EXPONENTIAL DECAY IN ONE-DIMENSIONAL TYPE II THERMOVISCOELASTICITY WITH VOIDS

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Abstract: In this paper we consider the one-dimensional type II thermoviscoelastic theory with voids. We prove that generically we have exponential stability of the solutions. This is a striking fact if one compares it with the behavior in the case of the classical thermoviscoelastic theory based on the classical Fourier law for which the decay is generically slower.

Keywords: Type II thermoelasticity, voids, exponential decay, viscosity.

1. Introduction

Elastic solids with microstructure have been one of the subjects of study in continuum mechanics for more than one hundred years. A subclass of these materials corresponds to elastic materials with voids. They were introduced by Nunziato and Cowin [5, 6, 25] in the 1970's. In this case the materials have a microstructure such that the mass in each point can be expressed as the product of the mass density by the volume fraction. The applications of these materials are of interest for solids with small distributed porous. Rocks, soils, woods, ceramics, pressed powders or biological materials such as bones are examples where this theory is considered. From the structural point of view they are a nice example of a combination between the macrostructure determined by the elastic part and the microstructure determined by the porous part. For this reason we could ask about the relevance of the macrostructural behavior into the microstructure and the other way around. We could consider this question when we consider a dissipative mechanism in one of the structures and to see how it damps the other. It is worth recalling that the first contribution in this sense was proposed in [28]. There, the author showed that generically the

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porous dissipation is not strong enough to guarantee the exponential decay of the solutions for a porous elastic structure. A large number of contributions have been proposed to clarify the decay of the thermomechanical perturbations for elastic solids with voids when damping effects are taken into account [1, 3, 7, 8, 13, 16, 17, 24, 26, 27]. It is accepted that generically we would need two dissipative mechanisms to guarantee the exponential decay of solutions. In particular when we consider the viscoelastic effect combined with the classical heat conduction we have the slow decay of solutions, because both dissipative effects are of macroscopic type. That is, two dissipative effects that are not sufficient to bring all the system to the exponential decay.

On the other hand the classical theory of heat conduction based in the Fourier law has deserved a big criticism in the second part of the last century. This is because (among other reasons) this theory allows the propagation of thermal waves instantaneously which violates the principle of causality. For this reason there has been a big interest in proposing alternative heat conduction theories in the last fifty years. Green and Lindsay [9] or Lord and Shulman [15] are two examples of thermoelastic theories based on the Cattaneo-Maxwell heat conduction equation [4]. We here want to recall the ones suggested by Green and Nagdhi. In the 1990's these authors proposed three new thermoelastic theories where the heat conduction contained an innovation. They called them type I, II and III, respectively [10, 11, 12]. It is known that the linear version of the first one agrees with the classical theory based on the Fourier law. The second one is known as *thermoelasticity without energy dissipation* because the heat propagation is not a dissipative process in that case. The third one is the most general and it contains the former two as limit cases. It is worth noting that in the theories of type II and III a new variable is considered between the independent variables which is the *thermal displacement*. This new kind of heat conduction suggests new systems of equations to study and understand. We can recall that these new theories have been also studied in the context of different problems [20, 21, 23].

In this paper we want to consider the case proposed by the thermoelastic solids with voids when the heat conduction is described by the type II theory. In [13] the authors proved that the porous dissipation is strong enough to guarantee the exponential decay of the solutions. This was an interesting result if we compare with the previous comments, because we have a case where only a dissipation mechanism is sufficient to bring the system to the exponential decay. The mechanical reason for this behavior is the presence of the thermal displacement. This variable determines new coupling between the macro and micro structures. This coupling is strong enough to guarantee this new behavior for the system. A similar behavior has been observed recently [22] for the type III thermoelasticity with voids where the authors proved that generically the thermal dissipation is also enough to bring the whole system to the exponential decay. This is also because the thermal displacement plays a role of *transmission band* between the macro and micro structures. In general the use of the Green and Nagdhi theories allows us to obtain some new results which are different from the ones obtained in the classical theory as it was pointed out in [18, 19]. The aim of this article is to give another example of this behavior. We will consider the viscoelastic behavior on the macrostructure and we will prove the exponential decay of the solutions. We want to emphasize that although we obtain similar results to the ones proposed in [13, 22] our arguments will need different ways. In fact as in the two previous contributions we will use semigroup's arguments to prove the exponential stability. Part of the analysis is very similar and for this reason we will not make a detailed proofs of this first part (existence of the semigroup). However as the arguments are very different to prove the exponential decay of solutions we will give a detailed proof of this part. We believe that an important aspect to study in continuum thermomechanics is to distinguish the different consequences of the several theories. Our note addresses this purpose.

The plan of this note is the following: in the next section we recall the basic equations and conditions determining our problem as well as the existence of the semigroup. In Section 3 we prove the exponential decay of solutions. Some conclusions are proposed at the end of the paper.

2. Statement of the problem and well-posedness

In the context of the one-dimensional type II thermoviscoelasticity with voids the system of equations is determined by the evolution equations

$$\rho\ddot{u} = t_x, \tag{2.1}$$

$$J\ddot{\phi} = h_x + g, \tag{2.2}$$

$$\rho\dot{\eta} = q_x, \tag{2.3}$$

and the constitutive equations

$$t = \mu u_x + \gamma\phi - \beta\theta + \mu^* u_{xt}, \tag{2.4}$$

$$h = b\phi_x + m\psi_x, \tag{2.5}$$

$$g = -\gamma u_x + d\theta - \xi\phi, \tag{2.6}$$

$$\rho\eta = \beta u_x + a\theta + d\phi, \tag{2.7}$$

$$q = k\psi_x + m\phi_x. \tag{2.8}$$

As usual ρ is the mass density, J is the product of the mass density by the equilibrated inertia, t is the stress, h is the equilibrated stress, g is the equilibrated body force, q is the heat flux, η is the entropy and the variables u, ϕ and ψ are the displacement, the volume fraction and the thermal displacement, respectively.

After substitution of the constitutive equations into the evolution equations, we obtain the field equations for the one-dimensional problem

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + \gamma\phi_x - \beta\dot{\psi}_x + \mu^* u_{xxt} \\ J\ddot{\phi} = b\phi_{xx} + m\psi_{xx} - \xi\phi + d\dot{\psi} - \gamma u_x \\ a\ddot{\psi} = k\psi_{xx} + m\phi_{xx} - d\dot{\phi} - \beta\dot{u}_x \end{cases} \tag{2.9}$$

The parameters proposed in the system are related with the properties of the material. From now on, we assume that

$$b > 0,\ J > 0,\ \mu > 0,\ a > 0,\ \rho > 0,\ \mu\xi > \gamma^2,\ bk > m^2,\ \mu^* > 0. \tag{2.10}$$

Our assumptions are in agreement with the thermomechanical axioms and the empirical experiments. The condition on μ, ξ, b, k, m and γ can be understand with the help of the elastic stability. The condition on the viscosity μ^* is the natural one to guarantee the dissipation. The assumptions concerning mass density, the thermal capacity and the parameter J are also obvious.

It is well known that the parameter β relates the displacement to the temperature. Furthermore m relates the thermal displacement to the volume fraction. These two parameters are responsible for the strong coupling between the variables. We will prove that if the coupling is strong enough, the viscosity brings our system to the exponential stability.

To have the problem determined, we need to impose boundary and initial conditions. Thus, we assume that the solutions satisfy the boundary conditions

$$u_x(0,t) = u_x(\pi,t) = \phi(0,t) = \phi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0 \quad \text{for} \quad t > 0, \tag{2.11}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \phi(x,0) = \phi_0(x),\ \dot{\phi}(x,0) = \varphi_0(x),\\ &\psi(x,0) = \psi_0(x),\ \dot{\psi}(x,0) = \theta_0(x) \quad \text{for} \quad x \in [0,\pi]. \end{aligned} \tag{2.12}$$

The aim of this paper is to determine the asymptotic behavior in time of the solutions of the problem given by system (2.9), boundary conditions (2.11) and initial conditions (2.12).

We note that there are solutions (uniform in the variable x) that do not decay. To avoid these cases, we will also assume that

$$\int_0^\pi u_0(x)\,dx = \int_0^\pi v_0(x)\,dx = 0. \tag{2.13}$$

We consider the Hilbert space

$$\mathcal{H} = \big\{(u,v,\phi,\varphi,\psi,\theta) \in H^1_* \times L^2_* \times H^1_0 \times L^2 \times H^1_0 \times L^2\big\}, \tag{2.14}$$

where

$$L^2_* = \{f \in L^2, \int_0^\pi f(x)dx = 0\} \text{ and } H^i_* = L^2_* \cap H^i.$$

We note that our boundary conditions are different from the ones proposed in [13, 22]. The motivation for this is that for this system the mathematical analysis is easier in this case.

Taking into account that $\dot{u} = v$, $\dot{\phi} = \varphi$ and $\dot{\psi} = \theta$ and writing $D = \frac{d}{dx}$, we can restate system (2.9) in the following way:

$$\begin{cases} \dot{u} = v \\ \dot{v} = \dfrac{1}{\rho}\big(\mu D^2 u + \gamma D\phi - \beta D\theta + \mu^* D^2 v\big) \\ \dot{\phi} = \varphi \\ \dot{\varphi} = \dfrac{1}{J}\big(bD^2\phi + mD^2\psi - \xi\phi + d\theta - \gamma Du\big) \\ \dot{\psi} = \theta \\ \dot{\theta} = \dfrac{1}{a}\big(kD^2\psi + mD^2\phi - d\varphi - \beta Dv\big) \end{cases} \tag{2.15}$$

Moreover, if $U = (u,v,\phi,\varphi,\psi,\theta)$, then our initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U,\ \ U_0 = (u_0,v_0,\phi_0,\varphi_0,\psi_0,\theta_0),$$

where $\mathcal{A}$ is the following 6×6-matrix

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & \frac{\mu^*}{\rho}D^2 & \frac{\gamma}{\rho}D & 0 & 0 & -\frac{\beta}{\rho}D \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\gamma}{J}D & 0 & \frac{bD^2-\xi}{J} & 0 & \frac{m}{J}D^2 & \frac{d}{J} \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & -\frac{\beta}{a}D & \frac{m}{a}D^2 & -\frac{d}{a} & \frac{k}{a}D^2 & 0 \end{pmatrix} \tag{2.16}$$

and I is the identity operator.

We note that $\mathcal{D}(\mathcal{A})$ is given by the functions $(u, v, \phi, \varphi, \psi, \theta) \in \mathcal{H}$ such that $v \in H^1_*, \varphi, \theta \in H^1_0$,

$$\mu D^2 u + \gamma D\phi - \beta D\theta + \mu^* D^2 v \in L^2_*,$$

$$bD^2\phi + mD^2\psi - \xi\phi + d\theta - \gamma Du \in L^2,$$

$$kD^2\psi + mD^2\phi - d\varphi - \beta Dv \in L^2.$$

In view of the above conditions we can characterize the domain as the subspace of $\mathcal{H}$ such that $v \in H^1_*, \varphi, \theta \in H^1_0$, $\mu u + \mu^* v \in H^2_*$, $\phi, \psi \in H^2$.

It is clear that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$.

If $U^* = (u^*, v^*, \phi^*, \varphi^*, \psi^*, \theta^*)$, then

$$\begin{aligned} \langle U, U^* \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^\pi \Big(& \rho v \bar{v}^* + J\varphi\bar{\varphi}^* + a\theta\bar{\theta}^* + \mu u_x \bar{u}^*_x + b\phi_x\bar{\phi}^*_x + \xi\phi\bar{\phi}^* + \gamma(\phi\bar{u}^*_x + \bar{\phi}^* u_x) \\ & + k\psi_x\bar{\psi}^*_x + m(\phi_x\bar{\psi}^*_x + \bar{\phi}^*_x\psi_x)\Big) dx. \end{aligned} \tag{2.17}$$

As usual a superposed bar denotes the conjugate complex number. It is worth mentioning that this product is equivalent to the usual product in the Hilbert space $\mathcal{H}$.

Lemma 2.1. *For every $U \in \mathcal{D}(\mathcal{A})$, we have*

$$Re\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq 0.$$

Proof. If we consider the inner product, we can see that

$$\text{Re}\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{2} \int_0^\pi \mu^* |v_x|^2 dx.$$

As we assume that μ^* is positive the lemma is proved. □

Lemma 2.2. 0 *belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$).*

Proof. The proof of this lemma is standard. It can be done (for instance) in a similar way as lemma 3.1 of [13]. □

In view of these two lemmas and the fact that the domain of the operator is dense we can recall the Lumer-Phillips corollary to the Hille-Yosida theorem to conclude.

Theorem 2.3. *The operator given by matrix $\mathcal{A}$ generates a contraction C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t \geq 0}$ in $\mathcal{H}$.*

3. Exponential decay of the solutions

In this section we will prove the exponential decay of the solutions of our problem. Apart from the assumptions proposed above on the constitutive coefficients, from now on we also impose that $m \neq 0,\ \beta \neq 0$. These new assumptions say that the coupling between the three components of the problem is strong. In particular we note that for the classical theory the parameter m is not present.

To prove the main result of this section, we recall the characterization stated in the book of Liu and Zheng [14] that ensures the exponential decay.

Theorem 3.1. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(*i*) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(*ii*) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Lemma 3.2. *The operator $\mathcal{A}$ defined in (2.16) satisfies $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. We here follow the arguments given in the book of Liu and Zheng ([14], page 25). The proof can be obtained after the following three steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we see that for any real λ such that $|\lambda| < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I} - \mathcal{A} = (i\lambda_0\mathcal{I} - \mathcal{A})\Big(\mathcal{I} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A})^{-1}\Big)$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda, |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1} - M^{-1}, ||\mathcal{A}^{-1}||^{-1} + M^{-1})$.

(iii) Let us assume that the intersection of the imaginary axis and the spectrum is nonempty. Then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi,\ |\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \psi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{3.1}$$

Writing this condition term by term we get

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{3.2}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\mu D^2 u_n + \gamma D\phi_n - \beta D\theta_n + \mu^* D^2 v_n\right) \to 0 \text{ in } L^2, \tag{3.3}$$

$$i\lambda_n \phi_n - \varphi_n \to 0 \text{ in } H^1, \tag{3.4}$$

$$i\lambda_n \varphi_n - \frac{1}{J}\big(- \gamma D u_n + b D^2\phi_n - \xi\phi_n + m D^2\psi_n + d\theta_n\big) \to 0 \text{ in } L^2, \tag{3.5}$$

$$i\lambda_n \psi_n - \theta_n \to 0 \text{ in } H^1, \tag{3.6}$$

$$i\lambda_n\theta_n - \frac{1}{a}\left(-\beta Dv_n + mD^2\phi_n - d\varphi_n + kD^2\psi_n\right) \to 0 \text{ in } L^2. \tag{3.7}$$

In view of the dissipative term for the operator, we see that

$$v_n \to 0 \text{ in } H^1. \tag{3.8}$$

Then $\lambda_n u_n$ also tends to zero in H^1. If we consider the integral of (3.3) and take into consideration the boundary conditions at $x = 0$ and the fact that u_n and v_n tend to zero in H^1 we obtain

$$\gamma\phi_n - \beta\theta_n \to 0 \text{ in } L^2.$$

We multiply now λ_n^{-1}(3.3) by $D\psi_n$. We can note that $\mu u_n + \mu^* v_n$ is in H^2 and the regularity we need for θ_n and ϕ_n is satisfied. We find

$$\langle \mu D^2 u_n + \mu^* D^2 v_n, \lambda_n^{-1} D\psi_n\rangle + \gamma\langle D\phi_n, \lambda_n^{-1} D\psi_n\rangle - \beta\langle D\theta_n, \lambda_n^{-1} D\psi_n\rangle \to 0.$$

Note that

$$\langle \mu D^2 u_n + \mu^* D^2 v_n, \lambda_n^{-1} D\psi_n\rangle = -\langle \mu D u_n + \mu^* D v_n, \lambda_n^{-1} D^2\psi_n\rangle = -\mu\langle Du_n, \lambda_n^{-1} D^2\psi_n\rangle - \mu^*\langle Dv_n, \lambda_n^{-1} D^2\psi_n\rangle.$$

From (3.5) and (3.7) we see that $\lambda_n^{-1} D^2\psi_n$ is bounded and therefore we see that

$$\gamma\lambda_n^{-1}\langle D\phi_n, D\psi_n\rangle - i\beta||D\psi_n||^2 \to 0.$$

In particular, we see that $\langle D\phi_n, D\psi_n\rangle$ tends to an imaginary number which agrees with the limit of $i\beta\lambda_n\gamma^{-1}||D\psi_n||^2$.

In the particular case when γ vanishes we obtain directly that $D\psi_n \to 0$.

We now want to prove that $D\psi_n$ also tends to zero. To this end we multiply (3.5) by ϕ_n. We see that

$$i\lambda_n J\langle\varphi_n, \phi_n\rangle + b||D\psi_n||^2 + m\langle D\psi_n, D\phi_n\rangle + \xi||\psi_n||^2 - d\langle\theta_n, \phi_n\rangle \to 0.$$

In view of the previous convergences and taking the imaginary part we obtain that $D\psi_n$ tends to zero. From (3.7) and after multiplication by ψ_n we then obtain that θ_n tends to zero in L^2. Now, we want to prove that ϕ_n tends to zero in H^1. To this end we can multiply (3.7) by ϕ_n. In view of the fact that $\langle i\lambda_n\theta_n, \phi_n\rangle = \langle i\theta_n, \lambda_n\phi_n\rangle \to 0$ and that $\langle\varphi_n, \phi_n\rangle$ tends to an imaginary number we also obtain the convergence of ϕ_n. Now, we multiply (3.5) by ϕ_n and we also obtain that φ_n tends to zero. We have thus obtained a contradiction and the lemma is proved.

□

Lemma 3.3. *The operator $\mathcal{A}$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. Let us assume that the thesis of the lemma is not true. Then, there exist a sequence of λ_n such that the absolute value is unbounded and a sequence of vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \psi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that (3.1) holds. Using again the dissipative terms we see that Dv_n and $\lambda_n Du_n$ tend to zero.

We multiply again λ_n^{-1}(3.3) by $D\psi_n$. Using the same argument as in the previous lemma we see that

$$\gamma\lambda_n^{-1}\langle D\phi_n, D\psi_n\rangle - i\beta||D\psi_n||^2 \to 0.$$

As λ_n becomes unbounded and $D\phi_n$, $D\psi_n$ are bounded, we see that $D\psi_n$ tends to zero. From this point on we can follow the last part of the proof of the previous lemma to arrive at a contradiction that shows that the our thesis is correct.

□

The two previous lemmas give rise to the following result.

Theorem 3.4. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 3.2, Lemma 3.3 and Theorem 3.1. □

It is worth noting that the behavior of the solutions for this model completely differs from the behavior in the one-dimensional classical thermoviscoelasticity with voids, where slow decay is observed. The exponential stability obtained in our case is a consequence of the strong coupling between the porosity and the temperature. This coupling is not present in the classical model and plays a role of transmission band. This behavior is another striking effect of the type II thermoelasticity.

4. Conclusion

In this paper we have proved that, under suitable hypotheses on the different constitutive parameters, the solutions of the system of equations that models the type II thermoviscoelasticity with voids decay exponentially. This behavior differs significantly from the one obtained for the classical theory.

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