Two-Side Boundary Value Problems in Distance-Regular Graphs ⋆

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Abstract. In this work we analyze regular boundary value problems on a distance-regular graph associated with Schrödinger operators in the case that the boundary has two vertices. Moreover, we obtain the Green matrix for each regular problem. In each case, the Green matrix is given in terms of two families of orthogonal polynomials, one of them corresponding with the distance polynomials of the distance-regular graph.

Key words: Jacobi Matrices, Orthogonal Polynomials, Distance-regular Graphs, Boundary Value Problems.

1 Introduction

In this work we analyze linear two-side boundary value problems on a distance-regular graph as any distance-regular graph can be seen as the covering of a weighted path. In spite of its relevance, the Green matrix on a path has been obtained only for some boundary conditions, mainly for Dirichlet conditions or more generally for the so–called Sturm–Liouville boundary conditions, see [5,7]. Recently, some of the authors have obtained the Green function on a path for general boundary value problems related to Schrödinger operators with constant conductances and potential, [3]. In this work, we show that the Green matrix associated to any regular boundary problem can be obtained in terms of two families of orthogonal polynomials, one of them given by the so-called distance polynomials that are closely related with the intersection array of the graph.

Our study is similar to what is known for boundary value problems associated with ordinary differential equations, [6, Chapters 7,11,12]. The boundary value problems here considered are of two types that correspond to the cases in which the boundary has two vertices. We show that it is possible to obtain

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explicitly such solutions in terms of the chosen orthogonal polynomials. As an immediate consequence of this property, we can easily characterize those boundary value problems that are regular.

2 Definitions and notation

A network $\Gamma = (V, E, c)$ is composed by a set of elements V called vertices, a set of pairs of vertices E called edges, and a symmetric map $c : V \times V \to [0, \infty)$ named conductance, associated to the edges. The order of the network is $n + 1$, the number of its vertices. The Laplacian matrix of the network Γ is the matrix $\mathcal{L} \in \mathcal{M}_{n+1\times n+1}$ whose elements are $(\mathcal{L})_{ij} = -c(i, j)$ for $i \neq j$, $(\mathcal{L})_{ii} = \sum_{j=0}^{n} c(i, j)$ and 0 otherwise, for any $0 \leq i, j \leq n$. The Schrödinger matrix $\mathcal{L}_Q$ on Γ with potential Q is a perturbation of the Laplacian matrix defined as $\mathcal{L}_Q = \mathcal{L} + Q$, where $Q = diag(q_0, \dots, q_n)$, $q_i \in \mathbb{R}$. Through the paper, $\vec{u}$ stands for the $n + 1$–upla $\vec{u} = (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$.

Given $F \subset V$, the Schrödinger equation on F with data $\vec{f}$ is the equation

$$(\mathcal{L}_Q\vec{u}^T)_i = \vec{f}_i^T, \quad i \in F, \quad \vec{u}, \vec{f} \in \mathbb{R}^{n+1}, \tag{1}$$

and the equation $(\mathcal{L}_Q\vec{u}^T)_i = 0$, $i \in F$, $\vec{u} \in \mathbb{R}^{n+1}$ is called the corresponding homogeneous Schrödinger equation on F.

On the other hand, the boundary of F, $\delta(F)$, is the set of vertices of $V \setminus F$ connected to a vertex in F. A linear boundary condition $\vec{\mathcal{B}}$ on $\delta(F)$ is the equation $\vec{\mathcal{B}}\vec{u}^T = g$, where $\vec{\mathcal{B}}$, $\vec{u} \in \mathbb{R}^{n+1}$, $g \in \mathbb{R}$. A two-side boundary value problem on F consist in finding the vector $\vec{u} \in \mathbb{R}^{n+1}$ satisfying the Schrödinger equation and two linear boundary conditions $\vec{\mathcal{B}}_1, \vec{\mathcal{B}}_2 \in \mathbb{R}^{n+1}$ on $\delta(F)$, i.e., $\mathcal{L}_Q\vec{u}^T = \vec{f}$, $\vec{\mathcal{B}}_1\vec{u}^T = g_1$, $\vec{\mathcal{B}}_2\vec{u}^T = g_2$, for a given $\vec{f} \in \mathbb{R}^{n+1}$, g_1, $g_2 \in \mathbb{R}$.

Some of the authors studied in [3] boundary value problems in a path P_{n+1}, considering as $F = \{1, \dots, n-1\}$, with conductances $c(i, i+1) = c(i+1, i)$, and $c(i, j) = 0$ otherwise, for any $0 \leq i, j \leq n$. In this case, a vector $\vec{u} \in \mathbb{R}^{n+1}$ is a solution of the Schrödinger equation with data $\vec{f} \in \mathbb{R}^{n+1}$ on F, iff

$$c(i, i+1)(u_i - u_{i+1}) + c(i, i-1)(u_i - u_{i-1}) + q_i u_i = f_i, \quad 1 \leq i \leq n-1. \tag{2}$$

By using the usual techniques for solving second order difference equations (see for instance [1]), given two solutions $\vec{u}$, $\vec{v} \in \mathbb{R}^{n+1}$ of the homogeneous Schrödinger equation on F, their wronskian is $w[\vec{u}, \vec{v}](i) = u_i v_{i+1} - v_i u_{i+1}$, for any $0 \leq i \leq n-1$, and two solutions $\vec{u}$, $\vec{v} \in \mathbb{R}^{n+1}$ are linearly independent iff their wronskian is not null. The Green matrix $\mathcal{G}_H \in \mathcal{M}_{n+1\times n+1}$ of the homogeneous Schrödinger equation on F is defined as follows:

$$(\mathcal{G}_H)_{ij} = \frac{1}{c(j, j+1)w[\vec{u}, \vec{v}](j)}[u_i v_j - u_j v_i], \quad 0 \leq i, j \leq n,$$

for two linearly independent solutions $\vec{u}$ and $\vec{v}$ of the homogeneous Schrödinger equation.

3 Jacobi matrices in distance-regular graphs

In a previous work [2], the authors define the Schrödinger matrix associated to a family of orthogonal polynomials in a weighted path of $n+2$ vertices, P_{n+2}, which is a Jacobi matrix, and study BVP associated to it. In this work we extend the problem to distance-regular graphs.

Consider a distance-regular graph $\Gamma = (V, E)$ of order n and degree δ. Let $\Gamma_i(u)$ denote the set of vertices of Γ that are at distance i from u, then $k_i = |\Gamma_i(u)|$ and it holds $k_i c_i = k_{i-1} b_{i-1}$, for any $0 \le i \le d$. The intersection matrix of Γ is the non-symmetric Jacobi matrix

$$i(\Gamma) = \begin{pmatrix} a_0 & b_0 & \dots & 0 & 0 \\ c_1 & a_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{d-1} & b_{d-1} \\ 0 & 0 & \dots & c_d & a_d \end{pmatrix},$$

where $a_i + b_i + c_i = \delta$, $0 \le i \le d$, $a_0 = 0$, $c_1 = 1$, $b_0 = \delta$. For a given $0 \le i \le d$, let A_i be the i-distance matrix of the graph, that is, the matrix whose elements $a^i_{uv} = 1$ if the vertices u, v are at distance i. Note that $A_0 = I$ and $A_1 = A$ is the adjacency matrix of the graph. Now recall (see [4]) that in a distance-regular graph these matrices are polynomial, that is, $A_i = p_i(A)$. These polynomials are called the distance polynomials and are a family of orthogonal polynomials, as they satisfy the following recurrence relation

$$p_i(x) = \left(\frac{x}{c_i} - \frac{a_{i-1}}{c_i}\right) p_{i-1}(x) - \frac{b_{i-2}}{c_i} p_{i-2}(x), \quad 1 \le i \le d, \tag{3}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and $p_1(x) = x$.

On the other hand (see [5]), any distance-regular graph Γ can be seen as the covering of a weighted path P_{d+1}, with conductances $c(i, i-1) = b_i$, $c(i, i) = a_i$ and $c(i, i+1) = c_i$, for any $0 \le i \le d-1$, and 0 otherwise. Therefore, considering a vertex $u \in V(\Gamma)$ and the set of vertices of the graph at maximum distance from u, that is $\Gamma_d(u)$, let F' be the rest of the vertices of the graph. To solve a BVP in the boundary $\delta(F')$ of the distance-regular graph is equivalent to solve it in the boundary of the set $F = \{1, \dots, d-1\} \subset V(P_{d+1}) = \{0, \dots, d\}$. Observe that the adjacency matrix of the path is just $i(\Gamma)$, however is not a symmetric matrix. In order to obtain the Schrödinger matrix associated to P_{d+1} (which must be symmetric), we multiply $i(\Gamma)$ on the left side by a diagonal matrix $H = diag(k_0^{-1}, \dots, k_d^{-1})$. We point out that this technique is also possible for Jacobi matrices, because the difference equations associated to them are second order difference equations. The weighted path associated to this matrix is the one having as conductances $c(i, i+1) = c_{i+1}/k_i$, for any $0 \le i \le d-1$.

Therefore, for any $x \in \mathbb{R}$ we define the Schrödinger matrix $\mathcal{L}_Q(x)$ associated to a distance-regular graph, with potential $Q(x) = diag(q_0(x), \dots, q_d(x))$ as the matrix

$$\mathcal{L}_Q(x) = \begin{pmatrix} (x-a_0)/k_0 & -b_0/k_1 & 0 & \dots & 0 \\ -c_1/k_0 & (x-a_1)/k_1 & -b_1/k_2 & \dots & 0 \\ 0 & -c_2/k_1 & (x-a_2)/k_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & & (x-a_d)/k_d \end{pmatrix},$$

where $q_i(x) = \frac{x-a_i}{k_i} - \frac{c_{i+1}}{k_i} - \frac{c_i}{k_{i-1}}$ is the potential of each vertex, for any $0 \le i \le d-1$. Observe that the matrix is symmetric, as it holds $b_{i-1}/k_i = c_i/k_{i-1}$, for $1 \le i \le d-1$.

Now consider the following two families of orthogonal polynomials: the distance polynomials $\{p_n\}_{n=-1}^d$ and $\{r_n\}_{n=-1}^d$, with $r_0(x) = 1$, $r_1(x) = r_{-1}(x) = ax + b$. By choosing as $b_{-1} = 1$, $a = 1/2$ and $b = 0$ in the family $\{r_n\}_{n=-1}^d$, we also consider $x \neq 0$, we get the following result.

Lemma 1. *The vectors $\vec{p} = (p_0(x), \dots, p_d(x))$, $\vec{r} = (r_0(x), \dots, r_d(x)) \in \mathbb{R}^{d+1}$ form a basis of the solution space of the homogenous Schrödinger equation iff $x \neq 0$. Their wronskian is $w[\vec{p}, \vec{r}](n) = x/2$, for any $0 \le n \le d-1$. The Green matrix $\mathcal{G}_H$ of the homogenous Schrödinger equation is determined by*

$$(\mathcal{G}_H)_{ij} = \frac{2}{x}[p_i(x)r_j(x) - p_j(x)r_i(x)], \quad 0 \le i,j \le d, \quad x \in \mathbb{R}. \tag{4}$$

Thus, the general solution $\vec{y}$ of the Schrödinger equation on F with data $\vec{f} \in \mathbb{R}^{d+1}$ is given for any $0 \le i \le d$ by

$$(\vec{y})_i = \alpha p_i(x) + \beta r_i(x) + \sum_{k=1}^{i} (\mathcal{G}_H)_{ij} f_j, \quad \alpha, \beta \in \mathbb{R}.$$

4 Green matrix of two side boundary value problems

In this section we study problems with two side boundary conditions. Recall that a two side boundary value problem on F consists in finding $\vec{u} \in \mathbb{R}^{d+1}$ such that

$$\mathcal{L}_Q \vec{u}^T = \vec{f}^T \quad \text{on } F, \quad \vec{\mathcal{B}_1} \vec{u}^T = g_1, \quad \vec{\mathcal{B}_2} \vec{u}^T = g_2, \tag{5}$$

for given $\vec{f} \in \mathbb{R}^{d+1}$ and $g_1, g_2 \in \mathbb{R}$, where the boundary conditions $\vec{\mathcal{B}_1}$ and $\vec{\mathcal{B}_2}$ are linearly independent, i.e., the rank of the following matrix is 2

$$\begin{pmatrix} \vec{\mathcal{B}_1} \\ \vec{\mathcal{B}_2} \end{pmatrix} = \begin{pmatrix} m_{10} \; m_{11} \; 0 \dots 0 \; m_{1d-1} \; m_{1d} \\ m_{20} \; m_{21} \; 0 \dots 0 \; m_{2d-1} \; m_{2d} \end{pmatrix}.$$

Let μ_{ij} be the determinant of each 2×2 submatrix, $\mu_{ij} = m_{1i}m_{2j} - m_{2i}m_{1j}$ for all $i, j \in B = \{0, 1, d-1, d\}$ and 0 otherwise. Besides $\mu_{ii} = 0$ and $\mu_{ij} = -\mu_{ji}$ for any $i, j \in B$. On the other hand consider the following associated BVP

$$\mathcal{L}_Q \vec{u}^T = \vec{0} \quad \text{on } F, \quad \vec{\mathcal{B}_1} \vec{u}^T = g_1, \quad \vec{\mathcal{B}_2} \vec{u}^T = g_2, \tag{6}$$

$$\mathcal{L}_Q \vec{u}^T = \vec{f}^T \quad \text{on } F, \quad \vec{\mathcal{B}_1} \vec{u}^T = 0, \quad \vec{\mathcal{B}_2} \vec{u}^T = 0, \tag{7}$$

$$\mathcal{L}_Q \vec{u}^T = \vec{0} \quad \text{on } F, \quad \vec{\mathcal{B}_1} \vec{u}^T = 0, \quad \vec{\mathcal{B}_2} \vec{u}^T = 0. \tag{8}$$

The last problem (8) is called homogeneous BVP. The BVP (5) is regular iff the homogeneous BVP (8) has the null vector as its unique solution. It follows by standard arguments that the BVP (5) is regular iff for each $\vec{f} \in \mathbb{R}^{d+1}$ has a unique solution. Moreover the homogeneous BVP problem (8) has a unique solution $\vec{y} = \alpha \vec{v}_1 + \beta \vec{v}_2$, where $\vec{v}_1$, $\vec{v}_2$ are two independent solutions of the homogeneous Schrödinger equation, iff the following matrix is regular

$$\begin{pmatrix} \vec{\mathcal{B}_1} \vec{v}_1 & \vec{\mathcal{B}_1} \vec{v}_2 \\ \vec{\mathcal{B}_2} \vec{v}_1 & \vec{\mathcal{B}_2} \vec{v}_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore the homogeneous BVP problem (8) is regular iff the determinant of this matrix, that is, the boundary polynomial $P_{\mathcal{B}}(x) = \mathcal{B}_1 \vec{v}_1 \cdot \mathcal{B}_2 \vec{v}_2 - \mathcal{B}_2 \vec{v}_1 \cdot \mathcal{B}_1 \vec{v}_2$ is not null, and hence, it also holds that the BVP (5) is regular.

Now by using Lemma(1), we consider $\vec{p} = (p_0(x), \ldots, p_d(x))$ and $\vec{r} = (r_0(x), \ldots, r_d(x))$ two independent solutions of the homogeneous Schrödinger equation. We compute the boundary polynomial in this case

$$P_{\mathcal{B}}(x) = \mathcal{B}_1 \vec{p} \cdot \mathcal{B}_2 \vec{r} - \mathcal{B}_2 \vec{p} \cdot \mathcal{B}_1 \vec{r} = \sum_{i,j \in B} \mu_{ij} p_i(x) r_j(x) = \frac{x}{2} \sum_{\substack{i<j \\ i,j \in B}} \mu_{ij} (\mathcal{G}_H)_{ij}.$$

Furthermore the unique solution for problem (5) can be obtained as the sum of two solutions $\vec{u} = \vec{u}_1 + \vec{u}_2$, respective unique solutions of both problems (6) and (7). Nevertheless, the following lemma shows that any general BVP (5) can be restricted to a semi-homogenous one of type (7).

Lemma 2. *Let $\vec{\mathcal{B}_1}$, $\vec{\mathcal{B}_2}$ be two linear boundary conditions and let $\vec{u}_p \in \mathbb{R}^{d+1}$ such that $\vec{\mathcal{B}_1} \vec{u}_p^T = g_1$, $\vec{\mathcal{B}_2} \vec{u}_p^T = g_2$. Then $\vec{u} \in \mathbb{R}^{d+1}$ is a solution of the boundary value problem $\mathcal{L}_Q \vec{u}^T = \vec{f}$ on F, $\vec{\mathcal{B}_1} \vec{u}^T = g_1$, $\vec{\mathcal{B}_2} \vec{u}^T = g_2$ if and only if $\vec{v} = \vec{u} - \vec{u}_p$ is the solution of the boundary value problem $\mathcal{L}_Q \vec{v} = \vec{f} - \mathcal{L}_Q \vec{u}_p^T$ on F, $\vec{\mathcal{B}_1} \vec{v}^T = 0$, $\vec{\mathcal{B}_2} \vec{v}^T = 0$.*

Therefore, just by considering the vector $\vec{u}_p = \alpha \vec{\varepsilon}_0 - \beta \vec{\varepsilon}_1 - \gamma \vec{\varepsilon}_{d-1} - \delta \vec{\varepsilon}_d$ where $\vec{\varepsilon}_i$ is the i-th caracteristic vector, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, the boundary value problem (5) can be restricted to the semi-homogeneous boundary problem (7) with

$$\mathcal{L}_Q\vec{v}^T = \vec{f} + \left(-\beta\frac{(x-a_1)}{k_1} + \alpha\frac{c_1}{k_0}\right)\vec{\varepsilon}_1 + \beta\frac{c_2}{k_1}\beta\vec{\varepsilon}_2 + \gamma\frac{b_{d-2}}{k_{d-1}}\beta\vec{\varepsilon}_{d-2} + \left(-\gamma\frac{(x-a_{d-1})}{k_{d-1}} + \delta\frac{b_{d-1}}{k_d}\right)\vec{\varepsilon}_{d-1},$$

on F and boundary conditions $\vec{\mathcal{B}_1}\vec{v}^T = \vec{\mathcal{B}_2}\vec{v}^T = 0$. Thus, we focus on solving regular BPV of type (7).

The solution of any regular BVP (7) can be obtained by considering its resolvent kernel, *i.e.*, the matrix $\mathcal{G}_Q(x) \in \mathcal{M}_{V\times F}$ such that fixing a column $s \in F$

$$\sum_{k=0}^{d} \mathcal{L}_Q \cdot (\mathcal{G}_Q(x))_{\cdot s} = \vec{\varepsilon}_s, \quad \vec{\mathcal{B}_1} \cdot (\mathcal{G}_Q(x))_{\cdot s} = \vec{\mathcal{B}_2} \cdot (\mathcal{G}_Q(x))_{\cdot s} = 0.$$

This matrix is named the *Green matrix* for Problem (5). Notice that for any $\vec{f} \in \mathbb{R}^{d+1}$ the unique solution of problem (7) is given by

$$\vec{u}_k = \sum_{s=1}^{d-1} (\mathcal{G}_Q(x))_{ks} \cdot \vec{f}_s,$$

for any $k \in V$.

Theorem 1. *The BVP (5) is regular iff*

$$P_{\mathcal{B}}(x) = \frac{x}{2} \sum_{\substack{i<j \\ i,j\in\{1,\dots,d-1\}}} \mu_{ij}(\mathcal{G}_H(x))_{ij} \neq 0$$

The Green matrix of the BVP problem (5), for any $s \in F$, $k \in V$, is the matrix whose ks-element, $(\mathcal{G}_Q(x))_{ks}$, is given by

$$\frac{x}{2P_{\mathcal{B}}(x)}\left[\frac{k_{d-1}}{c_d}\mu_{d-1d}(\mathcal{G}_H(x))_{sk} + \sum_{i=0}^{1}(\mathcal{G}_H(x))_{ik}\left(\sum_{j=d-1}^{d}\mu_{ij}(\mathcal{G}_H(x))_{sj}\right)\right] + \begin{cases} 0 & k \le s, \\ (\mathcal{G}_H(x))_{ks} & k \ge s. \end{cases}$$

5 Common two side boundary value problems

In what follows we study the more usual boundary value problems appearing in the literature with proper name; that is, unilateral, Dirichlet and Neumann problems, or more generally, Sturm-Liouville problems.

The pair of boundary conditions $(\vec{\mathcal{B}_1}, \vec{\mathcal{B}_2})$ is called *unilateral* if either $m_{1,j} = m_{2,j} = 0$, for any $j \in \{d-1, d\}$ (initial value problem) or $m_{1,i} = m_{2,i} = 0$, for any $i \in \{0,1\}$ (final value problem). Therefore for the initial value problem we have that only $\mu_{01} \neq 0$ and $P_{\mathcal{B}}(x) = \frac{x}{2}\mu_{01}(\mathcal{G}_H)_{01} = -\mu_{01}$, and for the final value problem only $\mu_{d-1,d} \neq 0$ and $P_{\mathcal{B}}(x) = \frac{x}{2}\mu_{d-1d}(\mathcal{G}_H)_{d-1d} = \frac{x}{2}\frac{k_{d-1}}{c_d}\mu_{d-1d}$. Observe that in both cases since the boundary conditions are linearly independent, both unilateral boundary problems are regular. Therefore, any unilateral pair is equivalent to either (u_0, u_1) for initial value problems, or (u_{d-1}, u_d) for final value problems.

Corollary 1. *The Green matrix for the initial boundary value problem is given by*

$$(\mathcal{G}_Q(x))_{ks} = \begin{cases} 0 & k \leq s, \\ (\mathcal{G}_H)_{ks} & k \geq s. \end{cases}$$

Whereas the Green matrix for the final boundary value problem is

$$(\mathcal{G}_Q(x))_{ks} = \frac{2}{xP_{\mathcal{B}}(x)}\frac{k_{d-1}}{c_d}\mu_{d-1d}(\mathcal{G}_H)_{sk} + \begin{cases} (\mathcal{G}_H)_{ks} & k \leq s, \\ 0 & k \geq s, \end{cases}$$

for any $s \in F$, $k \in V$.

The boundary conditions are called *Sturm-Liouville conditions*, when $m_{1j} = m_{2i} = 0$, for $i \in \{0,1\}$, $j \in \{d-1, d\}$; that is, when

$$\vec{\mathcal{B}_1}\vec{u} = au_0 + bu_1 \quad \text{and} \quad \vec{\mathcal{B}_2}\vec{u} = cu_{d-1} + du_d, \tag{9}$$

where $a, b, c, d \in \mathbb{R}$ are such that $(|a| + |b|)(|c| + |d|) > 0$. The most popular Sturm-Liouville conditions are the so-called *Dirichlet boundary conditions*, that correspond to take $b = c = 0$, and *Neumann boundary conditions*, that correspond to take $b = -a$ and $d = -c$.

Corollary 2. *Given $a, b, c, d \in \mathbb{R}$ such that $(|a| + |b|)(|c| + |d|) > 0$ and the Sturm-Liouville boundary conditions, then*

$$P_{\mathcal{B}}(x) = \frac{x}{2}\Big[\big(a+bp_1(x)\big)\big(cr_{d-1}(x)+dr_d(x)\big) - \big(a+br_1(x)\big)\big(cp_{d-1}(x)+dp_d(x)\big)\Big],$$

and the Green matrix for the Sturm-Liouville BVP is for any $0 \leq k \leq s \leq d-1$ and $1 \leq s$; whereas

$$\begin{aligned}(\mathcal{G}_Q(x))_{ks} = \tfrac{2}{xP_{\mathcal{B}}(x)} & \Big[\big(a + bp_1(x)\big)r_k(x) - \big(a + br_1(x)\big)p_k(x)\Big] \\ & \cdot \Big[\big(dr_d(x) + cr_{d-1}(x)\big)p_s(x) - \big(dp_d(x) + cp_{d-1}(x)\big)r_s(x)\Big],\end{aligned}$$

for any $d \geq k \geq s \geq 1$ and $s \leq d-1$.

As a consequence, the boundary polynomial for the Dirichlet problem is

$$P_{\mathcal{B}}(x) = \frac{1}{2}xad\Big[(1+p_1(x)r_d(x)) - (1+r_1(x)p_d(x))\Big],$$

and hence it is regular iff $r_d(x) \neq p_d(x)$, and the Green's matrix is given by

$$\mathcal{G}_Q(x)_{ks} = \begin{cases} \dfrac{\big(p_k(x)-r_k(x)\big)\big(r_d(x)p_s(x)-p_d(x)r_s(x)\big)}{p_1(x)\big(r_d(x)-p_d(x)\big)}, & k \leq s, \\[2ex] \dfrac{\big(p_s(x)-r_s(x)\big)\big(r_d(x)p_k(x)-p_d(x)r_k(x)\big)}{p_1(x)\big(r_d(x)-p_d(x)\big)}, & k \geq s. \end{cases}$$

Finally, for Neumann boundary problem, the boundary polynomial is

$$P_B(x) = \frac{x}{2}ac\Big[\big(1-p_1(x)\big)\big(r_{d-1}(x)-r_d(x)\big) - \big(1-r_1(x)\big)\big(p_{d-1}(x)-p_d(x)\big)\Big]$$

and the ks-element of the Green matrix, $(\mathcal{G}_Q(x))_{ks}$, for the Neumann problem is

$$\frac{\Big[(1-r_1(x))p_k(x)-\big(1-p_1(x)\big)r_k(x)\Big]\Big[r_s(x)\big(p_d(x)-p_{d-1}(x)\big)-p_s(x)\big(r_d(x)-r_{d-1}(x)\big)\Big]}{p_1(x)\Big[\big(1-r_1(x)\big)\big(p_d(x)-p_{d-1}(x)\big)-\big(1-p_1(x)\big)\big(r_d(x)-r_{d-1}(x)\big)\Big]},$$

for any $0 \leq k \leq s \leq d-1$ and $1 \leq s$; whereas

$$\frac{\Big[(1-r_1(x))p_s(x)-\big(1-p_1(x)\big)r_s(x)\Big]\Big[r_k(x)\big(p_d(x)-p_{d-1}(x)\big)-p_k(x)\big(r_d(x)-r_{d-1}(x)\big)\Big]}{p_1(x)\Big[\big(1-r_1(x)\big)\big(p_d(x)-p_{d-1}(x)\big)-\big(1-p_1(x)\big)\big(r_d(x)-r_{d-1}(x)\big)\Big]},$$

for any $d \geq k \geq s \geq 1$ and $s \leq d-1$.

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