

While solving some cases using this code we have noticed, while observing \( Z_R \) and \( Z_I \) in a particular point \((r_i, z_j)\), that there were only two types of solutions. Those solutions are of the following form (notice that the expressions are written using the form we chose before (39-40), where the complex Fourier coefficients \((F_m)\) are decomposed into \(n_\theta\) independent real valued functions \((f_l)\)).

A.

\[ Z_R(r_i, z_j) = (f_{2m-1}(r_i, z_j) + if_{2m}(r_i, z_j))e^{im\theta} + (f_{2m-1}(r_i, z_j) - if_{2m}(r_i, z_j))e^{-im\theta} \]

\[ Z_I(r_i, z_j) = (f_{2m}(r_i, z_j) - if_{2m-1}(r_i, z_j))e^{im\theta} + (f_{2m}(r_i, z_j) + if_{2m-1}(r_i, z_j))e^{-im\theta} \]  

(62)

B.

\[ Z_R(r_i, z_j) = (f_{2m-1}(r_i, z_j) + if_{2m}(r_i, z_j))e^{im\theta} + (f_{2m-1}(r_i, z_j) - if_{2m}(r_i, z_j))e^{-im\theta} \]

\[ Z_I(r_i, z_j) = (-f_{2m}(r_i, z_j) + if_{2m-1}(r_i, z_j))e^{im\theta} + (-f_{2m}(r_i, z_j) - if_{2m-1}(r_i, z_j))e^{-im\theta} \]  

(63)

From here, to simplify the expressions we will write \( a = f_{2m-1}(r_i, z_j) \) and \( b = f_{2m}(r_i, z_j) \). Now working on the case A, applying trigonometric identities and this change of notation we achieve:

\[ Z_R = 2a \cos(m\theta) - 2b \sin(m\theta) \]

\[ Z_I = 2b \cos(m\theta) + 2a \sin(m\theta) \]  

(64)

If we substitute this expressions into (61) and manipulate a little bit we arrive to:
\[ X = 2(2a \cos(m\theta) - 2b \sin(m\theta)) \cos(\omega t) - 2(2b \cos(m\theta) + 2a \sin(m\theta)) \sin(\omega t) \]
\[ = 4a \cos(m\theta) \cos(\omega t) - 4a \sin(m\theta) \sin(\omega t) \]
\[ - 4b \sin(m\theta) \cos(\omega t) - 4b \cos(m\theta) \sin(\omega t) \]
\[ = 4a \cos(m\theta + \omega t) - 4b \sin(m\theta + \omega t) \]  
(65)

This last result is the important one. If you look carefully you will notice this is a function that describes a rotating wave: \( f(m\theta + \omega t) \). Furthermore it is describing a rotating wave that is travelling clockwise, ergo it is a retrograde wave.

Now if we do the analogous case for B, we arrive to a similar result:
\[ X = 4a \cos(m\theta - \omega t) - 4b \sin(m\theta - \omega t) \]  
(66)

in this second case, we also have a rotating wave, but of the form: \( f(m\theta - \omega t) \). So we are in front of a rotating wave that is travelling anti-clockwise, hence a prograde wave.

So basically this method consists in finding out on which case we are for each conjugated pair of eigenvalues. Requesting the program to write a and b for \( Z_R \) and \( Z_I \), and looking at its signs is direct way to denote the prograde/retrograde character. In this particular work this method was automated, giving a positive sign to \( \omega \) for retrograde case and a negative \( \omega \) for prograde case. The fact that we have the real sign of \( \omega \) gives us the ability to identify more easily if there are changes on the modes (when increasing the rotation rate for example) by analysing the frequencies discontinuities.

Notice that the frequencies \( (\omega) \) on the graphic refer to the imaginary part of the dominating eigenvalue and not the angular rotation of the solution. To know the real angular velocity of the solutions \( (\omega_{RW}) \) on the cylinder system of reference you have to divide this frequency by its mode number \( (m) \): \( \omega_{RW} = \frac{\omega}{m} \).

### 3.4.1 Drawing mode eigenfunctions

From \( Z_R \) and \( Z_I \) found for each discretized point of the cylinder \( (r_i, z_j) \) we are able to make a plot to observe the shape of the perturbation appearing for each mode in all parts of the cylinder. We may use expression (60) to construct our real eigenvectors for each point and for a particular time. But as we found the instability on its bifurcation we have enough by finding the solution at \( t = 0 \) because we know they are rotating waves rotating at \( \omega \). So if we want to plot the eigenfunction appearing at \( t = 0 \), we can start from (65) or (66) obtaining the same expression for the two cases:
\[ X = 4a \cos(m\theta) - 4b \sin(m\theta) = 2Z_R \]  
(67)
So by drawing the corresponding value of $Z_R$ at each point of the cylinder we directly obtain a proportional solution of that mode perturbation at $t = 0$. Remember that $X$ is a vector containing 5 different fields. So we are able to draw all of them by using its corresponding $Z_R$.

There is a special case that needs to be treated in a different way. It’s the mode 0 case, which is not a rotating wave, it consist of a pulsation periodic on time. We will start from the retrograde case, despite this reasoning can be done analogously from the prograde case and the results only varies in a time translation. So by making $m = 0$ at (65) we obtain:

$$X(t) = 4acos(\frac{2\pi}{T}t) - 4bsin(\frac{2\pi}{T}t)$$ (68)

where $w$ can be written in function of the period as $w = \frac{2\pi}{T}$. So we obtain:

$$X(t = 0) = 4a$$
$$X(t = T/4) = -4b$$
$$X(t = T/2) = -4a$$
$$X(t = 3T/4) = 4b$$ (69)

being able to draw the eigenfunction of the solution in different moments of its period by simply use the real valued functions ($f_I$) found when computing the Fourier coefficients. It is directly seen that if we apply $m = 0$ at (64) we obtain $Z_R = 2a$ and $Z_I = 2b$. So we can again draw the eigenfunctions by using directly $Z_R$ and $Z_I$ with the appropriate sign.
4 Results

4.1 Selection of parameters

Before presenting and commenting our results, we will discuss the reason of simulating in the ranks we did. As explained, most of the parameters used are chosen to model a water-ethanol mixture. We used the following parameters:

- Separation ratio: $S = -0,1$
- Prandtl number: $\sigma = 7$
- Lewis number: $Le = 0,01$
- Adimensionalized rotation: $\Omega_t = 1 - 1000$

We chose this rank for $\Omega_t$ because in this rank we can still consider the term containing $Ce$ on the momentum equation (22) as 0. In the term we are talking about, there is a $\Omega_t$ squared multiplying the $Ce$, so that there will be a rotation rate for which it would be wrong to neglect this term. Using typical data of a fluid experiment of our characteristics ($\kappa = 8,636 \cdot 10^{-8}$ and $l = 1,3 \cdot 10^{-3}$) we would have a $Ce$:

$$Ce = \frac{\kappa^2}{g l^3} = \frac{(8,636 \cdot 10^{-8})^2}{9,81(1,3 \cdot 10^{-3})^3} = 3,46 \cdot 10^{-7}$$

So when we are at $\Omega_t = 1000$ we have that $\Omega_t^2 Ce \approx 0,1$ and this term begins to have an important weight in the equation; in consequence, we decided to stop the simulations there. From here we should take into account the $Ce$ term and solve a more difficult problem that has a non-trivial basic state (the basic solution consists on a large-scale circulation that needs to be calculated).

To have an idea of what rotation ranks we are talking about in the real problem, with dimensions, we calculated them in revolutions per minute (rpm) using the before mentioned $l$ and $\kappa$ values.

$$\Omega_D = \Omega_t \frac{\kappa}{l^2} \frac{60}{2\pi} \text{ [rpm]}$$

If our adimensional rotating rank is $\Omega_t = 1 - 1000$, we can conclude that the real rotation velocity rank of our experiment will be $\Omega_D = 0,49 - 490 \text{ rpm}$, a moderate rotation.

Apart from that, we had to decide the amount of discretized points used for each simulation in order to obtain sufficiently good results while the simulation time remains into acceptable ranks. Remember that we defined the following...
parameters: number of azimuthal points ($nt$), number of radial points ($nr$) and number of vertical points ($nz$). As we worked using the linear stability analysis for each mode we analysed, we used a $nt$ bigger than two times the mode analysed so that the Fourier expansion could be expanded to that mode. So we had fewer azimuthal points when analysing smaller modes, producing faster simulations for them. On the other hand we decided to do a convergence study to decide which $nr$ and $nz$ we would use to generate the stability curves we wanted to calculate. To do that we calculated the Rayleigh for which $\lambda_R = 0$ (the first eigenvalue crosses 0) on a specific $\Omega_t$ for different $nr$ and $nz$.

![Figure 3: Convergence study for the case $\Gamma = 1$. Relative error calculated from Rayleigh results using as reference solutions the cases $nr=128$ $nz=128$.](image-url)
For the case $\Gamma = 1$ one can observe that solutions with bigger $nz$ achieve more accurate solutions, while increasing $nr$ does not seem to improve anything. This happens because the vertical solution structure is complex and requires more points to be defined. So for this case we decided to use $nr = 32$ and $nz = 64$ for all the simulations. So we will assume relative errors around 5% in the worst case scenario.

For $\Gamma = 2.76$ we observe that for Mode 1 and $\Omega_t=490$ case, we found the real solution just by using $nr = 32$ and $nz = 32$. Using these number of points in other cases we saw that the relative errors do not exceed 3%, error small enough to opt for these $nr$ and $nz$. 

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<tr>
<td>32 50</td>
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<td>0.00</td>
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<tr>
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4.2 Aspect Ratio = 1

On this section we will present the results obtained in simulations for the smaller aspect ratio case, $\Gamma = 1$, starting with the stability curves and then showing various figures to understand the spatial structure of its first appearing instabilities. Then we do the same on the next section for the larger aspect ratio case, $\Gamma = 2$.

Notice that from here we will refer to $\Omega_t$ also as ROTK.

4.2.1 Linear Stability Analysis results

In Figure 5 there are the stability curves computed on the ranks discussed above for each mode from 0 to 6. They show for which value of $Ra$ and $\Omega_t$ (adimensional rotation velocity) we have the first eigenvalue crossing the 0 axis on each mode, in other words, where the mode becomes unstable. The modes above number six appear at bigger Rayleigh, hence we did not compute their curves. There are only two modes that predominate in this case: the axisymmetric mode 0 for lower rotation rates and the mode 2 that dominates from $\Omega_t = 146$ to the end of our exploration ranks. For the non-rotating case the critical Rayleigh is $Ra_c = 2578$ and as we increase rotation rate the critical Rayleigh becomes larger.

![Figure 5: Stability curves for each mode (0-6) for $S = -0.1$, $Le = 0.01$, $\sigma = 7$ and $\Gamma = 1$.](image)

The next plot, Figure 6, shows the frequencies for each of the modes (0-6). It shows the imaginary part of the eigenvalue crossing the 0 axis and its sign denotes if it is a prograde (-) or a retrograde (+) solution. It is clear that all the non-axisymmetric solutions are prograde. Around ROTK=500 there is a
frequency jump on mode 0, denoting a change of the dominating eigenvalue. Despite this implies a change on the structure of this mode, it is not studied because at those rotating rates this mode is no longer dominating. Remember that the frequencies ($\omega$) on the graphic refer to the imaginary part of the dominating eigenvalue and not the angular rotation of the solution.

![Frequency vs. ROTK](image)

Figure 6: Frequencies for each mode (0-6) for $S = -0.1$, $Le = 0.01$, $\sigma = 7$ and $\Gamma = 1$.

The following figure (Figure 7a) shows the critical Rayleigh curve for this case. This is, in fact, obtained by making the minimum of Figure 5. This line states the border from which instabilities appear. If a water-ethanol mixture is in Rayleigh and ROTK conditions that are over the line we will certainly have instabilities. Figure 7b shows the corresponding frequency of the first appearing modes. We can see how we have a negative $w$ for mode 2 that seems to stabilise for bigger rotation rates around $w = -2.2$. The $w$ sign of mode 0 is arbitrary.
Once we have looked at the stability analysis we will focus on the geometrical structure of the dominating modes, in this first case mode 0 and mode 2. We also decided to plot the third mode due to its closeness to the onset. Even though it is never the critical solution, it probably will affect to the real solution.
To create the plots we used the method described on section 3.4.1 of this work. We opted to draw the temperature field and the concentration one to be able to visualise a bit their structure. As we are talking of a 3D solution the more comfortable way to show results was to plot some cuts. For each dominating mode we did at least one transversal cut (at the middle of the cylinder) and one vertical cut (made where the pick is maximum) to picture its vertical structure. Moreover, we made an isosurface plot of the two fields to make it easier to understand the 3D structure. It is important to notice that the actual value of the fields in those plots is not defined so that the colorbar is there just to facilitate the readers interpretation and to know where the zero is in the temperature field case. This is said because is known from an eigenvalue problem (51) that the solutions found can be multiplied by any factor and still be solution.

4.2.2 Mode 0

Here we present the structure of mode 0. This is a special case because this solution is not a rotating wave but is a periodic pulsating wave as we state at (68). On Figure 8 is clearly seen that we are in front of an axisymmetric solution, profiles of the perturbation are the same for any chosen radial direction. Applying the method described in (63) we have drawn the profile of the pulsating perturbation for different times of the pulsation's period, see Figure 9. Remember that $T = \frac{2\pi}{\nu}$. On both fields the absolute maximum is found at the centre of the cylinder. About its vertical structure (see Figure 10), we see that the concentration field is a bit more complex than the temperature one. On the concentration case we have a variation on the structure near the cylinder plates, where we go from having 2 maximal peaks to only one.
Figure 8: Mode 0 eigenfunctions found on the onset of the instability at ROTK=101 for the temperature field (a) and the concentration field (b) at transversal cuts made in at half of the cylinder at t=0.

Figure 9: Profile of the eigenfunctions (mode 0) found on the onset of the instability at ROTK=101 for the temperature field (a) and the concentration field (b) at half of the cylinder for different times of the pulsation’s period.
Figure 10: Mode 0 eigenfunctions found on the onset of the instability at $\text{ROTK}=101$ for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 0$.

Figure 11: Isosurface plot for two different values of the Mode 0 eigenfunctions found on the onset of the instability at $\text{ROTK}=101$ for the temperature field (a) and the concentration field (b).
4.2.3 Mode 2

Here we present the structure of the second mode at the onset of the instability, calculated for ROTK=400 and Rayleigh=4506. It is a prograde rotating wave and its angular velocity is $\omega_{RW} \approx 1.5$. As one can observe on Figure 12, this solution has its peaks near the cylinder walls. As we will comment later, if this occurs then it is called a wall-mode. From the vertical cut we see that the maximum peaks stay near the walls along all the height and larger perturbations are found mid-height of the cylinder.

![Figure 12](image)

Figure 12: Mode 2 eigenfunctions found on the onset of the instability at ROTK=400 for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.
4.2.4 Mode 3

This mode is not on the onset of the instability, however it appears close enough to take it in consideration, specially for the biggest rotations. We calculated it at the same rotation rate as the case before, $\text{ROTK}=400$ and $\text{Rayleigh}=5074$. 

Figure 13: Mode 2 eigenfunctions found on the onset of the instability at $\text{ROTK}=400$ for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 117^\circ$ and $\theta = 90^\circ$ respectively.

Figure 14: Isosurface plot for two different values of the Mode 2 eigenfunctions found on the onset of the instability at $\text{ROTK}=400$ for the temperature field (a) and the concentration field (b).
It is also a prograde rotating wave with an angular velocity $\omega_{RW} \approx 2$. We observe that it follows the same patterns as mode 2 case but having a three peaks instead of two. See figures 15-17.

Figure 15: Mode 3 eigenfunctions found near the onset of the instability at ROTK=400 for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.

Figure 16: Mode 3 eigenfunctions found near the onset of the instability at ROTK=400 for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 146.25^\circ$ and $\theta = 112.5^\circ$ respectively.
4.3 Aspect Ratio = 2.76

We present in this section the results of the second case we worked on, $\Gamma = 2.76$. This case shows much more diverse results than the previous one, nonetheless they maintain some similarities such as an increase of Rayleigh (necessary to produce an instability) when increasing the rotation velocity. Differences between both cases and its causes will be discussed on the discussion section.

4.3.1 Linear Stability Analysis results

For this case there are more dominating modes. In fact we had to simulate up to mode 8, after checking that from that one, the following modes do not appear on the onset of the instability on our ranks. The first mode to become unstable is mode 1 (from ROTK=0-81), followed by mode 4 (from ROTK=81-96), next is mode 0 (from ROTK=96-180), and the last two are mode 7 (from ROTK=180-470 & 880-1000 ) and mode 8 (from ROTK=470-880).
Figure 18: Stability curves for each mode (0-8) for $S = -0.1$, $Le = 0.01$, $\sigma = 7$ and $\Gamma = 2.76$. It shows for which value of Ra and ROTK (adimensional rotation velocity) we have the first eigenvalue crossing the 0 axis on each mode.

As before, on this case, all non-axisymmetric solution are rotation waves rotating on the prograde (anticlockwise) direction except for modes 6, 7 and 8 at lower rotation rates where they show retrograde character (see Figure 20). Remember that mode 0 does is not a rotating wave and the sign of its frequency is arbitrary. On this case we also see that there are a lot of frequency jumps. However, none of those jumps occur while the mode is dominating, so we do
not find different structures within the same dominant mode.

Figure 20: Frequencies for each mode (0-8) for $S = -0.1$, $Le = 0.01$, $\sigma = 7$ and $\Gamma = 2.76$. Same colour legend used on the previous figure. It shows the imaginary part of the eigenvalue crossing the 0 axis and its sign denotes if its prograde(-) or a retrograde(+) solution.

Now, as we did for the previous case we plot the critical Rayleigh curve in function of the dimensionless rotation (Figure 21a). We denote dominating mode number by the colour of the line. The frequencies of the dominating modes are also plotted in Figure 21b.
Figure 21: (a) $Ra(\Omega_t)$ and (b) $w(\Omega_t)$ for $S = -0.1$, $Le = 0.01$, $\sigma = 7$ and $\Gamma = 1$. First appearing instabilities. Colour indicates its mode (see the legend).
4.3.2 Mode 1

The first mode that arises at the lowest rotation rates is mode 1. We made the plot using the eigenfunctions calculated in a simulation made at the onset of the instability for ROTK=40. Looking at its structure we observe that it is a body-mode, because the perturbation peaks are not localised on the walls. Besides, it has spiral character (see Figure 22). Six convection cells are created on both fields as it can be seen in Figure 23, with bigger peaks as closer the centre. The vertical structure is quite simple so only a middle height cylinder was made. This prograde rotating wave will approximately rotate at $\omega_{RW} \approx 6.5$.

![Figure 22: Mode 1 eigenfunctions found on the onset of the instability at ROTK=40 and Rayleigh=2112.5 for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.](image)
Figure 23: Mode 1 eigenfunctions found on the onset of the instability at ROTK=40 and Rayleigh=2112.5 for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 54^\circ$ and $\theta = 135^\circ$ respectively.

Figure 24: Isosurface plot for two different values of the Mode 1 eigenfunctions found on the onset of the instability at ROTK=40 and Rayleigh=2112.5 for the temperature field (a) and the concentration field (b).
4.3.3 Mode 4

This mode is found at the onset of the instability during a short range of rotation rate, ROTK=81-96. It is also a body-mode even though it has secondary peaks near the boundaries, having small features of a wall-mode. It is a prograde rotating wave too and also has spiral character. On Figure 26 we see that two convection cells are formed, the largest one in the middle of radius and the smallest totally adjacent and touching the wall. As it happens at the majority of the structure plots we did, the larger perturbations occur at mid-height of the cylinder. Its rotation velocity (always in the cylinder reference system) will be approximately $\omega_{RW} \approx 1.9$.

![Mode 4 eigenfunctions](image)

Figure 25: Mode 4 eigenfunctions found on the onset of the instability at ROTK=90 and Rayleigh=2401.9 for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.
Figure 26: Mode 4 eigenfunctions found on the onset of the instability at ROTK=90 and Rayleigh=2401.9 for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 90^\circ$ and $\theta = 72^\circ$ respectively.

Figure 27: Isosurface plot for two different values of the Mode 4 eigenfunctions found on the onset of the instability at ROTK=90 and Rayleigh=2401.9 for the temperature field (a) and the concentration field (b).
4.3.4 Mode 0

Next it appears an axisymmetric solution, mode 0. We treated this solution the same way we did for $\Gamma = 1$ case. So it is a periodic pulsating wave. Looking at Figures 28-29 is clear that it is a body-mode. On Figure 29 one can see the profile of the perturbation for different times of its period, in this case

$$T = \frac{2\pi}{w} = \frac{2\pi}{8} \approx 0.79$$

(non dimensional time).

Figure 28: Mode 0 eigenfunctions found on the onset of the instability at $\text{ROTK}=140$ and $\text{Rayleigh}=2805.7$ for the temperature field (a) and the concentration field (b) at transversal cuts made in at half of the cylinder at $t=0$. 
Figure 29: Profile of the eigenfunctions (mode 0) found on the onset of the instability at ROTK=140 and Rayleigh=2805.7 for the temperature field (a) and the concentration field (b) at half of the cylinder for different times of the pulsation period.

Looking at the vertical structure (Figure 30) we see that there are seven convection cells for both fields, having the bigger perturbations in the mid-height of the cylinder and its maximum on its centre.
Figure 30: Mode 0 eigenfunctions found on the onset of the instability at ROTK=140 and Rayleigh=2805.7 for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 0$.

Figure 31: Isosurface plot for two different values of the Mode 0 eigenfunctions found on the onset of the instability at ROTK=140 and Rayleigh=2805.7 for the temperature field (a) and the concentration field (b).
4.3.5 Mode 7

This mode dominates at ROTK=180-470 and ROTK=880-1000. It is a pro-grade rotating wave. The wave approximately rotates, in the cylinder system of reference, at the angular velocity $\omega_{RW} \approx 0.5$. This is a clear example of a wall-mode. If you take a look at Figures 32-33 you can see how all the perturbations concentrate around the walls and there is practically no perturbation in the centre of the cylinder.

Figure 32: Mode 7 eigenfunctions found on the onset of the instability at ROTK=400 and Rayleigh=4715.4 for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.
Figure 33: Mode 7 eigenfunctions found on the onset of the instability at ROTK=400 and Rayleigh=4715.4 for the temperature field (a) and the concentration field (b) at a radial cut made for $\theta = 126^\circ$ and $\theta = 36^\circ$ respectively.

Figure 34: Isosurface plot for two different values of the Mode 7 eigenfunctions found on the onset of the instability at ROTK=400 and Rayleigh=4715.4 for the temperature field (a) and the concentration field (b).
4.3.6 Mode 8

This mode is very similar to mode 7, with the difference that it has one more peak. It is also a rotating wave going in the same direction as the other non-axisymmetric modes commented above. Its angular velocity will approximately be \( \omega_{RW} \approx 0.5 \), the same as for mode 7.

![Mode 8 eigenfunctions](image)

Figure 35: Mode 8 eigenfunctions found on the onset of the instability at \( \text{ROTK}=700 \) and \( \text{Rayleigh}=6917.1 \) for the temperature field (a) and the concentration field (b) at the transversal cuts made at half of the cylinder.

![Mode 8 eigenfunctions](image)

Figure 36: Mode 8 eigenfunctions found on the onset of the instability at \( \text{ROTK}=700 \) and \( \text{Rayleigh}=6917.1 \) for the temperature field (a) and the concentration field (b) at a radial cut made for \( \theta = 0^\circ \) and \( \theta = 0^\circ \) respectively.
Figure 37: Isosurface plot for two different values of the Mode 8 eigenfunctions found on the onset of the instability at ROTK=700 and Rayleigh=6917.1 for the temperature field (a) and the concentration field (b).
5 Discussion

In this work we have presented results for the onset of convection in a water-ethanol binary mixture confined in a rotating cylinder. By doing this work we wanted to better know the behaviour of a binary fluid and specially understanding the effects of rotation. The results reported in this work are for \( \Gamma = 1 \) and \( \Gamma = 2.76 \) container. The other parameters refer to the water-ethanol mixture and they are \( S = -0.1, \sigma = 7, Le = 0.01 \) and \( Ce = 0 \). We made the Linear Stability Analysis for each mode, by obtaining the critical Rayleigh with which the bigger real part of the eigenvalue equals 0 (more exactly we chose an epsilon of \( 10^{-5} \)). This has been obtained by applying a Secant method on the code for each perturbation mode studied and each rotation rate (\( \Omega_t \)).

As one can observe in Figure 7 there are only two modes that dominate for the smaller aspect ratio case \( \Gamma = 1 \). On the lower rotation rates there appears an axisymmetric mode, mode 0. This is a pulsating mode as we found in (68). As we increase the rotation rate up to \( \Omega_t = 146 \) another mode appears on the onset of the instability, mode 2 which maintains until the end of our rank exploration. This mode is a nonaxisymmetric precessing mode and we can catalogue it as a wall mode because of its geometrical profile (see Figures 12-14), having its maximums near the container walls. On the contrary, mode 0 is a body mode having its maximums inside the container, more specifically in the centre, as one can see on Figures 8-11.

Note that as the rotation rate increases the critical Rayleigh numbers for nonaxisymmetric modes increase, but more slowly than for the axisymmetric mode. This is in concordance with Isabel Mercader’s work on ”Binary fluid convection in a rotating cylinder”\[2\], where it is noted that this behaviour is similar to the case of a pure fluid. The fact that wall-modes critical Rayleigh grows less on \( \Omega_t \) than for the body-mode is due the destabilising effect of the side walls. Having boundaries creates these type of modes (wall-modes) that could never be found in the non-boundary case.

The second mode is the first instability appearing for bigger rotation rates even though there is another mode very close way to this one, mode 3 (see figure 5). This mode is similar to the second one, is also a rotation wave rotating on the prograde (anticlockwise) direction, but with different geometry and a bigger rotation speed. Although this third mode is not on the onset, in this case we decided to draw it because it may affect the real instability solution due to its closeness to the onset of the instability.

Commenting a little on the instability spatial structure we can confirm that mode 0 is an axisymmetric mode with its maximum on the cylinder centre both in the temperature case and in the concentration one. On figure 9 we can observe its temporal behaviour of the temperature and concentration profile in a period of its pulsation. Only a transversal plot made at the half of the cylinder
was necessary due to the fact that the vertical structure is simple, specially in the case of temperature. On the concentration case we have a variation on the structure near the cylinder plates, where we go from having 2 maximal peaks to only one. For the second mode case we clearly see it is a wall-mode. On the temperature case the peaks are directly stuck on the sidewalls while concentration peaks stay close the walls but without touching them. In Figure 13 we can see that the vertical structure is simple and the bigger perturbations occur near medium height of the cylinder. For mode 3 we have practically the same structure as mode 2, but with one more peak.

The next case studied, $\Gamma = 2.76$, is far different from the case seen earlier. First of all, we can see that there is a greater diversity of instabilities. This may be explained by the fact that the container is wider, causing its walls to have a weaker effect and letting other modes to develop before wall-modes totally dominate due to the rotation. In fact wall-modes appear from $\Omega_t = 180$ while in the previous case they appeared at $\Omega_t = 146$. Another difference is that the instability modes appearing are bigger, even a $k=8$ mode is found. As we were saying, for this case we also found body-modes dominating for lower rotations and wall-modes dominating on the bigger rotations. Also we found a periodic pulsating axisymmetric mode dominating for $\Omega_t = 96 - 180$. A big difference between the perturbation characteristics is that for $\Gamma = 2.76$ all the dominating non-axisymmetric body-modes have spiral character while for $\Gamma = 1$ they do not.

Comparing the graphics $Ra(\Omega_t)$ (see Figure 38) is directly seen that for lower rotation rates a bigger Rayleigh is needed to achieve an instability for the $\Gamma = 1$ case than for $\Gamma = 2.76$ case. The results on rotation 0 and low rotation rates are qualitatively in concordance with the results presented on the work "Pattern selection near the onset of convection in binary mixtures in cylindrical cells" presented by Arantxa Alonso, Isabel Mercader and Oriol Batiste[3], where it is shown that critical Rayleigh tends to diminish when aspect ratio is increased. On the other hand, for bigger rotation rates the opposite happens. When we focus on the bigger rotation rates is easier to obtain an instability for $\Gamma = 1$ case, defining easy as having to use a lower Rayleigh, which means for example less temperature gradient between plates. This may be explained by the fact that on those rotation rates the modes that predominate are wall-modes and it’s obvious that this type of modes will form easier on a narrower cylinder.
Figure 38: Comparison of $Ra_c(\Omega_t)$ for the two cases studied: $\Gamma = 1$ (blue) and $\Gamma = 2.76$ (orange).

We chose to do the case $\Gamma = 2.76$ specifically cause it was the Aspect ratio used for Isabel Mercader at “Binary fluid convection in a rotating cylinder” [2]. On that work an He3-He4 binary mixture was considered so parameters different from ours were used: Prandtl number $\sigma = 0.755$, Lewis number $\tau = 0.067$ and several Separation ratio values. Although the parameters of the two works are quite different, we can still extract some interesting conclusions from that paper. It can be useful to predict how our fluid will act if we increase the Separation ratio for example. From [2], using the parameters commented above the following results were extracted:

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</table>

Table 1: Several results from the paper “Binary fluid convection in a rotating cylinder” [2] for $\Omega_T = 717$, Prandtl number $\sigma = 0.755$ and Lewis number $\tau = 0.067$.

Observing these results we can conclude that the closer to zero separation ratio is, bigger dominating mode number we have. For our particular experiment ($S = -0.1$) and using the same Aspect ratio we found that mode 8 was dominating for the same rotation rate.
6 Conclusions

On this work a complete linear stability analysis was made for a water-ethanol mixture. Stability curves and geometrical representation of the solutions were obtained and studied. From the results obtained we clearly saw the stabilising effect that produce cylinder walls when rotation is considered. The results have been compared to other similar work as Net, Mercader et al.[2] where the same problem was studied for another binary mixture, 3He-4He. Same behaviour of the modes were found, where critical Rayleigh increase with the rotation, and more specifically wall-modes increase less than body-modes when rotation becomes greater.

There are still many subjects to be studied about this specific problem. For example from the stability analysis we did a simplification of it by considering the centrifugal term as null, so basically the rotation effect considered is all due to the gravitational and Coriolis effect. We already explained that the error done was minimal for our rotation rates. However studying the same problem taking into account the centrifugal term would let us know the solution for much bigger ranks and in addition we could visualise the basic conductive solutions arising because of the centrifugal force.

On the other hand it would be really interesting to perform a time-evolution simulation for different conditions to visualise how the dominating solutions, that we find out, would evolve in the real experiment. Specially in interesting points found on our stability analysis, as it could be for example a situation in which two modes coexist on the onset of the instability, point known as codimension-two bifurcation, and where interesting and different evolving patterns use to appear. We can’t know, by our work, if when crossing the onset of the stability, we are in a supercritical Hopf bifurcation (the bifurcating stable periodic solutions are beyond the bifurcation) or in a subcritical Hopf bifurcation (the unstable bifurcating periodic solutions exist before the bifurcation). On the non-rotating study of binary mixtures [3] it was found that the perturbations arising were subcritical, which is the case where we could not foresee how the perturbation would behave in time without doing the temporal evolution. However, is a pending task to make a time-evolution simulation of this problem to know how those instabilities behave in time.

To conclude I just want to say that this project has been for me a very pleasant headache. Just from the starting when I spent a month only reading and trying to understand the literature and how that pseudo-spectral collocation method worked. Then trying to do the simulations without making any stupid error. Also the patience of having to wait days and days to obtain the desired results. And the hopeless feel when you realise that the simulation you had been waiting weeks to finish had an error on the code and all the simulation has to be repeated. I have learned a lot during this months. I have learned not only
concrete things of fluid dynamics, but also many other things as working with Linux terminal, new coding tools from Fortran and Matlab, etc. For all of this I would specially like to thank Arantxa and Isabel for guiding me, solving all the problems I found on the way (that were a few) and for the easy and friendly treatment you gave me.
References


