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**Study of minimum length, supersonic nozzle design using the Method of Characteristics**

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Declaration of Originality

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Abstract

This report outlines the pertinent theory and methods for supersonic nozzle design using the method of characteristics. The programmes developed design ideal nozzle contours, using numerical methods based on the gas properties, for desired exit velocities. Ideal contours include the most rapid expansion without inducing shock waves followed by the straightening section to provide uniform exit conditions. Then based on the ideal results, the contours can be truncated and the shape modified to induce minor shocks and only sacrifice minimal thrust to save length and thereby weight.
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Nomenclature

\( \dot{m} \)  Mass flowrate
\( \gamma \)  Isentropic expansion factor
\( \mu \)  Mach angle
\( \omega \)  Prandtl-Meyer function
\( \theta \)  Flow turning angle
\( A \)  Area
\( a \)  Local speed of sound
\( C_F \)  Thrust Coefficient
\( g_0 \)  Acceleration due to gravity
\( h \)  Enthalpy
\( I^+ \)  Invariant inclined at \( \theta + \mu \)
\( I^- \)  Invariant inclined at \( \theta - \mu \)
\( I_t \)  Total Impulse
\( I_{sp} \)  Specific Impulse
\( k \)  Compression factor
\( m^+ \)  Length along \( I^+ \)
Contents

$m$ Length along $I$

$N$ Number of characteristic lines

$P$ Pressure

$r_t$ Throat radius

$V$ Volume

$v$ Velocity

$x$ Axial position

$y$ Radial position relative to throat radius

CTIC Compressed Truncated Ideal Contour

F Force/Thrust

II Mach line

M Molecular mass/Mach number

MOC Method of Characteristics

PDE Partial Differential Equation

R Universal gas constant

T Reference temperature

$t$ time

TIC Truncated Ideal Contour

TOC Thrust Optimised Contour

TOP Thrust Optimised Parabolic
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1. Introduction

In general terms, propulsion is the act of accelerating a body. Rocket propulsion, generally, is governed by the type of propellant used and the design of the nozzle. For solid and liquid propellants, nozzles are used to accelerate the exhaust gases. De Laval (convergent-divergent) nozzles are typical propelling nozzles used to expand the exhaust gases and accelerate the flow to hypersonic velocities. The role of the engine is to convert the propellant to high pressure gas. The nozzle then converts the sonic, high pressure, high temperature gas to high velocity gas at almost ambient (local atmospheric) pressure to produce thrust as described by Newton’s third law of motion.

1.1. Background

Gustaf de Laval developed the de Laval nozzle in the 19th century for use in steam turbines before Robert Goddard used the design with early rocket engines. Walter Thiel used the shape in the German V-2 rocket and since then it has been used in almost every modern rocket engine.

The performance of a rocket is evaluated in terms of the thrust it can produce and also the Specific Impulse ($I_{sp}$) of the rocket. Specific Impulse is used to understand the efficiency of a rocket engine. Total Impulse is defined as the total thrust $F$ as a function of time, integrated over the rocket’s burn time $t$.

$$I_t = \int_0^t F \, dt \quad (1.1)$$
1. Introduction

Specific Impulse is the impulse per unit of propellant. It is measured in seconds by dividing by acceleration due to gravitational \( g_0 \).

\[
I_{sp} = \frac{F}{mg_0}
\]  

(1.2)

The attainable Specific Impulse is primarily a function of the propellant mix. However, the practical limits of chamber pressures and expansion ratios, related to the nozzle, impose limits on the performance that can be achieved.

Rocket performance is optimal when the nozzle exit pressure is equal to ambient pressure of its surroundings. If the exit pressure is greater than the ambient pressure, the flow is underexpanded. In the case of the exit pressure being less than the ambient pressure, the flow is said to be overexpanded. Minor overexpansions result in minor performance compromises. However, if the pressure is 60%, or less, of ambient pressure, flow separation may occur with potentially disastrous consequences.

In a vacuum, or at very high altitude, it is impossible to match the exit pressure to the ambient pressure. Nozzles with larger area ratios are generally more efficient but a larger area ratio means a longer nozzle and more vehicle mass for the engine to accelerate. Thus, a length which optimises performance, a point of diminishing returns, must be found.

As well as the area ratio of the nozzle, the shape effects the performance of the nozzle. Conic nozzles typically employ a half-angle of \( \sim 15^\circ \) which yields an efficiency of \( \sim 98\% \). Bell nozzles (parabolic shape) return a greater efficiency (\( \sim 99\% \)). Compared to the cone nozzle, they are usually shorter and lighter. Bell nozzles are often used on launch vehicles where minimising weight is crucial. There is a theoretically optimal shape for Bell nozzles to give maximum exhaust velocity for a given nozzle length. However, slightly shorter nozzles are often used in order to reduce external drag and weight in exchange for only slightly compromised exhaust speed.


1. Introduction

1.1.1. The Ideal Rocket

Assuming an ideal rocket can be very useful to obtain an estimate of a rocket’s performance. For chemical rockets, the actual performance is typically between 1 and 6% less than the value calculated under ideal assumptions. Gas flow in rocket engine nozzles can be analysed in one-dimension, as an ideal rocket, but many assumptions are made in order to do so [3]:

1. The propellant mixture is homogeneous.
2. All of the fluid in the nozzle is gaseous and any liquid or solid mass is negligible.
3. The propellant mixture obeys the ideal gas law.
4. The flow is adiabatic, thus there is no heat transfer across the walls of the rocket.
5. Friction and boundary layer effects are negligible.
6. No shock waves or discontinuities are present in the nozzle.
7. There is steady-state flow in the nozzle.
8. Exhaust gases leaving the nozzle have axial velocity.
9. Gas parameters are uniform across any plane normal to the axis of the nozzle.
10. Frozen flow exists in the nozzle chamber, meaning chemical equilibrium is established and the gas composition does not change in the nozzle.
11. Cryogenic propellants are assumed to be at their boiling points while stored propellants are assumed to be at room temperature.

The above assumptions allow the compilation of a simple, quasi-one-dimensional model. The following equation (where subscripts 1 and 2 refer to the inlet and outlet respectively) can be used to calculate the exit gas velocity \( v \) of an ideal rocket as a function of the pressure \( p \) [3].

\[
v_2 = \sqrt{\frac{2\gamma RT}{\gamma - 1} M \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{\gamma - 1}{\gamma}} \right]}
\]  

(1.3)
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Where:
R is the universal gas constant,
γ is the isentropic expansion factor,
T is the reference temperature of the gas,
M is the molecular mass of the gas.

1.2. Aims and Objectives

As a simple one-dimensional model is already provided, this thesis aims to create a programme to solve the three-dimensional partial differential equations (PDE) which govern the behaviour of fluids in a nozzle. The purpose is to solve, with greater accuracy the velocities throughout the nozzle flow and use the results to design a minimum length nozzle. The method of characteristics will be employed to solve the system of PDEs. The specific study objectives are:

— Create a programme to solve the exit velocity of a gas in a simple cone shape nozzle using the two-dimensional form.
— Create a programme to design a minimum length Bell nozzle for a given gas exit velocity using the two-dimensional form.
— Create a programme to design a minimum length Bell nozzle for a given gas exit velocity using the three-dimensional form.

1.2.1. Limitations

The main limitation of this study is the time constraints placed upon it. As a single-semester thesis, the total allowable time-frame from conception to completion is five months. The developed code is limited to nozzle design for supersonic gas. The code produced in this study is also limited by it’s handling of shock cases: minor shocks can be dealt with by the code, while retaining accuracy, however large shocks cannot be handled by the programme.
1. Introduction

The Method of Characteristics does not deal with the boundary layer of the nozzle flow and thereby, the heat transferred from the gas to the nozzle. However, the energy dissipated through heat loss is generally $< 1\%$, meaning a minor loss in $I_{sp}$.

1.3. Methodology

This report comprises five chapters:

— *Chapter 1 - Introduction* gives an overview of the project by defining the objectives and limitations of the study.
— *Chapter 2 - Literature Review* lays out the information pertinent to compiling the codes for this thesis.
— *Chapter 3 - Method* presents the steps taken to develop the two-dimensional and three-dimensional codes and resulting nozzles.
— Results and numerical analyses are presented in *Chapter 4 - Results and Discussion*.
— *Chapter 5 - Conclusions* provides the conclusion of the study and recommendations for further study.

Appendices follow the report and present detailed results and explanations relevant to the report’s content.
2. Literature Review

2.1. Nozzles

According to fable, the principles of rocketry were first tested more than 2,000 years ago. Ancient Greek stories tell of Archytas, a philosopher and mathematician, who demonstrated a wooden pigeon that flew around, on wires, by propulsive steam. Heron of Alexandria, believed to have created the first steam engine and the first coin operated vending machine, used a sphere, atop boiling water, with a number of holes in it to create thrust. The steam entered the sphere and was forced through two L-shaped exhaust pipes to make the ball rotate.

Historians believe the first modern rockets were created by the Chinese in the first century A.D. and they were akin to modern fireworks. Until the modern era, rockets were largely used for military purposes.

Three men are credited as the “fathers of rocketry”. Konstantin E. Tsiolkovsky, Robert Goddard and Hermann Oberth are credited with having defining roles in modern rocketry and spaceflight. Tsiolkovsky published his famous rocket equation in 1903, revolutionising aerospace. Goddard is known for his role in the first liquid-fuelled rocket, while Oberth is the man who created the German V-2 rocket. The V-2 missile became the first man-made object to cross the Kármán Line into space. Oberth is the only one of the three who lived to see rockets being used for space exploration.

Sputnik 1 was the first artificial satellite launched into space. The Soviet Union launched Sputnik in 1957 before launching Yuri Gagarin into space, making him the first human to orbit Earth, four years later. The rapid innovation brought on by the space race between the Soviet Union and the United States came to a dramatic climax on July 20, 1969 when Neil
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Armstrong and Buzz Aldrin became the first people to walk on the moon. The space race ended in the 1970s and aerospace innovation stagnated. Recently there has been a significant resurgence of the space industry thanks to the popularisation and success of private space corporations.

Each one of these events endeavoured to maximise thrust and specific impulse through propellant and nozzle design. The aerospace community is continuously trying to develop better nozzles in order to convert the maximum amount of energy from the propellant reaction, to kinetic energy in the engine’s nozzle.

2.1.1. Nozzle Principles

As explained in Chapter 1, the modern rocket engine ignites the propellant in the chamber to create pressure. The only escape for the particles is through the nozzle. From the chamber, the nozzle converges to accelerate the particles to sonic velocity. This constricted part of the nozzle, with the smallest cross-section, is known as the throat. The nozzle then expands to accelerate the flow to supersonic speeds, creating thrust. It is the expanding section which is the main focus of modern rocket propulsion studies and this study will focus on.

The nozzle’s area ratio, the ration of the exit area to the throat area, determines the exit pressure of the exhaust gas. An idealised nozzle expands exhaust gases, isentropically, to the same pressure as the ambient pressure beyond the nozzle exit. The issue for engineers when designing nozzles, is that ambient pressure is a function of altitude. Conventional nozzles are designed to perform optimally at a single mid-range altitude and have one single, uninterrupted contour. At low altitudes, conventional nozzles are over-expanded and under-expanded at high altitudes (Figure 2.1).

A solution to this issue, common in modern spaceflights, is the use of two-stage rockets. Vast amounts of propellant are needed to hoist the huge masses of rockets and their cargo from the Earth’s surface. As the rockets reach high altitudes and low pressures, the first stage detaches, shedding the massive fuel tanks and the engines designed for atmospheric use. The second stage engines then start-up, making use of nozzles specifically
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![Diagram of rocket nozzles](image)

Figure 2.1.: Simplified depictions of exhaust gases from three different rocket nozzles [3].

designed to be used in the vacuum of space. Of course, the problem of optimal design altitudes still exists for first stage engines and there are many solutions offered by the engineering community. Double-bell nozzles, aerospike nozzles and many more offer designs which can adjust their shape or are designed to operate at more than one optimal altitude. However, the Bell nozzle has been ubiquitous in rocket design from the V-2 until the SpaceX Falcon Heavy.

### 2.1.2. Nozzle Design

Two primary categories of nozzles exist, Conical Contour Nozzles and Thrust Optimised Contour (TOC) nozzles. Conical nozzles are self-explanatory in that they are a cone shape defined by their half angle. Bell shape
2. Literature Review

(De Laval) nozzles provide faster expansion of the exhaust gases, allowing the desired velocity to be reached in a shorter distance, making the nozzles lighter and shorter than conical nozzles. Bell nozzles entail more manufacturing complexity than conical nozzles.

The Method of Characteristics (MOC) is used to design TOC nozzles. The streamlines of the expanding gas are calculated to produce a bell shaped nozzle (Figure 2.2). In the nozzle, the streamlines follow the contours of the nozzle walls, expanding through small expansion waves.

![Figure 2.2: Method of Characteristics nozzle designs [8].](image)

MOC designed nozzles make use of the uniform conditions at the throat and at the exit of a minimum length nozzle. The nozzle throat is uniform at Mach singularity and the nozzle exit velocity is imposed by the design dependent on the desired engine thrust.

As the bell shape of the nozzle decreases in area, small compression waves may propagate within the flow, presenting as mild shocks. The idealised MOC cancels such shocks, thus minimising energy loss [3]. Minimum length nozzles are ones which end at the point when uniform exit velocity has been met. However, these nozzles are often too long, and thus too heavy for most spaceflight applications.

Truncated Ideal Contour (TIC) nozzles are compressed versions of TOC nozzles. Two methods of creating TIC nozzles exist. The first is to specify the length as a fraction of a conical nozzle with a half-angle of 15° and then create a contour. The second option is to design a minimum length...
2. Literature Review

Bell nozzle and truncate the design using a Rao parabolic approximation. A Thrust Optimised Parabolic (TOP) nozzle is created using the second method and manages to increase the thrust potential of a rocket while reducing the length [3].

2.1.3. Thermodynamic Relations

Mathematical tools for determining rocket engine design parameters are furnished by thermodynamic relations. Such tools are used to evaluate and compare the performance of various rocket systems. Nozzle size and shape can be determined for any system that uses the expansion of gases.

Conservation of energy can be applied to the adiabatic process inside a nozzle. As the processes this report considers do not involve shocks or friction, the change in entropy is zero. The work performed by the gas at a velocity $v$ plus the internal energy constitutes the enthalpy in the process. From these assumptions and the expression for the enthalpy of an ideal gas, we get the following equation for stagnation enthalpy per unit mass.

$$h_0 = h + \frac{v^2}{2} J = \text{constant}$$  \hspace{1cm} (2.1)

It is also important to note the continuity equation of the principle of conservation of mass, which states the mass flowrate at any section $x$ is equal to the mass flowrate at any other section $y$ for steady flow with a single inlet and outlet, such as a nozzle. The mass flowrate can be written in terms of the cross-sectional area $A$, the velocity $v$ and the volume $V$.

$$\dot{m}_x = \dot{m}_y = Av/V$$  \hspace{1cm} (2.2)

The ideal gas law will also be important to note going forward.

$$PV = nRT$$  \hspace{1cm} (2.3)
2. Literature Review

For isentropic flow, the following relations hold between any two points $x$ and $y$, where $\gamma$ is the constant ratio between the specific heat capacity $c_p$ and the specific heat at constant volume $c_v$.

$$\frac{T_x}{T_y} = \left(\frac{P_x}{P_y}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{V_y}{V_x}\right)^{\gamma^{-1}}$$

(2.4)

The area ratio for a nozzle with isentropic flow can eventually be expressed in terms of the Mach number at any to given points within the nozzle.

$$\frac{A_y}{A_x} = \frac{M_x}{M_y} \sqrt{\left(\frac{1 + M_y^2(\gamma - 1)/2}{1 + M_x^2(\gamma - 1)/2}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

(2.5)

Figure 2.3 demonstrates the relationship between area ratio, temperature ratio and pressure ratio as a function of Mach number.

Figure 2.3 shows that the contraction in the chamber may be as small as 3 to 5 times the throat area. The effect of the nozzle’s rapid expansion is evident then as pressure and temperature drop while Mach increases. The other remarkable aspect of the graph is the minor influence of the constant $\gamma$, represented as $k$ in the image.

2.1.4. Isentropic Nozzle Flow

As demonstrated in Figure 2.3, the gas pressure and temperature drop dramatically in the nozzle to create the supersonic speeds desired. This process is reversible, “essentially isentropic” [3]. From Equation 2.1, the following expression for the nozzle exit velocity $v_2$ can be found.

$$v_2 = \sqrt{2J(h_1 - h_2) + v_1^2}$$

(2.6)

With constant $\gamma$ Equation 2.6 can be rewritten to give:
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Figure 2.3: Relationship between area ratio, temperature ratio and pressure ratio as a function of Mach number in a bell shape nozzle. [3]

\[
v_2 = \sqrt{\frac{2\gamma}{\gamma - 1} R T_1 \left[ 1 - \left( \frac{P_2}{P_1} \right)^{(\gamma - 1)/\gamma} \right] + v_1^2}
\]  

(2.7)

\(v_1\) refers to the velocity of the chamber while \(v_2\) refers to a point in the nozzle. Generally \(v_1\) is comparatively small and can be neglected to give an equation frequently used in nozzle analysis.
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\[ v_2 = \sqrt{\frac{2\gamma RT_1}{\gamma - 1} \left[ 1 - \left( \frac{P_2}{P_1} \right)^{(\gamma-1)/\gamma} \right]} \]  
(2.8)

### 2.1.5. Thrust and Thrust Coefficient

Thrust is generated by the reaction force on the rocket structure which comes from the propellant momentum flux. With supersonic flow, the pressure at the nozzle exit plane may be different to the ambient pressure and the pressure thrust component adds to the momentum thrust:

\[ F = \dot{m}v_2 + (P_2 - P_3)A_2 \]  
(2.9)

It is clear from Equation 2.9 the maximum thrust is obtained when the nozzle is operating in a vacuum and \( P_3 = 0 \). Below the vacuum of space, Equation 2.9 gives the variation of thrust with altitude.

The thrust coefficient is obtained by dividing the thrust by the chamber pressure \( (P_1) \) and the throat area \( (A_t) \). The thrust coefficient \( (C_F) \) is then:

\[ C_F = \sqrt{\frac{2\gamma^2}{\gamma - 1} \left( \frac{2}{\gamma + 1} \right)^{(\gamma+1)/2} \left[ 1 - \left( \frac{P_2}{P_1} \right)^{(\gamma-1)/\gamma} \right] + \frac{P_2 - P_3 A_2}{P_1 A_t}} \]  
(2.10)

### 2.2. Real Nozzles

In reality, nozzle flow is two-dimensional but axisymmetric. Temperatures and velocities are not uniform across the section of a simple bell shape nozzle and are usually higher in the centre. For example, in an ideal nozzle, the surface where the Mach number is 1, is a plane but in a real nozzle, the surface is slightly curved downstream of the throat. The average velocity \( v_2 \)
2. Literature Review

for an axisymmetric nozzle can be determined as a function of the radius $r$.

$$ (v_2)_{\text{average}} = \frac{2\pi}{A_2} \int_0^{r_2} v_2 r \, dr $$  \hspace{1cm} (2.11)

The assumptions related to ideal nozzles listed in Section 1.1.1 of this report are approximations that allow relatively simple algorithms and calculations for analysing real rockets and the associated phenomena.

Compared to the ideal nozzle, real nozzles have energy losses and not all of the chamber energy can be converted to kinetic energy. [3] provides a list of the following 10 losses:

1. In conical nozzles, the divergence of the flow leaving the nozzle causes a loss which can be reduced for bell shaped contours.
2. Pressure losses in the chamber, as a result of small areas compared to the throat area (nozzle contraction ratios), cause pressure losses in the chamber, marginally reducing the thrust and exhaust velocity.
3. The effect of the boundary layer may reduce the flow velocity, reducing the effective exhaust velocity by 0.5% to 1.5%.
4. As discussed in Section 2.2.2, solid or liquid particles in the gas may result in losses of up to 5%.
5. 0.5% losses can be induced by chemical reactions in nozzle flow, changing the gas properties.
6. Transient operations endure lower performance.
7. During operation, the throat diameter may be eroded, decreasing the nozzle expansion ratio. Decrease in thrust will be proportional to the decrease in area ratio.
8. Non-uniform gas composition may result in incomplete mixing, turbulence or incomplete combustion regions.
9. The value of $\gamma$ may not remain perfectly constant and could have an effect of losing between 0.2% and 0.7% of thrust.
10. Thrust losses for fixed expansion nozzles may be up to 15% during a portion of the flight, compared to a nozzle with altitude compensation.
2. Literature Review

2.2.1. Boundary Layer

Due to wall friction, real nozzles have a viscous boundary layer where the gas velocity is significantly lower than the free-stream velocity. Adjacent to the wall, the gas velocity is zero and the boundary layer can be imagined to comprise successive layers of increasing velocity, limited at the inviscid flow region velocity. Crucially, part of the boundary layer closest to the wall is laminar and subsonic but, further from the wall, the supersonic region can become turbulent. [7]

Due to viscous friction, the local temperature in the boundary layer can be substantially higher than the free-stream temperature. Immediately next to the wall, the temperature will be lower because of heat transfer to the wall. Figure 2.4 describes the velocity and temperature profiles of the boundary layer.

A good theoretical analysis of rocket nozzle boundary layers has not yet developed. Fortunately, however, the effect of the boundary layer on nozzle performance is relatively small. The loss in $I_{sp}$ rarely exceeds 1% [3].

2.2.2. Multiphase Flow

As mentioned, the gas flow in the nozzle can contain small liquid droplets and/or solid particles. Such droplets/particles must be accelerated by the gas flow. Multiphase flow is a common phenomenon in solid propellants and gelled liquid propellants containing aluminium powder. The powder can form small oxide particles in the exhaust.

Typically, particles with diameters smaller than 0.005mm or less will have similar temperatures and velocities as the surrounding gas. Larger particles are harder to accelerate as their mass is proportional to the cube of their diameter while the drag force needed to accelerate them is only proportional to the square of their diameter. Therefore, larger particles do not travel as fast as the gas and thermal equilibrium is not achieved. If the amount of particles is small or the size of the particles is small, the amount of energy needed to accelerate the particles is negligible.
2. Literature Review

The loss in specific impulse is generally less than 2% for values of $\beta$ (the percent of particles) less than 6% and particle diameters less than 0.01 mm. Values of $\beta$ larger than 6% and particle diameters greater than 0.015 mm can lead to reductions in $I_{sp}$ of 10% to 20%. [3]

In nozzles with very high area ratios, operating in a vacuum, the propellant ingredients which are normally gas, can be condensed. The condensing gases can become liquid droplets as the temperature drops sharply in the nozzle. The performance consequences are small providing the particles are few and small. It is also possible to precipitate fine particles of snow, in solid phase $\text{H}_2\text{O}$. [3]
2. Literature Review

2.2.3. Other Phenomena and Losses

In reality, the combustion process is not steady and chamber pressure may oscillate by up to 5% and the process can still be considered relatively steady. Flow properties, including velocity, volume, temperature and pressure, oscillate with time. Thus the calculated values are simply averages. It is difficult to theoretically assess the losses due to unsteady propellant burning. Experimentally, the losses have been found to be negligible for smooth-burning engines. [3]

Gas composition changes a small amount in the nozzle as chemical reactions occur in flowing gas. Thus, the assumption of uniform equilibrium gas composition is not fully valid. The thermal energy carried out of the nozzle is not available for conversion to useful propulsive energy. The loss can be minimised by reducing the nozzle exit temperature $T_2$. Even with the reduced exit temperature, the loss is significant.

2.3. Computational Methods

2.3.1. Prandtl-Meyer

In fluid dynamics, the Prandtl-Meyer function can be used to determine the angle through which transonic flow can turn. As flow passes from sonic to supersonic flow may turn isentropically. The Prandtl-Meyer function is expressed, for an ideal gas, as a function of the flow Mach number.

$$v(M) = \int \frac{\sqrt{M^2 - 1}}{1 + M^2(\gamma - 1)/2} \frac{dM}{M} \quad (2.12)$$

Which can be expressed as:

$$v(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} tan^{-1} \left( \sqrt{\frac{\gamma - 1}{\gamma + 1}} (M^2 - 1) - tan^{-1} \sqrt{M^2 - 1} \right) \quad (2.13)$$
2. Literature Review

For isentropic expansion: \( \nu(M_2) = \nu(M_1) + \theta \) where the subscripts 1 and 2 denote conditions before and after the turn respectively, and \( \theta \) is the angle of the turn relative to the theoretically extended straight wall, as shown in Figure 2.5.

![Diagram](image.png)

Figure 2.5.: Depiction of the variables associated with the Prandtl-Meyer function.

2.3.2. Euler Equations

The Euler equations in fluid dynamics are a set of hyperbolic equations governing inviscid and adiabatic flow. They can be obtained by linearising the Navier-Stokes equations and represent conservation of mass and the balance of momentum and energy. In supersonic flow, the Euler equations are explicitly hyperbolic, meaning the flow is determined, solely by the upstream conditions. In this hyperbolic case, the MOC can be utilised to calculate the nozzle flow and, crucially, the region of the flow. The MOC is the method of choice for modern rocket nozzle design. The method can be used to generate contours and determine performance parameters.
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2.3.3. The Method of Characteristics for Ideal Gas

Two-Dimensional or Axisymmetric

The relevant assumptions for two-dimensional flow modelling with the MOC are as follows:

— The flow is irrotational.
— The fluid flow is isentropic.
— Flow is entirely supersonic.
— The fluid is a perfect gas.
— The effect of gravity on the flow is negligible.

Prandtl-Meyer Flow. In a two-dimensional, supersonic flow regime, small disturbances manifest themselves as pressure waves along Mach lines. These lines are also lines of constant perturbation potential. The change in direction and magnitude of the velocity produced by a pressure wave has a direction normal to the Mach line, according to the disturbances.

![Figure 2.6: Streamlines and Mach lines according to linear theory. The mean Mach line (II) is midway between the upstream and downstream flow Mach lines (II_1 and II_2 respectively).](image)

Figure 2.6 shows the organisation of the single unit process which the relevant calculations can be reduced to using the aforementioned considerations. The figure shows an initially uniform parallel flow disturbed by a small turn. The effect of the turn is propagated along the Mach line (II), making the
2. Literature Review

Mach angle $\mu$ to the original flow. The velocity change can be determined from the known turning angle $\theta$ of the wall and the fact that the vector change is normal to the Mach line $II$.

The theory can be extended to the flow around a curved wall, as in Figure 2.7. The curve is approximated as a series of straight walls with corners between them.

![Figure 2.7: Flow around a curved wall by modified linear theory.](image)

A pressure wave propagates from each corner with enough strength, so as to preserve the continuity at the wall. The conditions at “2” and the direction of the Mach line $II_a$ are determined by the mean flow conditions between “1” and “2”. The flow field is calculated using stepwise increments in the same fashion. Such flow in which pressure waves of only one family appear is known as Prandtl-Meyer flow. Two important features of Prandtl-Meyer flow are: along each straight Mach line, all flow properties are uniform; and the velocity at any point depends only on initial conditions and the local flow direction. These two properties work together, mathematically.

**The Physical Characteristics.** The hodograph plane is used to plot the velocity of a particle as a function of time. Using polar coordinates $V$ and $\theta$ for hodograph gives:

$$ u = V \cos\theta; \quad v = V \sin\theta \quad (2.14) $$

Implementing these relationships into the equation for characteristic directions included in Appendix A gives the identities shown in Equation 2.15,
following rearrangement making use of standard trigonometric identities [6].

\[
\left( \frac{dy}{dx} \right)_I = \tan(\theta - \mu); \quad \left( \frac{dy}{dx} \right)_{II} = \tan(\theta + \mu)
\] (2.15)

In conjunction with Figure 2.8, Equation 2.15 shows that the two physical characteristics, \( I^- \) and \( I^+ \), are inclined at the Mach angle \( \mu \), to velocity vector \( \vec{v} \).

Aligning the velocity vector parallel to the x-axis, we obtain the values \( u = V, v = 0, M = V/a, \theta = 0 \); Equation A.7 becomes:

\[
\left( \frac{dy}{dx} \right)_{I^+,I^-} = \pm \frac{\sqrt{V^2 - 1}}{1 - \frac{V^2}{a^2}} = \pm \frac{1}{\sqrt{M^2 - 1}} = \mp \tan \mu
\] (2.16)
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Thus, it is clear that the characteristic $I^+$ is inclined at the Mach angle $\mu$ below the velocity vector and the physical characteristic $I^-$ is inclined at $\mu$ above the velocity vector.

**Numerical Methods.** As we have seen the Mach angle is defined as

$$\mu = \sin^{-1} \frac{1}{M}$$

The Prandtl-Meyer function $\omega$ is a function of the Mach number.

$$d\omega(M) = \sqrt{M^2 - 1} \frac{du}{u}$$  \hspace{1cm} (2.17)

Appendix B contains the derivation of the Prandtl-Meyer function which is found to be:

$$\omega(M) = K \arctan \frac{\sqrt{M^2 - 1}}{K} - \arctan \sqrt{M^2 - 1}$$  \hspace{1cm} (2.18)

Where $K$ is a constant defined by:

$$K = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$  \hspace{1cm} (2.19)

The inclination of the characteristics are then defined by the following equations where $m^+$ and $m^-$ are lengths along the characteristics.

$$\frac{\partial}{\partial m^+}(\theta + \omega) = \frac{\sin \mu \sin \theta}{r}$$  \hspace{1cm} (2.20)

$$\frac{\partial}{\partial m^-}(\theta - \omega) = -\frac{\sin \mu \sin \theta}{r}$$  \hspace{1cm} (2.21)

For two-dimensional calculations, $r \to \infty$. Therefore, the following constant relationships hold.
2. Literature Review

\[ \theta + \omega = \text{constant along } m^+ \equiv I^+ (\text{inclined } \theta - \mu) \]
\[ \theta - \omega = \text{constant along } m^- \equiv I^- (\text{inclined } \theta + \mu) \]  

(2.22)

2.4. Truncated Nozzles

The length of nozzle expansion sections, designed by the MOC for given design Mach numbers, is minimum. The straightening section is responsible for obtaining uniform exit conditions. In order to reduce the nozzle losses, the straightening section controls the interaction of the exhaust gases with the surrounding atmospheric fluid. Thus, the efficiency of the nozzle depends on the straightening section.

2.4.1. Truncated Ideal Contour nozzles (TIC)

The ideal nozzle produced by MOC codes is very long and consequently very heavy. The reason for the long length is to obtain uniform exit velocity, producing a one-dimensional exhaust profile. As the slope of the nozzle contour is very small towards the end of the nozzle, the contribution is negligible. To reduce the weight and external drag of the nozzle, the contour can be truncated, producing a TIC nozzle.

[1] proposed a graphical technique for optimising nozzle contours. A set of ideal nozzle contours is plotted with lines representing the dimensionless contours along with lines showing constant surface area, exit diameter and thrust coefficient. Figure 2.9 shows a set of MOC contours plotted with thrust coefficient. In the figure, lines of constant length, area and radius are included and the thrust coefficient curve is exaggerated for illustrative purposes. Point A shows the largest truncation and represents a nozzle of maximum performance. Point B is found when the constant surface area is tangent to the thrust coefficient and represents the optimum nozzle length for a given surface area. To obtain a nozzle of maximum thrust for a given expansion ratio, point C is the point where the thrust coefficient is tangent to a line of constant radius.
2.4.2. Compressed Truncated Ideal Contoured Nozzles (CTIC)

As the name states, CTIC nozzles are compressed versions of TIC nozzles. For the same flow profile, it was proposed by Gogish [9] in 1966 that CTIC nozzles would have higher performance than a Rao nozzle of the same flow profile. The procedure to axially compress a TIC nozzle yields more rapid initial expansion, followed by a more severe turn back than the TIC nozzle. The compressed contour results in the propagation of strong compression waves into the flow field. With strong enough compressions, the characteristic lines will converge and create an oblique shock wave, running from left to right. Oblique shocks are often visible in the exhaust jet of jet engines and rocket engines, presenting themselves as Mach/shock diamonds before being dissipated by mixing (see Figure 2.10).
The static pressure will then increase as the flow crosses the shock wave. Inducing a shock near the nozzle wall will increase the pressure on the wall, thus increasing the thrust produced by the nozzle. Gogish considered this method in his proposal but later studies, namely that of Hoffman [4], showed the CTIC nozzle was not more efficient than the Rao nozzle. However, the performance differential was very small for some designs indicating that an optimal CTIC nozzle can be a viable design for certain applications.

2.4.3. Thrust Optimised Contoured Nozzles (TOC)

Nozzle contours can be designed using the calculus of variations. A complex method of designing the optimum nozzle contour and exit area for given length and ambient pressure was developed in Germany during the late 1950s. The method was then simplified by Rao before being commonly adapted. As a result, in the west, the resultant nozzle is commonly referred to as a Rao nozzle.

To produce a TOC nozzle the procedure is as follows. For given throat curvature $r_{td}$ and a variety of $\theta_N$ (see Figure 2.11). Given design parameters the points P and N can be found by satisfying the following conditions:

1. Mass flow across PE is equal to the mass flow across NP.
2. The nozzle gives maximum thrust.
With N and P known, the expansion region TNKO is fixed and the contour line is constructed by selecting points P', P'', etc. along the kernel region NK. A series of control surfaces P'E', P''E'', etc. can then be generated to compute E', E'', etc. along the nozzle contour.

The TOC nozzle produced by [5] had the same performance as the TIC nozzle produced in the same work, for the same initial conditions and gas properties.

The method produces a shock free flow in the upper region NPE. Thus, the effect desired by Gogish in the CTIC nozzle, returning increased wall pressure, is not a feature of the TOC nozzle. By definition, an ideal nozzle is produced when point P is located at point K. Right-running shocks are produced downstream of point P, when P≠K. This is because a more extreme flow turning is induced and the compression waves will coalesce into a shock wave.

2.4.4. Simply Truncated Nozzles

The TIC and TOC nozzles are similar in shape. However, the TOC nozzle has a greater initial expansion and a more severe turn compared to the
2. Literature Review

TIC nozzle. This means, for the TOC nozzle, a greater wall angle and Mach number downstream of the throat but lower wall angles and Mach number at the nozzle exit, relative to the TIC nozzle (see Figure 2.12). While the performance is unaffected, the difference in flow structure affects the separation and side-load characteristics [5].

Figure 2.12: Comparison of TIC (Ideal Truncated) and TOC (Rao-Shmyglevsky) nozzle flow structures [5].
3. Methodology

3.1. Nozzle Design

As previously discussed, to accelerate flow to supersonic speeds, convergent-divergent nozzles are used. This study specifically deals with the divergent section, beginning at the throat. One-dimensional analyses predict the flow properties as a function of $x$ the distance along the nozzle, through a nozzle of given shape. The one-dimensional analyses predict flow properties as an average for a nozzle cross-section and therefore, cannot deduce wall contours.

3.1.1. Two-Dimensional Method of Characteristics

The converging subsonic flow is multidimensional and therefore slightly curved. However, for simplicity, the sonic line is assumed to be straight. Also, due to the symmetry of the flow about the nozzle’s axis, the contour need only be calculated for one side of the nozzle. This symmetry applies to both two- and three-dimensional calculations.

For two dimensional analyses the relationships defined as constant in Equation 2.22 holds true along characteristic lines and the constants are referred to as invariants.

Conic Nozzle Design

In order to gain a better understanding of the MOC and the associated theory, a two-dimensional code to produce the profile of a conic nozzle
3. Methodology

was created. The algorithm to compute the characteristics in a conic nozzle forms the basis of the algorithms for two- and three-dimensional Bell nozzle designs with the MOC. The methodology implemented is as follows.

Initial conditions and gas properties are provided which include the half angle of the nozzle, the number of characteristic lines, the sonic throat conditions and the adiabatic coefficient of the gas. A unit throat radius has been employed in each analysis to non-dimensionalise the resulting radii. From the initial conditions, the invariants \( I^+ \) and \( I^- \) at the first point along the flow axis can be determined by determining omega, the Prandtl-Meyer function.

Prandtl-Meyer expansion allows the computation of the Prandtl-Meyer function at the intersection of the first characteristic line and the nozzle wall, giving the invariants at that point.

With the first and last invariants \( I_1^+ \) and \( I_N^+ \) known, it is assumed that the increase in invariant values \( I^+ \) is linear and thus all invariants \( I_i^+ \) can be calculated in the kernel zone (see Figure 3.1). With the knowledge that \( \theta = 0 \) at the axis, the invariants \( I^- \) are simply calculated as \( I_i^- = -I_i^+ \) at each characteristics’ intersection with the axis.

As the value of \( \theta \) at the wall is imposed, \( \omega \) can be obtained as simply \( \omega_i = \theta - I_i^- \). The positive invariants \( I^+ \) in the transition zone are then determined from definitions. The invariants throughout the flow field are
3. Methodology

then known and the remaining values of $\theta$, $\omega$, Mach number and $\mu$ can easily be found.

The final step is to determine the position of each point in Cartesian coordinates. The inclination of the invariants is defined in Equation 2.22. Thus, the position of the nodes can be found by simple trigonometry.

The length of the nozzle is the minimum which provides uniform exit velocity. This condition is met when the final characteristic intersects the wall of the nozzle, line BC in Figure 3.1.

An example of the results for the simple conic nozzle design is displayed in Figure 3.2.

![Conic nozzle with 20° half angle and 20 characteristic lines.](image)

Figure 3.2.: Conic nozzle with 20° half angle and 20 characteristic lines.
3. Methodology

**Minimum Length Nozzle Design**

The code developed to design a simple, conic MOC nozzle was further developed to create a code to design minimum length supersonic nozzles with uniform exit conditions in two-dimensions.

The initial conditions for the minimum length nozzle are largely the same, however, instead of supplying a wall angle, the exit Mach number is imposed.

The following steps follow the same algorithm but instead of assuming a linear distribution of invariants, the Mach distribution was assumed linear and the invariants were calculated based on the local Mach number, to provide greater accuracy in calculations.

A simple example of the resulting plot is displayed in Figure 3.3.

![Figure 3.3: Minimum length two-dimensional nozzle with $M_E = 3$ and 20 characteristic lines.](image)
3. Methodology

The code was validated by comparing results to a similar code produced. Further validations were done by checking the mass flowrate parameter and the thrust produced by the nozzle against one-dimensional calculations.

The two-dimensional MOC produces a wedge-shaped nozzle with a rectangular outlet (see Figure 3.4). Thus the length of the nozzle is expected to be significantly greater than that of the three-dimensional axisymmetric nozzle. Wedge nozzles are uncommon and represent a weight disadvantage compared to the axisymmetric nozzle. The nozzles produced with the two-dimensional code in this work are calculated for a unit breadth and unit throat height.

![Figure 3.4: Example of a typical wedge nozzle.](image)

3.1.2. Three-Dimensional Method of Characteristics

In three-dimensional MOC, the relationships known as invariants up to this point, are no longer constant. Thus, the MOC becomes significantly more complex. The method employed is as follows.

Initial conditions are imposed as in the two-dimensional cases. However, $M_E$ is not imposed but $M$ at the end of the Prandtl-Meyer expansion fan is imposed and through simple trial and error it can be used to impose $M_E$ at the nozzle exit. In addition to entering a desired number of characteristic lines, a number of subdivisions in the first characteristic can be added to
increase the accuracy of the computation. These inlet subdivisions can be seen in Figure 3.5.

![Figure 3.5: Five subdivisions of the first characteristic.](image)

From the initial conditions, $I^+$ at the throat and $I^+$ at the exit can be determined as well as the initial Prandtl-Meyer expansion fan properties.

The kernel zone is then computed iteratively by assuming an initial $\theta$ based on the results of the expansion fan. The bisection method of convergence is used to iterate the angles and thus position associated with each point. At the axis, the same iterations are carried out but with the condition $\theta = 0$ imposed.

In the transition zone similar iterations are carried out, as with the kernel zone. However, no initial value for $\theta$ is known, as there is from the expansion fan in the kernel zone. Thus, the iterations are done within another iterative solver to deduce the value of $\theta$ at the wall, from an initial guess. The error for these particular iterations is typically of the order of $1e^{-10}$.
3. Methodology

Figure 3.6.: Minimum length three-dimensional nozzle with $M_E = 3$, 20 characteristic lines and 5 inlet subdivisions.

The results are then validated by comparing the values to the ones predicted by a one-dimensional analysis. Figure 3.6 shows a sample of the results obtained by the three-dimensional MOC code.

3.2. Nozzle Truncation

As explained in Section 2.4.2, CTIC nozzles can be created from the resulting contours of MOC designed, minimum length nozzles.

To recreate the example plot in Figure 2.9, a number of cases were examined to produce a variety of nozzle contours. The thrust coefficient $C_F$ along the contour walls was then calculated. The nozzle contours were then plotted
3. Methodology

and the thrust coefficients were plotted for constant values across each nozzle contour so that the number of $C_F$ data points was equal to the number of nozzle contours produced.

The thrust coefficients used were determined by the $C_F$ at the nozzle exits so that the contours would be tangential to $C_F$.

With the plot in Figure 2.9 reproduced, the nozzle could be truncated at the axial distance corresponding to the inflection of the chosen thrust coefficient parabola. This resulted in a TIC nozzle.

Creating a CTIC nozzle from a TIC nozzle involves compressing the TIC wall contour. To compress the contour in the axial direction, Equation 3.1 was used with an imposed compression constant $k$. Of course, the radius of each point remains unchanged.

\[
x_{i,new} = \frac{x_i - x_1}{x_N} k
\]  

(3.1)
4. Results & Discussion

As mentioned in Section 3.1.1, due to the symmetry of the characteristic lines, the contours need only be calculated for one side of the nozzle. Thus, the contour graphs displayed here are only half of the nozzle.

The gas properties for each contour generated were the same in all cases ($\gamma = 1.25$).

4.1. Two-Dimensional Nozzles

4.1.1. Ideal Contours

By computing the nozzle contours for $M_E = 3$ with varying mesh density, it is observed that the length of the nozzle decreases with the refinement of the mesh. Figures 4.1 through 4.6 show the plots of the contours for various mesh densities. The half-height of the nozzle is also seen to decrease with the increase in mesh density.

Nozzle length is seen to vary from 26.47 throat radii for 10 characteristic lines, to 22.95 throat radii for 200 characteristic lines. The result becomes steadier with the refinement of the mesh. Maximum nozzle half height varies from 5.04 throat radii for 10 characteristic lines to 4.88 throat radii for 200 characteristic lines. Again, the result becomes steadier as the mesh is increasingly refined.
4. Results & Discussion

Figure 4.1.: Numerical output for Mach 3 and 10 characteristic lines.

Figure 4.2.: Numerical output for Mach 3 and 30 characteristic lines.
4. Results & Discussion

Figure 4.3: Numerical output for Mach 3 and 50 characteristic lines.

Figure 4.4: Numerical output for Mach 3 and 75 characteristic lines.
4. Results & Discussion

Figure 4.5.: Numerical output for Mach 3 and 100 characteristic lines.

Figure 4.6.: Numerical output for Mach 3 and 200 characteristic lines.
4. Results & Discussion

As Mach number increases, the initial expansion becomes more severe. More rapid expansions lead to longer straightening sections to obtain uniform exit conditions. This can be observed in Figures 4.7 through 4.10. As the figures show, a greater fraction of the nozzle length is occupied by the straightening section as the Mach number increases while a greater fraction of the nozzle radius is occupied by the expansion section.

Figure 4.7.: Numerical output for Mach 2.
4. Results & Discussion

Figure 4.8: Numerical output for Mach 3.

Figure 4.9: Numerical output for Mach 4.
4. Results & Discussion

These relationships lead to the exponential increase between Mach number and nozzle length depicted by the graph in Figure 4.11.

Figure 4.10.: Numerical output for Mach 5.
4. Results & Discussion

![Nozzle Length vs. Mach Number](image)

Figure 4.11: Relationship between Mach number and nozzle length.

Further validating the value of increased mesh density, the error, as calculated with respect to propellant mass fraction, decreases with the increase in number of characteristic lines (see Figure 4.12).
4. Results & Discussion

Figure 4.12: Error in propellant mass fraction calculations versus number of characteristic lines.
4. Results & Discussion

4.1.2. Truncated Ideal Contour

As explained in Section 2.4, having plotted the nozzle contours with curves of constant thrust coefficient, the nozzles can be truncated. For the ideal two-dimensional contours obtained, this plot is shown in Figure 4.13.

The plot is then used to obtain a TIC nozzle of thrust coefficient 1.65 based on the ideal contour nozzle produced for design $M_E = 3$. Truncating the nozzle at the inflection point of the thrust coefficient curve produces a TIC nozzle of length $x/r_t \approx 13$. 

Figure 4.13.: Zoomed ideal nozzle contours with lines of constant thrust coefficient.
Figure 4.14: Characteristic lines for a truncated ideal contour (TIC) nozzle obtained by truncating the ideal contour nozzle presented in Figure 4.8.

The TIC nozzle produced, outputs a thrust coefficient of $C_F = 1.64$ and the exit Mach number at the wall is $M_E = 2.72$ rather than 3, as a result of the shortened straightening section. The lowest Mach number at the exit is $M_E = 2.47$.

The dimensionless surface area of the ideal nozzle was 442.96. The surface of the TIC nozzle is 231.52, a reduction in surface area of 47.73%. The nozzle length has reduced from $x/r_t = 23.4$ to $x/r_t = 13.4$, a reduction in length of 42.74%.
4. Results & Discussion

4.1.3. Compressed Truncated Ideal Contour

Making use of a compression factor $k = 0.75$, in conjunction with Equation 3.1, the following compressed contour is obtained (highlighted green in Figure 4.15).

![Figure 4.15: Representation of the wall contours for an ideal contour nozzle, a TIC nozzle and a CTIC nozzle based on the ideal contour for uniform exit velocity $M_E = 3$, in two dimensions.](image)

The length of the two-dimensional CTIC nozzle is $x/r_1 \approx 12.66$. 
4. Results & Discussion

4.2. Three-Dimensional Nozzles

4.2.1. Ideal Contours

Section 3.1.2 describes the use of inlet ray subdivisions to increase the accuracy of the model. The overall length of the nozzle is found to increase with the number of inlet characteristic subdivisions. Figures 4.16 through 4.19 show the numerical outputs for $M_E = 3$ with varying number of inlet subdivisions.

Figure 4.16: Numerical output for Mach 3 exit velocity and no inlet characteristic subdivisions.
4. Results & Discussion

Figure 4.17: Numerical output for Mach 3 exit velocity and two inlet characteristic subdivisions.

Figure 4.18: Numerical output for Mach 3 exit velocity and four inlet characteristic subdivisions.
4. Results & Discussion

Figure 4.19: Numerical output for Mach 3 exit velocity and nine inlet characteristic subdivisions.

Similar results are found by increasing the number of characteristic lines. As the number of characteristic lines increases, the overall nozzle length increases. This observation is contrary to the change in length observed in the two-dimensional model. Figures 4.20 through 4.23 show the numerical outputs for $M_E = 3$ with varying number of characteristic lines.
4. Results & Discussion

Figure 4.20.: Numerical output for Mach 3 exit velocity and 10 characteristic lines.

Figure 4.21.: Numerical output for Mach 3 exit velocity and 30 characteristic lines.
4. Results & Discussion

Figure 4.22.: Numerical output for Mach 3 exit velocity and 50 characteristic lines.

Figure 4.23.: Numerical output for Mach 3 exit velocity and 100 characteristic lines.
4. Results & Discussion

4.2.2. Truncated Ideal Contour

As with the two-dimensional contours the ideal nozzle contours are plotted with lines of constant thrust coefficient, to truncate the ideal nozzle produced by the code.

Truncating the ideal nozzle of design $M_E = 3$ at the inflection of $C_F = 1.6$ produces the TIC nozzle shown in Figure 4.25.
4. Results & Discussion

Figure 4.25.: Characteristic lines for a truncated ideal contour (TIC) nozzle, in three dimensions, obtained by truncating the ideal contour nozzle presented in Figure 4.22.

The thrust coefficient at the exit of the TIC nozzle in Figure 4.25 is found to be $C_T = 1.58$, compared to $C_T = 1.65$ at the exit of the ideal contour nozzle. The truncated nozzle represents a shortening of almost 64%, from $x/r_t \approx 9.85$ to $x/r_t \approx 3.52$. The Mach number suffers a reduction from uniform exit $M_E = 3.00$ to $M_E = 2.52$ at the wall of the TIC nozzle. The dimensionless surface area of the ideal contour nozzle was 73.24 and the surface area of the TIC nozzle is 19.07. The total surface area of the nozzle has reduced by 74% by truncating the nozzle.
4. Results & Discussion

4.2.3. Compressed Truncated Ideal Contour

Making use of Equation 3.1 with $k = 0.75$, the TIC nozzle obtained was compressed to produce the CTIC nozzle displayed against the ideal contour and the TIC in Figure 4.26.

![Graph showing contour comparisons](image)

Figure 4.26: Representation of the wall contours for an ideal contour nozzle, a TIC nozzle and a CTIC nozzle based on the ideal contour for uniform exit velocity $M_E = 3$.

The three-dimensional CTIC nozzle is of length $x/r_t \approx 2.77$. 
4. Results & Discussion

4.3. Nozzle Parameters

Table 4.1 shows the axial length of the six contours produced for $M_E = 3$. The difference in length between the two-dimensional nozzles and the three-dimensional nozzles is significant. On average, the three-dimensional nozzles are 32.13% of the length of the two-dimensional nozzles.

<table>
<thead>
<tr>
<th>Length ($x/r_t$)</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal</td>
<td>23.4</td>
<td>9.753</td>
</tr>
<tr>
<td>TIC</td>
<td>13.41</td>
<td>3.52</td>
</tr>
<tr>
<td>CTIC</td>
<td>12.66</td>
<td>2.77</td>
</tr>
</tbody>
</table>

Table 4.1.: Lengths of the six contours generated for $M_E = 3$.

The surface area of the nozzles is unit-less as it is calculated in terms of the non-dimensionalised axial length and radius, $x/r_t$ and $r/r_t$ respectively. The surface areas are relevant as they relate to the mass of the nozzle, although not directly as the nozzle thickness is usually not uniform along the nozzle length. For each of the relevant contours, the surface area is shown in Table 4.2.

<table>
<thead>
<tr>
<th>Surface Area</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal</td>
<td>187.6</td>
<td>75.24</td>
</tr>
<tr>
<td>TIC</td>
<td>100.93</td>
<td>22.66</td>
</tr>
<tr>
<td>CTIC</td>
<td>72.47</td>
<td>18.79</td>
</tr>
</tbody>
</table>

Table 4.2.: Surface Area of each of the six nozzles defined by the contours generated for $M_E = 3$.

As the MOC theory collapses for the CTIC nozzles produced, more complex analysis mechanisms are required to compute the flow. Computational Fluid Dynamics models using transient hybrid solvers can be utilised to assess the performance of the CTIC nozzles as well as observing the effect of the oblique shocks propagated and their locations within the nozzle. Unfortunately, such analyses are outside the scope of this work. According to the theory, it is expected that the CTIC nozzles presented here would produce
4. Results & Discussion

a Mach number greater than the corresponding TIC nozzle, with lesser surface area. The external drag may not be significantly reduced despite the reduction in surface area as the angle at which the surface intersects the external flow is increased. However, the external drag represents a minor contribution to overall nozzle performance. Consequently, the CTIC nozzles included previously will be disregarded hence forward.

<table>
<thead>
<tr>
<th>Mach Number</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>TIC</td>
<td>2.72</td>
<td>2.52</td>
</tr>
</tbody>
</table>

Table 4.3.: Exit Mach number of ideal and TIC nozzles defined by the contours generated for $M_E = 3$.

The nozzles presented here have been truncated with reference to arbitrarily chosen thrust coefficients, merely as an example to use for analysis, as they have no intended application. In actual design processes, nozzles are generally truncated in accordance with design length or area ratios imposed. Table 4.3 details the change in Mach number as a result of the truncation of the ideal contour nozzles.

There is a more significant compromise in $M_E$ for the axisymmetric nozzle compared to the two-dimensional, wedge nozzle. This is a product of the fact that the ideal contour of the axisymmetric nozzle is significantly shorter than that of the ideal wedge nozzle.

To understand the effect of truncating the nozzles more comprehensively, it is worth assessing the change in $C_F$.

<table>
<thead>
<tr>
<th>Thrust Coefficient</th>
<th>2D</th>
<th>3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal</td>
<td>1.66</td>
<td>1.65</td>
</tr>
<tr>
<td>TIC</td>
<td>1.64</td>
<td>1.58</td>
</tr>
</tbody>
</table>

Table 4.4.: Thrust coefficient at the wall exit of ideal and TIC nozzles defined by the contours generated for $M_E = 3$.

As with the Mach number, the three-dimensional nozzle suffered greater losses in truncation. This is to be expected as the portion of the length
4. Results & Discussion

truncated for the axisymmetric nozzle was greater than the fraction of the two-dimensional contour truncated in creating the TIC nozzle.

It is important to note that the losses as a result of truncating the ideal contour nozzles is much less than the amount of length and surface area sacrificed.

The codes created can be used to analyse the effect of many variables considered when designing a nozzle, whether it is for a wind tunnel, a rocket nozzle or a supersonic aircraft. Using a model for an axisymmetric nozzle with $M_E = 3$, the effect of variation in $\gamma$ was examined.

![Figure 4.27](image)

Figure 4.27: Maximum nozzle height relative to the throat radius, as a function of $\gamma$. Results obtained for three-dimensional calculations with $M_E = 3$ imposed.

Figures 4.27 and 4.28 show an inverse relationship between length and height, as $\gamma$ increases. While the nozzle’s area ratio decreases, the length
4. Results & Discussion

increases with $\gamma$, meaning the nozzle becomes longer and narrower, with a slower expansion with higher values of $\gamma$. This is a result of the gas expanding less rapidly as the specific heat of the gas at constant pressure increases more than the specific heat at constant volume.

Figure 4.28.: Nozzle length as a function of $\gamma$. Results obtained for three-dimensional calculations with $M_E = 3$ imposed.

As mentioned in the theory section of this report, the MOC breaks down at hypersonic exit velocities. The error between the thrust coefficient $C_F$ calculated for the MOC nozzle and the expected $C_F$ calculated with the one-dimensional model begins to increase rapidly when the velocity crosses Mach 5. The increased error is displayed in Table 4.5.
4. Results & Discussion

<table>
<thead>
<tr>
<th>$M_E$</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5.37</td>
</tr>
<tr>
<td>6</td>
<td>10.76</td>
</tr>
<tr>
<td>7</td>
<td>17.6</td>
</tr>
<tr>
<td>8</td>
<td>25.91</td>
</tr>
<tr>
<td>9</td>
<td>35.61</td>
</tr>
</tbody>
</table>

Table 4.5.: Error in Thrust Coefficient for two-dimensional model with 50 characteristic lines.

The two dimensional model was used with fifty characteristic lines to assess the hypersonic error as the axisymmetric model cannot compute contours for $M_E \geq 8$. At $M_E \approx 8$, the code does not solve and at velocities greater than Mach 8, the code returns an error in iterating the value of $theta$ at the nozzle wall.
5. Conclusions

5.1. Summary

The Method of Characteristics is well defined, having been the preferred method of defining supersonic nozzle contours since the 1960s. In this report, the relevant theory and methods have been outlined before detailing the results obtained for the MOC calculations carried out. It has been explained, that the codes were validated and the methods by which the validations were done.

Having solved the system of PDEs to produce nozzle contours, the results have shown the relationships between nozzle design features and the variables involved in the calculations, which are not clear from examination of the equations. The capabilities of the method of characteristics have also been investigated, leading to the confirmation that the method breaks down above and below the supersonic range of exit velocities. As a greater fraction of the flow becomes hypersonic/subsonic, the calculation error increases.

After producing ideal contours, the nozzles designed by the programmes developed, were truncated in accordance with the theory laid out. The truncated nozzles were then compressed. However, the compressed truncated nozzles could not be analysed with the same methods as the ideal and simply truncated nozzles. It would have been interesting to investigate the effect of varying the compression factor in the equation proposed. However, as mentioned in the Introduction of this report, a significant constraint placed on the project was the time available to complete it.
5. Conclusions

5.2. Recommendations for Future Work

The method of characteristics for ideal nozzle design is well understood. However, there are conflicting reports on the validity of CTIC nozzles. The original proposal by Gogish [9] predicted the CTIC nozzle to be superior to the Rao nozzle. This was later debunked by Hoffman [4] although he did find the CTIC nozzle to be a very close alternative to the Rao nozzle with the performance being 0.04% to 0.34% lower for the CTIC nozzle in the parametric analysis carried out. It seems that an optimal CTIC design method has yet to be defined.
Bibliography


Appendix A.

Characteristic Directions
Appendix A. Characteristic Directions

Two-dimensional steady, isentropic, irrotational flow is expressed by the differential equation of the velocity potential.

\[(c^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2 - \phi_y^2)\phi_{yy} = 0 \quad (A.1)\]

\(u\) and \(v\) are the \(x\) and \(y\) components of the velocity vector, respectively and are given by:

\[u = \phi_x; \quad v = \phi_y \quad (A.2)\]

The local velocity of sound is related to the velocity potential by the following energy equation, where \(c_0\) is the sound velocity at the stagnation temperature.

\[c^2 = c_0^2 - \frac{k - 1}{2}(\phi_x^2 + \phi_y^2) \quad (A.3)\]

The differential equation, providing a solution to Equation A.1, is of the general form:

\[A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} = D \quad (A.4)\]

Two pairs of simultaneous, ordinary, first order differential equations define the characteristic curves of Equation A.1.

\[\left(\frac{dy}{dx}\right)_{l^+} = \frac{B + \sqrt{B^2 - AC}}{A} \quad (A.5)\]

\[\left(\frac{dy}{dx}\right)_{l^-} = \frac{B - \sqrt{B^2 - AC}}{A}\]
Appendix A. Characteristic Directions

\[
\left( \frac{d\phi_y}{d\phi_x} \right)_{l^+} = \frac{B + \sqrt{B^2 - AC}}{C} + \frac{D}{C} \left( \frac{dy}{d\phi_x} \right)_{l^+} \\
\left( \frac{d\phi_y}{d\phi_x} \right)_{l^-} = \frac{B - \sqrt{B^2 - AC}}{C} + \frac{D}{C} \left( \frac{dy}{d\phi_x} \right)_{l^+}
\]

(A.6)

Looking at Equations A.1 and A.4, the coefficients A, B, C and D are clear.

\[ A = c^2 - u^2; \quad B = -uv; \quad C = c^2 - v^2; \quad D = 0 \]

Substituting these coefficients into Equations A.5 and A.6, the following definitions of characteristic directions at each point in the physical plane are obtained.

\[ \left( \frac{dy}{dx} \right)_{l^+, l^-} = \frac{-uv \pm \sqrt{u^2 + v^2 - 1}}{1 - \frac{u^2}{c^2}} \]

(A.7)

At each point in the flow field, the characteristic directions are defined by Equation A.7. It is clear from the equation that characteristics are only real for supersonic flow \((u^2 + v^2 > c^2)\).

Using polar coordinates \((V\) and \(\theta)\) for the hodograph plane, it is obtained:

\[ u = V \cos \theta; \quad v = V \sin \theta \]

Introducing these to Equation A.7 and making use of the following relationships, Equation A.8 is obtained.

\[ V/c = M; \quad M = 1/\sin \mu; \quad \sqrt{M^2 - 1} = 1/\tan \mu; \]

\[ \left( \frac{dy}{dx} \right)_{l^+, l^-} = \frac{-\cos \theta \sin \theta \pm \frac{1}{\tan \mu}}{1 - \frac{\cos^2 \theta}{\sin^2 \mu}} \]

(A.8)
Appendix A. Characteristic Directions

Rearranging Equation A.8 with the help of typical trigonometric identities, the following relationships are obtained, as mentioned in Section 2.3.3, *The Physical Characteristics.*

\[
\left( \frac{dy}{dx} \right)_I = \tan(\theta - \mu); \quad \left( \frac{dy}{dx} \right)_{II} = \tan(\theta + \mu) \quad \text{(A.9)}
\]
Appendix B.

Derivation of the Prandtl-Meyer Function
Appendix B. Derivation of the Prandtl-Meyer Function

\[ d\omega = \sqrt{M^2 - 1} \frac{du}{u} = \sqrt{M^2 - 1} \left( \frac{dM}{M} + \frac{1}{2} \frac{dT}{T} \right) \quad (B.1) \]

But

\[ c_p T + \frac{u^2}{2} = \text{constant} \quad (B.2) \]

Thus,

\[ T = \frac{T_0}{1 + \frac{\gamma - 1}{2} M^2} \quad (B.3) \]

Leading to

\[ \frac{dT}{t} = -\frac{\gamma - 1}{2} M \frac{dM}{1 + \frac{\gamma - 1}{2} M^2} \quad (B.4) \]

\[ d\omega = \frac{\sqrt{M^2 - 1}}{M} \left( 1 - \frac{\gamma - 1}{2} \frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) dM = \frac{\sqrt{M^2 - 1}}{M} \frac{dM}{1 + \frac{\gamma - 1}{2} M^2} \quad (B.5) \]

Equation B.5 is then integrated with \( \omega = 0 \) at \( M = 1 \) to give:

\[ \omega = K \arctan \frac{\sqrt{M^2 - 1}}{K} - \arctan \sqrt{M^2 - 1} \quad (B.6) \]

Where \( K \) is a constant defined by:

\[ K = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \quad (B.7) \]