On identifying codes in line digraphs

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Extended Abstract

1 Description

We consider simple digraphs without loops or multiple edges. In the line digraph LD of a digraph D, each vertex represents an arc of D. Thus, the set of vertices of LD is $V(LD) = \{uv : (u,v) \in A(D)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$. For any integer $k \geq 1$, the *k-iterated line digraph* $L^k D$ is defined recursively by $L^k D = LL^{k-1} D$, where $L^0 D = D$.

A large known family of digraphs obtained with the line digraph technique is the family of Kautz digraphs. The *Kautz digraph* of degree d and diameter k is defined as the $(k-1)$-iterated line digraph of the symmetric complete digraph of $d+1$ vertices K_{d+1}, that is, $K(d,k) \cong L^{k-1} K_{d+1}$.

For a given integer $\ell \geq 1$, a vertex subset C of the set of vertices $V(D)$ is a $(1, \leq \ell)$-*identifying code* in the digraph D if it is a dominating set and for all distinct subsets $X, Y \subset V(D)$, with $1 \leq |X|, |Y| \leq \ell$, we have

$$N^-[X] \cap C \neq N^-[Y] \cap C. \quad (1)$$

A $(1, \leq 1)$-identifying code is known as an *identifying code*.

In [3], Charon, Hudry and Lobstein proved that, given an oriented graph D and an integer k, the decision problem of the existence of a $(1, \leq 1)$-identifying code of size at most k in D is NP-complete, even when we are restricted to strongly connected, oriented, and bipartite digraphs without cycles.

In [1], the authors studied the $(1, \leq \ell)$-identifying codes in digraphs. They proved that if D is a digraph admitting a $(1, \leq \ell)$-identifying code, then $\ell \leq \min\{d^-(u) + 1 \mid u \in V(D) \text{ and } d^+(u) \leq 1\}$ and gave some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 1$ to admit a $(1, \leq \ell)$-identifying code for $\ell = \delta^-, \delta^- + 1$. They also gave in [2] an upper bound on ℓ for graphs and digraphs. In this work, we study $(1, \leq \ell)$-identifying codes in line digraphs, where $\ell \geq 1$ is an integer.

2 Results

Line digraphs were characterized by Heuchenne's condition: A digraph D is a line digraph if and only if it has no multiple arcs, and for any pair of vertices u and v, either $N^-(u) \cap N^-(v) = \emptyset$ or $N^-(u) = N^-(v)$. Using this characterization we prove the next proposition.

Proposition 1 *The line digraph of a strongly connected digraph of order at least* 3 *admits an identifying code.* □

In the following theorem, we give sufficient and necessary conditions for a line digraph to admit a $(1, \leq 2)$-identifying code.

Theorem 2 *Let LD be a line digraph different from a 4-cycle and such that the vertices of in-degree 1 (if any) does not lay on a digon. Then, LD admits a* $(1, \leq 2)$*-identifying code if and only if LD satisfies the following conditions:*

(i) There are no 3-cycles with at least 2 vertices of in-degree 1;

(ii) There do not exist four vertices x, x', y and y' such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$;

(iii) There do not exist two vertices $x, y \in V(LD)$ such that $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$, and $N^-(x') \cap N^-(y') \neq \emptyset$.

Corollary 3 *Let L^kD be a k-iterated line digraph with minimum in-degree $\delta^- \geq 2$.*

(i) If $k \geq 2$, then L^kD admits a $(1, \leq 2)$-identifying code.

(ii) If $k = 1$ and $\delta^- \geq 3$, then LD admits a $(1, \leq 2)$-identifying code.

Corollary 4 *For each $n \geq 3$, the Kautz digraph $K(n, 2) = LK_{n+1}$ admits a $(1, \leq 2)$-identifying code.*

By Corollary 3 (iii), the Kautz digraph $K(2, 2)$ is isomorphic to LK_3. Then, the condition $k \geq 2$ in Corollary 3 (i) is necessary.

Proposition 5 *Let LD be a line digraph with minimum in-degree $\delta^- \geq 2$, then LD does not admit a $(1, \leq 3)$-identifying code.*

Foucaud, Naserasr, et al. [4] characterized the digraphs that only admit as identifying code the whole set of vertices. As a consequence, if LD is a line digraph with minimum in-degree $\delta^- \geq 2$, then $\overrightarrow{\gamma}^{ID}(LD) \leq |V(LD)| - 1$, where $\overrightarrow{\gamma}^{ID}(D)$ denotes the minimum size of an identifying code of a digraph D. Next, we establish better upper bounds on $\overrightarrow{\gamma}^{ID}(LD)$.

For each vertex $v \in V(D)$, we denote $\omega^-(v) = \{(u, v) \in A(D)\}$ and $\omega^+(v) = \{(v, u) \in A(D)\}$.

Definition 6 *Given a digraph D, a subset C of $A(D)$ is an* arc-identifying code *of D if C satisfies both conditions:*

- *an arc-dominating set of D, that is, for each arc $uv \in A(D)$, $(\{uv\} \cup \omega^-[u]) \cap C \neq \emptyset$, and*
- *an arc-separating set of D, that is, for each pair $uv, wz \in A(D)$ $(uv \neq wz)$, $(\{uv\} \cup \omega^-[u]) \cap C \neq (\{wz\} \cup \omega^-[w]) \cap C$.*

Hence, a line digraph LD admits a $(1, \leq \ell)$-identifying code if and only if D admits a $(1, \leq \ell)$-arc-identifying code. As a consequence, the minimum size of an identifying code of a digraph D, $\overrightarrow{\gamma}^{ID}(LD)$, is equivalent to the minimum size of an arc-identifying code of its line digraph LD.

Let D be a digraph. We denote $V^+_{\geq 2}(D) = \{v \in V(D) : d^+(v) \geq 2\}$, and $V^+_1(D) = \{v \in V(D) : d^+(v) = 1\}$. Hence, in particular, if D is a strongly connected digraph, $V(D) = V^+_1(D) \cup V^+_{\geq 2}(D)$.

Theorem 7 *Let D be a strongly connected digraph with minimum in-degree $\delta^- \geq 2$. Then,*

$$\overrightarrow{\gamma}^{ID}(LD) \geq |A(D)| - |V(D)|.$$

Theorem 8 *Let D be a strongly connected digraph of order at least 3, and let $C \subseteq A(D)$. Then, C is an arc-identifying code of D if and only if C satisfies the following conditions:*

(i) For all $v \in V(D)$, $|\omega^+(v) \setminus C| \leq 1$, and if $|\omega^+(v) \setminus C| = 1$, then $\omega^-(v) \cap C \neq \emptyset$;

(ii) For all $uv \in C$, if $vu \in C$ or $|\omega^+(v) \setminus C| = 1$, then $((\omega^-(v) \cup \omega^-(u)) \setminus \{uv, vu\}) \cap C \neq \emptyset$.

Now we present an algorithm for constructing an arc-identifying code of a strongly connected oriented (that is, without digons) graph with minimum in-degree $\delta^- \geq 2$.

Algorithm 9 *Constructing an arc-identifying code C of a strongly connected digraph D with minimum in-degree $\delta^- \geq 2$ and without digons.*

1: *Let* $U^- := \{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}$, $U := \emptyset$ *and* $C := \emptyset$
2: ***while*** $U^- \setminus U \neq \emptyset$ ***do***
3: *let* $v \in U^- \setminus U$ *and* $f \in N^-(v)$
4: *replace* U *by* $U \cup \{v\}$ *and* C *by* $C \cup \{fv\}$
5: ***end while***
6: *let* $X := V_1^+(D)$ *and* $Y := U^-$
7: *let* $xy \in A(D)$ *such that* $x \in V(D) \setminus X$ *and* $y \in V(D) \setminus Y$
8: *replace* Y *by* $Y \cup (N^+(x) \setminus \{y\})$, X *by* $X \cup \{x\}$ *and* C *by* $C \cup (\omega^+(x) \setminus \{xy\})$
9: ***while*** $Y \neq V(D)$ ***do***
10: ***while*** $N^-(y) \setminus X \neq \emptyset$ ***do***
11: *let* $t \in N^-(y) \setminus X$ *and let* $z \in N^+(t) \setminus \{y\}$
12: *replace* Y *by* $Y \cup (N^+(t) \setminus \{z\})$, X *by* $X \cup \{t\}$, C *by* $C \cup (\omega^+(t) \setminus \{tz\})$, t *by* x *and* z *by* y
13: ***end while***
14: ***if*** $N^-(y) \setminus X = \emptyset$ ***then***
15: *choose an arc* uv *of* D *such that* $v \notin Y$
16: *replace* Y *by* $Y \cup (N^+(u) \setminus \{v\})$, X *by* $X \cup \{u\}$, C *by* $C \cup (\omega^+(u) \setminus \{uv\})$, u *by* x *and* v *by* y
17: *return to 3*
18: ***end if***
19: ***end while***
20: ***if*** $Y = V(D)$ ***then***
21: ***while*** $X \neq V(D)$ ***do***
22: *let* $u \in V(D) \setminus X$ *and let* $v \in N^+(u)$
23: *replace* C *by* $C \cup (\omega^+(u) \setminus \{uv\})$, X *by* $X \cup \{u\}$
24: ***end while***
25: ***end if***
26: *return* C

Theorem 10 *Let D be an oriented and strongly connected graph with minimum in-degree $\delta^- \geq 2$. Then, Algorithm 9 produces a subset $C \subset A(D)$ with*

$$|C| = |A(D)| - |V(D)| + |\{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}|,$$

satisfying the requirements of Theorem 8.

As a consequence of Theorems 7 and 10, we can conclude the following corollary.

Corollary 11 *Let D be a strongly connected oriented graph with minimum in-degree $\delta^- \geq 2$. Then, the following assertions hold.*

(i) $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)| + |\{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}|$ *if* $\delta^+ = 1$*;*

(ii) $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$ *if* $\delta^+ \geq 2$.

Next, we also present a result for all Hamiltonian digraphs of minimum degree at least two, not necessarily oriented.

Theorem 12 *Let D be a Hamiltonian strongly connected digraph with minimum in-degree $\delta^- \geq 3$ and out-degree $\delta^+ \geq 2$. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.*

Moreover, using Theorem 12 and the 1-factorization of Kautz digraphs obtained by Tvrdík [5], we conclude the following result.

Theorem 13 *The identifying number of a Kautz digraph $K(d,k)$ is $\gamma^{ID}(K(d,k)) = d^k - d^{k-2}$ for $d \geq 2$ and $k \geq 2$.*

References

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