Study of the computational performance of different models for pantograph/catenary dynamics

Llicenciat en Enginyeria Mecànica – Final Project

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La dinàmica del pantògraf / catenària és un dels camps més importants per estudiar a l'hora de millorar els dissenys de trens elèctrics. La interacció dinàmica entre el pantògraf i la catenària és un criteri decisiu per al desenvolupament d'un sistema de pantògrafs / catenàries eficient en termes de manteniment i per a la velocitat màxima del tren. S'han proposat diversos models matemàtics, que tracten l'anàlisi de la força de contacte entre el pantògraf i la catenària, utilitzant diferents tècniques per resoldre el problema. L'objectiu principal d'aquesta tesi és proporcionar un estudi que contingui el rendiment computacional de cada model i la seva exactitud. Es fa un estudi detallat d’uns quants models rellevants i es mostra la seva comparació mitjançant diferents solucionadors. A més, es proporcionen els mèrits i els demèsters de cada model i es dóna una decisió sobre el model de millor rendiment.

Paraules clau - Força de contacte, mètodes d'elements finits (FEM), dinàmica de pantògrafs / catenàries, funció de Green, feix d'Euler-Bernoulli.
Resumen (ESP)

La dinámica del pantógrafo / catenaria es uno de los campos más importantes para estudiar cuando se trata de mejorar los diseños de los trenes eléctricos. La interacción dinámica entre el pantógrafo y la catenaria es un criterio decisivo para el desarrollo de un sistema eficiente de pantógrafo / catenaria en términos de mantenimiento y para la velocidad máxima del tren. Varios modelos matemáticos, que tratan el análisis de la fuerza de contacto entre el pantógrafo y la catenaria, se han propuesto utilizando diferentes técnicas para resolver el problema. El objetivo principal de esta tesis es proporcionar un estudio que contenga el rendimiento computacional de cada modelo y su precisión. Se lleva a cabo un estudio detallado de algunos modelos relevantes y se muestra su comparación con diferentes solucionadores. Además, se proporcionan los méritos y deméritos de cada modelo y se da una decisión sobre el mejor modelo de rendimiento.

Palabras clave: fuerza de contacto, métodos de elementos finitos (FEM), dinámica de pantógrafo / catenaria, función de Green, haz de Euler-Bernoulli.
Abstract (ENG)

The pantograph/catenary dynamics is one of the most important fields to study when it comes to improvement of the designs of electric trains. The dynamic interaction between the pantograph and the catenary is a decisive criterion for the development of efficient pantograph/catenary system in terms of maintenance and for the maximum train speed. Several mathematical models, which deal with the analysis of the contact force between the pantograph and the catenary, have been proposed using different techniques to solve the issue. The main aim of this thesis is to provide a study that contains the computational performance of each model and its accuracy. A detailed study of a few relevant models is carried out and its comparison using different solvers is shown. Furthermore, the merits and demerits of each model are provided and a decision on the best performing model is given.

Key words- Contact force, Finite element methods (FEM), Pantograph/Catenary dynamics, Green’s function, Euler-Bernoulli beam.
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1. Aim & Scope:

To study the computational efficiency of different computational models of the pantograph/catenary interaction problem. The models that will be considered in the benchmark comparison are a 3D finite element method (FEM) model, and a 2D semi-analytical model. All the models will be revised and tested and parametric studies to compare the efficiency and the accuracy of the algorithms will also be performed. Finally, a decision on the best performing algorithm will be provided.

As per the aim, the main objective of this study is to review existing models and suggest the best algorithm for pantograph/catenary interaction problems. Therefore, the focus of the thesis is:

- To establish an understanding about the advantages of the rigid catenary system over the traditional catenary system
- To review the FEM models and compare the results of both the 2D and 2D FEM models.
- To compare the solutions of different ode solvers and the time take by the ode algorithms for different finite elements per span.
- To review a semi-analytical model and compare the results obtained with the FEM models.
- Decision on the best working model based on the accuracy of the results and the time taken to compute the results.

However, the thesis does not consider the results between the modelling of the traditional catenary with the rigid catenary system. All the models reviewed only consider the catenary as a rigid beam. The thesis focuses on the time taken by the ode solvers only and does not make modifications on how the pantograph/catenary are modelled.
2. Justification

Various mathematical models have been proposed to model the interaction between the pantograph and the catenary. It is important to simulate the dynamic interaction virtually so as to reduce the overall cost of maintenance of the two systems. However the simulations have to be accurate and computationally efficient in order to be reliable. In order to achieve this, investments are to be made on powerful computers which is not deemed wise or compromise on the accuracy to save time which will reduce the reliability of the model drastically. When proposing new models for the interaction, it is imperative to understand the existing models and their shortcomings. Furthermore to test the reliability of new models it is best to compare the results with the existing models. Therefore the main aim of this thesis is to review the existing models of pantograph/catenary interaction and make comparisons between them to determine the best working model based on its computational performance and accuracy. This will provide a base for future interaction models which can be tested.

The thesis starts off with an introduction to topics related to the project. Then a literature review which gives a history of all the research carried out with respect to pantograph/catenary interaction. Chapter 1 gives a brief introduction to finite element methods, rigid multibody dynamics and pantograph/catenary dynamics. Chapter 2 deals with the study of both 3D and 2D FEM models and their comparison. Chapter 3 deals with the study of semi-analytical model and its comparison with the FEM models. Then finally, a review of all the models with a decision on the best model is given in the Conclusions section.
3. Introduction

Electric trains have come a long way in the history of transportation. Power transmission for trains is a vital area of study for better performance and reduced maintenance costs. The pantograph/catenary system is an elaborate design which can ensure good quality of electricity transmission at relatively high speeds. As the name implies, the pantograph/catenary system contains two sub-systems, the pantograph, mounted on the roof of train set and the catenary, the overhead power line suspended above the track. The electric current is collected by the pantograph from the catenary and returned through the train wheels back to the rails. As the rails and wheels are always in good contact during operation, the contact between pantograph and catenary becomes one of the key issues for the reliability of railway operation.

Based on its structure the catenary is divided into two types- the traditional catenary and the rigid catenary. Traditional catenary system consists of typically three parts- the contact wire, droopers, and the messenger wire (Figure 1). The contact wire is made of copper or copper alloy due to the good electric conductivity of copper. Because of gravity, the wires cannot keep level by themselves and they tend to droop, so the contact wire is kept level with the help of droopers of different lengths at specific positions [1]. To spread the wear uniformly along the width of the carbon strips the catenary is arranged in a zigzag pattern (when viewed from the top) as shown in Figure 2. However such a system is susceptible to heavy cross winds which in turn affect the interaction between the pantograph and catenary.
In recent years, a new catenary system has been specially designed for railway tunnels, called rigid catenary. Rigid catenary comprises a hollow aluminium profile rail, with an opening in the underside which acts as a pincer or pliers, thereby clamping the copper contact wire in place (Figure 3(a) & 3 (b)). The aluminium rails come in lengths of between 5 and 12 m and are connected to each other by means of bridles, located and bolted on the inside of the profile, thus forming longer spans called overlap sections [2]. Each rail is suspended from the tunnel roof arch, its correct positioning being ensured by a support which allows for its vertical and lateral position to be regulated. These supports also allow for any longitudinal displacement that might occur as a result of thermal expansion. The advantage of this system is that it has a very high bending stiffness and is not easily deformed by cross winds. The maintenance cost of said system is comparatively lower and provides good overhead clearance.
The pantograph, as stated earlier, is a current collecting device which when in contact with the contact wire transfers the electric power from the contact wire to the drive motor (Figure 4). It consists of a lifting device, guide rods, lower and upper arms and the pantograph head. The supporting frame is powered by the pneumatic cylinder on the base and is vertically raised up from the folded position to a certain range of working height while pushing the pan-head against the catenary with a roughly constant uplift force. The pan-head is wide enough to cover all possible positions of the catenary in the lateral direction.

The pantograph is typically described using its four degrees of freedom and they are (Figure 5):

- Raising the pan head
- Bounce of the panhead
- Roll
- Pitch
Due to the importance of the contact force between the catenary and the pantograph, a significant amount of research studies has been conducted in the last decade to improve its efficiency. A major problem regarding the contact force is that if it is too less then there is loss of power and if it is too high then there is wear of the carbon strips on the pantograph head which in turn leads to arcing and again leading to the same problem of loss of power[3]. The main requirement for the measurement of contact force is to ensure uninterrupted power supply and long service life of the carbon strips which in turn reduces maintenance cost of the trains[]. Therefore for the design of efficient pantograph catenary system it is required to maintain sufficient contact for good power transmission.. The contact force is composed essentially of a static force component given by the application force and a dynamic component dependent on the running speed and the vibrational behaviour of the catenary and the pantograph.
4. Literature Review

The pantograph catenary interaction has been the key point of research in this field. The focus has been mostly on the variation of contact force between the pantograph and catenary as it is the decisive criterion for quality of power transmission. Earlier models of pantograph catenary interaction depended on the basic dynamic principles. Poetsch et al [4] published paper which analysed the different models used for the pantograph/catenary system. The most basic design models that were used are lumped-mass system, Infinite string model and drooper-reflection model.

Earlier mathematical models of catenary were related to the traditional catenary system. Most of the research was involved in the enhancement of high speed train system. The traditional catenary system was modelled using simple standard elements (one-dimensional wires and droppers, registration arms, insulators and supports). The simplest model for catenary is the string model. It considers the contact wire as a string with constant mass per unit length and its simplicity lies in its solution of partial differential equations (PDE’s). This model also took into account the propagation of the disturbance throughout the length of the catenary.

Formulation of the catenary as a rigid beam was put forth by Cook et al [5]. This theory considers the bending stiffness and the mass per unit length of the beam. Unlike Timoshenko’s beam theory, the Euler-Bernoulli formulation does not take into account the rotation of the beam about its bending axis. Therefore The Euler-Bernoulli beam is considered to be stiffer. This became the foundation for the modelling of the catenary.

A comparison of different models for the catenary was carried out by Hedayati Kia et al [6]. In this paper the author describes different models used for the catenary. The variation of contact force in catenary is caused by the variation of stiffness along the span of the catenary. The catenary is modelled using a combination of Euler-Bernoulli formulation and Timoshenko beam formulation. The pantograph models being compared are the low order model which considers a linear relation between the rigid bodies and a multibody system which considers the rotation of the pan-head. From the results it is shown that for low speeds the static model of catenary can gives the reasonable result for
contact force estimation. For high speed catenary systems the effect of wave propagation along the span of the catenary must also be considered for contact force calculation. It is also seen that FEM based Euler-Bernoulli-Timoshenko formulation is useful for high speed catenary and the multibody system is most suitable for modelling the pantograph.

Another model for the pantograph/catenary interaction was proposed by Seo et al [7]. The catenary was modelled using nonlinear continuous beam element, which was based on an absolute nodal coordinate formulation. The pantograph was modelled using rigid multibody dynamics. The results obtained from the simulation were then compared with experimental data obtained from a running high-speed train.

A study on the recent developments in pantograph/catenary dynamics was present by Pombo and Ambrosio[8]. The paper provides an overview of the different computational methods and also provides minimum requirements for each model development. In the paper it is found that using Euler-Bernoulli beam formulation is adequate enough for describing the catenary. For modelling the pantograph, it is ideal to use lumped-mass system as it provides all the required information. When it comes to the contact model, the penalty method is ideal to monitor the contact force.

Pombo and Ambrosio proposed a model which used FEM and rigid multibody dynamics but also included the influence of the roughness of the track and other environmental influences [9, 10]. The studies were carried out on high speed trains travelling at roughly 300km/h. They also tested for the influence of multiple pantographs connected to the catenary and its effects on current collection. The paper shows that the leading pantograph interferes with the performance of the trailing pantograph. The leading pantograph creates excitations on the catenary which causes the trailing pantograph to experience high amplitudes of contact force and loss of contact. Furthermore wind loads have been shown to raise the pantograph which increases the contact force. The surface roughness of the railway track has small effects on the contact force.

Qian et al [11] proposed a model where again, the catenary was modelled using Euler-Bernoulli formulation and the pantograph was modelled as a simple multibody spring mass damper system. However the coupled these two system through the induced friction
during the contact. This model also tests for the effect of the wind loads on the contact force and as a result the wind load has a minor effect on the interaction. The results showed that larger the friction coefficient stronger is the friction coupling, however this lead to decrease in the damping coefficient which in turn lead to unstable vibrations.

An advanced 3D model of the pantograph catenary interaction was proposed by Jesus Benet et al [12]. A realistic model was developed which could account for some details not found in traditional models like the lateral displacement of the contact wire, or a lateral wind load actuating on the catenary. Furthermore, the dynamic equations of the pantograph have been formulated newly, considering this pantograph as an articulated multibody system, by using independent coordinates and symbolic computation. The model also includes a procedure to find the drooper lengths for different contact wire configurations and can be used for different catenaries.

Luka Skrinjar et al [13] put forth a paper which gave a detailed analysis of the contact models that could be used for the pantograph/catenary interaction. The paper reviews two types of contact groups-point contact and cylindrical contact. 20 models have been reviewed under point contact and 10 models under cylindrical contact. The effects of the different hysteresis-damping models on the presented general contact-force models are compared. The paper gives insight into the different models used and also its preferred applications.

Carnicero et al [14] presented a paper which deals with the influence of the railway track irregularities and its effects on the pantograph/catenary interaction. They have simulated over 180 tracks with their respective pantograph/catenary system and obtained general conclusions on the influence of the track irregularities. It is been found that the track irregularities provide negligible effect on the pantograph/catenary interaction. The results show that track perturbations provide less than 5% difference in the contact force. However it shows a difference to the displacement of the contact point on low quality tracks.
5. Chapter 1: Introduction to Key Concepts

In this chapter a brief introduction to finite element methods, rigid multibody dynamics and pantograph/catenary dynamics is given.

5.1 Introduction to Finite Element Methods

Finite element method (FEM) is a numerical method used to accurately solve complex engineering problems. The basic idea behind FEM is to replace complex problems with a simpler one. In other words we divide any given system into subsystems so that each individual subsystem can be solved separately. The subsystems can also be further divided to simplify the problem. As a result the solutions obtained are approximate in nature. However the accuracy of the results can be altered by spending more computational effort[15]. The basic process for FEM is the same irrespective of the nature of the system (Figure 6)-
Figure 6: Steps involved in FEM[21]

Depending on the type of mesh and the number of cells created the computational time will vary. Higher the number of finite elements longer is the computational time. FEM has been used in all the fields solve various problems. Several softwares of design and analysis have derived their source code from FEM. Solving any system will not only involve time but also affects the cost of production of said system.

5.2 Introduction to Rigid Multibody Dynamics

Multibody systems are considered as interconnected systems of rigid bodies. In general a theory is presented for the analysis of multibody systems having arbitrary connections. This theory allows us to simulate the kinematic and dynamic behaviour of multibody systems like simple 4-bar mechanism and also complex systems like industrial robots. Multibody dynamics is a useful tool required for the analysis of complex systems. Its uses basic algebraic and differential equations which are formulated using standard theories
such as Newton’s Laws of Motion, D’Alembert’s principle, Hamilton’s principle, Lagrange’s equation etc. Simple differential equations can sometimes be solved analytically. However when it comes to complex systems where there are multiple rigid bodies then solving the equations analytically becomes extremely tedious. In most cases the complex systems includes non-linear elements and hence it is best to turn to numerical methods to obtain the solution.

Multibody systems have two branches- kinematic systems and dynamic systems. In the field of kinematics all the motions of the system such its velocity, position and acceleration are the key parameters to be found out. It does not take into account the forces acting on the system. Dynamic systems take into account the forces acting on the rigid bodies and are principally solved by bringing the system to equilibrium. [16]

The procedure for solving any mechanical system using multibody dynamics is similar in most cases. First we take any system and break it down to rigid bodies and links. We can treat the individual bodies of multibody system as a free particle, as considered by Newton, or as rigid bodies, as considered by Euler. The motion of these bodies is defined by their kinematics and the resultant dynamic behaviour is described by the equilibrium of the forces applied on these bodies. The governing equations of motion for a mechanical system can be derived and expressed in a variety of forms, depending mainly upon the type of coordinates being employed. The governing equations can then be solved either by direct methods or by numerical means.

Rigid multibody dynamics helps establishing the degrees of freedom of any complex system. We can obtain various types of information pertaining to the system such as dynamic stress, velocity, acceleration of any rigid body. It also plays a key role in coupling two individual systems. For analysing the systems computer-aided design has been implemented to formulate the system of equations and solve it under various initial conditions. There are limitations of rigid multibody dynamics especially in the field of biomechanical research. Normally, the joints between the rigid bodies are simplified into simple connections. However the joints are more complex in realistic cases
5.3 Introduction to Pantograph/catenary dynamics

As stated earlier the interaction between the pantograph and catenary is an important factor that affects the overall functioning of the system. The pantograph lifts the pan-head up against the catenary to establish good frictional contact. If the contact force is too high this will result in the abrasion of the carbon strips of the pan-head which in turn leads to increase in maintenance cost. If there is less contact between the pantograph and catenary causes’ poor current transmission and in worst cases electrical discharges will erode the carbon strips and causes heavy damage to pantograph. (Figure 7)

![Figure 7(a) Arcing between pantograph and catenary](image1)  (b) Damage in pantograph  
![Figure 7(c) Excessive wear on carbon strips of pantograph](image2)

The contact force between the pantograph and catenary does not remain constant because of irregularities of the surface of the pantograph and catenary, structural errors and other
environmental disturbances. Therefore range of satisfactory contact force must be defined. The key parameters involved in the study of contact force are[1]:

1. Mean contact force
2. Standard deviation of contact force
3. Statistical maximum and minimum contact force
4. Statistical occurrence of loss contact force
5. Statistical occurrence of low contact force below specified safety margin.

Although we are able to calculate the above data, it does not represent real life situations. The theories behind the models make a few assumptions to make calculations simpler and different types of pantograph/catenary systems experience different environmental disturbances. Therefore it is of utmost importance to monitor the contact force under real world situations.

6. Chapter 2: Finite Element Models

In this chapter a study of both 3D and 2D FEM models with their comparison is given

6.1 3D FEM model

6.1.1 Catenary model:

The catenary used here is the rigid catenary as shown in Figure 8.
Figure 8: Schematic diagram of rigid catenary

It consists of a box frame beam structure made of aluminium and it is fixed to the ceiling of the tunnel via supports. To provide more stiffness the catenary is further supported by the bracing wires. $Y_c(x, t)$ is the displacement in the vertical plane and $Z_c(x, t)$ is the displacement in the horizontal plane. The catenary when viewed from the top is staggered in shape as shown in Figure 2. This accounts a uniform distributed wear of the carbon strips of the pantograph. The catenary is modelled using Euler Bernoulli Beam theory and solved using finite element methods. The mass matrix $M_c$ and stiffness matrix $K_c$ are determined by eq (1) and eq (2) respectively.

$$M_c = \frac{\rho AL}{78} \begin{bmatrix} 39 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 39 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 39 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 39 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L^2 \end{bmatrix}$$
Where $L$ is the length of Euler-Bernoulli element, $A$ is the area of the element, $\rho$ is the density of the Euler beam and $E$ is the Young’s Modulus of the beam.

The damping matrix of the catenary is obtained using Rayleigh’s equation:

$$
\mathbf{C}_c = \alpha \mathbf{M}_c + \beta \mathbf{K}_c
$$

Where $\alpha$ and $\beta$ are the damping coefficients of the catenary.

### 6.1.2 Pantograph model:

The pantograph is modelled using rigid multibody dynamics as shown.
It is 4DOF system and the governing equation describing this system is as follows

$$
\begin{bmatrix}
   m_{p1} & 0 & 0 & 0 \\
   0 & m_{p2} & 0 & 0 \\
   0 & 0 & J_p & 0 \\
   0 & 0 & 0 & m_{p3} \\
\end{bmatrix}
\begin{bmatrix}
   \dot{y}_{p1} \\
   \dot{y}_{p2} \\
   \dot{\varphi}_p \\
   \dot{y}_{p3} \\
\end{bmatrix}
+ \begin{bmatrix}
   c_{p1} + c_{p2} & -c_{p2} & 0 & 0 \\
   -c_{p2} & c_{p2} + c_{p3} & c_{p3} z_H & 0 \\
   0 & c_{p3} z_H & c_{p2} a^2 + c_{p3} z_H^2 & 0 \\
   0 & -c_{p3} & -c_{p3} z_H & c_{p3} \\
\end{bmatrix}
\begin{bmatrix}
   \dot{y}_{p1} \\
   \dot{y}_{p2} \\
   \dot{\varphi}_p \\
   \dot{y}_{p3} \\
\end{bmatrix}
+ \begin{bmatrix}
   0 \\
   0 \\
   0 \\
   0 \\
\end{bmatrix}
= \begin{bmatrix}
   F_w \\
   0 \\
   0 \\
   0 \\
\end{bmatrix},
$$

(4)

6.1.3 Description of code:

The code uses a launcher which creates different structs to store the data for the catenary, the pantograph, and simulation data. The launcher calls the function which simulates the pantograph/catenary interaction. In the function that simulates the interaction, the code calls other sub-functions.

The catenary model is setup in an individual function using the data of the catenary, supports and bracing wires. The setup of the catenary also includes the initial deflection of the catenary due to its self weight. First the stiffness matrix $K_g$ and $M_g$ are calculated for each element of the catenary and then added to the global stiffness matrix. Furthermore, the stiffness and mass matrices are sparsed so as to save memory. While calculating the stiffness and mass matrices of each element, the coordinates system is also transformed to the one relative to the catenary. This change in coordinate system is due to the staggered arrangement of the catenary between supports (Figure 2). The stiffness and mass of both bracing wires and each support are also added to the global matrices. The global damping matrix is obtained from equation (1).

The pantograph model, like the catenary one, is setup using an individual function. During the setup of the pantograph the data of the train vehicle could also be considered. Sometimes the pantograph/catenary interaction force can be influenced by the wheel/rail contact force.
After the setup of the catenary and the pantograph models, the initial values of the degrees of freedom of all the nodes of the catenary and the pantograph are calculated. The differential equation is setup for the catenary. The model is then simulated for affixed period of time. The variation for contact force in the vertical axis (y) and horizontal axis (z) is plotted with time.

6.1.4 Comparison of Ode Solvers:

The computational efficiency and the accuracy of the code depend mainly on the number of finite elements per span and the type of solver used. The following table shows the time taken by different ode solvers for the same number of elements per span. Self time is the time spent within a particular function. For example the time spent in executing the ode algorithms. The total time is time taken for computing the entire function. In other words if a function calls more sub-function total time considers the time spent in the sub-functions as well while self time does not.

<table>
<thead>
<tr>
<th>ODE solver</th>
<th>Total time (s)</th>
<th>Self time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ode 15s</td>
<td>90.524</td>
<td>40.047</td>
</tr>
<tr>
<td>ode 23s</td>
<td>4803.925</td>
<td>75.737</td>
</tr>
<tr>
<td>ode 23t</td>
<td>192.890</td>
<td>53.358</td>
</tr>
<tr>
<td>ode 23tb</td>
<td>124.821</td>
<td>37.759</td>
</tr>
</tbody>
</table>

Table 1: Time take by different ode solvers for 3D FEM for 20 elements per span

Since the problem pantograph/catenary interaction is stiff, ode 45 is not use as it would take too long for the solver to handle stiff problems. The algorithms ode15s, ode23s, ode23t and ode23tb are all used for moderate to high stiff problems. It is clearly seen that ode15s is by far the best solver when it comes to time consumed in this case. The ode 23s takes too long to compute even thought the computation is done for 20 elements per span and therefore it should be avoided. The algorithm ode23s is based on a single step implicit method, which uses a second order formula to advance the step and a third order formula to estimate the error. It recomputes the Jacobian with each step, thereby making it quite expensive in terms of function evaluations.[17,18]
The different ode solvers use different methods to evaluate the solution of the differential equation. Figure 10 (a) shows the difference in the solutions obtained for the vertical displacement of the catenary using three different ode solvers. From the figure it is clear that the difference in the solution obtained from the solvers used are very minute.

![Figure 10(a): Comparison of ode solvers for 20 elements per span](image)

On increasing the elements per span to 50 the plot in Figure 10(b) is obtained. On observing the plot it is seen that has the number of elements per span increases the solutions of each ode solver converge (Figure10(c)). Furthermore the values of displacement are more refined when compared to the ones obtained from 20 elements per span.
Figure 10(b): Comparison of ode solvers for 50 elements per span.
6.1.5 Advantages:

- The 3D FEM model takes into account the effect of vibration in the horizontal plane (z axis) on the contact force between the catenary and pantograph.
- The model calculates each degree of freedom with respect to each node of the catenary. It also takes into account the fact that apart from the first and the last node every other node is shared between two consecutive elements.
- By taking into consideration of the staggered arrangement of the catenary the code advertently ensured a uniform wear of the carbon strips of the pan head.

6.1.6 Disadvantages:

- The 3D model does not take into account the influence environmental factors, such as the aerodynamics forces, cross winds, etc., on the variation in the contact interaction.
- Since the 3D FEM model is a stiff problem this will cause ode 45 to take longer to compute the results, although for most differential equations ode45 is the best solver to be used. Therefore ode15s is used.
- The maximum order of the solver is limited to 2. However higher orders can be used but the computational time increases drastically.

6.2 2D FEM model

6.2.1 Catenary Model:

The model used for this code is very similar to 3D FEM model. The only difference is that the horizontal axis ($z_c(x,t)$) is not considered. As a result the staggered arrangement of the catenary is neglected for the results.

6.2.2 Pantograph model:

The pantograph model remains the same as in 3D model (Figure 9) i.e. a lumped-mass system is used to define the pantograph. Even the governing equation for the pantograph is the same as equation (4).

6.2.3 Code description:

The procedure for the simulation of the interaction remains the same as in 3D model. The main aim of this code is to find if the vibrations in the horizontal axis has any effect on
the variation of contact force. This is a more simplified model compared to 3D FEM and as a result the computation time is expected to be lower.

6.2.4 Comparison of ode solvers:

The same set of solvers used for 3D model is used for 2D model. Ode 15s, ode23s, ode23tb, ode23t are used and the computational time for the selected solvers is as follows (table 2):

<table>
<thead>
<tr>
<th>Ode solver</th>
<th>Total time (s)</th>
<th>Self time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ode15s</td>
<td>89.434</td>
<td>39.345</td>
</tr>
<tr>
<td>ode23s</td>
<td>4578.841</td>
<td>73.162</td>
</tr>
<tr>
<td>ode23t</td>
<td>80.294</td>
<td>23.347</td>
</tr>
<tr>
<td>ode23tb</td>
<td>239.596</td>
<td>45.987</td>
</tr>
</tbody>
</table>

Table 2: Time take by different ode solvers for 2D FEM for 20 elements per span

It can be seen that using ode 23t for 2D model is computationally faster. Again ode 23s takes too long to compute even though there is a significant difference in the time taken for the solution in 3D and 2D model. All the solvers take considerably lesser time for the 2D model when compared to 3D model. This is because the calculations with respect to the horizontal axis are not considered for 2D model.

A similar comparison of the ode solvers which was carried out in 3D model is done for 2D model as well. Figure 11(a) shows the solution for 20 elements per span. The solutions of ode 23t and ode23tb almost overlap each other(Figure 11(b)). This occurs as both the solvers uses the trapezoidal rule to solve the differential equations.
Figure 11(a): Comparison of the ode solvers for 20 elements
Figure 11(b): Comparison of the ode solvers for 20 elements (zoomed plot)

The plot was carried out for 50 elements per span (Figure 11(c)). A similar observation seen in 3D model is made for 2D model as well. The solutions are more refined and the solution of different ode’s converges with each other (Figure 11(d)).

Figure 11(c): Comparison of the ode solvers for 50 elements
6.2.5 Advantages:

- The 2D FEM model is based on simple differential equations and does not require the transformation of the coordinate axes as the horizontal axis is neglected.
- The 2D model is less time consuming compared to 3D model for the same number of finite elements per span.
- At higher elements per span the solution of the 2D model is almost the same as that of the 3D model. Therefore for simplicity and computational efficiency, 2D model is preferred.

6.2.6 Disadvantages:

- As mentioned earlier the 2D model does not take into account the environmental factors affecting the contact interaction such the cross winds, vibrations along the horizontal plane etc.
- The 2D model does not take into account the roughness of the surface of the catenary and the effect of vibrations induced by this unevenness.
- Even though the results of the 2D model are the same as 3D model, for realistic cases 3D model is given preference.
6.3 3D FEM Model vs 2D FEM Model

3D FEM model is more realistic when compared to the 2D model. It takes into account the effect of vibrations along the horizontal plane (z axis) on the contact force. However this results in an addition of 2 more DOF to the catenary i.e. displacement along z axis ($z_c$) and rotation about z axis ($\theta_z$). This in turn will increase the computation time and also the complexity of the program. By considering the z axis, the staggered arrangement of the catenary must also be taken into account for the calculation of the global stiffness, mass and damping matrices. As a result a change of coordinate axis from the local coordinates to the global coordinates is required. This is not seen in the 2D model. 2D FEM model is simplistic version of the pantograph catenary interaction. Since the horizontal axis is eliminate the calculations for the differential equations become much simpler. However the solutions contain inaccuracies as the effect of the environmental conditions such as cross winds are not considered.

Figure 12(a) shows the comparison of ode 15s for 3D FEM and 2D FEM model. There is a clear difference in the values obtained for the two models (Figure 12(b)). This could be the result of the constraints on the ode solvers and the fact that less number of elements was used for this analysis and also the fact that the disturbances along z axis are neglected for the final solution.

![Figure 12(a): Comparison of ode 15s of 3D FEM and 2D FEM models for 20 elements per span.](image)
Figure 12(b): Comparison of ode 15s of 3D FEM and 2D FEM models for 20 elements per span (zoomed plot).

The same plot was done for higher number of elements (Figure 12(c)) and it was observed that the difference between the two solvers were comparatively lower (Figure 12(d)). It indicates that the 2D model approximately gives the same solution as that of the 3D model at higher number of elements per span. This indicates that the vibrations along the horizontal axis can considered negligible. However, when environmental forces such as heavy cross winds and other aerodynamic forces are applied then the Z axis plays an important role.
Figure 12(c) Comparison of ode 15s of 3D FEM and 2D FEM models for 50 elements per span.

Figure 12(d) Comparison of ode 15s of 3D FEM and 2D FEM models for 50 elements per span (zoomed plot).
<table>
<thead>
<tr>
<th>FEM Model</th>
<th>Amount of FE per span</th>
<th>Total time(s)</th>
<th>Self time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D FEM Model</td>
<td>20</td>
<td>90.524</td>
<td>40.047</td>
</tr>
<tr>
<td>3D FEM Model</td>
<td>50</td>
<td>457.430</td>
<td>359.125</td>
</tr>
<tr>
<td>2D FEM Model</td>
<td>20</td>
<td>89.434</td>
<td>39.345</td>
</tr>
<tr>
<td>2D FEM Model</td>
<td>50</td>
<td>436.792</td>
<td>337.740</td>
</tr>
</tbody>
</table>

Table 3: Difference in time taken for Ode 15s for different number of finite elements per span for the FEM models

From Figure 12(b) and 12(d) it is clear that the error in values between 3D FEM and 2D FEM decreases as the number of elements increase. However for this increase in accuracy, time is sacrificed. Table 3 gives the time taken by ode 15s for different number of elements for both 3D FEM and 2D FEM models. As seen from Figure 12(d) for higher number of elements the solution of the 2D FEM and 3D FEM models converge. However by reducing the error % from approximately 0.215% to 0.0248% the time is increased by approximately 350s (around 6 min). Therefore, it can be concluded that, even at low number of elements per span the results are accurate enough.
7. Chapter 3: Semi-analytical model

In this chapter a study of semi-analytical model and its comparison with the FEM models is given.

7.1 2D Semi-analytical model

7.1.1 Theory Description:

Semi-analytical models are based on elastodynamic models of basic systems, as beams, plates and half-spaces. As the name implied they are not numerical but also not completely analytical, since numerical integration should be performed in most of the cases. Thus, for the present problem, the catenary is not modelled using the finite element method in the semi-analytical model that is presented in this section. Instead, the catenary is assumed to be an infinite Euler-Bernoulli beam over a periodic spring-damper foundation. In order to compute the Green’s functions related to this catenary model, the system is subjected to a unitary moving harmonic load.

The equation describing the catenary is as follows:

\[ EI \frac{\partial^4 z_c}{\partial x^4} + \rho A \frac{\partial^2 z_c}{\partial x^2} + \sum_{n=-\infty}^{\infty} \left( k_s + c_s \frac{\partial}{\partial t} \right) Z_r \delta(x - x_n) = \cos(\tilde{\omega}t) \delta(x - vt) \]

(5)

where, \( EI \) is the bending stiffness of the beam, \( \rho A \) is the mass per unit length of the beam and \( z \) is the vertical displacement of the beam. \( k_s \) and \( c_s \) are the stiffness and damping of each support respectively.

Fourier transforms are used to transform the equations and the variables from the space-time domain to the wavenumber-frequency domain. The response Green’s function of the catenary can be written as follows:

\[ H(k_x, \tilde{\omega}) = \frac{2\pi}{EIk_x^4 - \rho A(\tilde{\omega} + k_xv)^2} - \frac{k_s + i(\tilde{\omega} + k_xv)c_s}{EIk_x^4 - \rho A(\tilde{\omega} + k_xv)^2} \sum_{n=1}^{\infty} Z_r n e^{i k_x x_n} \]

(6)
where $\tilde{\omega}$ is the circular frequency with respect to the moving frame of reference, $k_x$ is the wavenumber. $Z_{rn}$ is the response of the catenary in the frequency domain at the support positions, defined by $x_n$, and $N$ is a finite amount of supports to be considered.

The contact model used for this interaction is the Hertz contact model [19]. The Hertz contact model is defined by the equation:

$$F_{p/c} = k_h \delta^{3/2}$$

(7)

where $F_{p/c}$ is the contact force and $k_h$ is the hertz stiffness and $\delta$ is the deflection of the pantograph/catenary contact. Once the pantograph is coupled with the catenary through this interaction model, the contact force is computed and the necessary results are plotted.

7.1.2 Code description:

The code follows a similar approach as that of 2D and 3D FEM model. It uses a launcher which has the structs containing all the necessary data. The launcher calls forth a function to calculate the contact force.

In the function contact_force, the force is calculated into 2 parts- Dynamic force and Static force. The dynamic force is calculated using the following equation:

$$F_d = (H_p + H_c + k_h^{-1})^{-1} E_c$$

(8)

Where $F_d$ is the dynamic force in space frequency domain. $H_p$ is the response of the pantograph in frequency domain, $H_c$ is the response of the catenary in frequency domain, $k_h$ is the linearised hertz stiffness constant and $E_c$ is the unevenness of the catenary in the frequency domain.

From the results the variation of contact force is plotted against range of predefined frequencies. The contact force is also converted to time domain using inverse Fourier transform and plotted against the simulation. These results are then compared with the results obtained from 3D and 2D FEM models.
7.1.3 Results obtained:

The contact force has been obtained in the frequency domain and then has been transformed to the time domain by means of inverse Fourier transform. The force was plotted against time. Figure 13(a) and 13(b) shows the variation of contact force with respect to angular frequency and time respectively. The following plots are carried out for a frequency range of 0-500 Hz taking 1024 sampling points for the wave-numbers. The results of the semi-analytical code depends upon the frequency range and range of wave numbers.

If the sampling points for the wave number are too less than, the model would not be able to define the response of the catenary completely. By taking large number of sampling points for a predetermined wave-number range, the results become more accurate. However, the time taken to compute the response of the catenary, Green’s function, for each frequency increases.

![Figure 13(a) Contact force vs wm](image-url)
Figure 13(b) Contact force vs time
7.2 2D Semi-Analytical model vs FEM models:

The results of the semi-analytical model are compared with the FEM models. The modelling of the pantograph is same in all 3 models. The catenary is defined an Euler Bernoulli beam in all the models but it is not discredited in semi-analytical model. It is considered as an infinite beam of uniform properties. The semi analytical model does not account for the uplift force exerted by the pantograph, instead the model only represent the dynamic contact of panhead with the catenary. Therefore in the code an uplift force is assumed and added to the static force vector. The FEM models, however, accounts for the initial configuration of the system and then computes the dynamic properties once the transient nature of the interaction begins.

The contact force of 3D FEM model is shown in figure 14(a). When compared to the semi-analytical model the nature of variation is different. The variation in the FEM model shows irregular patterns while in the semi-analytical model there is regular variation of contact force. The magnitudes of the force are also different for the two models. The FEM model records a maximum force of approximately 1190N, while over 300N is recorded in the semi-analytical code.

![Figure 14 (a) Contact force vs time for 3D FEM model](image-url)
Figure 14(b) and Figure 14(c) contains the variation of contact force vs frequency. When compared with the semi-analytical model there is a similarity in the patterns between the two models. The 3D FEM model has a much larger number of sampling points and a
larger range of frequencies. The semi-analytical model has lower number of sampling points and lower range of frequencies. The dissimilarities between the graphs could be the result from the way each model calculates the dynamic parameters. The FEM model was calculated entirely in the time domain, so no importance was given to the frequency and wave number domain. While the semi-analytical model is entirely calculated in space-frequency domain along with specified frequency range and wave number range. One more difference is in the contact model used, the semi-analytical model uses the Hertz contact model and the FEM model, although using the same contact model, defines the deflection of the pantograph/catenary interaction in a different manner. The FEM models consider the distance between the train roof and the tunnel roof and the height of the pantograph while it is not considered in the semi-analytical model. These two parameters play an important role in the magnitude of static force. The initial configuration of the pantograph and catenary is not defined in the semi-analytical model as the simulation considers an infinite catenary setup and the results of the contact force are obtained by considering one section of the catenary in time. Hence the abrupt peaks in the contact force at the beginning and at the end indicates there are minor errors in the code. The FEM models define the initial position of the train and the catenary setup, thereby clearly defining the transition between the static and dynamic case.

The time taken by the semi-analytical for the above results took approximately one hour, while the FEM model for 120 elements per span took 1 hour 40 minutes. This shows that for computing the results the semi-analytical model is faster. Since the semi-analytical code is recently developed, more research is required to be carried out to improve the accuracy and the reliability.

7.2.1 Advantages:

- The semi-analytical model does not discretize the beam into elements. The catenary is considered as an infinite Euler-Bernoulli beam. This reduces the number of equations to be solved.
- The model gives the variation of the contact force at different range of frequencies.
• For accurate results the computational time of the semi analytical model is faster when compared to FEM model.

• Since the model analyses only a particular section of an infinite beam, focus is given to the dynamic interaction between the pantograph and the catenary.

7.2.2 Disadvantages:

• The results of the model do not match the results of the FEM model.

• The model does not account for the uplift force and stiffness of the other parts of the pantograph.

• The abrupt increases in the contact force at the beginning and the end of the simulation does not represent the dynamic interaction between the pantograph and the catenary.

• The model does not consider the height of the pantograph and the distance between the tunnel roof and the train roof, which affect the magnitude of the static force applied.
7. Conclusion & Discussion

In this thesis all the selected models have been reviewed and tested so as to improve its computation time and accuracy. Both the 3D FEM and 2D FEM models have been compared with respect to the time taken for different elements per span and also each model is tested for different ode solvers. This helps in establishing the ideal ode solver to be used under any given circumstances. Furthermore, the 3D model though, time consuming, and gave good results with respect to the variation of the contact force. It took into account all the degrees of freedom of the catenary along with the pantograph. The 2D FEM model was comparatively quicker but did not simulate a realistic case i.e. the 3D model simulations are similar to real world situation. Nevertheless the results obtained from 2D model are reliable with good accuracy. The results of both 2D FEM and 3D FEM are compared for different number of elements per span. However for higher elements per span it is seen that the results obtained from the 2D FEM is similar to 3D FEM. Therefore it can be concluded that for higher number of elements per span 2D FEM is preferable.

The semi-analytical method uses the concept of Green’s function to obtain the response of the interaction in time and frequency domain. Based on its comparison with the FEM models the variation of the contact forces in time domain shows a different trend and the magnitudes are also different. At very low wave numbers and very low frequency, the governing equation of the system is not defined clearly. The semi-analytical code does not define the static force clearly as it only models the interaction between the panhead and the contact wire of the catenary. Hence the constant uplift force applied by the pantograph on the catenary is assumed.

So far the best working model is the 3D FEM model as it accurately shows the variation of the contact force with respect to time. It also takes into account the initial configuration of the interaction. However, when compared to the semi-analytical model the 3D model takes more time to compute. The semi-analytical code has its own shortcomings but when these faults are rectified it can prove to be a better model when compared to 3D FEM model.
8. Reference


19. V. L. Popov *Contact Mechanics and Friction Physical Principles and Applications*.
